Extremes from meta distributions and the shape of the sample clouds

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Structure of the talk

- Introduction: motivation & main questions
- Meta distributions: formal definition & some properties
- Preliminary results
  - review of useful definitions
  - consequences of relaxing some of the underlying assumptions
- Setting up the framework & assumptions
- Key steps to compute the limit set
- Main result
- Sensitivity of the limit shape
- Concluding remarks & further questions

Motivation: *Why multivariate extremes?*

Quantitative Risk Management (QRM):

- a new field of research

Key ingredients:

- Regulatory framework for financial institutions: Basel Accords and Solvency II
- Quantile-based risk measures: Value-at-Risk (VaR) and Expected Shortfall (ES)
- Extremes matter: high quantiles ($\alpha \in \{99\%, 99.9\%, 99.97\%\}$)
- Dependence matters: risk aggregation, diversification/ concentration
- Dimensionality matters: high-dimensional portfolios

Conclusion:

- Extremal behaviour in multivariate models

Motivation: *A few examples*

Some concrete underlying models:

- multivariate normal (KMV model)
- multivariate Student t
- and refinements
  - (mixture of) Gaussian copula(s) with exponential marginals (Li model)
  - meta-t with normal marginals
  - etc.

$\longrightarrow$ The subprime crisis questions some of these developments/models

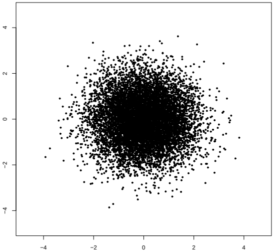
Why meta distributions?

- Asymptotic independence of coordinatewise maxima as a shortcoming of the multivariate Gaussian model
- Go beyond normality by introducing stronger tail dependence while preserving normal marginals
- A typical example:
  - Start with a multivariate Student t distribution (tail dependence and heavy tails)
  - Transform each coordinate so that the new distribution has normal marginals (light tails)
  - The new distribution is referred to as meta distribution with normal marginals based on the original t distribution (tail dependence and light tails)

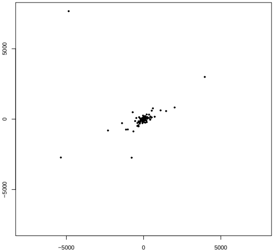
Extremes and asymptotic shape of sample clouds and level sets

- Global shape of sample clouds vs. classical EVT of coordinatewise maxima
- The limit shape describes the relation between maximal observations in different directions
- Relation of the shape of density level sets and the shape of risk regions to the conditional laws (cf. *Barbe, P. (2003)*)

Examples of sample clouds

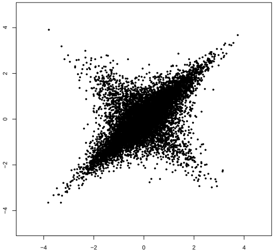


standard normal



elliptic Cauchy with dispersion matrix

$$\Sigma = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$$



meta-Cauchy with normal marginals

Main questions of this talk

- The shape of level sets $\{g > c\}$ for the meta density g depends on the level c.

  Does the shape converge as $c \downarrow 0$?

  What is the limit shape of the level sets?

- Can sample clouds from the meta density g be scaled to converge?

  What is the limit shape of scaled sample clouds?

- Properties of the limit set?

Definition (Meta distribution)

- Random vector $\mathbf{Z}$ in $\mathbb{R}^d$ with df F and continuous marginals F_i, $i = 1, \ldots, d$
- $G_1, \ldots, G_d$: continuous df's on $\mathbb{R}$, strictly increasing on $\mathrm{I_i} = \{0 < \mathrm{G_i} < 1\}$
- Define transformation:

$$K(x_1, \ldots, x_d) = \big(K_1(x_1), \ldots, K_d(x_d)\big), \quad K_i(s) = F_i^{-1}\big(G_i(s)\big), \quad i = 1, \ldots, d$$

- The df $G = F \circ K$ is the meta distribution (with marginals G_i) based on original df F
- $\mathbf{X}$ is said to be a meta vector for $\mathbf{Z}$ (with marginals G_i) if $\mathbf{Z} \stackrel{\mathrm{d}}{=} K(\mathbf{X})$
- The coordinatewise map $K = K_1 \otimes \cdots \otimes K_d$ which maps $\mathbf{x} = (x_1, \ldots, x_d) \in \mathrm{I} = \mathrm{I}_1 \times \cdots \times \mathrm{I_d}$ into the vector $\mathbf{z} = (K_1(x_1), \ldots, K_d(x_d))$ is called the meta transformation

