Extreme VaR scenarios in higher dimensions

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Abstract

The dependence scenario yielding the worst possible Value-at-Risk at a given level α for $X_1 + \cdots + X_n$ is known for $n = 2$. In this paper we investigate this problem for higher dimensions. We provide a geometric interpretation highlighting the shape of the dependence structures which imply the worst possible scenario. For a portfolio $(X_1, \ldots, X_n)$ with given uniform marginals, we give an analytical solution sustaining the main result of Rüschendorf (1982). In general, our approach allows for numerical computations.

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1 Introduction

For a portfolio $(X_1, \ldots, X_n)$ of risks with given marginal distributions, we consider the problem of finding the worst possible Value-at-Risk at the level α for $X_1 + \cdots + X_n$, which we refer to as $\mathrm{wVaR}_\alpha(\sum_{i=1}^n X_i)$. This question has been widely studied in the literature, often formulated in terms of the best possible lower bound for the distribution function of the sum; see for instance Section 6.2 in McNeil et al. (2005) and references therein. In risk management this question is motivated by the fact that the worst-case scenario does not occur under comonotonic dependence; see Fallacy 3 in Embrechts et al. (2002). We do not emphasize this issue further. Recent publications on this subject, which also widely discuss the role of comonotonicity, are Denuit et al. (2005), Embrechts et al. (2003), Embrechts and Puccetti (2006) and Embrechts et al. (2005), where the problem is considered for non-decreasing

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functionals. While the above cited papers provide bounds on $\mathrm{wVaR}_\alpha(\sum_{i=1}^n X_i)$ and fully explain the two-dimensional situation finding a worst dependence scenario in terms of copulas, they all fail to catch the nature of the copula solving the problem in higher dimensions. In this paper we describe this extreme dependence scenario extending some geometrical arguments introduced in Embrechts et al. (2005) for $n = 2$. This allows us to numerically answer the question at hand and, for uniform marginal distributions, to provide an analytical solution equivalent to that presented in Rüschendorf (1982). The latter is the only known analytical result for continuous marginals. Some applications of our results are given in Section 4.

2 Preliminaries and fundamental results

We briefly summarize the basic tools used in the literature and recall the fundamental results on the problem of bounding the Value-at-Risk. All the theorems are formulated for the sum of risks assuming no information about their interdependence. For further discussions regarding more general functionals and the assumption of partial dependence information, we refer to the papers cited in the introduction.

2.1 Value-at-Risk and copulas

For risk management purposes we assume $X_1, \ldots, X_n$ to have distribution functions $F_1, \ldots, F_n$ with losses represented in their right tails.

Definition 1 *Let X be a random variable with distribution F_X. For $0 < \alpha < 1$ the Value-at-Risk at probability level α of X is its α-quantile, i.e.* $\mathrm{VaR}_\alpha(X) := F_X^{-1}(\alpha) := \inf\{x \in \mathbb{R} \colon F_X(x) \geq \alpha\}$.

In risk management applications, typical values for α are 0.95 or 0.99 in the case of market or credit risk and $\alpha = 0.999$ for operational risk.

Given the joint distribution function $F(\boldsymbol{x}) = \mathbf{P}(X_1 \leq x_1, \ldots, X_n \leq x_n), \boldsymbol{x} \in \mathbb{R}^n$, the problem of calculating $\mathrm{VaR}_\alpha(\sum_{i=1}^n X_i)$ reduces to a computational issue. In what follows we assume full knowledge about the marginals but no prior information on the dependence structure. In this context, the idea of copula allows for a precise formulation of the problem separating F into one part describing the dependence structure and another part containing the information on the marginals. We refer to Nelsen (1999) for the basic results about copulas.

Definition 2 *An n-dimensional copula C is a distribution function restricted to $[0,1]^n$ with uniform-$(0,1)$ marginals. We denote their class by $\mathfrak{C}^n$.*

Remark 1 A copula can be equivalently defined as a function $C : [0,1]^n \to [0,1]$

satisfying $C(u_1,\dots,u_{i-1},0,u_{i+1},\dots,u_n)=0$, $C(1,\dots,1,u_i,1,\dots,1)=u_i$ and $\sum_{i_1=1}^2\cdots\sum_{i_n=1}^2(-1)^{i_1+\cdots+i_n}C(u_{1,i_1},\dots,u_{n,i_n})\geq 0$ for $\boldsymbol{u},\boldsymbol{v}\in[0,1]^n$ with $\boldsymbol{u}\leq\boldsymbol{v}$ and $u_{j,1}=u_j$, $u_{j,2}=v_j$, $j=1,\dots,n$.

Sklar's Theorem yields that, for a $C\in\mathfrak{C}^n$ and marginals $F_1,\dots,F_n$, the function $F(\boldsymbol{x}):=C(F_1(x_1),\dots,F_n(x_n))$ is a distribution with these marginals. Conversely, for any joint distribution function with given marginals, there is a copula linking them. It is unique if the marginals are continuous. Any copula C lies between the so called lower and upper Fréchet bounds $W(\boldsymbol{u}):=(\sum_{i=1}^n u_i-n+1)^+$ and $M(\boldsymbol{u}):=\min_{1\leq i\leq n}u_i$ implying countermonotonic (if $n=2$), respectively comonotonic dependence for the coupled random variables. Taking $\Pi(\boldsymbol{u}):=\Pi_{i=1}^n u_i$ we obtain independence. Finally, we want to stress that the lower Fréchet bound is not a copula for $n\geq 3$.

2.2 Bound on wVaR and known optimality results

Let F^- denote the left-continuous version of a distribution function F, i.e. $F^-(x)=\mathbf{P}(X<x)=F(x-)$. For $C\in\mathfrak{C}^n$, univariate distributions $F_1,\dots,F_n$ and $s\in\mathbb{R}$ we define

$$\sigma_{C,+}(F_1,\dots,F_n)(s):=\int_{\left\{\sum_{i=1}^n x_i<s\right\}}dC(F_1(x_1),\dots,F_n(x_n)),$$
$$\tau_{C,+}(F_1,\dots,F_n)(s):=\sup_{x_1,\dots,x_{n-1}\in\mathbb{R}}C\left(F_1(x_1),\dots,F_{n-1}(x_{n-1}),F_n^-\left(s-\sum_{i=1}^{n-1}x_i\right)\right),$$

where $\sigma_{C,+}(F_1,\dots,F_n)(s)=\mathbf{P}(X_1+\cdots+X_n<s)$ for a portfolio $(X_1,\dots,X_n)$ with marginals $F_1,\dots,F_n$ and copula C. The following result yields distributional bounds for $\sigma_{C,+}(F_1,\dots,F_n)(s)$ and $\mathrm{VaR}_\alpha(\sum_{i=1}^n X_i)$ when no information about the underlying dependence structure is available. A more general version can be found in Embrechts et al. (2003, Theorem 3.1) and Embrechts and Puccetti (2006, Theorem 3.1), where results are given for non-decreasing functionals in the presence of partial information.

