Third Nomura Lecture

From Dutch Dykes to Value-at-Risk: Extreme Value Theory and Copulae as Risk Management Tools

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En in alle gewesten
Wordt de stem van het water
Met zijn eeuwige rampen
Gevreesd en gehoord

(Marsman, 1938)

Throughout the centuries, the Dutch coast has been suffering numerous disastrous floods:

1421, 1570, 1775, 1825, 1916, 1925, 1953, 2002*

1421: The St. Elisabeth flood

1570: The All Saints flood

* Not complete

"Springtij en orkaan veroorzaken nationale ramp. Nederland in grote watersnood."

(De Yssel - en Lekstreek, 6/2/53)

January 31 – February 1, 1953* (February flood):

- $1\,836$ people killed
- $72\,000$ people evacuated
- $49\,000$ houses and farms flooded
- $201\,000$ cattle drowned
- 500 km of sea dykes destroyed, ≥ 400 breaches
- $200\,000$ ha of land flooded

* Antwerp (Schoten), February 3, 1953

The Delta Project

- coastal defence against flooding
- required dyke height at location l: $h_\delta(l)$
- safety margin: **MYSH(l)**= Maximum Yearly Sea Height at location l:

$$P(\text{MYSH}(l) > h_\delta(l)) \quad \text{“small”}$$

where “small” means:

* 10^{-4} at $l =$ Randstad
* $4^{-1} \times 10^{-3}$ at $l =$ Delta area and the North

- similar safety requirements for rivers but with safety values in the range of 10^{-3}–10^{-2}
- solution for $l =$ Randstad: $h_\delta(l) = \text{NAP} + 5.14$ m

Hence, we can reformulate the problem as:

- "small" $= 1 - \alpha \in (0, 1)$ (small: EVT)
- $l =$ fixed (different l: copulae)
- MYSH$(l) = X$, a rv with df F (co-variables: EVT, copulae, non-stationarity)
- $h_\delta(l) = q_\alpha$

$$\boldsymbol{q_\alpha = F^{-1}(\alpha)} \qquad (\alpha - \text{quantile})$$

The February flood happened as a coincidence of several factors

- very low pressure over Scotland
- very strong North-Westerly storm
- "springtij" (extremely high tide)

All contributing to "the perfect storm scenario"

THE REGULATORY ENVIRONMENT

Financial Industry

- Basel Accord for Banking Supervision (1988)
  - Cooke ratio, “haircut” principle, too coarse
- Amendment to the Accord (1996)
  - VaR for Market Risk, Internal models, Derivatives, Netting
- **Basel II** (1998 – 2007)
  - Three Pillar approach
  - Increased granularity for Credit Risk
  - Operational Risk

THE REGULATORY ENVIRONMENT

Insurance

- Solvency 1 (1997)
  - Solvency margin as % of premium (non-life), of technical provisions (life)
- **Solvency 2** (2000–2004)
  - Principle-based (not rule-based)
  - Mark-to-market (/model) for assets and liabilities (ALM)
  - Target capital versus solvency capital
  - Explicit modelling of dependencies and stress scenarios
- **Integrated Risk Management**

Value-at-Risk (Amendment to the Basel Accord, 1996)

Ingredients and Definition:

- $V(\tau)$: value of portfolio at time τ

- loss rv: $L_{[\tau,\tau+\Delta t]} = -(V(\tau+\Delta t) - V(\tau))$

- loss df: $P\left(L_{[\tau,\tau+\Delta t]} \leq x\right)$ (unconditional)

  $P\left(L_{[\tau,\tau+\Delta t]} \leq x | \mathcal{F}_t\right)$ (conditional)

  (P & L: Profit and Loss Distribution)

  often: $\tau_t = t \times \Delta t$, $t \in \mathbb{N}$ ($\Delta t = 1$ day, 10 days, 1 year)

- $L_{t+1} = L_{[\tau_t,\tau_{t+1}]} = -\left(V_{t+1} - V_t\right)$

- the mapping: $V_t = f(\tau_t, \boldsymbol{Z}_t)$, $\boldsymbol{Z}_t' = (Z_{1,t}, \ldots, Z_{d,t})$

  (in practice, d is large, $d \geq 1\,000$, say)

- risk factor changes: $\boldsymbol{X}_t = \boldsymbol{Z}_t - \boldsymbol{Z}_{t-1}$