Meta density

Proposition

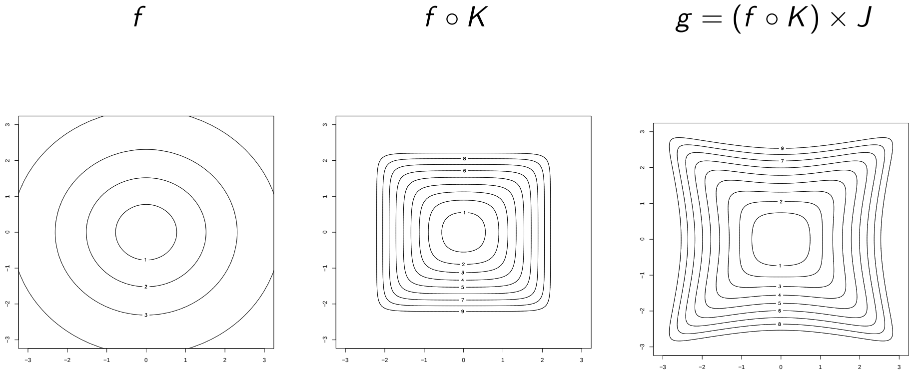
If

- Original vector $\mathbf{Z}$ has a density, f
- Marginals of meta distribution have densities, g_i

then the meta distribution has a density, g, and g is of the form

$$g(\mathbf{x}) = f(K(\mathbf{x})) \prod_{i=1}^{d} \frac{g_i(x_i)}{f_i(z_i)} \qquad z_i = K_i(x_i),\ x_i \in \mathrm{I_i} = \{0 < \mathrm{G_i} < 1\}$$

Level sets: from original density to meta density



Preliminary results

Useful definitions

Definition (weak asymptotic equivalence)

$\tilde{h}(\mathbf{x}) \asymp h(\mathbf{x})$ for $\|\mathbf{x}\| \to \infty$ if

- $\tilde{h}$ and h are positive eventually
- both $\dfrac{\tilde{h}(\mathbf{x})}{h(\mathbf{x})}$ and $\dfrac{h(\mathbf{x})}{\tilde{h}(\mathbf{x})}$ are bounded outside a compact set

Useful definitions

Definition (univariate regular variation)

A measurable function h on $(0, \infty)$ is regularly varying at ∞ with index ρ (written $h \in RV_\rho$) if for $x > 0$

$$\lim_{t \to \infty} \frac{h(tx)}{h(t)} = x^\rho$$

Question:

What is the effect on the meta density when changing the original density into a density which is (weakly) asymptotic to it?

On asymptotic behaviour of multivariate functions

Proposition

Assume:

- F & $\tilde{F}$: multivariate df's with continuous marginals F_i & $\tilde{F}_i$, $i = 1, \ldots, d$
- $G_1, \ldots, G_d$ are strictly increasing df's on $\mathbb{R}$
- $F_i(-t) \in RV_{\rho^-}$ and $1 - F_i(t) \in RV_{\rho^+}$ with $\rho^{\pm} < 0$
- $\tilde{F}_i(-t) \sim F_i(-t)$ and $1 - \tilde{F}_i(t) \sim 1 - F_i(t)$ as $t \to \infty$

Then the meta transformations satisfy:

$$\frac{\|\tilde{K}(\mathbf{x}) - K(\mathbf{x})\|}{1 + \|K(\mathbf{x})\|} \to 0 \qquad \|\mathbf{x}\| \to \infty$$

Proposition

Assume:

- Densities f and $\tilde{f}$ are continuous and positive outside a bounded set
- Marginal densities f_i and $\tilde{f}_i$ are continuous
- $\tilde{f}(\mathbf{z}) \asymp f(\mathbf{z})$ for $\|\mathbf{z}\| \to \infty$
- $f(\mathbf{z}_n + \mathbf{p}_n) \asymp f(\mathbf{z}_n)$ if $\|\mathbf{z}_n\| \to \infty$ and $\|\mathbf{p}_n\|/\|\mathbf{z}_n\| \to 0$
- Meta densities g and $\tilde{g}$ have all marginals equal to a continuous positive symmetric density g_d