Proposition 1 *Let $(X_1,\dots,X_n)$ have marginals $F_1,\dots,F_n$ and copula $C\in\mathfrak{C}^n$. Then for every real s and every $\alpha\in(0,1)$ we have that*

$$\sigma_{C,+}(F_1,\dots,F_n)(s)\geq\tau_{W,+}(F_1,\dots,F_n)(s)\tag{1}$$

implying $\mathrm{VaR}_\alpha\left(\sum_{i=1}^n X_i\right)\leq\mathrm{wVaR}_\alpha\left(\sum_{i=1}^n X_i\right)\leq\tau_{W,+}(F_1,\dots,F_n)^{-1}(\alpha)$.

Note that for practical applications, the F_i's are assumed to be known but C is unknown. A long history exists about the sharpness of these bounds. Makarov (1981) provided the first result for the sum of two random variables. Later, using a geometric approach, Frank et al. (1987) restated the result using the copula language.

The pointwise best possible nature of the bounds in the two-dimensional case was finally proved in Williamson and Downs (1990) for non-decreasing functionals. Below we reformulate their optimality theorem for the sum. More historical references can be found in the introduction of Embrechts and Puccetti (2006).

Proposition 2 *Let (X_1, X_2) have marginal distributions F_1, F_2 and define $C_{\tilde{\alpha}} \in \mathfrak{C}^2$ for $\tilde{\alpha} \in [0, 1]$ as*

$$C_{\tilde{\alpha}}(u_1, u_2) := \begin{cases} \max\{\tilde{\alpha}, W(u_1, u_2)\} & \text{if } (u_1, u_2) \in [\tilde{\alpha}, 1]^2, \\ M(u_1, u_2) & \text{otherwise}. \end{cases} \tag{2}$$

Then, choosing $\tilde{\alpha} = \alpha(s) := \tau_{W,+}(F_1, F_2)(s)$, we obtain $\sigma_{C_{\tilde{\alpha}},+}(F_1, F_2)(s) = \alpha(s)$. Hence, for any $\alpha \in (0, 1)$, $\mathrm{wVaR}_\alpha (X_1 + X_2) = \tau_{W,+}(F_1, F_2)^{-1}(\alpha)$ is attained under $C_{\tilde{\alpha}}, \tilde{\alpha} = \alpha$.

Remarks 2

(a) Observe that, given some $C_{\mathrm{L}} \in \mathfrak{C}^2$, a similar result holds assuming partial information $C \geq C_{\mathrm{L}}$ on the unknown copula C and substituting $W(u_1, u_2)$ by $C_{\mathrm{L}}(u_1, u_2)$.
(b) Taking $C_{\mathrm{L}}(\boldsymbol{u}), \boldsymbol{u} \in [0, 1]^n, n \geq 3$ instead of $C_{\mathrm{L}}(u_1, u_2)$, (2) is not a copula. In the no information case, this immediately follows from the fact that the lower Fréchet bounds is not a copula for $n \geq 3$. In the presence of partial information, we refer to the example by Geiss and Päivinen reported in Embrechts and Puccetti (2006).

Without mentioning the idea of copulas, Rüschendorf (1982) gave the same result stated by Frank et al. (1987) extending it for the sum of n uniform random variables.

Proposition 3 *The best possible lower bound on the distribution of $\sum_{i=1}^n X_i$ with $(X_1, \ldots, X_n)$ having standard uniform marginals is $\min\{(2s/n - 1)^+, 1\}$ for $s \in (0, n)$. This implies $\mathrm{wVaR}_\alpha \left(\sum_{i=1}^n X_i\right) = n(1 + \alpha)/2$ for $\alpha \in (0, 1)$.*

Till now, this and a similar expression for binomial marginals are the only known analytical results for the multidimensional problem.

3 Worst Value-at-Risk scenarios for the multidimensional problem

The above result of Rüschendorf (1982) provides sharpness of the bounds for the n-dimensional problem for uniform marginals. An analytical generalization of (2), replacing $W(u_1, u_2)$ by $W(\boldsymbol{u})$, $\boldsymbol{u} \in [0, 1]^n$, $n \geq 3$, does not lead to sharp bounds for the multidimensional case. Below we take a more geometric approach. From this point of view, the problem at hand consists in maximizing the probability of the set $G_s := \{\boldsymbol{x} \in \mathbb{R}^n : x_1 + \cdots + x_n \geq s\}$. We transport the problem onto the unit square through $h : \mathbb{R}^n \to [0,1]^n$, $h(\boldsymbol{x}) := (F_1(x_1), \ldots, F_n(x_n))$ and denote

$$A_s := h(G_s) = \{\boldsymbol{u} \in [0,1]^n : F_1^{-1}(u_1) + \cdots + F_n^{-1}(u_n) \geq s\}. \tag{3}$$

By definition, for $s \in \mathbb{R}$, we have that $\mathrm{wVaR}_{\alpha(s)}(X_1 + \cdots + X_n) = s$, when $1 - \alpha(s) := \sup_{C \in \mathfrak{C}^n}(1 - \sigma_{C,+}(F_1, \ldots, F_n)(s))$. The worst Value-at-Risk dependence scenario at level α therefore solves the equality

$$\mathbf{P}_C\left(A_{\mathrm{wVaR}_\alpha(X_1+\cdots+X_n)}\right) = 1 - \alpha. \tag{4}$$

3.1 Geometrical properties of $C_{\tilde{\alpha}} \in \mathfrak{C}^2$ with $\tilde{\alpha} = \tau_{W,+}(F_1, F_2)(s)$

In the two-dimensional case, applying Proposition 2, we immediately see that $C_{\tilde{\alpha}}$ satisfies (4) if $\tilde{\alpha} = \alpha$. Moreover, for a uniform portfolio (U_1, U_2), Embrechts et al. (2005, Proposition 9) yields that this is the only copula putting measure $1 - \alpha$ on $A_{\mathrm{wVaR}_\alpha(U_1+U_2)}$ with $\mathrm{wVaR}_\alpha(U_1 + U_2) = 1 + \alpha$. Therefore, in this case the density of C_α in $A_{1+\alpha}$ is concentrated on the boundary $\underline{H}_\alpha = \underline{A}_{1+\alpha}$; see Figure 1 (left).

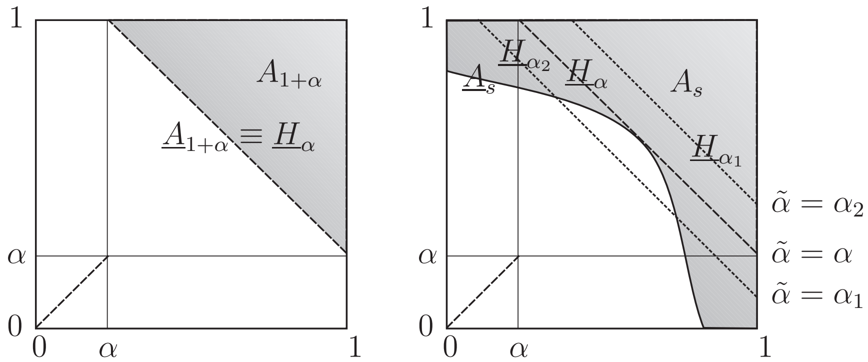

Fig. 1. Sets A_s and boundaries $\underline{A}_s$ for a two-dimensional uniform portfolio for $s = 1.25$ (left) and Lognormal$(0.4, 1)$ portfolio for $s = 4$ (right). Together we plot the support $\underline{H}_\alpha$ of $C_{\tilde{\alpha}}$ for $\tilde{\alpha} = \alpha = \tau_{W,+}(F_1, F_2)(s)$ and the (upper) supports for $\tilde{\alpha} = \alpha_1 < \alpha$ and $\tilde{\alpha} = \alpha_2 > \alpha$.

