MODELING DISTRIBUTIONS: EXTREME VALUE THEORY AND COPULAE

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SOME STATEMENTS ON EXTREMES AND CORRELATION

- "A natural consequence of the existence of a lender of last resort is that there will be some sort of allocation of burden of risk of extreme outcomes. Thus, central banks are led to provide what essentially amounts to **catastrophic insurance coverage** ... From the point of view of the risk manager, **inappropriate use of the normal distribution can lead to an understatement of risk**, which must be balanced against the significant advantage of simplification. From the central bank's corner, the consequences are even more serious because we often need to concentrate on the left tail of the distribution in formulating lender-of-last-resort policies. Improving the characterization of the distribution of extreme values is of paramount importance"

(Alan Greenspan, Joint Central Bank Research Conference, 1995)

SOME STATEMENTS ON EXTREMES AND CORRELATION

- "Extreme, synchronized rises and falls in financial markets occur infrequently but they do occur. The problem with the models is that they did not assign a high enough chance of occurrence to the scenario in which **many things go wrong at the same time** - the "perfect storm" scenario"

(Business Week, September 1998)

- "Regulators have criticised LTCM and banks for not "stress-testing" risk models against extreme market movements... The markets have been through the financial equivalent of several Hurricane Andrews hitting Florida all at once. Is the appropriate response to accept that it was mere bad luck to run into such a **rare event** - or to get new forecasting models that assume more storms in the future?"

(The Economist, October 1998, after the LTCM rescue)

SOME STATEMENTS ON EXTREMES AND CORRELATION

- "... The trading floor is quiet. But this masks their attempt at picking up the pieces with a new fund, JWM Partners. Now, Mr. Meriwether is preaching new gospel: World financial markets are bound to hit **extreme turbulences** again... Mr. Meriwether's crew, once bitten, also is betting on more liquid securities: "With globalisation increasing, you'll see more crises," he says. "Our whole focus is on the extremes now - **what's the worst that can happen to you in any situation** - because we never want to go through that again""

(John Meriwether, The Wall Street Journal, 21/8/2000)

SOME STATEMENTS ON EXTREMES AND CORRELATION

- "Over the last number of years, regulators have encouraged financial entities to use portfolio theory to produce dynamic measures of risk. VaR, the product of portfolio theory, is used for short-run day-to-day profit and loss exposures. Now is the time to encourage the BIS and other regulatory bodies to support studies on stress test and concentration methodologies. Planning for crises is more important than VaR analysis. And such **new methodologies** are the correct response to recent crises in the financial industry"

(Myron Scholes, American Economic Review, May 2000)

- "Someone told me that the bell curve is wrong"

(Banker, private communication, 1999)

