

# EXTREME VALUE THEORY: POTENTIAL AND LIMITATIONS AS AN INTEGRATED RISK MANAGEMENT TOOL

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ABSTRACT. Extreme Value Theory (EVT) is currently very much in the focus of interest in quantitative risk management. Originally conceived as the mathematical (probabilistic/statistical) theory for analysing rare events, it recently entered the risk management stage. In this paper I discuss some of the issues (mainly, but not exclusively) related to Value-at-Risk methodology. I try to come up with a virtues versus limitations assessment, both from an academic as well as from an end-user point of view.

## 1. INTRODUCTION

Without any doubt, Value-at-Risk (VaR) thinking has revolutionised Integrated Risk Management (IRM), both at the quantitative (obvious) and at the

qualitative (not so obvious) level. Originally conceived as a one-number summary of (short term) Market Risk, it is now being used in many different risk management systems like Credit Risk (Credit-VaR) and Operational Risk. Even the insurance world which could claim, through its actuarial skills, to be the master of risk, has imported VaR methodology in its more recent Asset-Liability platforms (as there are Dynamic Solvency Testing, Dynamic Financial Analysis, Risk Adjust Capital, ...). An yet, the VaR notion is an extremely simple one: suppose we have a random variable  $X$  denoting the presently unknown, future random realisation of a financial position, then the risk for the holder of that position is summarised through (minus) the  $\alpha$ -quantile  $VaR_\alpha(X)$  of the (again unknown) distribution function  $F_X(x)$ . Therefore:

$$VaR_\alpha(X) = -F_X^{-1}(\alpha), \quad (1)$$

where  $F_X^{-1}$  denotes the inverse function of  $F_X$  (this may not be uniquely defined for non-strictly increasing  $F_X$ , but in those cases, a slight modification yields an unambiguous definition; see Embrechts, Klüppelberg and Mikosch [6], p. 130). In Figure 1 we give a graphical presentation of (1).

In practice,  $F_X$  is referred to as the Profit-and-Loss (P&L) distribution function. Its calculation (better said: estimation) for market risk can be performed by essentially three methods: historical, analytical, Monte Carlo. There are of course numerous practical problems to be overcome before any relatively clean P&L can be presented, and hence VaR be calculated. Besides these, time scaling (the famous, or better said, infamous  $\sqrt{t}$ -rule) may be brought in to scale VaR from a 1-day to a  $t$ -day holding period. Regulators will typically insist on a 10-day period and a 99% confidence (i.e.  $\alpha = 0.01$ ) for market risk. Precisely the latter, low level of  $\alpha$  is the key which opens the door through which Extreme Value Theory (EVT) walks on the IRM stage. IRM (with or without VaR) is concerned about the estimation of rare events: a 1 in 100 event in the  $\alpha = 0.01$  case. Companies (such as banks) have a great interest in estimating these (and

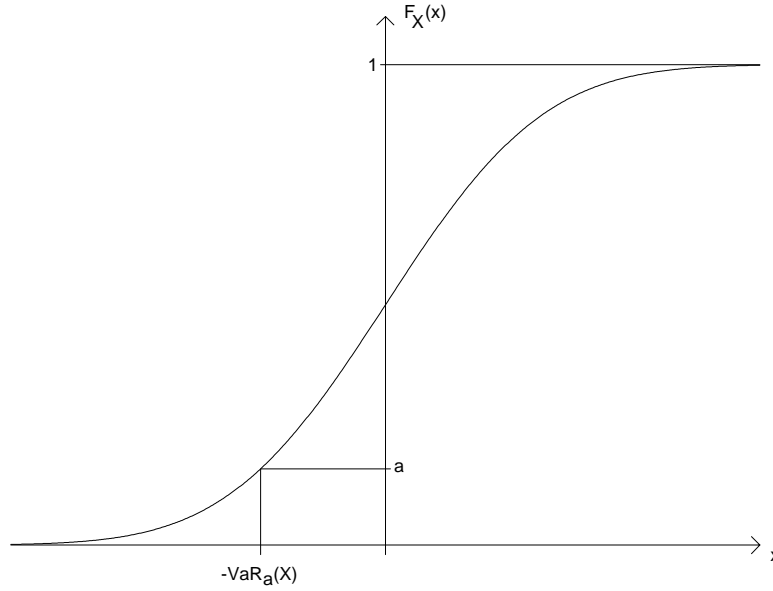


FIGURE 1. Value-at-Risk as the left  $\alpha$ -quantile of the P& L distribution function  $F_X$

