QUANTITATIVE RISK MANAGEMENT: CONCEPTS, TECHNIQUES AND TOOLS*

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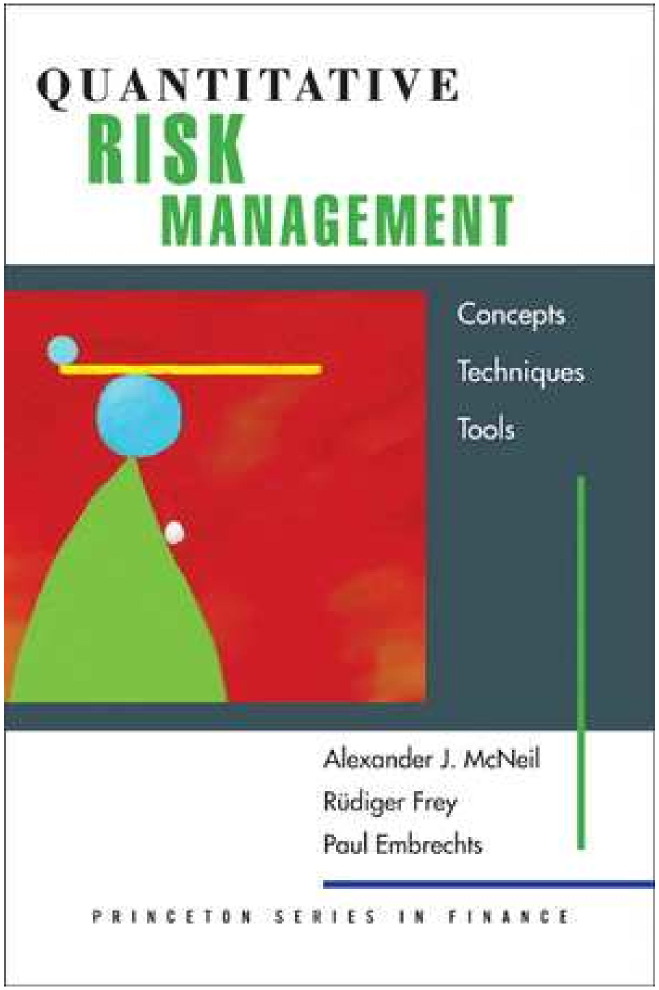


OUTLINE OF THE TALK

- Introduction
- The Fundamental Theorems of Quantitative Risk Management
- PE's Desert-Island Copula
- Example 1: Credit Risk
- Example 2: A Copula Model for FX-data
- Some Further References
- Conclusion

INTRODUCTION

For copula applications to financial risk management, it "all" started with the **RiskLab** report:

> **P. Embrechts**, **A.J. McNeil** and **D. Straumann** (1997[(1)], 1999[(2)], 2000[(3)]) Correlation and dependency in risk management: Properties and pitfalls.

and the **RiskMetrics** report:

> **D.X. Li** (1998, 2000) On default correlation: A copula function approach. Working paper 99-07, RiskMetrics Group.

(1) First version as RiskLab report

(2) Abridged version published in RISK Magazine, May 1999, 69-71

(3) Full length published in: Risk Management: Value at Risk and Beyond, ed. M.A.H. Dempster, Cambridge University Press, Cambridge (2000), 176-223

POTENTIAL COPULA APPLICATIONS

- Insurance:
  - Life (multi-life products)
  - Non-life (multi-line covers)
  - Integrated risk management (Solvency 2)
  - Dynamic financial analysis (ALM)

- Finance:
  - Stress testing (Credit)
  - Risk aggregation
  - Pricing/Hedging basket derivatives
  - Risk measure estimation under incomplete information

- Other fields:
  - Reliability, Survival analysis
  - Environmental science, Genetics
  - ...

FROM THE WORLD OF FINANCE

- "Extreme, synchronized rises and falls in financial markets occur infrequently but they do occur. The problem with the models is that they did not assign a high enough chance of occurrence to the scenario in which **many things go wrong at the same time** - the "perfect storm" scenario"

  (Business Week, September 1998)

- Consulting for a large bank, topics to be discussed were:
  - general introduction to the topic of EVT
  - common pitfalls and its application to financial risk management
  - the application of EVT to the quantification of operational risk
  - general introduction to the topic of copulae and their possible use in financial risk management
  - sources of information to look at if we want to find out more

  (London, March 2004)

THE REGULATORY ENVIRONMENT

- Basel Accord for Banking Supervision (1988)
  - Cooke ratio, "haircut" principle, too coarse
- Amendment to the Accord (1996)
  - VaR for Market Risk, Internal models, Derivatives, Netting
- **Basel II** (1998 – 2007)
  - Three Pillar approach
  - Increased granularity for Credit Risk
  - Operational Risk