Figure 1 highlights the geometric idea underlying the worst scenario $C_{\tilde{\alpha}}(u_1, u_2)$. The gray areas represent the sets A_s for a uniform portfolio $(s = 1 + \alpha = 1.25)$ and a Lognormal$(0.4, 1)$ portfolio $(s = 4)$, respectively. The boundary of A_s can be written as $\underline{A}_s := \{(F_1(t), F_2(s - t)), t \in \mathbb{R}\}$. We denote $\underline{H}_{\tilde{\alpha}}$ the support of $C_{\tilde{\alpha}}$ restricted to $[\tilde{\alpha}, 1]^2$. In general dimensions, we refer to the support restricted to $[\tilde{\alpha}, 1]^n$ as upper support. The solution of the problem for the uniform portfolio leads then to an optimizing copula, which upper support coincides with the boundary $\underline{A}_{1+\alpha}$.

Remark 3 The choice $\tilde{\alpha} = \tau_{W,+}(F_1, F_2)(s)$ in Proposition 2 implies that $\underline{H}_{\tilde{\alpha}}$ lies in A_s and is tangent to $\underline{A}_s$. Since $C_{\tilde{\alpha}} \in \mathfrak{C}^2$, the density on $\underline{H}_{\tilde{\alpha}}$ is uniformly distributed and proportional to its length $l(\underline{H}_{\tilde{\alpha}})$, say. Therefore C_α maximizes the density on A_s. In fact, a different choice of $\tilde{\alpha}$ would decrease the probability of A_s. Trivially $l(\underline{H}_{\alpha_2}) < l(\underline{H}_\alpha)$ for $\alpha_2 > \alpha$, whereas the shape of $\underline{A}_s$ implies

$$l(\underline{H}_{\alpha_1}) = l(\underline{H}_\alpha) + 2\sqrt{2}(\alpha - \alpha_1) - l(\underline{H}_\alpha \cap A_s^c) < l(\underline{H}_\alpha) \quad if\ \alpha_1 < \alpha.$$

3.2 Worst VaR scenario for a n-dimensional uniform portfolio

In this section we consider a uniform portfolio. Similar to the previous section, the uniform case will lead to an optimizing copula for general marginals.

The solution of the worst VaR question consists of maximizing the probability of a certain region of $\mathbb{R}^n$. As illustrated in the previous section, we transport the problem onto the n-dimensional unit cube and investigate the shape of the support of the copulas putting maximal measure on (3). For a n-dimensional uniform portfolio and $n-1 \le s \le n$, the region of the space where the probability has to be maximized is $A_s = \{\boldsymbol{u} \in [0,1]^n : \sum_{i=1}^n u_i \ge s\}$ with boundary $\underline{A}_s = \{\boldsymbol{u} \in [0,1]^n : \sum_{i=1}^n u_i = s\}$. For $0 \le s \le n-1$ the problem has a trivial solution. Because of the uniformity of the marginals, the upper support of a copula maximizing the probability of A_s has to lie in $A_s \cap [\tilde{\alpha}, 1]^n$ for some appropriate $\tilde{\alpha} \ge \alpha^\star := s - (n-1)$ with $\tilde{\alpha} = \alpha^\star$ when $n = 2$.

Theorem 1 *Let $\tilde{\alpha} \in [\alpha^\star, 1)$ and $C_{\tilde{\alpha}} : [0,1]^n \to [0,1]$ be a function with support in $\{\boldsymbol{u} \in [0,\tilde{\alpha})^n : u_1 = \cdots = u_n\} \cup \{\boldsymbol{u} \in [\tilde{\alpha}, 1]^n : \sum_{i=1}^n u_i \ge s\}$ for $s \in [n-1, n]$. A necessary condition for $C_{\tilde{\alpha}}$ to be a copula is that $\tilde{\alpha} = \bar{\alpha} := 2s/n - 1$, i.e. that the support in $[\tilde{\alpha}, 1]^n$ lies in $H_{\tilde{\alpha}} := \{\boldsymbol{u} \in [\tilde{\alpha}, 1]^n : \sum_{i=1}^n u_i \ge n(1 + \tilde{\alpha})/2\}$.*

Proof Assume $\tilde{\alpha} \in [\alpha^\star, \bar{\alpha}]$ and $C_{\tilde{\alpha}} \in \mathfrak{C}^n$ with corresponding measure $\mu_{\tilde{\alpha}}$. Let $S_i := \{\boldsymbol{u} \in [0,1]^n : \alpha^\star \le u_i \le \bar{\alpha}\}, i = 1, \ldots, n-1$, $S_n := \{\boldsymbol{u} \in [0,1]^n : 1 - (\bar{\alpha} - \alpha^\star) \le u_n \le 1\}$ and set $E_i = S_i \cap A_s^n, i = 1, \ldots, n$. By definition, $E_i \in E_n$ for all i. Since $C_{\tilde{\alpha}}$ is a copula with upper support in $\{\boldsymbol{u} \in [\tilde{\alpha}, 1]^n : \sum_{i=1}^n u_i \ge s\}$, $\mu_{\tilde{\alpha}}(E_1) = \cdots = \mu_{\tilde{\alpha}}(E_{n-1}) = \bar{\alpha} - \alpha^\star$ whereas $\mu_{\tilde{\alpha}}(E_n) = \bar{\alpha} - \alpha^\star$. It immediately follows that $\mu_{\tilde{\alpha}}(E_i) = 0$ for $i = 1, \ldots, n-1$ and $\tilde{\alpha} = \alpha^\star = \bar{\alpha}$. □

Remarks 4

(a) Geometrically, Theorem 1 implies that if $C_{\tilde{\alpha}} \in \mathfrak{C}^n$, the set

$$\underline{H}_{\tilde{\alpha}} := [\tilde{\alpha}, 1]^n \cap \underline{A}_{\frac{n}{2}(1+\tilde{\alpha})} = \left\{ \boldsymbol{u} \in [\tilde{\alpha}, 1]^n : \sum_{i=1}^n u_i = \frac{n}{2}(1 + \tilde{\alpha}) \right\} \quad (5)$$

is symmetric with respect to its center $((1 + \tilde{\alpha})/2, \ldots, (1 + \tilde{\alpha})/2)$.

(b) Observe that the analytic generalization of (2) with $W(u_1, u_2)$ replaced by $W(\boldsymbol{u})$ has upper support $\{\boldsymbol{u} \in [\tilde{\alpha}, 1]^3 : u_1 + u_2 + u_3 = 2\}$.

Next we extend the two-dimensional result of Embrechts et al. (2005, Proposition 9) to general dimensions and we provide the existence of a copula with support as in (5).