CORRELATION CONFUSION: IN WORDS

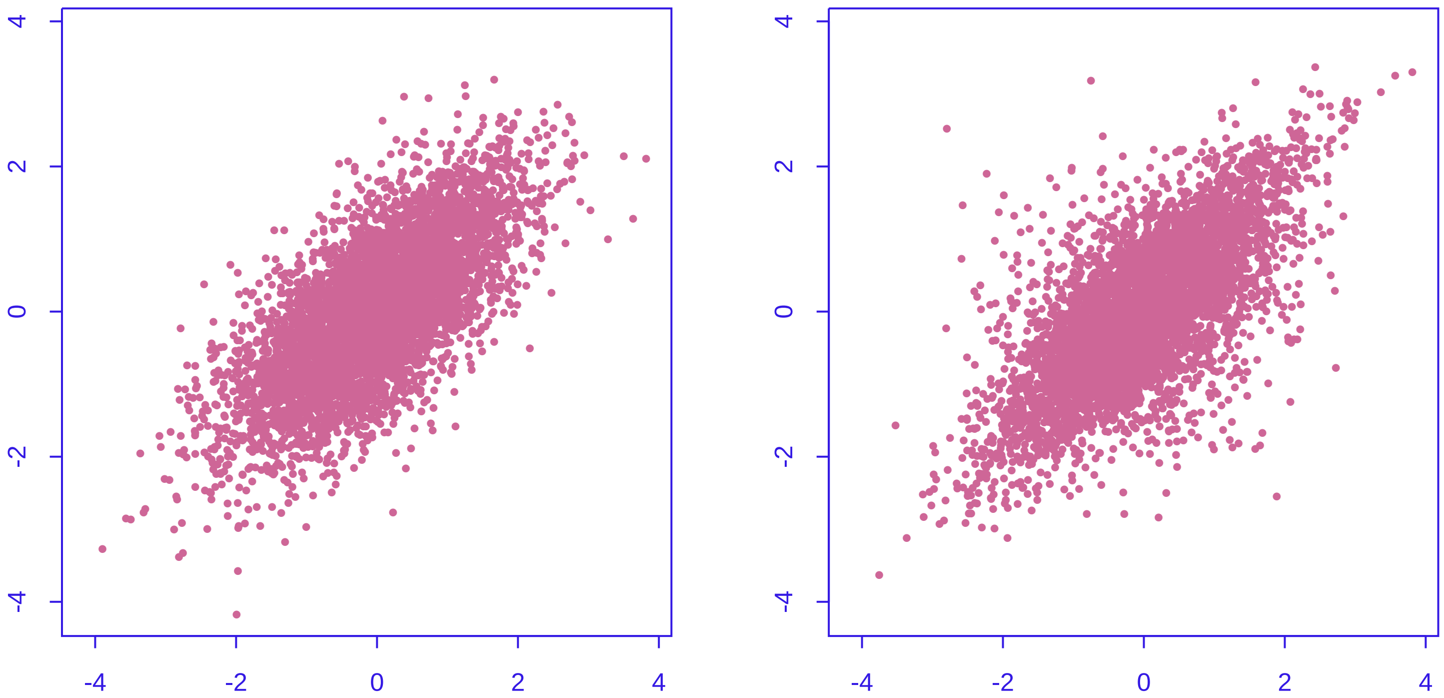
- "Among nine big economies, stock market correlations have averaged around 0.5 since the 1960s. In other words, for every 1 per cent rise (or fall) in, say, American share prices, share prices in the other markets will typically rise (fall) by 0.5 per cent"

(The Economist, 8th November 1997)

- "A correlation of 0.5 does not indicate that a return from stock-market A will be 50% of stockmarket B's return, or vice-versa... A correlation of 0.5 shows that 50% of the time the return of stockmarket A will be positivelly correlated with the return of stockmarket B, and 50% of the time it will not"

(The Economist (letter), 22nd November 1997)

CORRELATION CONFUSION: IN A PICTURE



Bivariate Normal
$\rho = 0.7$

t_4 copula ($\rho = 0.7$)
and Normal margins

MESSAGES FROM THE METHODOLOGICAL FRONTIER

- **Static case** (time fixed)
  - $d = 1$: Classical Extreme Value Theory (EVT)
    Peaks-over-threshold method (POT)
  - $d \geq 2$: Multivariate Extreme Value Theory (MEVT)
    Copulae

- **Dynamic case**
  - Extremes of stochastic processes in $d > 1$, only in rather special cases (Gaussian, Markov, ...)
  - Non-BSM models: Lévy driven price processes, incompleteness