If

- $F_i(-t) \in RV_{\rho^-}$ and $1 - F_i(t) \in RV_{\rho^+}$ with $\rho^{\pm} < 0$
- $\tilde{F}_i(-t) \sim F_i(-t)$ and $1 - \tilde{F}_i(t) \sim 1 - F_i(t)$ as $t \to \infty$

then the meta densities $\tilde{g}(\mathbf{x})$ and $g(\mathbf{x})$ satisfy: $\tilde{g}(\mathbf{x}) \asymp g(\mathbf{x}) \qquad \|\mathbf{x}\| \to \infty$

Framework & assumptions

A class of densities with level sets of the same shape (1/2)

- Set D: bounded convex open set containing the origin
- Gauge function of D: a unique function n_D with the properties

$$\{n_D < 1\} = D \qquad n_D(r\mathbf{z}) = r n_D(\mathbf{z}) \qquad r > 0,\ \mathbf{z} \in \mathbb{R}^d$$

- f_0: continuous, strictly decreasing positive function on $[0, \infty)$
- Then $f : \mathbf{z} \mapsto f_0(n_D(\mathbf{z}))$ is unimodal with convex level sets all of the same shape
- Assume f is a probability density

A class of densities with level sets of the same shape (2/2)

Note:

- If $f_0 \in RV_{-(\lambda+d)}$ then
  (i) f integrable
  (ii) marginal densities $f_i \in RV_{-(\lambda+1)}$

- (i) & (ii) remain true if
  - $f(\mathbf{z}) \sim f_0(n_D(\mathbf{z}))$ for $\|\mathbf{z}\| \to \infty$
  - D is a bounded star-shaped open set with continuous boundary

Definition (Standard set-up)

- f: a continuous density on $\mathbb{R}^d$, positive outside a bounded set
- $f(\mathbf{z}) \sim f_0(n_D(\mathbf{z}))$ for $\|\mathbf{z}\| \to \infty$, where
  - f_0: continuous, strictly decreasing
  - $f_0 \in RV_{-(\lambda+d)}$
  - D: bounded star-shaped open set containing the origin, with continuous boundary
- meta density g with marginal densities g_d satisfying
  - g_d: continuous, positive, symmetric
  - $g_d \sim e^{-\psi}$, a von Mises function; i.e.

$$\psi'(s) > 0, \quad \psi'(s) \to \infty, \quad (1/\psi')'(s) \to 0 \qquad s \to \infty$$

- Additional condition:

$$\psi \in RV_\theta, \quad \theta > 0 \qquad (\star)$$

- Remarks:
  - $(\star)$ is necessary to have a limit shape
  - $(\star)$ is satisfied for normal, Laplace, Weibull densities and densities of the form $g_d(s) \sim as^b e^{-ps^\theta}$, $s \to \infty$, $a, p, \theta > 0$

Derivation of the limit set

Main idea: The limit set may be described as the level set of a continuous function obtained from the meta density by scaling and power norming

Limit set

- Under the standard set-up & ($\bigstar$), level sets of g may be scaled to converge to a limit set, E
- There exists a compact set E such that

$$\frac{g(s\mathbf{u})}{g(s\mathbf{1})} \longrightarrow \begin{cases} \infty & \mathbf{u} \in \mathrm{int}(E) \\ 0 & \mathbf{u} \in E^c \end{cases} \qquad s \to \infty$$

- For a proper limit function for the quotient, use power norming to dampen exponential decrease by constructing functions

$$\left(\frac{g(s\mathbf{u})}{g(s\mathbf{1})}\right)^{\epsilon(s)} \qquad \text{where} \quad \epsilon(s) \to 0 \qquad s \to \infty$$

- Exponent $\epsilon(s)$ may be chosen so that χ_s converges to a continuous function uniformly on compact sets in $\mathbb{R}^d$

Limit set

- Write $g = e^{-\gamma}$
- Then

$$\left(\frac{g(s\mathbf{u})}{g(s\mathbf{1})}\right)^{\epsilon(s)} = \exp\left\{\left(\gamma(s\mathbf{1}) - \gamma(s\mathbf{u})\right)\epsilon(s)\right\}$$

- We show

$$\chi_s(\mathbf{u}) := \frac{\gamma(s\mathbf{1}) - \gamma(s\mathbf{u})}{\psi(s)/\lambda} \to \chi(\mathbf{u}) \qquad s \to \infty, \ \mathbf{u} \neq 0$$

That is,

$$\gamma(s\mathbf{1}) - \gamma(s\mathbf{u}) \sim \chi(\mathbf{u})\,\psi(s)/\lambda \qquad s \to \infty$$