Theorem 2 *Assume* $\bar{C}, \tilde{C} \in \mathfrak{C}^n$ *to have supports on* $\underline{H}_0^n$ *and* H_0^n*, respectively. Let* $\mu_{\tilde{C}}$ *be the measure induced by* $\tilde{C}$*. Then for* $H_0^{n+} := \{\boldsymbol{u} \in [0,1]^n : \sum_{i=1}^n u_i > n/2\}$ *and* $H_0^{n-} := \{\boldsymbol{u} \in [0,1]^n : \sum_{i=1}^n u_i < n/2\}$ *it holds that*

$$\mu_{\tilde{C}}(H_0^{n-}) = 0 \Leftrightarrow \mu_{\tilde{C}}(H_0^{n+}) = 0,$$

and the two copulas have the same support.

Proof Assume $\mu_{\tilde{C}}(H_0^{n-}) = 0$ and $\mu_{\tilde{C}}(H_0^{n+}) > 0$. Consider the independence copula Π with support $[0,1]^n$. Since $\bar{C}, \tilde{C} \in \mathfrak{C}^n$, there exist operators $\nu, \tilde{\nu} : \mathfrak{C}^n \to \mathfrak{C}^n$ with $\varphi := \tilde{\nu} \circ \nu^{-1} \neq Id$ such that $\bar{C} = \nu(\Pi)$ and $\tilde{C} = \tilde{\nu}(\Pi)$. It follows that $\mu_{\varphi(\bar{C})}(H_0^{n-}) = 0$ and $\mu_{\varphi(\bar{C})}(H_0^{n+}) > 0$. On the contrary, in order to preserve the uniformity of the marginals, any operator $\tilde{\varphi} : \mathfrak{C}^n \to \mathfrak{C}^n$ with $\mu_{\tilde{\varphi}(\bar{C})}(H_0^{n+}) > 0$, implies $\mu_{\tilde{\varphi}(\bar{C})}(H_0^{n-}) > 0$, which concludes the proof. □

Theorem 3 *Let* $C_{\tilde{\alpha}} : [0,1]^n \to [0,1]$ *have support* $\underline{H}_{\tilde{\alpha}}$ *as in* (5) *on* $[\tilde{\alpha}, 1]^n$*. Then there exists a sequence of copulas* $C_{N,\tilde{\alpha}} \in \mathfrak{C}^n, N \in 2\mathbb{N} + 1$ *such that*

$$C_{\tilde{\alpha}}(\boldsymbol{u}) := \begin{cases} \lim_{N \to \infty} C_{N,\tilde{\alpha}}(\boldsymbol{u}) & \text{if } \boldsymbol{u} \in [\tilde{\alpha}, 1]^n, \\ M(\boldsymbol{u}) & \text{otherwise}. \end{cases}$$

is a copula.

Proof Without loss of generality, we consider $\tilde{\alpha} = 0$ with $\underline{H}_0 = [0,1]^n \cap \underline{A}_{n/2}^n$. For $N \in 2\mathbb{N} + 1$ we consider the partition $I := [0,1] = \cup_{k=1}^N I_k$, where $I_k := [\frac{k-1}{N}, \frac{k}{N}]$. We identify the set $I_{k_1} \times \cdots \times I_{k_n}$ with the point $(k_1, \ldots, k_n)$ and define its measure as follows. For any $k = 1, \ldots, (N+1)/2$ and $1 \leq \bar{k} < k$ we set the functions

$$g_1(k) := |\{I_k \times I^{n-1}\} \cap H_0^{(N)}|, \quad g_2(k, \bar{k}) := |\{I_k \times I_{\bar{k}} \times I^{n-2}\} \cap H_0^{(N)}|,$$

where $H_0^{(N)} := \{(k_1, \ldots, k_n) \in \{1, \ldots, N\}^n : \frac{n}{2} - \frac{1}{N} < \sum_{i=1}^n k_i \leq \frac{n}{2} + \frac{1}{N}\}$. Then we define

$$f_0^{(N)}(k_1, \ldots, k_n) := \begin{cases} f^\star\left(\min_{1 \leq d \leq n} k_d\right) & \text{if } (k_1, \ldots, k_n) \in H_0^{(N)}, \\ 0 & \text{otherwise}, \end{cases}$$

where $f^\star(k) := (\frac{1}{N} - (n-1) \sum_{1 \leq \bar{k} < k} g_2(k, \bar{k}) f^\star(\bar{k}))(g_1(k) - (n-1) \sum_{1 \leq \bar{k} < k} g_2(k, \bar{k}))^{-1}$.

The above construction defines a copula on the grid $\{k_1, \ldots, k_n\}^N$. The set $I_k \times I^{n-1}$ denotes the k-th slice of $[0,1]^n$ along the first dimension. Therefore $g_1(k)$ counts the number of points on such a slice which also lie in $\underline{H}_0^{(N)}$. Similarly $g_2(k, \bar{k})$ counts the number of points on $\underline{H}_0^{(N)}$ which are on the k-th slice along the first dimension and on the $\bar{k}$-th slice along the second one. By the symmetry of the support, we could define these functions using any other two dimensions. Moreover, by definition, all the slices have width $1/N$. The idea is then to consider $I_0 \times I^{n-1}$ and weight each point in order to have total measure $1/N$. In doing this, by symmetry, we assign a measure to all the points lying on an equivalent slice for any of the other dimensions. We continue with $I_1 \times I^{n-1}$ assigning a weight only to the missing points. To do this we only have to take into account the missing points on slice k, i.e. $(n-1)\sum_{1 \le \bar{k} < k} g_2(k, \bar{k}) f^\star(\bar{k})$. By the symmetry of $\underline{H}_0^{(N)}$, we only evaluate slices $k = 1, \ldots, (N+1)/2$. Using $f^\star$ we finally assign probability weights to the points with respect of the marginal constraints.

For any N, by construction, the function $C_{N,0} : [0,1]^n \to [0,1]$ defined through

$$C_{N,0}(\boldsymbol{u}) := \sum_{k_1=1}^{k(u_1)} \cdots \sum_{k_n=1}^{k(u_n)} f_0^{(N)}(k_1, \ldots, k_n), \quad k(u) := \sup\left\{k \ge 1 : \frac{k}{N} \le u\right\}$$

is a copula. Setting $C_0(\boldsymbol{u}) := \lim_{N\to\infty} C_{N,0}(\boldsymbol{u})$ we then obtain that

$$\begin{aligned} C_0(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n) &= \lim_{N\to\infty} C_{N,0}(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n) \\ &\le \lim_{N\to\infty} 1/N = 0, \\ C_0(1, \ldots, 1, u_i, 1, \ldots, 1) &= \lim_{N\to\infty} C_{N,0}(1, \ldots, 1, u_i, 1, \ldots, 1) \\ &= \lim_{N\to\infty} k(u_i)/N = u_i, \end{aligned}$$

and for $\boldsymbol{u}, \boldsymbol{v} \in [0,1]^n$ with $\boldsymbol{u} \le \boldsymbol{v}$ and $u_{j,1} = u_j$, $u_{j,2} = v_j$, $j = 1, \ldots, n$

$$\begin{aligned} &\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C_0(u_{1,i_1}, \ldots, u_{n,i_n}) \\ &= \lim_{N\to\infty} \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C_{N,0}(u_{1,i_1}, \ldots, u_{n,i_n}) \ge 0. \end{aligned}$$

It follows that the conditions given in Remark 1 are satisfied and $C_0 \in \mathfrak{C}^n$. □

Remark 5 Similarly as in the above proof, it is possible to construct other copulas with support $\underline{H}_{\tilde{\alpha}}$. We denote the family of the copulas sharing this support by $\mathfrak{C}^n_{\tilde{\alpha}}$.

