Mathematical Models in Finance

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Black-Scholes price, option, future, financial derivative, Brownian motion, Itô stochastic calculus, change of measure, Girsanov transformation, martingale, equivalent martingale measure, securitization of risks, arbitrage, interest rate, bond, Value at Risk (VaR)

Contents

Glossary

American-type option. An option which can be exercised at any time up to the time of maturity of the option.

Arbitrage. The possibility to make unlimited profits without accompanying risks.

ART. Alternative Risk Transfer.

Black-Scholes price, Black-Scholes formula. An analytic option pricing formula developed by Black, Scholes and Merton in 1973.

Bond. A security paying regularly interest (cash bond) or a lump sum at maturity.

Brownian motion. The basic process in financial mathematics which describes the uncertainty of the market. In the Black-Scholes model its exponential (geometric Brownian motion) models price movements.

Call option. A contract which entitles, but not obliges one to buy a security at/by a future instant of time (maturity).

Cash bond. A continuously compounded bond which is appreciated at the instantaneous interest rate.

Change of measure. A mathematical technique which enables one to change the distribution of a stochastic process to another distribution. This step sometimes allows one to calculate certain formulae, such as expectations, in a simpler way.

Claim. A payment to be made in the future according to a contract such as an option or a future.

Conditional expectation. The expectation of a random variable given a σ-field representing gathered information.

Contingent claim. The value of a financial derivative; see claim.

Derivative. A security whose value depends on underlying market securities. Future values of a derivative are subject to uncertainty.

Discounting. Scaling a future price or reward down to make it comparable to present prices.

European-type option. An option which is exercised at time of maturity.

Equivalent measures. Two measures are equivalent if they vanish on the same sets.

Exercise date. A set future date at which an option may be exercised.

Exercise price. See strike price.

Exotic option. New (and therefore unusual) financial derivative which will become standard after some time or disappear from the market.

Filtration. An increasing family of σ-fields, describing an increasing stream of information, for example about the developments in the market.

Foreign exchange rate. The rate at which one currency is priced in terms of another.

Forward rate. Forward price of instantaneous borrowing.

Future. A contract with the obligation to sell or buy stock at a fixed price by/at an agreed date.

Geometric Brownian motion. The exponential of Brownian motion with linear drift. It is the basic price process in the Black-Scholes model.

Girsanov transformation. A mathematical theorem which allows one to change the distribution of Brownian motion with drift to the distribution of Brownian motion. It allows one to give an alternative interpretation of the Black-Scholes formula as an expectation under the changed distribution.

Heath-Jarrow-Morton. An interest rate model.

Hedge. A trading strategy to protect against market risks.

Incompleteness. Derivatives cannot be uniquely priced, nor perfectly hedged.

Instantaneous rate. Rate at which interest is paid.

Itô calculus. A stochastic calculus which gives the rules for integrating functions of Brownian motion. In particular, the Itô integral is an integral with respect to Brownian motion and the Itô formula is the chain rule of this calculus.

Martingale. A mathematical model for a fair game.

Maturity. Expiration date of a contract.

Numeraire. A basic security relative to which the value of another security is determined, for example the cash bond in the Black-Scholes model.

Option. A contract which entitles, but does not oblige one to buy/sell something by/at a future date.

Portfolio. A collection of securities.

Put option. A contract which entitles one to sell something by/at a future date (maturity).

Replicating strategy. A self-financing trading strategy which hedges a claim perfectly.

Security. A contract with a financial promise.

Self-financing. A trading strategy which changes the portfolio value only through price changes.

Short rate. See instantaneous rate.

Stochastic calculus. See Itô calculus.

Stochastic process. A collection of random variables indexed by time.

Strike price. A fixed price at which an asset may be bought or sold.

Trading strategy. The number of shares held in different securities as a function of time.

Transaction costs. A charge for buying or selling a security.

Underlying. A basic market security (stock, bond, currency, etc.) on which other securities can be based upon.

Volatility. The variability of the market price, described by the coefficient in front of Brownian motion in a stochastic differential equation.

Yield. Average interest rate of a bond.

Zero-coupon bond. A bond delivering a fixed amount of cash at maturity.

Summary

Over the last 25 years the financial markets have gone through an enormous development. The introduction of financial derivatives such as options and futures on underlyings (stock, bond, currencies) has led to a new quality of the securitization of financial risks. The basic idea of a financial derivative is to buy insurance for risky assets on the market, i.e., to find participants in the market who are willing to share the risks and the profits of future developments in the market which are subject to uncertainty.