- Hence

$$\mathrm{int}(E) = \{\chi > 0\} \qquad \partial E = \{\chi = 0\}$$

Next step: determine the limit function χ

The limit function χ

- First assume $f(\mathbf{z}) = f_0(\|\mathbf{z}\|_\infty)$ for a continuous strictly decreasing function $f_0 \in RV_{-(\lambda+d)}$

- Under the standard set-up & ($\bigstar$), it can be shown for $v = \|\mathbf{u}\|_\infty > 0$

$$\chi_s(\mathbf{u}) \to \chi(\mathbf{u}) = |u_1|^\theta + \cdots + |u_d|^\theta + \lambda - (\lambda + d)v^\theta \qquad s \to \infty$$

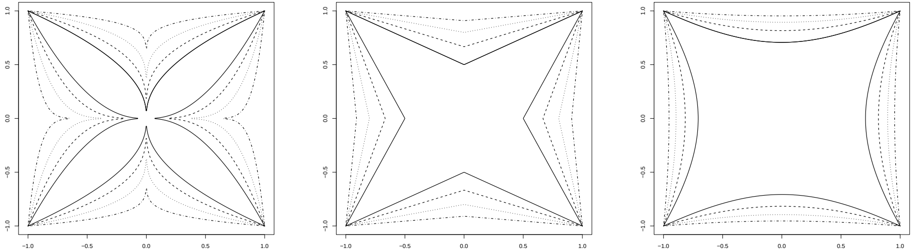
  - Convergence is uniform on $\Pi_r \setminus \epsilon B$ for any $r \geq 1$ and $\epsilon > 0$, where $\Pi_r = \{\mathbf{x} \mid |x_i| \leq x_d \leq r\}$ (upside down pyramid)

- Hence, the limit set is given by

$$E := E_{\lambda,\theta} := \{\mathbf{u} \in \mathbb{R}^d \setminus \{\mathbf{0}\} \mid |u_1|^\theta + \cdots + |u_d|^\theta + \lambda \geq (\lambda + d)\|\mathbf{u}\|_\infty^\theta\} \quad (\clubsuit)$$

Examples of limit sets

$\theta = 0.1$ $\theta = 1$ $\theta = 2$



Legend: $\lambda = 1$ (solid), $\lambda = 2$ (dashed), $\lambda = 4$ (dotted), $\lambda = 10$ (dotdash)

Limit sets for sample clouds

We show:

- For sample clouds from the meta density g there is a limit shape
- If
  - $\mathbf{X}_1$, $\mathbf{X}_2$,... is a random sample from meta density g
  - Scaling factor r_n is chosen s.t. $ng(r_n\mathbf{1}) \to 1$

then the scaled sample cloud $N_n = \{\mathbf{X}_1/r_n, \ldots, \mathbf{X}_n/r_n\}$ roughly fills out the limit set E

Definition (convergence of measures and sample clouds onto a set)

- E: a compact set in $\mathbb{R}^d$
- μ_n: finite measures

We say μ_n converge onto E if

- $\mu_n(\mathbf{p} + \epsilon B) \to \infty$ for any ϵ-ball centered in a point $\mathbf{p} \in E$
- $\mu_n(U^c) \to 0$ for all open sets U containing E

The finite point processes $N_n = \{\mathbf{X}_1/r_n, \ldots, \mathbf{X}_n/r_n\}$ converge onto E if

- $\mathbb{P}\{N_n(\mathbf{p} + \epsilon B) > m\} \to 1 \qquad m > 1,\ \epsilon > 0,\ \mathbf{p} \in E$
- $\mathbb{P}\{N_n(U^c) > 0\} \to 0$ for open sets U containing E

Proposition (criterion for convergence of sample clouds)

- N_n: an n-point sample cloud from a probability distribution π_n on $\mathbb{R}^d$
- N_n converges onto E if the mean measures $\mu_n = n\pi_n$ converge onto E

Main result

Theorem

Let:

- f & g_d satisfy assumptions of the standard set-up & ($\bigstar$)
- g: meta density with marginals g_d based on original density f
- $r_n > 0$ satisfies $g_d(r_n) \sim 1/n$
- $E = E_{\lambda,\theta}$: closed subset of $C = [-1, 1]^d$ defined in ($\clubsuit$)

Then:

- Level sets $\{g \geq 1/n\}$ scaled by r_n converge to E
- For any sequence of independent observations $\mathbf{X}_n$ from meta density g, the scaled sample clouds $N_n = \{\mathbf{X}_1/r_n, \ldots, \mathbf{X}_n/r_n\}$ converge onto E

How sensitive is the limit shape to small perturbations of the original density?