By Theorems 1, 2 and 3, any copula putting probability $1 - \tilde{\alpha} = 1 - (2s/n - 1)$ on $A_{n(1+\tilde{\alpha})/2}$ has support $\underline{H}_{\tilde{\alpha}}$ as in (5). Figure 2 illustrates these results in the three-dimensional case.

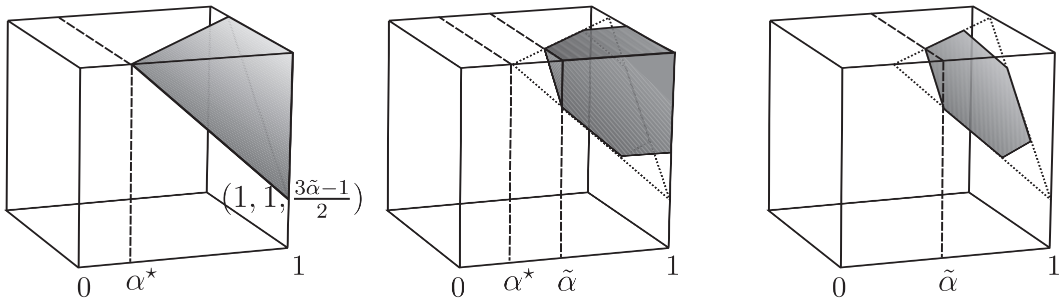

Fig. 2. Sets $A_{3(1+\tilde{\alpha})/2}$, $H_{\tilde{\alpha}}$ and $\underline{H}_{\tilde{\alpha}}$ for a three-dimensional uniform portfolio. The set $\underline{H}_{\tilde{\alpha}}$ with $\tilde{\alpha} = \alpha$ is the upper support of any copula leading to $\mathrm{wVaR}_\alpha(X_1 + X_2 + X_3)$.

As a consequence we obtain the result of Rüschendorf (1982) given in Proposition 3. We restate it here using the language of copulas.

Corollary 1 *Let $C_{\tilde{\alpha}} \in \mathfrak{C}^n_{\tilde{\alpha}}$ with $\tilde{\alpha} = \alpha$. Then*

$$\mathbf{P}_{C_\alpha}\{X_1 + \cdots + X_n < s\} = \alpha \tag{6}$$

for $s = n(1+\alpha)/2$ and the best possible lower bound on the distribution function of $X_1 + \cdots + X_n$ for uniform marginals is $\min\{(2s/n - 1)^+, 1\}$ for $s \in (0, n)$.

Proof The worst dependence scenario for Value-at-Risk at level α satisfies (4). Taking $C_{\tilde{\alpha}} \in \mathfrak{C}^n_{\tilde{\alpha}}$, we obtain $\mathbf{P}_{C_{\tilde{\alpha}}}(A_{n(1+\tilde{\alpha})/2}) = \mathbf{P}_{C_{\tilde{\alpha}}}(\underline{H}_{n(1+\tilde{\alpha})/2}) = 1 - \tilde{\alpha}$. Then equality (4) is satisfied for $\tilde{\alpha} = \alpha$ and $\mathrm{wVaR}_\alpha(X_1 + \ldots + X_n) = n(1+\alpha)/2$ which implies (6). □

3.3 Worst VaR scenario for a general portfolio

Relying on the solution for a the uniform case studied in the previous section, we provide an answer for a general portfolio with marginals $F_1, \ldots, F_n$. Although we illustrate the case of a three dimensional portfolio, our arguments remain valid in higher dimensions. We recall that, for a portfolio (X_1, X_2), the copula leading to the worst possible Value-at-Risk $\mathrm{wVaR}_\alpha(X_1 + X_2)$ is indeed the solution of the uniform case for $\tilde{\alpha} = \alpha$. This follows from the uniformity of the density on the upper support; see Figure 1 and Remark 3. In general, the worst value wVaR_α is not attained under a $C_{\tilde{\alpha}} \in \mathfrak{C}^n_{\tilde{\alpha}}$ with $\tilde{\alpha} = \alpha$.

Example 1 Consider the portfolio (X_1, X_2, X_3) for $X_i \sim \mathrm{Pareto}(1/\xi_i)$ with tail distribution function $\overline{F}_i(x) = (1+x)^{-1/\xi_i}$, $i = 1, 2, 3$. Assume $\xi_i = 0.7$, $i = 1, 2, 3$. In Figure 3 (left) we illustrate the surface $\underline{A}_s$ for $s = 21.4$ together with the upper support $\underline{H}_{\tilde{\alpha}}$ of $C_{\tilde{\alpha}}$ for $\tilde{\alpha} = 0.9$. On the right we plot $\underline{A}_s$ for $s = 22.7$ with $\underline{H}_{\tilde{\alpha}}$ for $\tilde{\alpha} = 0.895$. Computing the probability of A_s under these two dependence structures we obtain

$$\mathbf{P}_{C_{0.9}}(X_1 + X_2 + X_3 \geq 21.4) = \mathbf{P}_{C_{0.895}}(X_1 + X_2 + X_3 \geq 22.7) = 0.1$$

and therefore the Value-at-Risk of the sum at level $\alpha = 0.9$ under $C_{0.9}$ is smaller that under $C_{0.895}$.

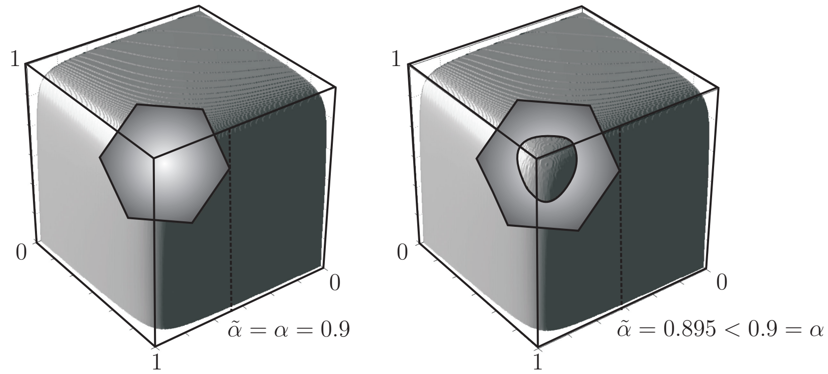

Fig. 3. Surfaces $\underline{A}_s$ for a three-dimensional Pareto(0.7) portfolio with $s = 21.4$ (left) and $s = 22.7$ (right). We plot $\underline{H}_{\tilde{\alpha}}$ for $\tilde{\alpha} = 0.9$ on the left and $\tilde{\alpha} = 0.895$ on the right.

The message coming from Example 1 is that choosing the upper support tangent to $\underline{A}_s$, i.e. $\tilde{\alpha} = \alpha$, in general does not imply a worst dependence scenario. This is due to the distribution of the density on the support. Indeed the marginal constraints imply that for $n > 2$ the density is not uniformly distributed but concentrated on the border of $\underline{H}_{\tilde{\alpha}}$ and more thinly when reaching the center $(n(1+\tilde{\alpha})/2, \ldots, n(1+\tilde{\alpha})/2)$. This can be easily seen in the three-dimensional case looking at the projection of $\underline{H}_{\tilde{\alpha}}$ on $[0, 1]^2$.