THE REGULATORY ENVIRONMENT

- Solvency 1 (1997)
  - Solvency margin as % of premium (non-life), of technical provisions (life)
- **Solvency 2** (2000–2004)
  - Principle-based (not rule-based)
  - Mark-to-market (/model) for assets and liabilities (ALM)
  - Target capital versus solvency capital
  - Explicit modelling of dependencies and stress scenarios
- **Integrated Risk Management**

THE FUNDAMENTAL THEOREMS OF QUANTITATIVE RISK MANAGEMENT (QRM)

- **(FTQRM - 1)** For elliptically distributed risk vectors, classical Risk Management tools like VaR, Markowitz portfolio approach, ... work fine:

  Recall:

  – $\mathbf{Y}$ in $\mathbb{R}^d$ is spherical if $\mathbf{Y} \stackrel{d}{=} U\mathbf{Y}$ for all **orthogonal** matrices U

  – $\mathbf{X} = A\mathbf{Y} + \mathbf{b}$, $A \in \mathbb{R}^{d \times d}$, $\mathbf{b} \in \mathbb{R}^d$ is called elliptical

  – Let $\mathbf{Z} \sim \mathrm{N}_d(\mathbf{0}, \mathbf{\Sigma})$, $W \geq 0$, **independent** of $\mathbf{Z}$, then

  $$\mathbf{X} = \boldsymbol{\mu} + W\mathbf{Z}$$

  is elliptical (multivariate Normal variance-mixtures)

  – If one takes

  $W = \sqrt{\nu/V}$, $V \sim \chi^2_\nu$, then $\mathbf{X}$ is multivariate t_ν

  W normal inverse Gaussian, then $\mathbf{X}$ is generalized hyperbolic

THE FUNDAMENTAL THEOREMS OF QUANTITATIVE RISK MANAGEMENT (QRM)

- **(FTQRM - 2)** Much more important!

  For non-elliptically distributed risk vectors, classical RM tools break down:

  - VaR is typically non-subadditive
  - risk capital allocation is non-consistent
  - portfolio optimization is risk-measure dependent
  - correlation based methods are insufficient

- **A**(n early) **stylized fact:**

  In practice, portfolio risk factors typically are **non-elliptical**

  Questions: - are these deviations relevant, important

  - what are tractable, non-elliptical models
  - how to go from static (one-period) to dynamic (multi-period) RM

SOME COMMON RISK MANAGEMENT FALLACIES

- **Fallacy 1:** marginal distributions and their correlation matrix uniquely determine the joint distribution

  True for elliptical families, **wrong** in general

- **Fallacy 2:** given two one-period risks X_1, X_2, VaR$(X_1 + X_2)$ is maximal for the case where the correlation $\rho(X_1, X_2)$ is maximal

  True for elliptical families, **wrong** in general (non-coherence of VaR)

- **Fallacy 3:** small correlation $\rho(X_1, X_2)$ implies that X_1 and X_2 are close to being independent

AND THEIR SOLUTION

- **Fallacy 1:** standard copula construction (Sklar)

- **Fallacy 2:** many related (copula-) publications

  Reference: P. Embrechts and G. Puccetti (2004). Bounds on Value-At-Risk, preprint ETH Zürich, `www.math.ethz.ch/~embrechts`

- **Fallacy 3:** many copula related examples

  An economically relevant example: Two country risks X_1, X_2

  – $Z \sim \mathsf{N}(0,1)$ independent of scenario generator $U \sim \mathsf{UNIF}(\{-1,+1\})$
  – $X_1 = Z$, $X_2 = UZ \sim \mathsf{N}(0,1)$
  – $\rho(X_1, X_2) = 0$
  – X_1, X_2 are strongly dependent
  – $X_1 + X_2 = Z(1+U)$ is **not** normally distributed

WHY ARE COPULAE USEFUL

- pedagogical: "Thinking beyond linear correlation"

- stress testing dependence: joint extremes, spillover, contagion, ...

- worst case analysis under incomplete information:

  given: $X_i \sim F_i$, $i = 1, \ldots, d$, marginal 1-period risks
  $\Psi(\mathbf{X})$: a financial position
  Δ: a 1-period risk or pricing measure

  task: find $\min \Delta(\Psi(\mathbf{X}))$ and $\max \Delta(\Psi(\mathbf{X}))$ under the above constraints

- eventually: finding better fitting dynamic models

THE BASIC MESSAGE FOR (STATIC) COPULA APPLICATIONS TO QRM

$\boldsymbol{X} = (X_1, \ldots, X_d)'$ one-period risks

$F_{\boldsymbol{X}}(\boldsymbol{x}) = P\left(X_1 \le x_1, \ldots, X_d \le x_d\right)$

$F_i(x_i) = P\left(X_i \le x_i\right)$, $i = 1, 2, \ldots, d$

$$F_{\boldsymbol{X}} \Longleftrightarrow (F_1, \ldots, F_d; C)$$

with copula C (via Sklar's Theorem)

"$\Rightarrow$" : finding useful copula models

"$\Leftarrow$" : stress testing

PE's DESERT-ISLAND COPULA

CLAIM: For applications in QRM, the most useful copula is the t-copula, $C^t_{\nu,\Sigma}$ in dimension $d \geq 2$ (McNeil, 1997)