The pricing of these financial instruments is based on an advanced mathematical theory, called Itô stochastic calculus. The basic model for an uncertain price is described by Brownian motion and related differential equations. The pricing of a European call option by Black, Scholes and Merton in 1973 (the Nobel prize winning Black-Scholes formula) was a breakthrough in the understanding and valuing of financial derivatives. Their approach has become the firm basis for modern financial mathematics which uses advanced tools such as martingale theory and stochastic control to find adequate solutions to the pricing of a world-wide enormously increasing number of derivatives.

1. A tutorial on mathematical finance without formulae

Financial mathematics has become one of the most recent success stories of mathematics and probability theory in particular. In contrast to many mathematical achievements which are known to specialists only, the Black-Scholes option pricing formula has gained popularity not only among practitioners in finance. Its fame has attracted the attention of economists, physicists, econometricians, statisticians, etc. Hundreds of popular articles in economics, physics, mathematics journals, and far beyond, have been written about this topic. BBC made a documentary about the Nobel prize winning formula. University professors explain the formula to high school students in order to convince them that mathematics is a topic worth studying....

Mathematical models have been used in economics for a long time. By now, operations research, econometrics and time series analysis constitute major parts of the curricula of business schools and economics departments. However, this article will not focus on these more classical topics but mainly on the approach which was started by Fisher Black (1938-1995), Myron Scholes [6] and Robert Merton [24] in 1973. What was so entirely new that Scholes and Merton were awarded the Nobel prize for economics in 1997?

In 1973 the Chicago Board of Trade (CBOT) started trading so-called *options*, *futures* and other *financial derivatives*. For example, a *European call option* is a ticket (a contract) which entitles its purchaser to buy one share of a *risky asset* (such as the stock of Microsoft) at a fixed price (*strike price*) at a known date in the future (*time of expiration* or *maturity*). It is natural to ask: how much would the purchaser of the call option be willing to pay? Clearly, there are various problems. The purchaser gains a positive amount of money only if the price of Microsoft at maturity is indeed above the strike price. Then he can buy the share at the lower strike price and sell it at the higher market price to somebody else. However, since the price of Microsoft is subject to uncertainty, one does not know in advance whether this price will exceed the strike price in the future. There is a chance that the share price at maturity will reach a level below the strike price. Then it would be cheaper to buy a share of Microsoft on the stock market. However, an option (in contrast to a future) is not a contract that obliges its purchaser to buy the share. His gain at maturity would be zero in this case.

Black, Scholes and Merton approached the problem of pricing an option in a physicist's way. They started by assuming a reasonable model for the price of a risky asset. The search for such a model has a long history. Empirical (statistical, econometric) research has shown that changes of prices in the future are hardly predictable by mathematical models. In the economics literature this fact runs under the name of "random walk hypothesis". A random walk is defined at discrete equidistant instants of time. In finance, however, one is mainly interested in modeling prices at every instant of time. We call this a continuous time model. *Brownian motion* is a natural analogue of a random walk in continuous time. It is a physical model for the movement of a small particle suspended in a liquid and has been studied in the physics literature since the beginning of the 20th century. One of the famous contributors to this theory was Albert Einstein [11]. Before Einstein, a young French PhD student, called Louis Bachelier [1], proposed in his 1900 thesis Brownian motion as a model for speculative prices. One of the imperfections of this model is that Brownian motion can assume negative values, and this might have been one of the reasons that his model had been forgotten for a long time. Only in the 1960ies the economist Samuelson (Nobel prize for economics in 1970) propagated the exponential of Brownian motion (so-called *geometric Brownian motion*) for modeling prices which are subject to uncertainty to his students at M.I.T.

In the work of Black, Scholes and Merton, geometric Brownian motion is the basic mathematical model for price movements. Moreover, they realized that Brownian motion is closely related to a deep mathematical theory, *stochastic* or *Itô calculus*, named after the Japanese mathematician Kiyosi Itô who developed this theory in the 1940ies. Classical calculus is about differentiation and integration of "smooth" functions. In contrast to the latter, paths of Brownian motion are extremely irregular (non-differentiable) functions and therefore classical calculus is not a suitable tool. Although Itô calculus had been known and used by certain physicists, engineers and other applied people for some time, it did not become very popular outside some groups of specialists. By now, everybody working in (theoretical or practical) finance knows about the basic rules of Itô calculus.