In Figure 1 (right) we take $\tilde{\alpha} < \alpha$. This implies that the measure on the upper support is greater than $1 - \alpha$. Contrary, for $\tilde{\alpha} > \alpha$, a portion of the support does not lie in A_s. In Remark 3 we discussed the two-dimensional situation, where the density is proportional to the length of the support and every choice of $\tilde{\alpha}$ different from α leads to a better scenario for the problem at hand. In the general case, cutting some portion of the support does not necessarily imply a better scenario. In fact the increment of probability on the boundary could compensate the reduction in some other region. For $\tilde{\alpha}$ sufficiently small, we lose too much density on A_s. Trivially, $\tilde{\alpha} > \alpha$ implies a better scenario. From the solution of the uniform problem given by Theorems 1, 2 and 3 and the distribution of the probability on its upper support, we immediately obtain the following result.

Theorem 4 *Let $(X_1, \ldots, X_n)$ be a portfolio with given marginals $F_1, \ldots, F_n$. Then $\mathrm{wVaR}_\alpha(X_1 + \cdots + X_n)$ is attained under a copula $C_{\tilde{\alpha}} \in \mathfrak{C}^n_{\tilde{\alpha}}$ for some $\tilde{\alpha} \leq \alpha$ depending on the marginal distributions. Using the same notation as in* (4)*, we have that*

$$\sup\{\mathbf{P}_{C_{\tilde{\alpha}}}\left(A_{\mathrm{wVaR}_\alpha(X_1+\cdots+X_n)}\right) : C_{\tilde{\alpha}} \in \mathfrak{C}^n_{\tilde{\alpha}},\ 0 < \tilde{\alpha} \leq \alpha\} = 1 - \alpha.$$

Remark 6 In contrast to the two-dimensional case, in dimensions higher than two a copula $C_{\tilde{\alpha}}$ leading to $\mathrm{wVaR}_\alpha(X_1 + \cdots + X_n)$ depends upon the choice of the marginals. In fact the region of the support where we lose probability is given by $\underline{H}_{\tilde{\alpha}} \cap \{\boldsymbol{u} \in [0,1]^n : F_1^{-1}(u_1) + \cdots + F_n^{-1}(u_n) < s\}$ and depends on $F_1, \ldots, F_n$.

4 Applications

In this section we apply Theorem 4 and compute the worst-possible Value-at-Risk for the sum at level α for a three-dimensional portfolio (X_1, X_2, X_3). The positions $X_i, i = 1, 2, 3$ are $\mathrm{Pareto}(1/\xi_i)$ distributed with tails $\overline{F}_i(x) = (1+x)^{-1/\xi_i}$. We solve the problem for $\alpha = 0.9, 0.95, 0.99$ (typically used for market or credit risk) and $\alpha = 0.995, 0.999$ (values used in operational risk) and this for various scenarios.

Scenario I: $X_i \sim \mathrm{Pareto}(1/\xi_i)$ with $\xi_1 = \xi_2 = \xi_3 = 0.7$,

Scenario II: $X_i \sim \mathrm{Pareto}(1/\xi_i)$ with $\xi_1 = 0.7504, \xi_2 = 0.6607$ and $\xi_3 = 0.2815$,

Scenario III: $X_i \sim \mathrm{Pareto}(1/\xi_i)$ with $\xi_1 = 1.1905, \xi_2 = 1.3889$ and $\xi_3 = 1.2195$.

The main features of these scenarios are: they are all heavy-tailed, homogeneous as in I, or heterogeneous as in II and III. Scenario II corresponds to a finite mean situation whereas III corresponds to an infinite mean model. The ξ-values chosen correspond to examples often encountered in risk management practice. For Scenario II and III; see for instance Moscadelli (2004). Based on Theorem 4 and the upper support $\underline{H}_{\tilde{\alpha}}$ of $C_{\tilde{\alpha}} \in \mathfrak{C}^n_{\tilde{\alpha}}$, we propose the following numerical procedure. For given $s \in \mathbb{R}$ and $\tilde{\alpha} \in (0,1)$, analogously as in the proof of Theorem 3, for $N \in 2\mathbb{N}+1$, we discretize the unit cube $[\tilde{\alpha}, 1]^3$ through

$$[\tilde{\alpha}, 1] = \cup_{k=1}^N I_k, \quad I_k := \left[\tilde{\alpha} + \frac{k-1}{N}, \tilde{\alpha} + \frac{k}{N}\right]$$

and we identify the set $I_{k_1} \times I_{k_2} \times I_{k_3}$ with the point $(k_1, k_2, k_3) \in \{1, \ldots, N\}^3$.

Further we consider the sets $A_s^{(N)}$ and $\underline{H}_{\tilde{\alpha}}^{(N)}$ as discretized versions of A_s and $\underline{H}_{\tilde{\alpha}}$, respectively. We let $\boldsymbol{w} \in \mathbb{R}^{N^3}$ be a vector containing the probability weights of the points in $[0,1]^3$. We then generate a vector $\boldsymbol{f} \in \mathbb{R}^{N^3}$ with entry one when the corresponding point lies on $[0,1]^3 \setminus A_s^{(N)} \cap \underline{H}_{\tilde{\alpha}}^{(N)}$ and zero elsewhere. Similarly we create a $N^3 \times 3N$ matrix A providing the marginal restrictions. Finally we solve the optimization problem

$$\min_{\boldsymbol{w}} \boldsymbol{f}^T \boldsymbol{w}, \quad A\boldsymbol{w} = \left(\tfrac{1}{N}, \ldots, \tfrac{1}{N}\right)^T, \quad \boldsymbol{w} \in [0,1]^{N^3}. \tag{7}$$

It follows that $s = \mathrm{wVaR}_\alpha(X_1 + X_2 + X_3)$ at level $\alpha = \tilde{\alpha} + \boldsymbol{f}^T \hat{\boldsymbol{w}}$, where $\hat{\boldsymbol{w}}$ is the solution of (7). Any copula leading to wVaR_α has support $\underline{H}_{\tilde{\alpha}}^{(N)}$.

We illustrate the above procedure for the Scenarios I, II and III. Together with the worst-case wVaR_α, in the Tables 1, 2 and 3 we provide the values under $C_{\tilde{\alpha}}$ with $\tilde{\alpha} = \alpha$ and for the comonotonic copula M for which $\mathrm{VaR}_\alpha(X_1 + X_2 + X_3) = \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2) + \mathrm{VaR}_\alpha(X_3)$.

$\downarrow C, \overset{\alpha}{\rightarrow}$	0.9	0.95	0.99	0.995	0.999	0.9999
M	13.0	21.4	72.3	119.4	374.7	1889.9
C_α	21.4	36.7	119.5	196.0	611.1	3074.7
$C_{\tilde{\alpha}}$	22.7	38.6	123.8	205.2	634.3	3120.0
$\tilde{\alpha}$	0.895	0.948	0.989	0.9948	0.9989	0.99989

Table 1
Values of $\mathrm{VaR}_\alpha(X_1 + X_2 + X_3)$ for scenario I under $C_\alpha, C_{\tilde{\alpha}}$ and M. In the last row we give the values of $\tilde{\alpha}$ yielding the worst dependence structure and $\mathrm{wVaR}_\alpha(X_1 + X_2 + X_3)$.