Its derived copulae include:

- skewed or non-exchangeable t-copula
- grouped t-copula
- conditional excess or t-tail-copula

Reference: S. Demarta and A.J. McNeil (2004). The t copula and related copulas, preprint ETH Zürich, `www.math.ethz.ch/~mcneil`

THE MULTIVARIATE SKEWED t DISTRIBUTION

The random vector $\boldsymbol{X}$ is said to have a multivariate skewed t distribution if

$$\boldsymbol{X} \stackrel{d}{=} \boldsymbol{\mu} + W\boldsymbol{\gamma} + \sqrt{W}\boldsymbol{Z}$$

where $\boldsymbol{\mu}, \boldsymbol{\gamma} \in \mathbb{R}^d$

$\boldsymbol{Z} \sim \mathsf{N}_d(\mathbf{0}, \Sigma)$

W has an inverse gamma distribution depending on ν

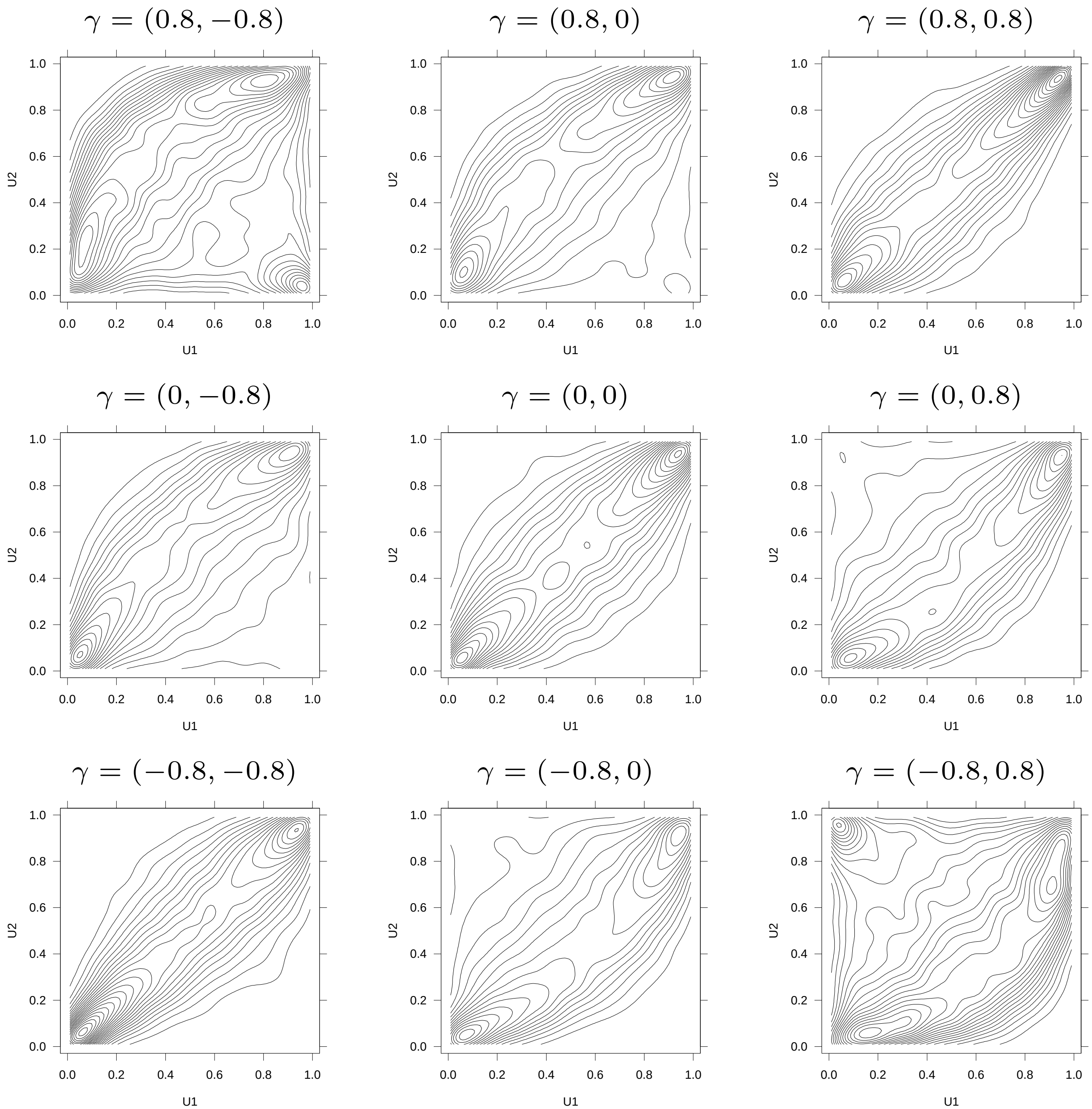
W and $\boldsymbol{Z}$ are independent

Density:

$$f(\boldsymbol{x}) = c\,\frac{K_{\frac{\nu+d}{2}}\left(\sqrt{(\nu+Q(\boldsymbol{x}))\boldsymbol{\gamma}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}}\right)\exp\left((\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}\right)}{\left(\sqrt{(\nu+Q(\boldsymbol{x}))\boldsymbol{\gamma}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}}\right)^{-\frac{\nu+d}{2}}\left(1+\frac{Q(\boldsymbol{x})}{\nu}\right)^{\frac{\nu+d}{2}}}$$

where $\quad Q(\boldsymbol{x}) = (\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu}), \qquad c = \frac{2^{1-(\nu+d)/2}}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{d/2}|\boldsymbol{\Sigma}|^{1/2}}$

and K_λ denotes a modified Bessel function of the third kind



$\gamma = (0.8, -0.8)$
$\gamma = (0.8, 0)$
$\gamma = (0.8, 0.8)$
$\gamma = (0, -0.8)$
$\gamma = (0, 0)$
$\gamma = (0, 0.8)$
$\gamma = (-0.8, -0.8)$
$\gamma = (-0.8, 0)$
$\gamma = (-0.8, 0.8)$
U1
U2

THE GROUPED t-COPULA

The grouped t-copula is closely related to a t-copula where different subvectors of the vector $\boldsymbol{X}$ have different levels of tail dependence

If

- $\boldsymbol{Z} \sim \mathsf{N}_d(\mathbf{0}, \Sigma)$
- G_ν denotes the df of a univariate inverse gamma, $\mathsf{Ig}\left(\frac{\nu}{2}, \frac{\nu}{2}\right)$ distribution
- $U \sim \mathsf{UNIF}(0,1)$ is a uniform variate independente of $\boldsymbol{Z}$
- $\mathcal{P}$ is a partition of $\{1, \ldots, d\}$ into m sets of sizes $\{s_k : k = 1, \ldots, m\}$
- ν_k is the degrees of freedom parameter associated with set of size s_k
- $W_k = G_{\nu_k}^{-1}(U)$

then

$$\boldsymbol{X} = \left(\sqrt{W_1}Z_1, \ldots, \sqrt{W_1}Z_{s_1}, \ldots \ldots, \sqrt{W_m}Z_{d-s_m+1}, \ldots, \sqrt{W_m}Z_d\right)'$$

has a grouped t copula (similarly, grouped elliptical copula)

THE TAIL LIMIT COPULA

Lower Tail Limit Copula Convergence Theorem: Let C be an exchangeable copula such that $C(v,v) > 0$ for all $v > 0$. Assume that there is a strictly increasing continuous function $K : [0,\infty) \to [0,\infty)$ such that

$$\lim_{v \to 0} \frac{C(vx,v)}{C(v,v)} = K(x), \qquad x \in [0,\infty).$$

Then there is $\eta > 0$ such that $K(x) = x^{\eta} K(1/x)$ for all $(0,\infty)$. Moreover, for all $(u_1,u_2) \in (0,1]^2$

$$C_0^{lo}(u_1,u_2) = G(K^{-1}(u_1), K^{-1}(u_2)),$$

where $G(x_1,x_2) := x_2^{\eta} K(x_1/x_2)$ for $(x_1,x_2) \in (0,1]^2$, $G := 0$ on $[0,1]^2 \backslash (0,1]^2$ and C_0^{lo} denotes the lower tail limit copula

Observation: The function $K(x)$ fully determines the tail limit copula

THE t LOWER TAIL LIMIT COPULA

For the bivariate t-copula $C^t_{\nu,\rho}$ with tail dependence coefficient λ we have for the t-LTL copula that

$$K(x) = \frac{x t_{\nu+1}\left(\frac{-(x^{1/\nu}-\rho)}{\sqrt{1-\rho^2}}\sqrt{\nu+1}\right) + t_{\nu+1}\left(\frac{-(x^{-1/\nu}-\rho)}{\sqrt{1-\rho^2}}\sqrt{\nu+1}\right)}{\lambda}$$

with $x \in [0,1]$, whereas for the Clayton-LTL copula, with parameter θ,

$$K(x) = \left((x^{-\theta}+1)/2\right)^{-1/\theta}$$

Important for pratice: "for any pair of parameter values ν and ρ the K-function of the t-LTL copula may be very closely approximated by the K-function of the Clayton copula for some value of θ" (S. Demarta and A.J. McNeil (2004))

EXAMPLE 1: the Merton model for corporate default (firm value model, latent variable model)

- portfolio $\{(X_i, k_i) : i = 1, \ldots, d\}$ firms, obligors

- obligor i defaults by end of year if $X_i \leq k_i$

  (firm value is less than value of debt, properly defined)