The main contribution of the fathers of the option pricing formula, however, was a totally new idea on the economics side. They argued that the seller of a European call option (usually a big financial institution) would not wait passively until time of maturity. On the contrary, if she was a rational person she would invest a certain amount of money in the same stock (e.g. Microsoft) and in a *riskless asset* (e.g. a savings account with fixed interest rate) according to a dynamic trading strategy such that the value of the portfolio at maturity would be exactly the value of the option at maturity: either zero if the share price was below the strike price or, otherwise, the positive difference between the share and the strike prices. A trading strategy which *replicates* the value of the option at maturity is called a *hedge*. The existence of such a hedge is a justification for the price of an option, but it is important in itself for financial practice. The amount of money which the seller of the option had to invest for her hedge would be a fair price for the option. Moreover, Black, Scholes and Merton argued that, if the option was sold at a price other than the Black-Scholes price, a rational person could use this fact to make unlimited profits without accompanying risk (so-called *arbitrage*).

The Black-Scholes price, expressed in the famous Black-Scholes formula, because of its convincing rationale, became a major success and was well accepted by practitioners in the financial markets. The *New York Times* (15 October, 1997) wrote: "Soon traders were valuing options in the floor of the exchange, punching half a dozen numbers into electronic calculators hard-wired with the formula. ... Mr. Black and Mr. Scholes became highly regarded at the exchange that when they visited, traders would give them a standing ovation."

Despite various imperfections of the underlying mathematical model, the Black-Scholes approach was a starting point for pricing other kinds of financial derivatives. For example, in practice European calls are less frequently traded than options which can be exercised at any time before or at time of maturity (American-type options). Moreover, an option, future, etc., does not necessarily have to be linked to a share price, but to a composite stock index such as the Dow Jones, Nikkei, Standard & Poors 500, DAX, etc., or to bond prices, foreign exchange rates, or any other underlying which is due to uncertainty. The basic aim of a financial derivative is *securitization of risks*; the Black-Scholes approach allows the seller and the purchaser of a properly priced derivative to hedge against future risks due to uncertainty of price movements.

The large variety of financial products which has been created by financial institutions became a challenge to mathematics and in particular to the specialists of Itô calculus. Since the end of the 1970ies, they have pushed forward the development of financial mathematics by exploiting the most advanced tools, in particular functional analysis, martingale theory, stochastic control, partial differential equations. By now, financial mathematics is a well established theory with a great future which is taught at mathematics and economics departments all over the world.

Clearly, the Black-Scholes world is an idealization of the real financial world. For example, the mathematical assumption of geometric Brownian motion as a model for a risky price is known to be in contradiction with real-life price data. Once in a while the stock market is shaken by shocks (due to political events, recessions, bursts of economic activity, etc.) resulting in unexpected price jumps. Such a behavior cannot happen in the Black-Scholes world. Over the last 20 years several events showed the limitations of the model. In October 1987 (Black Monday) a major crash affected the New York Stock Exchange causing financial losses of several billion U.S. dollars. Although it did not have a major impact on the market, the crash of Barings Bank made the world aware of the fact that financial derivatives can be very dangerous when handled by careless management. Quite recently, in October 1998, the turmoil around Long Term Capital Management (LTCM), a hedge fund worth hundreds of billions U.S. Dollars, with both Scholes and Merton as founding members, gave the public more reasons to have less confidence in a (seemingly perfect) mathematical formula. The events around LTCM caused, within a week's time, a 13.7 % loss of the U.S. Dollar against the Japanese Yen. *Newsweek* (19 October, 1998) asked: "The buck is bruised. So is another big hedge fund. What's going on?" By now, the derivatives business has exceeded an annual volume of $15 trillion. It is one of the financial fundamentals on which modern society is based.

Mathematical formulae can help to make rational decisions in finance. However, the above examples show quite clearly that too much trust in formulae, paired with wrong decisions of management, can lead to fatal consequences not only for one particular company, but for whole national economies. In view of the enormous amounts of money involved in derivatives it was realized quite early in the 1990ies that the financial industry had to facilitate the measurement of risk. In 1992 the so-called Basel Committee of the Bank for International Settlements (representing 27 European members plus the U.S., Canada, Japan, Australia, and South Africa) presented proposals to estimate market risk and to define the resulting capital requirements to be implemented in the banking sector. The European Union (EEC 93/6) approved a directive, effective January 1996, that mandates banks and investment firms to set capital aside to cover market risks. In the U.S., the Securities and Exchange Commission fulfills a similar regulatory function. Measuring and estimating financial risks in its various forms has become another challenge to mathematics, in particular statistics. Based on probabilistic models, various statistical methods have been developed to quantify financial risks. Among them, the Value at Risk (VaR) has become most popular. Companies such as *RiskMetrics* have specialized in advising the financial industry how to measure and estimate risks, and government regulators control financial institutions as to whether they satisfy certain risk standards.