SOME EXAMPLES

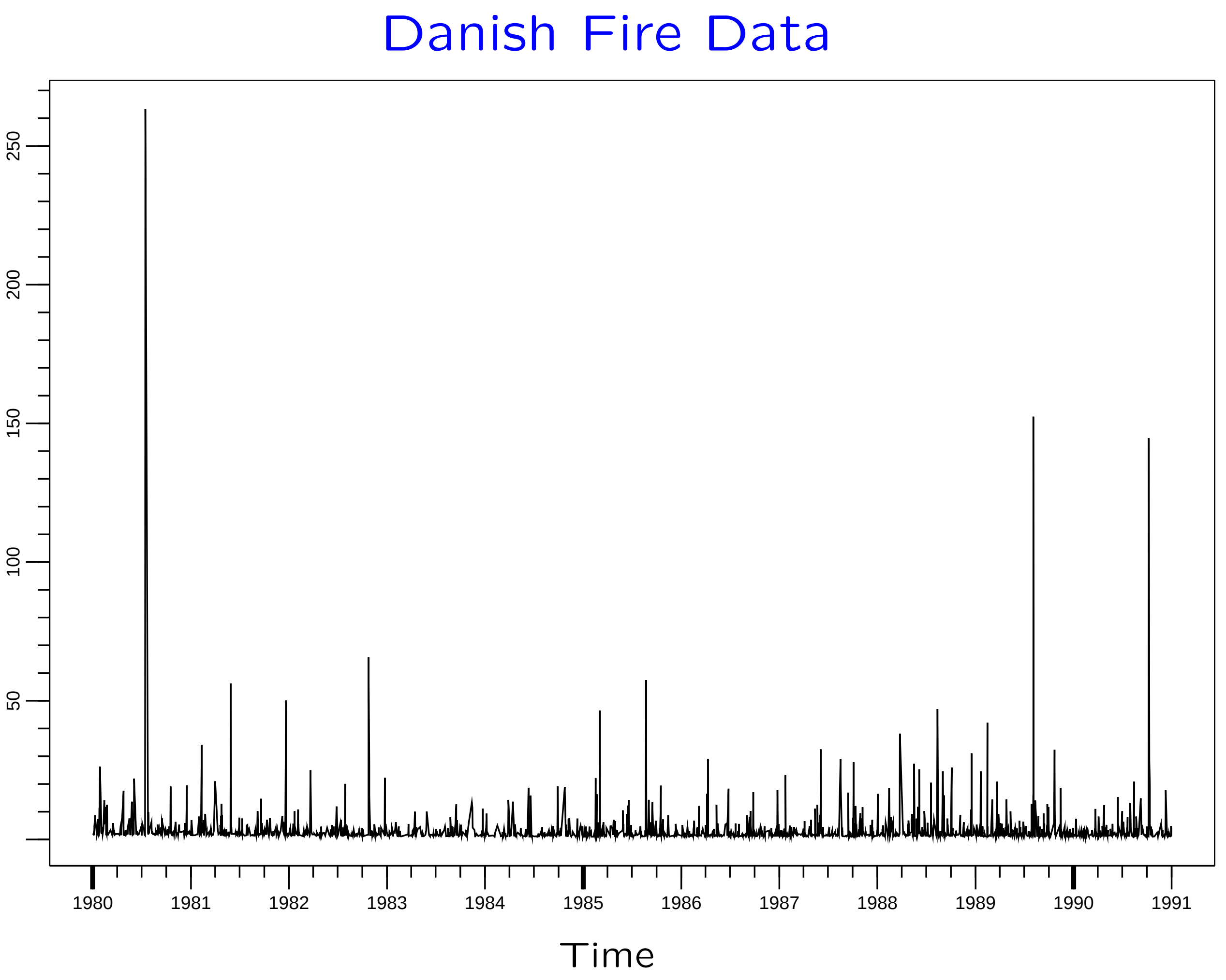
Example 1: EVT - POT

Example 2: MEVT

Example 3: Copulae

and their **Risk Management Consequences**

Example 1: EVT - POT



Example 1: EVT - POT

- **Notation:** $M_n = \max(X_1, \ldots, X_n)$, $X_i \sim F$, $F_u(x) = P(X-u \leq x | X > u)$

- **Fisher-Tippet theorem:** Let (X_n) be a sequence of iid rvs. If there exist norming constants $c_n > 0$, $d_n \in \mathbb{R}$ and some non-degenerate df H such that $c_n^{-1}(M_n - d_n) \xrightarrow{d} H$, then H belongs to the **Fréchet** $(\xi > 0)$, **Weibull** $(\xi < 0)$ or **Gumbel** $(\xi = 0)$ type of distributions

- **Balkema-de Haan-Pickands result:** For every $\xi \in \mathbb{R}$, $F \in \text{MDA}(H_\xi)$ if and only if $\lim_{u \uparrow x_F} \sup_{0<x<x_F-u} |F_u(x) - G_{\xi,\beta(u)}(x)| = 0$ for some positive function β and where $G_{\xi,\beta(u)}$ is the **generalized Pareto distribution** (GPD)

- **Tail estimation:** $\overline{F}(x) = P(X > x) \approx \frac{N_u}{n} \overline{G}_{\hat{\xi},\hat{\beta}}(x-u)$, $x \geq u$