other) risk measures well as they immediately translate into regulatory capital requirements. For instance, the regulatory (or market risk) capital for a bank using the so-called model approach equals:

$$MRC_t = \max \left\{ VaR^{t-1} + d_t ASR_{t-1}^{VaR}, \right. \\ \left. m_t \frac{1}{60} \sum_{i=1}^{60} VaR^{t-1} + d_t \frac{1}{60} \sum_{i=1}^{60} ASR_{t-1}^{VaR} \right\} \quad (2)$$

where  $VaR^s$  means the Value-at-Risk for day  $s$ ,  $d_t$  is a 0-1 indicator function related to the estimation of specific risk, the latter measured through an Additional Specific Risk measure which is VaR based. The multiplier  $m_t$  ( $\geq 3$ ) stands for the hotly disputed stress (or hysteria) factor. Formula (2) is currently in use within the Swiss Federal Banking Committee (EBK); see Jovic [12] for more details and further references. A standard text on VaR is Jorion [11]. In (2),

one very clearly sees the direct influence of VaR on regulatory capital. A similar discussion could have been given for the notion of Credit-VaR where extreme quantile estimates occur as Exceptional Losses (beyond the so-called Unexpected and Expected Losses).

## 2. VAR: SOME CRITICAL REMARKS

Whereas the development of quantitative risk measures, like VaR, over the past five years has been near to stormy, only fairly recently have academics and risk managers started asking critical questions about its concept. Some of the issues discussed are:

- Can VaR be used to allocate capital? This question is much related to the non-subadditivity of VaR for non-elliptical portfolios. See for instance the work on coherent risk measures by Artzner et al. [1] and the related discussion in Embrechts et al. [7].
- How can one optimally estimate VaR, including the calculation of confidence intervals for VaR?
- How to properly scale VaR, especially over (much) longer holding periods (a month to a year say)?
- How to model VaR dynamically in a stochastic volatility or regime-switching environment; see for instance McNeil and Frey [14] for a discussion on the former?
- Give improved risk measures beyond VaR: a standard example being tail conditional VaR:

$$E(-X \mid X \leq -\text{VaR}_\alpha(X)) . \tag{3}$$

The latter measure yields a non-trivial improvement on VaR for skew (typically non-multivariate normal) risks.

But independent from the VaR-discussion, other issues have recently emerged:

- How to handle financial risk in a truly multivariate way.
- How to measure dependence (not just linear correlation) in finance and insurance. A key issue, which is discussed in detail in Embrechts, McNeil and Straumann [7].
- How do securities behave in situations of market stress?
- What is sound methodology for stress scenario testing?
- And after all quantitative work is done, do we finally have a safer, more stable world financial system? For a short, critical discussion on this, see [4].

This is by no means a complete list of relevant IRM issues. The main point in raising them however is that EVT plays an important role in a methodological as well as practical discussion of each single one of them. Let me end this brief discussion on VaR with a non-technical version of what we termed the Fundamental Theorems of Risk Management in Embrechts, McNeil and Straumann [8], full details are to be found in [7]. The notion of a coherent risk measure comes from Artzner et al. [1].