An exciting new development is the birth of bank-assurance: we witness a convergence of financial and actuarial thinking. One relevant buzz-word is ART: Alternative Risk Transfer. Examples are Catastrophe Bonds; the coupon payment (and possibly the principal re-payment) is contingent on the (non-)occurrence of a catastrophic event. Think for the latter of an earthquake or hurricane, say. Other examples are energy and weather derivatives. For a very accessible introduction on the art of ART, see [32].

Readers wanting an elementary introduction to financial mathematics, avoiding the use of Brownian motion, can turn to texts like [9] and [18], where the basic concepts are presented via discrete trees; at any instant of time the price process can visit only a finite number of states (typically up or down). Very readable texts introducing the Brownian motion based models are for instance [2, 21, 25, 27]. References to more advanced literature are given at the end of the sections below. In [3, 23] some interesting historical discussions on the subject of derivatives and financial risk can be found. Nick Dunbar in [10] gives an excellent historical account of the events leading up to and surrounding the LTCM crash. Finally, for those wanting a more detailed understanding of the Barings case, [22] is a must.

2. The pricing of financial derivatives by mathematical means

2.1. The approach by Black, Scholes and Merton. The price P_t of a risky asset, the share price of a particular stock, say, is described by a geometric Brownian motion

$$P_t = P_0 \exp\left\{\sigma B_t + \mu\, t - 0.5\, \sigma^2\, t\right\} ,$$

where $(B_t, t \geq 0)$ stands for Brownian motion (a stochastic process with independent stationary increments, continuous sample paths and such that B_t has a normal distribution with mean zero, variance t), μ is the *mean rate of return* and $\sigma > 0$ is the *volatility.* In particular, for fixed t, P_t has a lognormal distribution. The larger σ the stronger the oscillations of P_t around its mean value. Hence, σ describes the degree of variability of the price. Alternatively, P_t satisfies the *Itô stochastic differential equation* (SDE)

$$dP_t = P_t\,[\mu\, dt + \sigma\, dB_t] .$$

The above SDE has to be interpreted as *Itô integral equation* :

$$P_t = P_0 + \mu \int_0^t P_s\, ds + \sigma \int_0^t P_s\, dB_s ,$$

where the first integral is an ordinary Riemann integral, the second an Itô integral. This means that it stands for the limit of the sums

$$\int_0^t P_s\, dB_s \approx \sum_i P_{t_{i-1}}\,[B_{t_i} - B_{t_{i-1}}] \tag{1}$$

along any partitions $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$ of the interval $[0, t]$ with $\max_i(t_i - t_{i-1}) \to 0$ as $n \to \infty$. Since the paths of Brownian motion are not of bounded variation, unlike the Riemann-Stieltjes integral, the limit *cannot* be taken along a fixed Brownian path, but in the mean square sense. For the definition of the Itô integral it is crucial that the sums on the right-hand side of (1) have to be constituted in a non-anticipating way, i.e., by taking the integrand at the beginning of the time intervals and assuming that it does not depend on future values of geometric Brownian motion, i.e., future values of the price.

The value β_t of a riskless asset (a *cash bond* such as a savings account) with *riskless interest rate* $r > 0$ evolves according to

$$\beta_t = \beta_0\, \mathrm{e}^{rt} .$$

At time t, the portfolio consists of a_t shares in stock and b_t shares in bond (a_t, b_t may assume any real values; a negative value of a_t corresponds to short sales of stock, a negative b_t to borrowing money at the riskless interest rate r). Hence the portfolio has value

$$V_t = a_t\, P_t + b_t\, \beta_t .$$

A crucial assumption is that the trading strategy (a_t, b_t) is *self-financing*:

$$dV_t = a_t\, dP_t + b_t\, d\beta_t , \tag{2}$$

i.e., changes of wealth are due only to changes in prices. The latter equation has again to be interpreted in the Itô sense:

$$V_t = V_0 + \int_0^t a_s\, dP_s + \int_0^t b_s\, d\beta_s ,$$

where the first integral is defined in a way similar to (1).