Example 1: EVT - POT

- For ($\xi > 0$),

$$F \in \mathrm{MDA}(H_\xi) \Longleftrightarrow \overline{F}(x) = x^{-1/\xi} L(x)$$

with L slowly varying. This means that for $x > 0$,

$$\frac{\overline{F}(tx)}{\overline{F}(t)} = \frac{P(X > tx)}{P(X > t)} \longrightarrow x^{-1/\xi}, \; t \to \infty$$

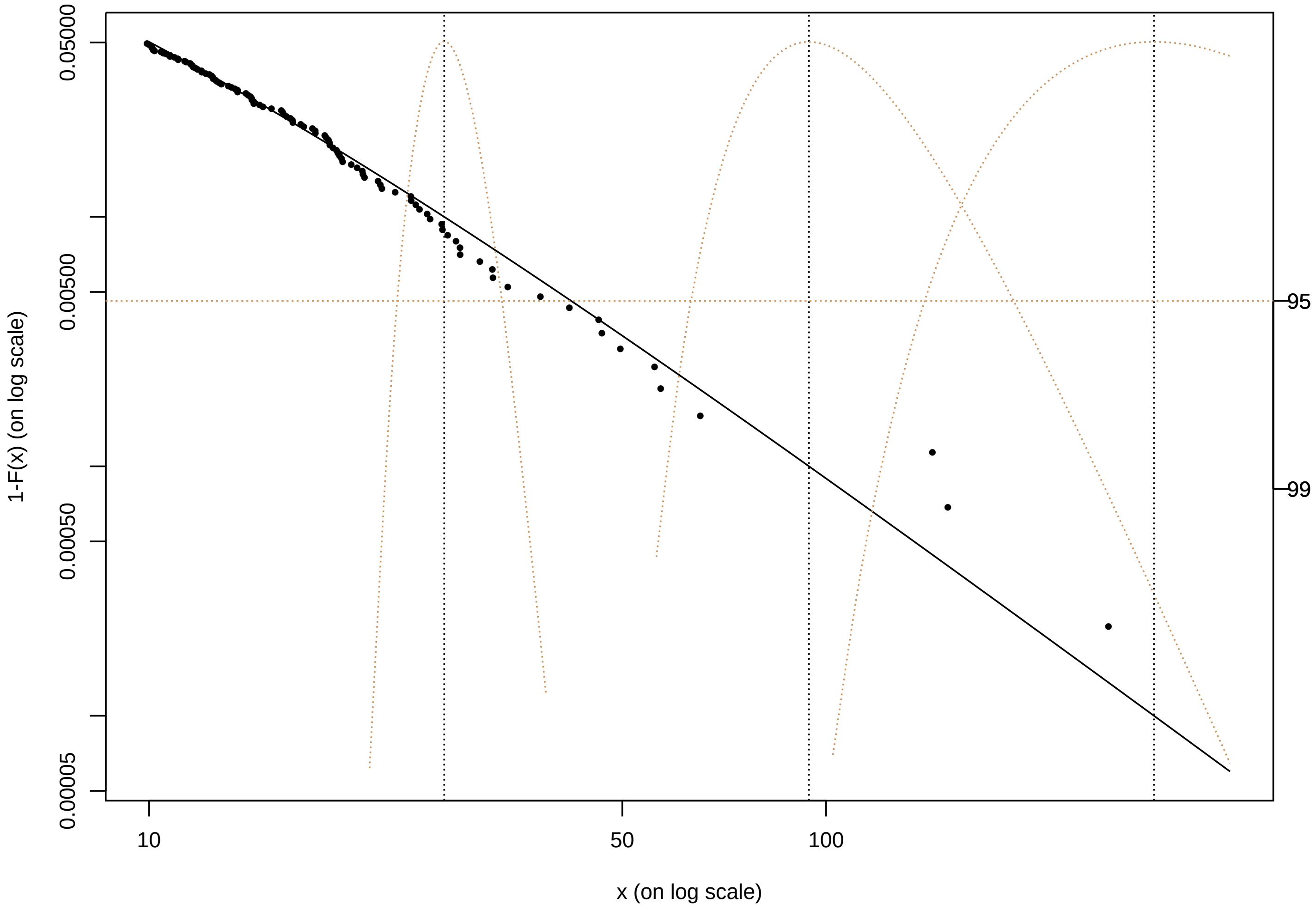
- A graphical device for checking the above condition:

plot $\log(\overline{F}_n(x))$ **versus** $\log(x)$

where F_n is the empirical distribution function of $(X_1, \ldots, X_n)$ and check for