- modelling joint default: $P(X_1 \leq k_1, \ldots, X_d \leq k_d)$
  - classical Merton model: $\boldsymbol{X} \sim N_d(\boldsymbol{\mu}, \Sigma)$
  - KMV: calibrate k_i via "distance to default" data
  - CreditMetrics: calibrate k_i using average default probabilities for different rating classes
  - Li model: X_i's as survival times are assumed exponential and use Gaussian copula

- hence standard industry models use Gaussian copula

- improvement using t-copula

- standardised equicorrelation ($\rho_i = \rho = 0.038$) matrix Σ calibrated so that for $i = 1, \ldots, d$, $P(X_i \leq k_i) = 0.005$ (medium credit quality in KMV/CreditMetrics)

- set $\nu = 10$ in t-model and perform $100\,000$ simulations on $d = 10\,000$ companies to find the loss distribution

- use VaR concept to compare risks

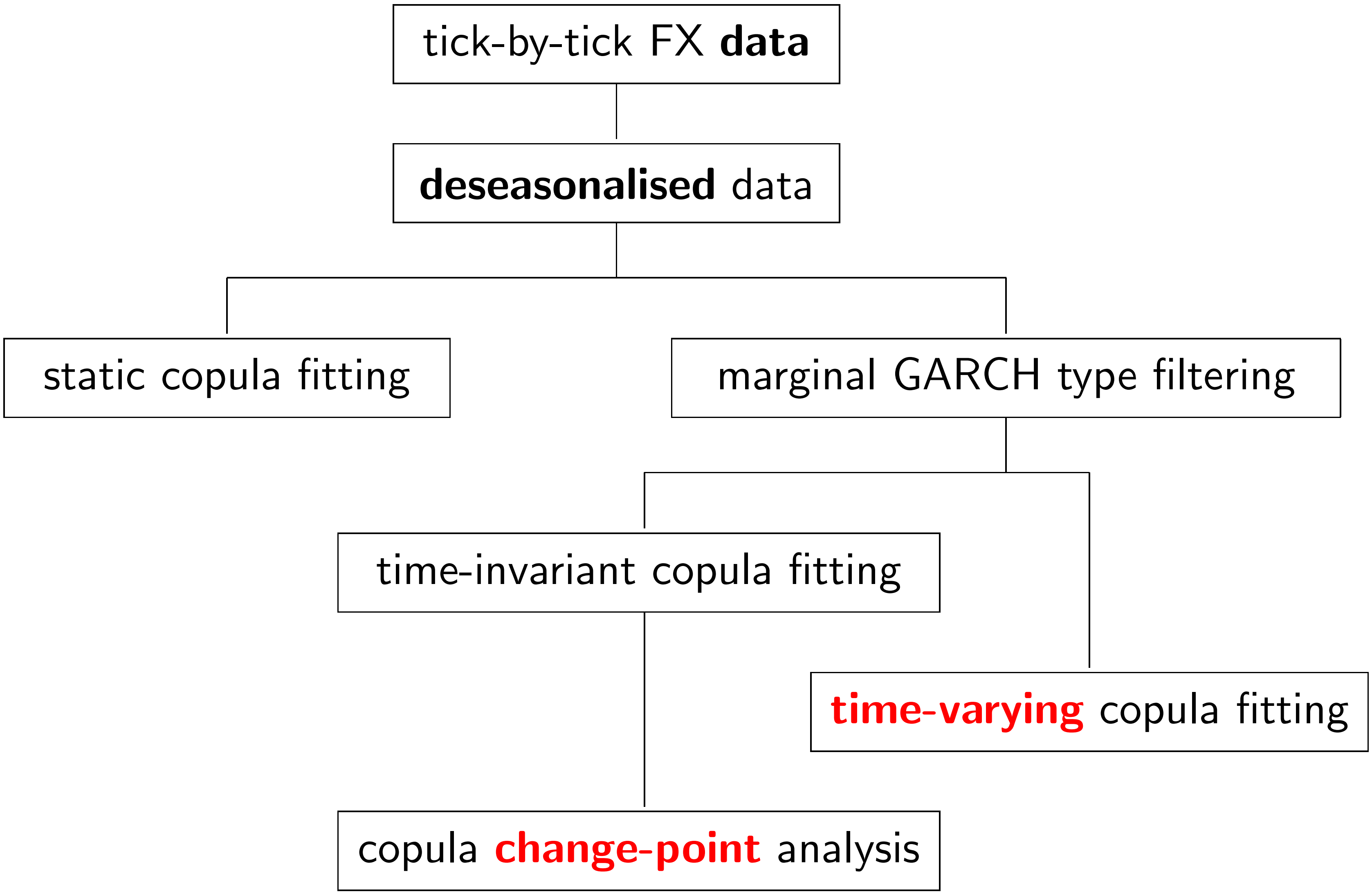
Results:

	min	25%	med	mean	75%	90%	95%	max
Gaussian	1	28	43	49.8	64	90	109	131
t	0	1	9	49.9	42	132	235	3 238