At time of maturity T, the value of a *European call option* with strike price K is given by

$$(P_T - K)_+ = \max(0, P_T - K)\,,$$

i.e., the purchaser of the option gains $\$(P_T - K)_+$. For simplicity, we leave out the premium paid for the option. The purchaser of a *European put option* is entitled (but not obliged) to sell one share of stock at the strike price K at maturity, i.e., she gains $\$(K - P_T)_+$. The terminal value of an option can also be based on the whole Brownian path. For example, an *Asian call option* entitles one to a payment of $\$(T^{-1}\int_0^T P_s ds - K)_+$. Such a contract may typically be used in the oil market. The value of a financial derivative is called a *contingent claim*. It is a random variable H which depends on the development of the price P_t up to maturity.

The classical way of valuing a contingent claim is to assume that V_t can be represented as a smooth function of P_t and t:

$$V_t = u(T-t, P_t) \quad \text{such that} \quad V_T = u(0, P_T) = H\,,$$

i.e., the contingent claim only depends on the price at maturity. An application of the chain rule of Itô calculus, the *Itô formula* or *Itô lemma*, combined with the self-financing property (2), leads to the partial differential equation

$$u_t(t,x) = 0.5\,\sigma^2\,x^2\,u_{xx}(t,x) + r\,x\,u_x(t,x) - r\,u(t,x)\,, \quad u(0,x) = H\,, \quad x > 0\,, \quad t \in [0,T]\,.$$

For the case of a European call option, i.e., $H = (P_T - K)_+$, the latter equation has solution

$$\begin{aligned} u(t,x) &= x\,\Phi(g(t,x)) - K\,\mathrm{e}^{-rt}\,\Phi(h(t,x))\,, \quad \text{where} \\ g(t,x) &= [\sigma^2\,t]^{-0.5}\,\left[\log(x/K) + (r + 0.5\,\sigma^2)\,t\right]\,, \\ h(t,x) &= g(t,x) - \sigma\,t^{0.5}\,, \end{aligned}$$

and $\Phi(x) = [2\pi]^{-0.5}\int_{-\infty}^x \mathrm{e}^{-0.5\,y^2}\,dy$ is the standard normal distribution function. The Black-Scholes price of the contingent claim $(P_T - K)_+$ is the value of the portfolio at time zero, i.e.,

$$V_0 = u(T, P_0) = P_0\,\Phi(g(T,P_0)) - K\,\mathrm{e}^{-rT}\,\Phi(h(T,P_0))\,. \tag{3}$$

A self-financing strategy (a_t, b_t), a *hedge*, can be derived from the function $u(t,x)$ as well.

On 15 October, 1997, equation (3) was to be found in the *New York Times*. It was one day after Scholes's and Merton's nomination for the Nobel prize; (3) is the celebrated *Black-Scholes option pricing formula*. On September 17, 1998, the same formula appeared in *The Observer*. This time the reason behind its appearance was the LTCM debacle. The heading now reads: "Is this really the key to future wealth? Win bigger, lose bigger."

2.2. Pricing by change of measure. The above approach is similar to a physicist's approach who wants to describe the movement of a certain object by solving suitable differential equations. The new dimension of this approach is the uncertainty of the trajectory of the price movement. However, there exists a purely probabilistic approach to the pricing problem as well. This was early on recognized by Davis Kreps, Michael Harrison and Stan Pliska [16, 17] who pointed at the relationship with martingale theory. Their approach has the advantage that the contingent claim H may depend on the whole price movement up to time T, including all kinds of *exotic options.*

They observed that, according to a representation theorem of Itô, *every* discounted derivative of geometric Brownian motion has the stochastic integral representation:

$$e^{-rT}H = H_0 + \int_0^T a_t^{(H)}\, d[e^{-rt}P_t]\,, \tag{4}$$

for some constant H_0 and a process $(a_t^{(H)})$ which at time t only depends on the price movement up to time t. As a next step, one observes that the discounted price $(e^{-rt}P_t)$ can be transformed into a martingale, given one changes the underlying probability measure.