**The First Fundamental Theorem of IRM:** For elliptically distributed portfolios, VaR is a sub-additive risk measure. The mean-variance optimal portfolio is equivalent to the optimal portfolio with respect to any other coherent risk measure. Hence VaR is fine! □

For those readers not familiar with elliptical distributions, think of densities taking constant value on ellipsoids, e.g. the multivariate normal and  $t$  distributions. However, a much more important result (and again one which can only be fully understood with sufficient EVT knowledge) is:

**The Second Fundamental Theorem of IRM:** For non-elliptically distributed portfolios, most (if not all) of the standard results from the previous theorem do not hold, in particular, VaR is more than questionable. □

Examples where non-ellipticity occurs is for P&Ls having with very small probability, large tail-losses, or indeed very asymmetric P&Ls as typically occur

in credit RM. Let me just give one example from [7] from the realm of the Second Fundamental Theorem. That linear correlation lies at the heart of portfolio theory, nobody will deny (see for instance CAPM, ATP, ...). However, at the same time, linear correlation

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} \quad (4)$$

is a ubiquitous measure of dependence in modern finance and insurance which is not always well understood by practitioners, and its use is problematic in certain situations. The following quotes from the *Economist* are telling:

- Quote 1 (*The Economist*, 8th November 1997): “Among nine big economies, stock market correlations have averaged around 0.5 since the 1960s. In other words, for every 1 percent rise (or fall) in, say, American share prices, share prices in other markets will typically rise (fall) by 0.5 percent.”

To which as a reaction came:

- Quote 2 (*The Economist* (letter), 22nd November 1997): “A correlation of 0.5 does not indicate that a return from stockmarket A will be 50% of stockmarket B’s return, or vice versa ... A correlation of 0.5 shows that 50% of the time the return of stockmarket A will be positively correlated with the return of stockmarket B, and 50% of the time it will not.”

Hence the “... not always well understood ...” above. Another fallacy which often creeps up in various disguises is the statement (belief) that VaR for a position  $X + Y$  say (among all possible portfolios  $(X, Y)$ ) is largest for  $(X, Y)$  with maximal correlation (the so-called comonotonic case). Again this is true in the elliptical world, but not in the more realistic non-elliptical one. See [7] for an example and further discussion.

### 3. EVT: VIRTUES AND LIMITATIONS

The claim made above is that EVT offers an interesting (in many ways, the correct) view on the quantitative measurement of risk, especially for skew P&Ls (corresponding to non-elliptical portfolios). Of course, this sentence as it stands is too vague. In [6] the basic one-dimensional theory was summarised with a view towards finance and insurance. From their introduction (p. 6) we quote: “Though not providing a risk manager in a bank with the final product he or she can use for monitoring financial risk on a global scale, we will provide that manager with stochastic methodology needed for the construction of various components of such a global tool”. I will not attempt to present EVT in “a short summary” form; see [6], [13], [9] and the references therein. The key point is that EVT gives the theory for describing extremes (maxima, minima, longest runs, longest time, ...) of random phenomena. In its easiest form, it yields the canonical theory for the (limit) distribution of normalised maxima of independent, identically distributed random variables

$$M_n = \max(X_1, \dots, X_n) . \quad (5)$$

This theory stands in contrast to the theory of averages (the normal world, Brownian motion, thinking about quantiles as multiples of standard deviations, ...) which is about

$$S_n = X_1 + \dots + X_n .$$

Once the right limit results for (5) have been worked out, estimation of extremal events (as for instance presented under Section 2) can be obtained; see again the above references for details. You may look at [5] for a discussion of multivariate extreme value theory. In its more advanced form, EVT describes the behaviour of extremal events for stochastic processes, evolving dynamically in time and space. The main virtue of EVT is that it gives the user a critical view on and a methodological toolkit for issues like skewness, fat tails, rare events, stress

scenarios, .... There should be no discussion on this; the following two quotes (see [6], p. VII) precisely encapsulate these virtues. Richard Smith: “There is always going to be an element of doubt, as one is extrapolating into areas one doesn’t know about. But what EVT is doing is making the best use of whatever data you have about extreme phenomena”. Jonathan Tawn: “The key message is that EVT cannot do magic – but it can do a whole lot better than empirical curve-fitting and guesswork. My answer to the sceptics is that if people aren’t given well-founded methods like EVT, they’ll just use dubious ones instead.”