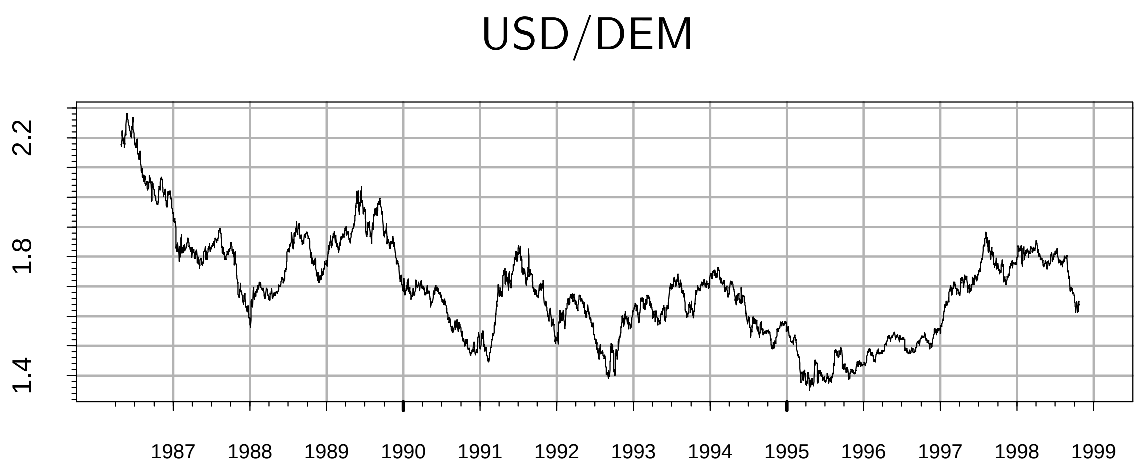
- more realistic t-model: block-t-copula (Lindskog, McNeil)

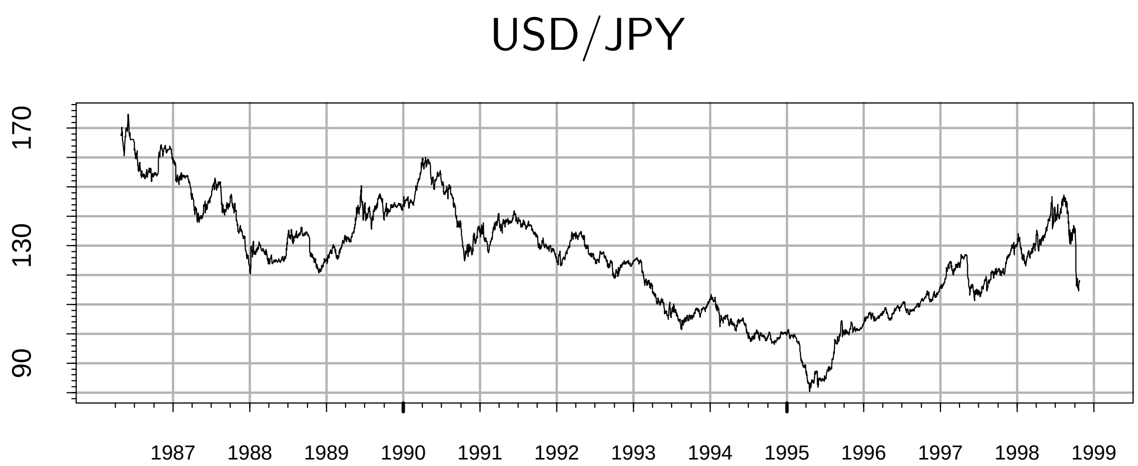
- has been used for banking and (re)insurance portfolios

Example 2: High-Frequency FX data



FX DATA SERIES





Olsen data set: Bivariate 5 minutes logarithmic middle prices

$$\bar{p}_t = \frac{1}{2}\left(\log\, p_{t,bid} + \log\, p_{t,ask}\right)$$

ASYMPTOTIC CLUSTERING OF BIVARIATE EXCESSES

- Extreme tail dependence copula relative to a threshold t:

$$C_t(u,v) = P(U \leq F_t^{-1}(u), V \leq F_t^{-1}(v) | U \leq t, V \leq t)$$

with conditional distribution function

$$F_t(u) := P(U \leq u | U \leq t, V \leq t),\ 0 \leq u \leq 1$$

- **Archimedean** copulae: there exists a continuous, strictly decreasing function $\psi : [0,1] \mapsto [0,\infty]$ with $\psi(1) = 0$, such that