The notion of *martingale* is used in probability theory to describe a "fair game". This means the following. Assume the game is played continuously with value M_t at time t with information $\mathcal{F}_t$ about the game up to time t. (In probability theory the latter is described by an increasing family of σ-fields, called a *filtration.*) The game is *fair* if the expected value of future values M_{t+h}, $h > 0$, given the information $\mathcal{F}_t$, is the value of the game at time t: M_t. This means that the best prediction (in the mean square sense) of future values M_{t+h} of the game is just the present value M_t. In probabilistic terms this is written as conditional expectation $E(M_{t+h} \mid \mathcal{F}_t) = M_t$.

As a matter of fact, the latter expectation is taken under a *given* probability measure P. In the case of geometric Brownian motion a natural choice for P is the *Wiener measure.* Under this measure $(e^{-rt}P_t)$ is *not* a martingale with respect to the filtration $(\mathcal{F}_t)$, *unless* the mean rate of return μ and the interest rate r coincide. However, the latter can be achieved if one *changes the distribution P by a uniquely determined equivalent probability measure Q such that the Brownian motion with drift $\widetilde{B}_t = B_t + [\mu - r]\sigma^{-1}t$ turns into standard Brownian motion.* This change of measure is theoretically justified by the *Girsanov transformation.* Under Q, the process $(e^{-rt}P_t)$ has an Itô integral representation with respect to the Brownian motion $(\widetilde{B}_t)$, in particular it is a martingale, and therefore Q gained the name *martingale measure.* Under Q, the stochastic integral in (4) has mean zero, and so one obtains a simple expression for the fair price H_0 of the contingent claim H at time zero:

$$H_0 = E_Q(e^{-rT}H)\,,$$

where $E_Q(X)$ is the expectation of the random variable X with respect to the probability measure Q. In the case of a European call option with $H = (P_T - K)_+$ the latter formula can be evaluated giving the Black-Scholes formula (3).

In the above (Black-Scholes) case, life is nice: there is a unique price for each contingent claim H (the Q above is unique), H can be perfectly hedged or replicated. Reality is often not so nice: there are *transaction costs*, *portfolio constraints,* the price process typically shows *random jumps*, etc. In all these and many more cases, pricing (i.e. Q) is not any more unique and H cannot be perfectly hedged. Notions like *minimum-variance, quantile* or *super-hedging* all point at the impossibility to perfectly replicate H in such markets. The latter, so-called *incomplete* case fits into the search for more realistic models taking the stylized facts from econometrics into account (see Section 4 below).

The advanced mathematical theory of derivative pricing is treated in [5, 20, 26, 33]. The necessary theory on martingales and stochastic integration can be found in [19, 28, 30].

3. Interest rate models

The interest rate market tells us how the value of money today is linked to its value in the future. As for share prices, foreign exchange rates or stock indices, future values of interest rates are uncertain and therefore call for modeling by stochastic processes. In contrast to a share price, we do not expect an interest rate r_t to grow on average in an exponential way, but rather to fluctuate in a reasonable range around some fixed value. Therefore geometric Brownian motion is certainly not a good model in this context.

The function r_t of time t is called *instantaneous* or *short rate*. Although it is not directly observable, it is convenient to work with. (In practice, estimates of r_t are provided by professional agencies. Among them is the London Inter-Bank Offer Rate (LIBOR) which presents a set of daily interest rates for various currencies and maturities.) Various models for r_t, as solutions to SDEs driven by Brownian motion (B_t), have been proposed in the literature. Standard models are the following.

$$\begin{array}{llll} \text{Ho-Lee:} & dr_t = & \sigma\, dB_t & + \quad \theta_t\, dt\,, \\ \text{Vasiček:} & dr_t = & \sigma\, dB_t & + \quad (\theta - \alpha\, r_t)\, dt\,, \\ \text{Cox-Ingersoll-Ross} & dr_t = & \sigma_t\, \sqrt{r_t}\, dB_t & + \quad (\theta_t - \alpha_t\, r_t)\, dt\,. \end{array}$$

Here σ, θ, α are constants, θ_t, σ_t, α_t deterministic functions. The Ho-Lee and Vasiček models have the non-desirable property that r_t can become negative. The Vasiček and the Cox-Ingersoll-Ross models are mean-reverting, i.e., r_t fluctuates around θ and θ_t, respectively, a property which it shares with real-life interest rates.