$\downarrow C, \overset{\alpha}{\rightarrow}$	0.9	0.95	0.99	0.995	0.999	0.9999
M	9.1	16.0	53.3	87.9	278.3	1453.4
C_α	13.6	22.7	70.5	114.1	348.1	1749.9
$C_{\tilde{\alpha}}$	13.6	22.7	70.5	114.1	360.5	1981.0
$\tilde{\alpha}$	0.9	0.95	0.99	0.995	0.99865	0.999865

Table 2
Values of $\mathrm{VaR}_\alpha(X_1 + X_2 + X_3)$ for scenario II under $C_\alpha, C_{\tilde{\alpha}}$ and M.

$\downarrow C, \overset{\alpha}{\rightarrow}$	0.9	0.95	0.99	0.995	0.999	0.9999
M	53.6	135.1	1111.2	2754.2	22946.6	492468.4
C_α	130.7	320.4	2531.2	6161.3	48905	960782
$C_{\tilde{\alpha}}$	144.3	351.5	2700	6500	52000	980000
$\tilde{\alpha}$	0.89	0.947	0.989	0.9943	0.99885	0.99988

Table 3
Values of $\mathrm{VaR}_\alpha(X_1 + X_2 + X_3)$ for scenario III under $C_\alpha, C_{\tilde{\alpha}}$ and M.

Figures 4 and 5 show the densities on $A_s^{(N)} \cap \underline{H}_{\tilde{\alpha}}^{(N)}$ as functions of the parameter $\tilde{\alpha}$ for Scenarios I and II and levels $\alpha = 0.99$ and $\alpha = 0.9999$, respectively. The

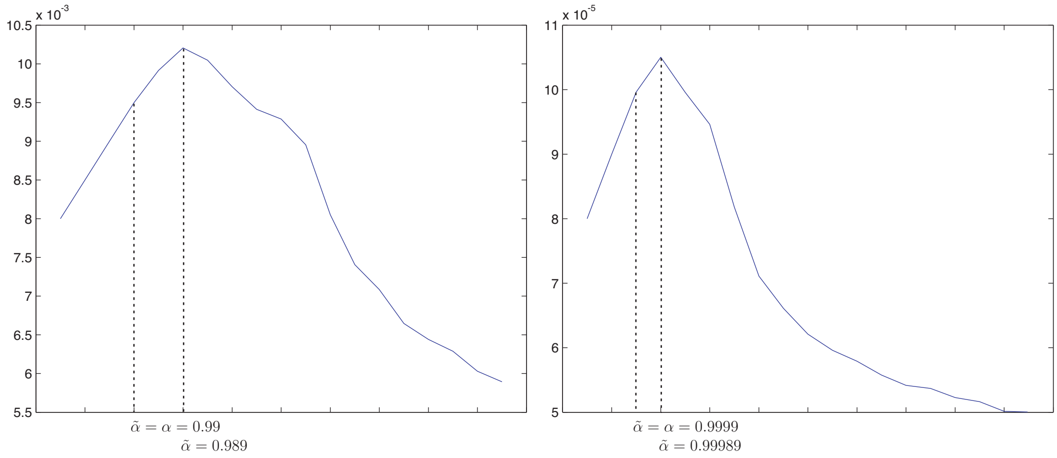

Fig. 4. Densities on $A_s^{(N)} \cap \underline{H}_{\tilde{\alpha}}^{(N)}$ for $s = 123.8$ $(\alpha = 0.99)$ (top) and $s = 3120$ $(\alpha = 0.9999)$ (bottom) as functions of $\tilde{\alpha}$ for Scenario I.

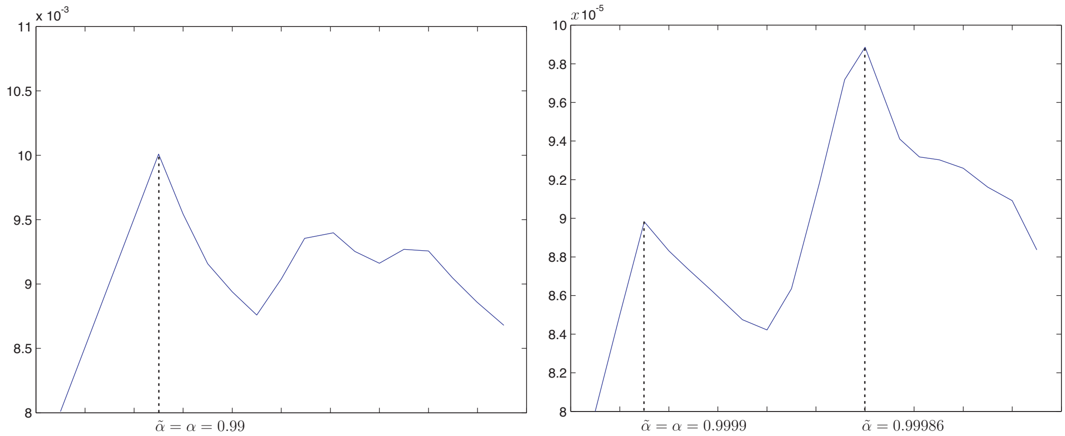

Fig. 5. Densities on $A_s^{(N)} \cap \underline{H}_{\tilde{\alpha}}^{(N)}$ for $s = 70.5.8$ $(\alpha = 0.99)$ (top) and $s = 1981$ $(\alpha = 0.9999)$ (bottom) as functions of $\tilde{\alpha}$ for Scenario II.

starting value for $\tilde{\alpha}$ is larger than α. We can observe that in both cases the densities increase linearly in $\tilde{\alpha}$ till reaching α. For the two scenarios we observe different behavior. For Scenario I, the densities continue to increase after α and, once a maximum is reached, they tend to zero. The $\tilde{\alpha}$ corresponding to this maxima, $\tilde{\alpha} = 0.989$ $(\alpha = 0.99)$ and $\tilde{\alpha} = 0.99989$ $(\alpha = 0.9999)$, give the worst dependence scenarios.

For Scenario II, the densities on $A_s^{(N)} \cap \underline{H}_{\tilde{\alpha}}^{(N)}$ have a first maximum in $\tilde{\alpha} = \alpha$ and a second one for some $\tilde{\alpha} > \alpha$. In the case $\alpha = 0.9$ the worst dependence scenario is implied by the first maximum and the upper support is tangent to $\underline{A}_s$. For $\alpha = 0.9999$, the second maximum dominates.