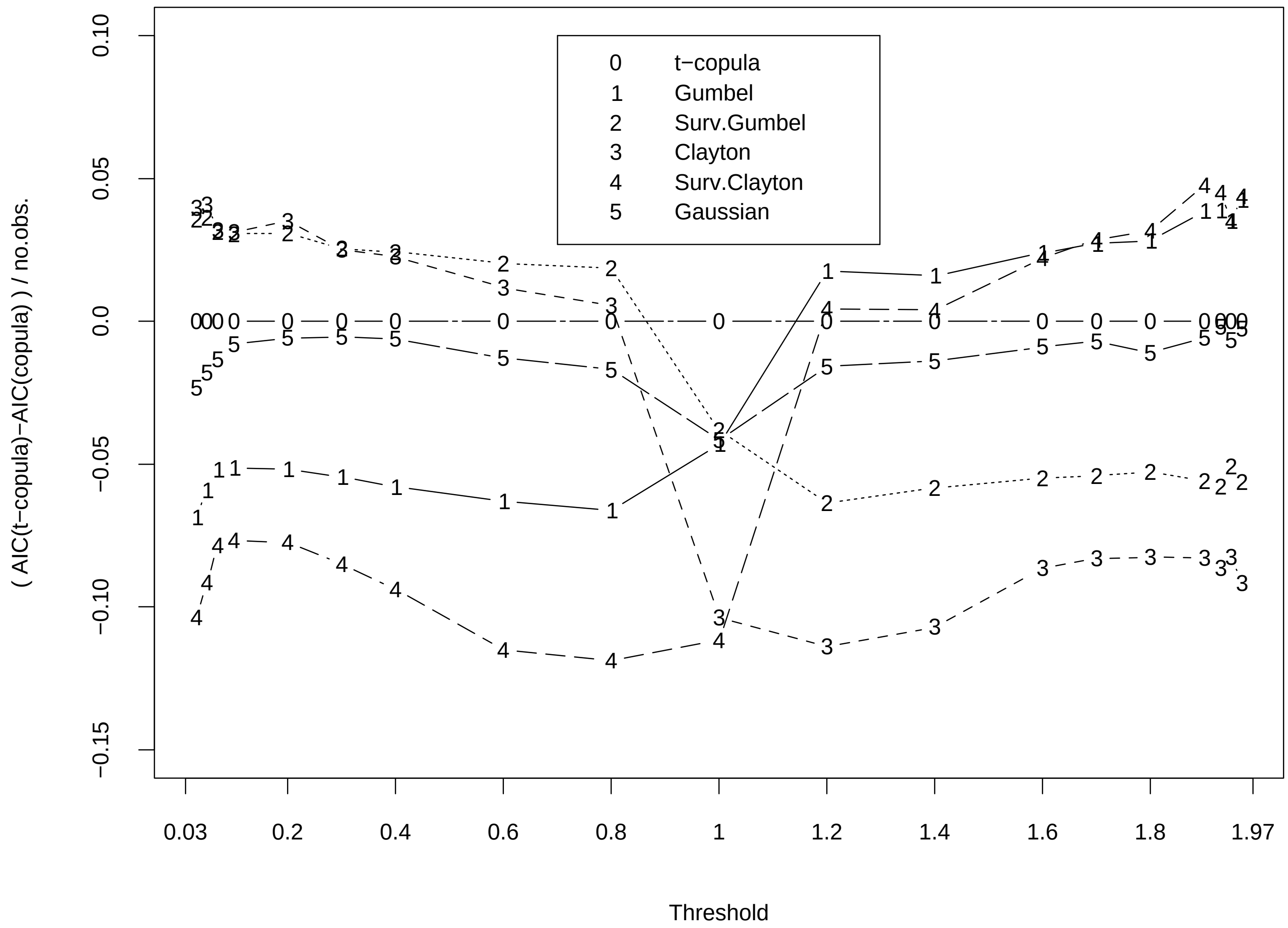
$$C(u,v) = \psi^{[-1]}(\psi(u) + \psi(v))$$

- For "sufficiently regular" Archimedean copulae (Juri and Wüthrich (2002)):