A basic interest bearing security is a *discount* or *zero-coupon bond*. It pays one dollar, say, at time of maturity T. The main question is: how much is this asset worth at time $t < T$? We write $P(t,T)$ for the price of a zero-coupon bond at time $t \le T$. Clearly, $P(T,T) = \$1$. Hence the price of a zero-coupon bond, in contrast to a share price, depends on two dates. This matters a lot because bonds with different maturities are dependent: the 20-year bond and the 10-year bond will behave in a very similar way on a short term basis, and therefore they will not be traded separately.

If the interest rate r_t were a constant: $r_t = r > 0$, the price of a T-bond at time $t < T$ would be

$$P(t,T) = \mathrm{e}^{\,-(T-t)\,r} = \exp\{-(T-t)\; R(t,T)\}\;, \quad \text{where} \quad R(t,T) = \frac{1}{T-t}\int_t^T r_s\, ds\,.$$

For non-constant r_t, $R(t,T)$ is the *average interest rate*. When considered as a function of t for fixed maturity T, it is called the *yield*, whereas it is the *yield curve* for fixed t and varying maturities T. The yield curve tells one about the *term structure* of the market, i.e., about the average return of bonds as a function of T.

For valuing bonds and bond options, another representation of $P(t,T)$ has become useful:

$$P(t,T) = \exp\left\{-\int_t^T f(t,u)\; du\right\}\,,$$

where $f(t,u)$ is the *forward rate*. It is the forward price of instantaneous borrowing at time u, satisfying

$$f(t,T) = R(t,T) + (T-t)\; R_T(t,T) \quad \text{and} \quad f(t,t) = r_t\,.$$

Given the forward rates $f(t,T)$ one can recover the prices $P(t,T)$ and the yields $R(t,T)$. A powerful interest rate model is due to Heath, Jarrow and Morton:

$$\text{Heath-Jarrow-Morton:} \quad df(t,T) = \quad \sigma(s,T)\, dB_t \quad + \quad \alpha(s,T)\, dt\,,$$

where the volatilities $\sigma(t,T)$ and $\alpha(t,T)$ may depend on the history of Brownian motion and on the rates up to t.

Many ideas for pricing interest rate models have been borrowed from stock models, so the idea of constituting a portfolio of a "risky" and a "riskless" asset in order to hedge against a contingent claim H with a self-financing trading strategy. The "riskless" asset is now represented by the *cash bond* which has value

$$\beta_t = \beta_0 \exp\left\{\int_0^t r_s\, ds\right\}.$$

Notice that, since (r_t) is a stochastic process, β_t is subject to uncertainty and therefore not really "riskless". The risky assets are the zero-coupon bonds with maturities $\theta \leq T$. The pricing of a contingent claim H related to the zero-coupon bond (for example, the claim of the European call option $H = (P(\theta,T) - K)_+$) is based on a self-financing strategy (a_t, b_t) for the portfolio of cash and zero-coupon bonds with value

$$V_t = a_t\, P(t,T) + b_t\, \beta_t\,, \quad t \leq \theta\,, \quad \text{and} \quad V_\theta = H\,, \quad \text{for some } \theta < T.$$

As for stock models, one can find an equivalent probability measure Q under which the fair price of the contingent claim H with maturity $\theta < T$ is given by $\beta_0 E_Q(\beta_\theta^{-1} H)$. However, the problem of finding an equivalent martingale measure is more complex because there is an infinite number of maturities $\theta \leq T$, and Q must not depend on θ. For this reason, one needs some subtle assumption which requires that there *exists* a probability measure Q eqivalent to P such that, for every $\theta \leq T$, the discounted process $(\beta_t^{-1} P(t,\theta), t \leq \theta)$ is a martingale.

The recent development of interest rates has been summarized in [4].

4. Financial time series models.

For the pricing of derivatives it is useful to consider continuous time models. Moreover, theoretical arguments tell one that a portfolio cannot be hedged if one only knows the prices at discrete instants of time. However, there are various other reasons to consider *financial time series models* $X_0, X_1, X_2, \ldots$, (the choice of the unit of time is free: seconds, minutes, weeks, months, etc.). One of them is that real-life data are not gathered continously but rather at discrete time points, and fitting a statistical model to them contributes to the understanding of the mechanism that have generated the data. Moreover, a good fit of a theoretical model to real-life data would also allow one to predict (in some sense) future values of the series.