Exploring sensitivity of the limit shape (1/4)

Example

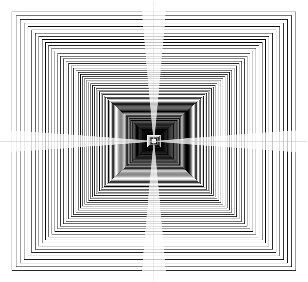
- Assume f: a density on $\mathbb{R}^2$ with square levels sets and Student t marginals $f_d(t) \sim 1/2t^2, \quad t \to \infty$
- Delete the mass on a strip $T = \{(x, y) \mid |x| \leq \frac{y}{\log y}, \ y \geq e\}$, and on sets obtained by reflections $(x, y) \mapsto (y, x), (-x, -y), (-y, -x)$
- Compensate for the lost mass by increasing f in compact neighbourhood of the origin
- We obtain a new density $\tilde{f}$; assume $\tilde{f}_d(t) \sim 1/2t^2, \quad t \to \infty$
- Choose $K_d(s) = e^s$, $s \geq s_0$
- Then $g_d(s) \sim e^{-s}/2, \quad s \to \infty$

Exploring sensitivity of the limit shape (2/4)

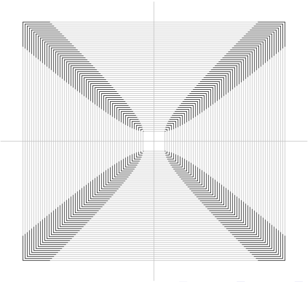
Example (cont'd)

- Edge of a square: $[-e^n, e^n] \times \{e^n\} \overset{\mathrm{K}^{-1}}{\mapsto} [-n, n] \times \{n\}$
- Strip T on the edge: $[-e^n/n, e^n/n] \times \{e^n\} \overset{\mathrm{K}^{-1}}{\mapsto} [-n + \log n, n - \log n] \times \{n\}$

Squares $C_n = e^n C$ in **z**-space



Squares $C_n = nC$ in **x**-space



Exploring sensitivity of the limit shape (3/4)

Example (cont'd)

We have:

- $\tilde{f}$ close to f such that $\tilde{g}$ vanishes everywhere except on a thin strip around the diagonals
- Scaled sample clouds from $\tilde{g}$ converge onto the cross consisting of the two diagonals of the standard square $C = [-1, 1]^2$
- Scaled sample clouds from $\tilde{f}$ and f converge to the same Poisson point process with intensity $h(\mathbf{w}) = 1/\|\mathbf{w}\|_\infty^3$
- Hence
  - Coordinatewise maxima from $\tilde{f}$ and f have the same limit $\Rightarrow$
  - Coordinatewise maxima from $\tilde{g}$ and g also have the same limit
  - But drastic change in the shape of the limit sets for $\tilde{g}$ and g

Exploring sensitivity of the limit shape (4/4)

Theorem

- F: a df on $[0, \infty)^d$ with marginals F_d continuous, strictly increasing
- $1 - F_d \in RV_{-\lambda}$
- Suppose dF does not charge set $T = T_\epsilon \setminus \epsilon^{-1} B$ for some $\epsilon > 0$, where

$$T_\epsilon = \{\mathbf{z} = (z_1, \ldots, z_d) \mid \frac{\min_i |z_i|}{r} < \frac{\epsilon}{\log r}, \ r = \|\mathbf{z}\|_2 > 1 + 1/\epsilon\}$$

- G: meta df with marginals G_d continuous, strictly increasing on $[0, \infty)$
- $1 - G_d \sim e^{-\psi}$, $\psi \in RV_\theta$, $\theta > 0$
- Scaling factor r_n: $1 - G_d(r_n) = 1/n$

Then measures $d\mu_n(\mathbf{u}) = ndG(r_n \mathbf{u})$ converge onto the set $E = \{t\mathbf{1} \mid 0 \le t \le 1\}$ in $[0, \infty)^d$

Concluding remarks

- Original and meta densities have the same copula, yet a relation between the shapes of their level sets is lost in the limit
- The limit set is unchanged if we replace the original density f by a density which is weakly asymptotic to f
- Sensitivity of the limit shape may be radical due to even slight perturbations of the original density
- The limit shape gives a very rough picture
- Next step: closer look at the edge of the scaled sample clouds under a more refined scaling

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