At the same time, all experts on EVT will point at its limitations; the general ones as in the statements above, or the more technical ones, a sample of which is listed below:

- In order to estimate way in the tails (beyond or at the limit of available data) one has to make mathematical assumptions on the tail model. These assumptions are very difficult (if at all) to verify in practice. Hence there is intrinsic model risk.
- Even for a (standard) EVT-VaR estimation, one has to set the “optimal” threshold above which the data are to be used for tail estimation. There is no canonical “optimal” choice!
- Non-linearities can ruin the heavy-tailed modeler’s day (Resnick [15]).
- Handling extremes for high dimensional portfolios is difficult (the curse of dimensionality); see [5] for a start.

And there are more: they are all true, but should be viewed in the light of the virtues alluded to above (the Smith, Tawn statements). See also [3] for an interesting discussion.

My summary would be: **if IRM is interested in the analysis of rare events, then EVT will play a small, though important role.**

So far I have restricted my discussion to IRM proper; of course in the construction of new derivatives, extremes may enter very naturally, and as such, an



understanding of the laws governing them becomes important. Some examples are:

- Options where the payout is a function of the largest value of an underlying security over a given period of time.
- Options triggered whenever a well specified extreme move (within one day, say) takes place.
- Alternative Risk Transfer products as there are catastrophe bonds.

In products of the above type, in some cases one may derive explicit analytic formulae, for others (most of them), careful simulation is called for. The word “careful” I chose because rare (or extreme) event simulation is an art on its own. By definition of “rare” we may need many runs to actually produce some of these events. The interested reader should consult Asmussen [2] and the references therein on the rare event simulation issue.

#### 4. CONCLUSION

Rather than reiterating (some of) the points made above, or delving deeper in some of the references, I would like to quote from someone from the banking industry who used EVT. In [10] John Gavin writes:

- **Conclusions: what EVT has to offer global risk management.**
  - **Complements** VaR model and Risk Factor Loss limits: exploits results from existing VaR model.
  - Is **Consistent** between risk factors and across markets, unlike scenario analysis.
  - **Smoothness**: avoids coarseness/bias in the tail estimates.
  - **Extrapolation**: can produce confidence intervals e.g. beyond 99% VaR; a “dangerous job but someone has to do it”.
  - Yields a useful **Risk language** for selling risk/probability concepts at all levels, “one day in a thousand”, “1 in 7 year event”.

- Yields a **Generic** approach towards operational risk.
- Both theoretical and computational **Tools** are available.
- **Context**: EVT alone is not enough, extra information is needed to add context.

— **Conclusions: the pitfalls**

- **EVT is not**:
  - \* A panacea for risk management, there are several theoretical issues that are unresolved, e.g. multiple dependent risk factors.
  - \* A way to blindly generate larger VaR numbers than those produced by historical simulation.
  - \* Going to revolutionise risk management.
- **EVT** implicitly assumes extreme losses are no worse than 10+ years ago.
- **Computation**
  - \* **Speed**: computation is fine for a few hundred time series at a time.
  - \* **Stability**: convergence of (Maximum Likelihood) estimated parameters is not guaranteed. Alternative estimators are available (method of moments, or other).
  - \* Might require **Monte Carlo simulations** when applied to portfolios. Parametric bootstrapping could be considered, but turns out to be computationally expensive.

Though the above may be a snapshot by a user of EVT in practice (based on an equity example in which the setting of consistent shock levels was the issue), it clearly shows how much EVT is (can be) alive within the wider IRM framework. I could take issue with some of the conclusions voiced above; however, I do agree with the overall (broader) assessment, and hence want to leave it at that. Only by real examples/uses as reported in Gavin [10], will we be able to find out where the real virtues and limitations within IRM lie.

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