In order to understand the different nature between the two scenarios, we look at the supports plotted in Figure 6. The idea is as follows. We set the upper support tangent to A_s (with s chosen such that $\tilde{\alpha} = \alpha = 0.9$) and we shift it by taking values of $\tilde{\alpha}$ smaller than α. The set $A_s^{(N)} \cap \underline{H}_{\tilde{\alpha}}^{(N)}$ is illustrated for $\tilde{\alpha} = 0.895$ and

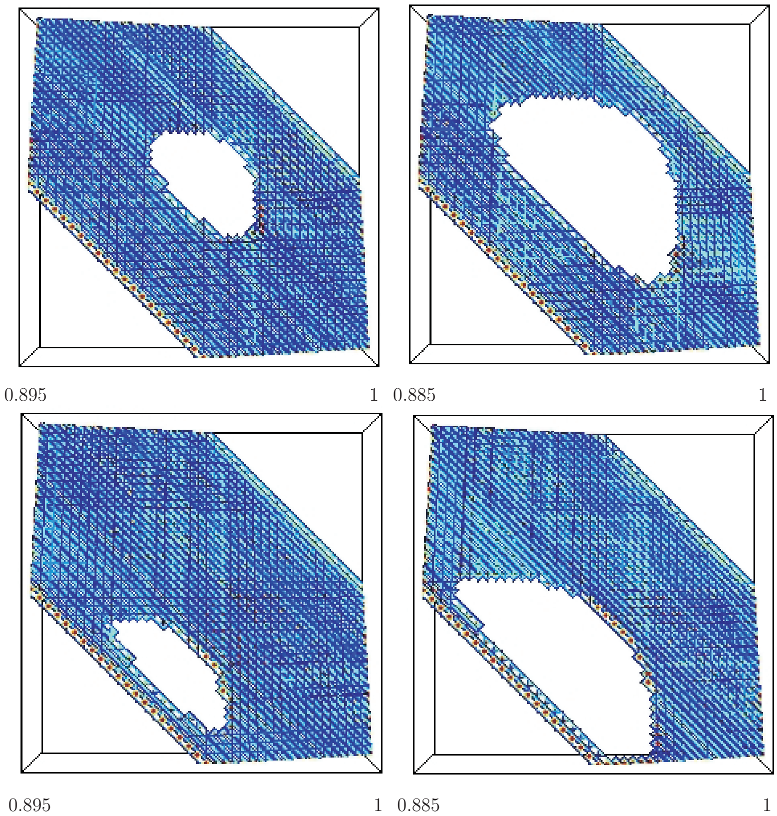

Fig. 6. Upper supports for Scenarios I (top) and II (bottom). In both cases we take $\alpha = 0.9$ and we consider $\tilde{\alpha} = 0.895$ (left) and $\tilde{\alpha} = 0.885$ (right).

$\tilde{\alpha} = 0.885$ under Scenarios I and II. Remark that a smaller $\tilde{\alpha}$ implies a larger cut of the support and an increment of the probability on $[\tilde{\alpha}, 1]^3$. At this point we recall that the density is not homogeneous on the support and more concentrated when reaching the border. The different dynamics observed in Figures 4 and 5 are due to the regions where the support is cut. In Scenario I (with equal marginals) the support loses probability in the center. Hence the probability on $A_s^{(N)} \cap \underline{H}_{\tilde{\alpha}}^{(N)}$ decreases only when $\tilde{\alpha}$ is small enough; see Figure 4. On the other hand, if the tail of one distribution dominates the others, the cut arises near the border. This is the case for Scenario II for instance, where the loss of probability can not be compensated for small adjustments of $\tilde{\alpha}$. With larger movements of the parameter, the cutted region includes the central region as in Figure 6 (bottom/right) and the probability on $A_s^{(N)} \cap \underline{H}_{\tilde{\alpha}}^{(N)}$ grows again. Besides the region where the loss of probability occurs, the shape of the set $\underline{A}_s$ plays a role. In particular, this explains the differences arising in Scenario II. For $\alpha = 0.99$, we observe a loss of probability for any small adjustment of $\tilde{\alpha}$, which is not compensated by the augmentation before the second maximum. The very sharp profile of $\underline{A}_s$ for $\alpha = 0.9999$ allows the initial loss to be compensated as illustrated in Figure 5 (bottom).

As further application of our methodology, we calculate $\mathrm{wVaR}_\alpha(X_1+X_2+X_3)$ for an homogeneous portfolio (X_1, X_2, X_3). We solve the problem for $\alpha = 0.9, 0.999$

and $X_i \sim \text{Pareto}(1/\xi), i = 1, 2, 3$ for different values of ξ. The following table gives the results of our numerical computations together with the scaling factors from $\text{wVaR}_{0.9}$ to $\text{wVaR}_{0.999}$ and from $\text{VaR}_{0.9}$ under $C_{0.9}$ to $\text{VaR}_{0.999}$ under $C_{0.999}$, respectively. We observe that the scaling curve grows exponentially as a function of the parameter ξ. It is moreover interesting to note that the scaling curve for the Value-at-Risk computed for $\tilde{\alpha} = \alpha$, i.e. with tangent upper support, does not differ significantly from the worst one.

ξ	$\text{wVaR}_{0.9}$	$\text{wVaR}_{0.999}$	$\frac{\text{wVaR}_{0.999}}{\text{wVaR}_{0.9}}$	$\text{VaR}_{0.9}^{(C_{0.9})}$	$\text{VaR}_{0.999}^{(C_{0.999})}$	$\frac{\text{wVaR}_{0.999}^{(C_{0.999})}}{\text{wVaR}_{0.9}^{(C_{0.9})}}$
0.7	22.7	634.3	27.9	21.4	611	28.6
0.8	31.1	1360	43.7	29.9	1310	43.8
0.9	43.8	2940	67.1	41.5	2806	67.6
1.0	60.8	6350	104.4	57	6006	105.4
1.1	84.3	13800	163.7	78	12850	164.7
1.2	116.0	30400	262.1	106.4	27490	258.4
1.3	160.4	65500	408.4	144.7	58805	406.4
1.4	221.0	145000	656.1	196.3	125793	640.8
1.5	304	310000	1019.7	266	269087	1011.6

Table 4
Values for $\text{wVaR}_{0.9}$, $\text{wVaR}_{0.999}$, $\text{VaR}_{0.9}$ under $C_{0.9}$ and $\text{VaR}_{0.999}$ under $C_{0.999}$ with the corresponding scaling factors.

Remark 7 The computational complexity of our numerical procedure increases exponentially with the dimension of the portfolio. Therefore, even if the values obtained are numerically not the exact worst-possible VaRs, in high dimensions the values obtained under C_α can be used as a first approximation for wVaR_α. More work on the numerical accuracy of the above procedure is called for.

5 Conclusion

In this paper we extend the geometrical properties of the copulae leading to the worst-possible Value-at-Risk at level α for the sum of two risks. These solutions depend upon the probability level α. We solve the problem for an n-dimensional portfolio and explain how, for $n \geq 3$, any worst-case scenarios $C_{\tilde{\alpha}}$ depends upon the choice of the marginals. In particular the worst scenarios are not obtained when the upper support of $C_{\tilde{\alpha}}$ is tangent to $\underline{A}_s$. However, when the dimension of the problem becomes high, the copulae with tangent upper support turn out to be useful in order to approximate wVaR_α. We conclude emphasizing that the results presented in this paper can be easily restated substituting A_s by $A_s^\psi := \{\boldsymbol{u} \in [0,1]^n : \psi(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)) \geq s\}$ corresponding to the Value-at-Risk optimization question for general increasing functionals ψ.

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