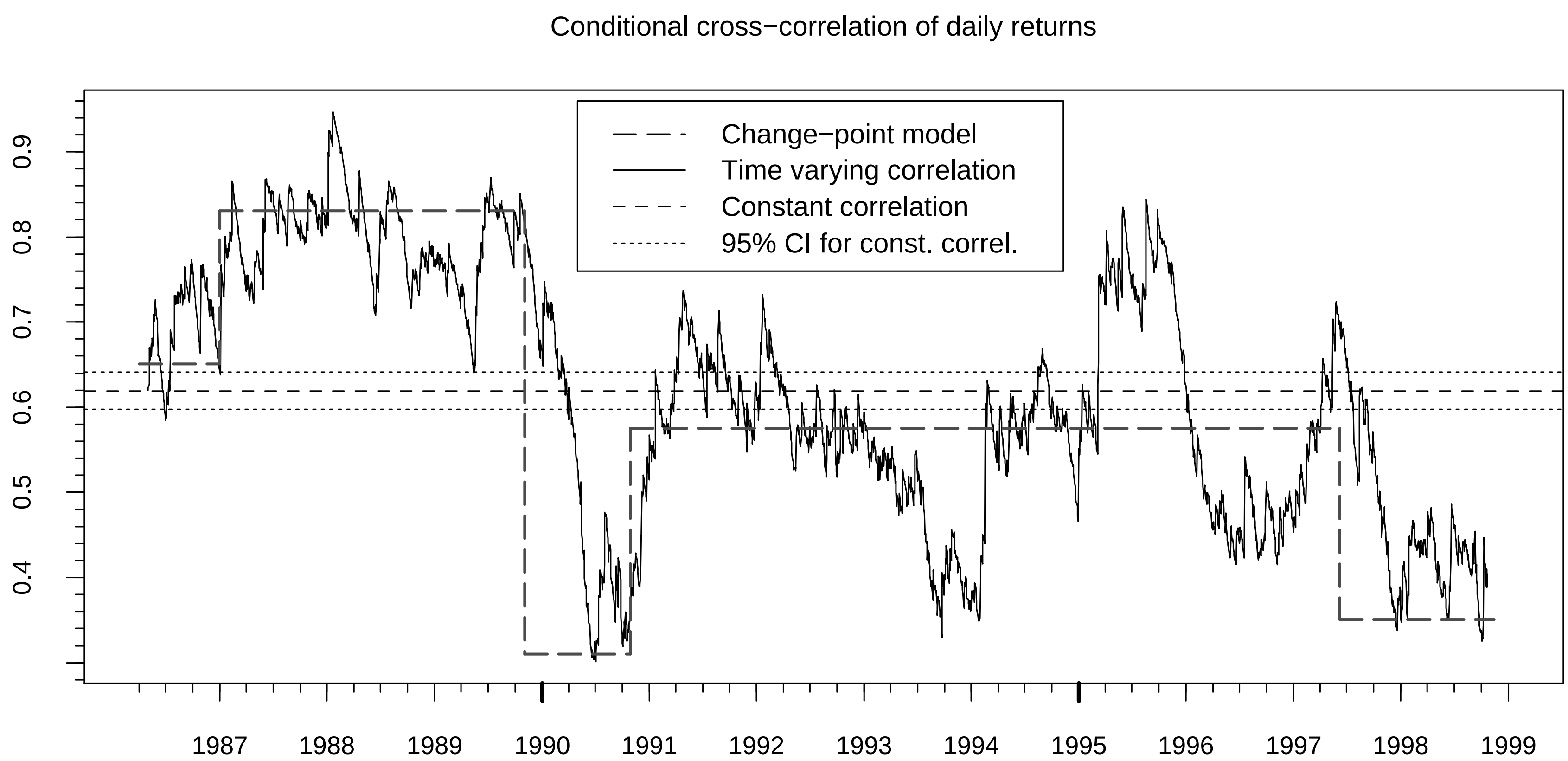
$$\lim_{t \to 0^+} C_t(u,v) = C_\alpha^{\mathbf{Clayton}}(u,v)$$

Juri, A. and M. Wüthrich (2002). Copula convergence theorems for tail events. Insurance: Math. & Econom., 30: 405–420

ASYMPTOTIC CLUSTERING OF BIVARIATE EXCESSES



Estimated t-copula conditional correlation of daily returns on the FX USD/DEM and USD/JPY spot rates



A. Dias and P. Embrechts (2004). Dynamic copula models and change-point analysis for multivariate high-frequency data in finance, preprint ETH Zürich

HINTS* FOR FURTHER READING

(* There exists an already long and fast growing literature)

Books combining copula modelling with applications to finance:

- Bluhm, C., Overbeck, L. and Wagner, C. *An Introduction to Credit Risk Modeling*. Chapman & Hall/CRC, New York, 2002

- Cherubini, U., E. Luciano and W. Vecchiato (2004). *Copula Methods in Finance*, Wiley, To appear

- McNeil, A.J., R. Frey and P. Embrechts (2004). *Quantitative Risk Management: Concepts, Techniques and Tools*. Book manuscript, To appear

- Schönbucher, P.J. *Credit Derivatives Pricing Models*, Wiley Finance, 2003

(and there are more)

HINTS FOR FURTHER READING

Some papers for further reading:

- Cherubini, U. and E. Luciano, Bivariate option pricing with copulas, *Applied Mathematical Finance* **9**, 69–85 (2002)

- Dias, A. and P. Embrechts (2003). Dynamic copula models for multivariate high-frequency data in finance. Preprint, ETH Zürich

- Fortin, I. and C. Kuzmics (2002). Tail-dependence in stock-return pairs: Towards testing ellipticity. Working paper, IAS Vienna

- Patton, A.J. (2002). Modelling time-varying exchange rate dependence using the conditional copula. Working paper, UCSD

- Rosenberg, J., Nonparametric pricing of multivariate contingent claims, NYU, Stern School of Business (2001), working paper

- van den Goorbergh, C. Genest and B.J.M. Werker, Multivariate option pricing using dynamic copula models, Tilburg University (2003), discussion paper No. 2003-122

(and there are many, many more)

CONCLUSION

- Copulae are here to stay as a risk management tool
- Dynamic models
- Calibration / fitting
- High dimensions ($d \geq 100$, say)
- Most likely application: credit risk
- Limit theorems
- Link to multivariate extreme value theory