In financial time series analysis the price series $P_t, t = 0, 1, \ldots$, (such as daily share prices of stock, daily foreign exchange rates, etc.) are transformed to log-returns

$$X_t = \log(P_t/P_{t-1}) = \log\left(1 + \frac{P_t - P_{t-1}}{P_{t-1}}\right), \quad t = 1, 2, \ldots.$$

The main goal of this transformation is to make prices independent of their unit and comparable to each other. Moreover, it is believed that, in contrast to the prices P_t, the log-returns X_t can be modeled by a *stationary process*, i.e., a process whose characteristics do not change when time goes by. The latter is a basic assumption in classical time series analysis.

Empirical research (see for example [12, 34]) shows that log-return series of foreign exchange rates, share prices, stock indices, etc., have various properties in common. Among them are the following "stylized facts":

- Very large and very small values of X_t occur more often than in a Gaussian white noise sequence. This means that the distribution of log-returns is *heavy-tailed*, i.e., the tails $P(X_t \leq -x)$ and $P(X_t > x)$ when compared to the tails of the normal distribution are much fatter for large x. This property also indicates that losses (as well as gains) can be much more severe than anticipated by the normal distribution.

- The sample autocorrelations of the X_ts are negligible at almost all lags, even for time series worth several years of data, whereas for longer time series the sample autocorrelations of the absolute values $|X_t|$ and squares X_t^2 decay rather slowly to zero. This indicates that the sequence (X_t) is uncorrelated (white noise) whereas $(|X_t|)$ and (X_t^2) seem to be heavily dependent, even over longer periods of time. In the latter case, one says that there is *long range dependence* or *long memory* in the time series. (A plausible alternative explanation of the latter sample autocorrelation behavior is non-stationarity of the time series.)

- Large and small values of X_t occur not separated in time, as one would expect for a sequence of indendent random variables, but in clusters, i.e., if one unusually large/small value of X_t occurs, various other values of a comparable size appear shortly afterwards.

It is the aim of financial time series analysis to explain these and other features of log-returns by a physical model. As a matter of fact, none of the standard models described below captures *all* stylized facts.

Standard models for log-returns are given by equations of the form

$$X_t = \mu + \sigma_t\, Z_t\,, \quad t = 0, 1, 2, \ldots\,, \tag{5}$$

where μ is a constant, σ_t is a function of past log-returns X_s, the past noise variables Z_ss and past volatilities σ_s. (We focus on one-dimensional series; vector-valued X_ts can be defined in a similar way; see [8] For ease of notation, assume $\mu = 0$. The volatility σ_t is supposed to be independent of Z_t, and the noise (Z_t) is often a sequence of independent, identically distributed symmetric random variables with variance 1. The symmetry expresses the belief that one cannot predict whether future price changes are positive or negative. This is in agreement with empirical research.

Among the many models for σ_t, two have become most popular: the *generalized autoregeressive conditionally heteroscedastic* (GARCH) processes of Tim Bollerslev [7] and the *stochastic volatility* (SV) processes; cf. [31]. The GARCH models are given by (5) and the recurrence equations

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2\,,$$

where α_i and β_j are non-negative parameters. Notice that for Z_t being standard normal, the conditional distribution of X_t, given the past values $X_{t-1}, \dots, X_{t-p}$ and $\sigma_{t-1}, \dots, \sigma_{t-q}$, has a normal distribution with mean zero and variance σ_t^2. The latter can be calculated and so one can give a distributional forecast of the values X_t. For example, there is a 95% chance that X_t assumes values in $[-1.96\sigma_t, 1.96\sigma_t]$. A similar statement can be made for SV models in which case (σ_t) are (Z_t) are mutually independent, and $\log \sigma_t$ is often assumed to follow an ARMA process.

This simple forecasting procedure is one of the reasons why models of type (5) have become popular. Moreover, GARCH and SV models are easily fitted to real-life data by statistical estimation procedures, and they frequently give a reasonably good fit provided the time series is not too long. GARCH and SV models can, to some extent, explain the clustering of large and small values in the log-return series (X_t) as well as the uncorrelatedness and heavy-tailedness of the X_ts. The GARCH model fails to explain the long memory property of the $|X_t|$s and X_t^2, which according to (5) would translate into long memory of the volatility process. Certain SV models can explain the long memory of log-returns, but the fact that the volatility sequence (σ_t) evolves completely independently from the noise (Z_t) and certain statistical difficulties in fitting data have not made them as popular as GARCH models, where the noise (Z_t) gets fed back into the volatility.

Surveys on ARCH and SV models are given in [8, 13, 14, 15, 31].

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