- the loss operator maps risks into losses:

$$l_{[t]}(\boldsymbol{x}) = -\left(f\left(\tau_{t+1}, \boldsymbol{Z}_t + \boldsymbol{x}\right) - f\left(\tau_t, \boldsymbol{Z}_t\right)\right), \quad \boldsymbol{x} \in \mathbb{R}^d$$

  hence

$$L_{t+1} = l_{[t]}\left(\boldsymbol{X}_{t+1}\right)$$

- for f smooth (e.g. European Call case, Black-Scholes):

$$L_{t+1} \approx L_{t+1}^{\Delta} := -\left(f_\tau\left(\tau_t, \boldsymbol{Z}_t\right) \Delta t + \sum_{i=1}^{d} f_{z_i}\left(\tau_t, \boldsymbol{Z}_t\right) X_{t+1,i}\right)$$

  ↑ the Greeks (delta, "vega", theta, rho)

Main approaches to (market) risk management:

(M1) Variance-covariance (normal) method

Assumption: $\boldsymbol{X}_{t+1} \sim \mathsf{N}_d(\boldsymbol{\mu}, \Sigma)$

hence $L^{\Delta}_{t+1} \sim \mathsf{N}_1\left(-ct - \boldsymbol{b}_t'\boldsymbol{\mu}, \boldsymbol{b}_t'\Sigma\boldsymbol{b}_t\right)$

(M2) Historical simulation

No assumptions!

Use empirical data: $\boldsymbol{x}_{t-n+1}, \dots, \boldsymbol{x}_t$ and

construct pseudo portfolio values $\left\{\tilde{l}_s = l_{[t]}(\boldsymbol{x}_s) : s = t-n+1, \dots, t\right\}$

yielding a histogram estimate of the P&L

(M3) Monte-Carlo simulation

Similar to (M2), but use historical data to calibrate some $d-$dimensional model for the law of $\boldsymbol{X}_{t+1}$ from which we simulate (Monte-Carlo) pseudo portfolio values

How to measure risk?

$\rho : \mathcal{M} = \{\text{all portfolios}\} \longrightarrow \mathbb{R}$

satisfying certain axioms (ADEH-coherence) yielding for $L \in \mathcal{M}$:

$$\rho(L) = \rho_{\mathcal{P}}(L) = \sup \left\{ \mathbb{E}^Q(L) : Q \in \mathcal{P} \right\}$$

Examples in practice are:

- $\rho_1(L) = \text{VaR}_\alpha(L) = F_L^{-1}(\alpha) \qquad (L_{t+1} = L, \text{ say})$

  Value-at-Risk at confidence α (not always coherent)

- $\rho_2(L) = \text{ES}_\alpha(L) = \mathbb{E}(L | L > \text{VaR}_\alpha(L))$

  Expected shortfall (easy adaptation is always coherent)

Basel Accord (Amendment) for Market Risk

- Internal models
- Regulatory capital based on 99% VaR with 10 day holding period
- Internal capital allocation using 95% VaR, 1 day
- Important issues:
  - scaling: $\sqrt{10}$–rule
  - backtesting: RC based on $k\,\mathrm{VaR}_{99}^{10}$, $3 \le k \le 4$
  - other risk categories (credit, operational)
  - risk aggregation

Remark: Within Solvency 2 (Insurance) use ES_α

FROM THE WORLD OF FINANCE

- "Extreme, synchronized rises and falls in financial markets occur infrequently but they do occur. The problem with the models is that they did not assign a high enough chance of occurrence to the scenario in which **many things go wrong at the same time** - the "perfect storm" scenario"

(Business Week, September 1998)

- Consulting for a large bank, topics to be discussed were:
  - general introduction to the topic of EVT
  - common pitfalls and its application to financial risk management
  - the application of EVT to the quantification of operational risk
  - general introduction to the topic of copulae and their possible use in financial risk management
  - sources of information to look at if we want to find out more

(London, March 2004)

EVT AND COPULAE

- Extreme Value Theory (EVT): stochastic models for

$$M_n = \max(X_1, \ldots, X_n) \qquad (d = 1)$$

$$P(X - u \le x | X > u), u \to \infty \qquad (d = 1, \text{ POT-method})$$

(non-obvious) generalization to $d \ge 2$

- Copulae: multivariate df C on $[0,1]^d$ with uniform marginals

$$\begin{aligned} F_{\boldsymbol{X}}(\boldsymbol{x}) &= P\left(X_1 \le x_1, \ldots, X_d \le x_d\right), \qquad X_i \sim F_i, i = 1, \ldots, d \\ &= C\left(F_1(x_1), \ldots, F_d(x_d)\right) \end{aligned}$$

$F_{\boldsymbol{X}} \Longleftrightarrow (F_1, \ldots, F_d; C)$ (Sklar)

"$\Rightarrow$" : new copula models

"$\Leftarrow$" : stress testing

WHY ARE COPULAE USEFUL

- pedagogical: "Thinking beyond linear correlation"
- stress testing dependence: joint extremes, spillover, contagion, ...
- worst case analysis under incomplete information:

  given: $X_i \sim F_i$, $i = 1, \ldots, d$, marginal 1-period risks

  $\Psi(\mathbf{X})$: a financial position

  Δ: a 1-period risk or pricing measure

  task: find $\min \Delta(\Psi(\mathbf{X}))$ and $\max \Delta(\Psi(\mathbf{X}))$ under the above constraints

- eventually: finding better fitting dynamic models

THE FUNDAMENTAL THEOREMS OF QUANTITATIVE RISK MANAGEMENT (QRM)

- **(FTQRM - 1)** For elliptically distributed risk vectors, classical Risk Management tools like VaR, Markowitz portfolio approach, ... work fine:

  Recall:

  – $\mathbf{Y}$ in $\mathbb{R}^d$ is spherical if $\mathbf{Y} \stackrel{d}{=} U\mathbf{Y}$ for all **orthogonal** matrices U

  – $\mathbf{X} = A\mathbf{Y} + \mathbf{b}$, $A \in \mathbb{R}^{d\times d}$, $\mathbf{b} \in \mathbb{R}^d$ is called elliptical

  – Let $\mathbf{Z} \sim \mathrm{N}_d(\mathbf{0}, \mathbf{\Sigma})$, $W \geq 0$, **independent** of $\mathbf{Z}$, then

  $$\mathbf{X} = \boldsymbol{\mu} + W\mathbf{Z}$$

  is elliptical (multivariate Normal variance-mixtures)

  – If one takes

  $W = \sqrt{\nu/V}$, $V \sim \chi^2_\nu$, then $\mathbf{X}$ is multivariate t_ν

  W normal inverse Gaussian, then $\mathbf{X}$ is generalized hyperbolic

THE FUNDAMENTAL THEOREMS OF QUANTITATIVE RISK MANAGEMENT (QRM)

- **(FTQRM - 2)** Much more important!

  For non-elliptically distributed risk vectors, classical RM tools break down:

  - VaR is typically non-subadditive
  - risk capital allocation is non-consistent
  - portfolio optimization is risk-measure dependent
  - correlation based methods are insufficient

- **A**(n early) **stylized fact:**

  In practice, portfolio risk factors typically are **non-elliptical**

  Questions:

  - are these deviations relevant, important
  - what are tractable, non-elliptical models
  - how to go from static (one-period) to dynamic (multi-period) RM

EXAMPLE: the Merton model for corporate default (firm value model, latent variable model)

- portfolio $\{(X_i, k_i) : i = 1, \ldots, d\}$ firms, obligors

- obligor i defaults by end of year if $X_i \leq k_i$

  (firm value is less than value of debt, properly defined)

- modelling joint default: $P(X_1 \leq k_1, \ldots, X_d \leq k_d)$

  – classical Merton model: $\boldsymbol{X} \sim \mathrm{N}_d(\boldsymbol{\mu}, \Sigma)$
  – KMV: calibrate k_i via "distance to default" data
  – CreditMetrics: calibrate k_i using average default probabilities for different rating classes
  – Li model: X_i's as survival times are assumed exponential and use Gaussian copula

- hence standard industry models use Gaussian copula

- improvement using t-copula

- standardised equicorrelation ($\rho_i = \rho = 0.038$) matrix Σ calibrated so that for $i = 1, \ldots, d$, $P(X_i \leq k_i) = 0.005$ (medium credit quality in KMV/CreditMetrics)

- set $\nu = 10$ in t-model and perform $100\,000$ simulations on $d = 10\,000$ obligors to find the loss distribution

- use VaR concept to compare risks

  Results:

	min	25%	med	mean	75%	90%	95%	max
Gaussian	1	28	43	49.8	64	90	109	131
t	0	1	9	49.9	42	132	235	3 238

- more realistic t-model: block-t-copula (Lindskog, McNeil)

- has been used for banking and (re)insurance portfolios

Challenges (see `www.math.ethz.ch/~embrechts`)

EVT

- extremes in economics and the economics of extremes
- EVT applications to operational risk
- non-stationarity
- multivariate EVT: $P(\boldsymbol{X}|\boldsymbol{X} \in H_\alpha)$ for $P(\boldsymbol{X} \in H_\alpha) \to 0$

Copulae

- dynamic models
- canonical limit theorems
- optimization results
- calibration within specific RM applications
- properties of statistical estimates

De meeste salicheyt hanght
aen de hoochte van eenen dijck

(Andries Vierlingh, Tractaet van Dyckagie, 1578)

Regulators have criticised LTCM and banks for not stress-testing risk models against extreme market movements. ... The markets have been through the financial equivalent of several Hurricane Andrews hitting Florida all at once. Is the appropriate response to accept that it was mere bad luck to run into such a rare event - or to get new forecasting models that assume more storms in the future?

(The Economist, October 1998, after the LTCM rescue)