(ultimate) linearity

Example 1: EVT - POT



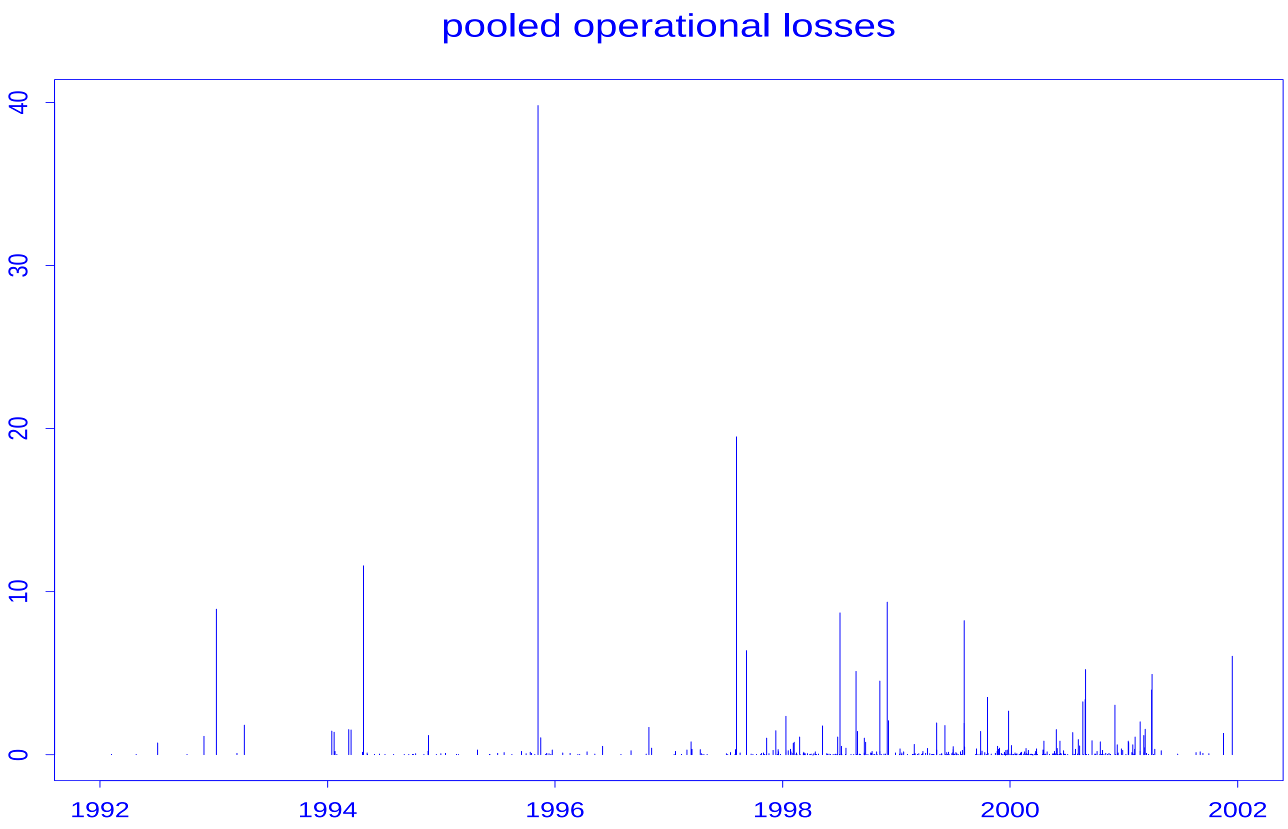
p	Quantile	ES
99.00%	27.28	58.24
99.90%	94.33	191.53
99.99%	304.90	610.13

EVT software: EVIS (`www.math.ethz.ch/~mcneil`)

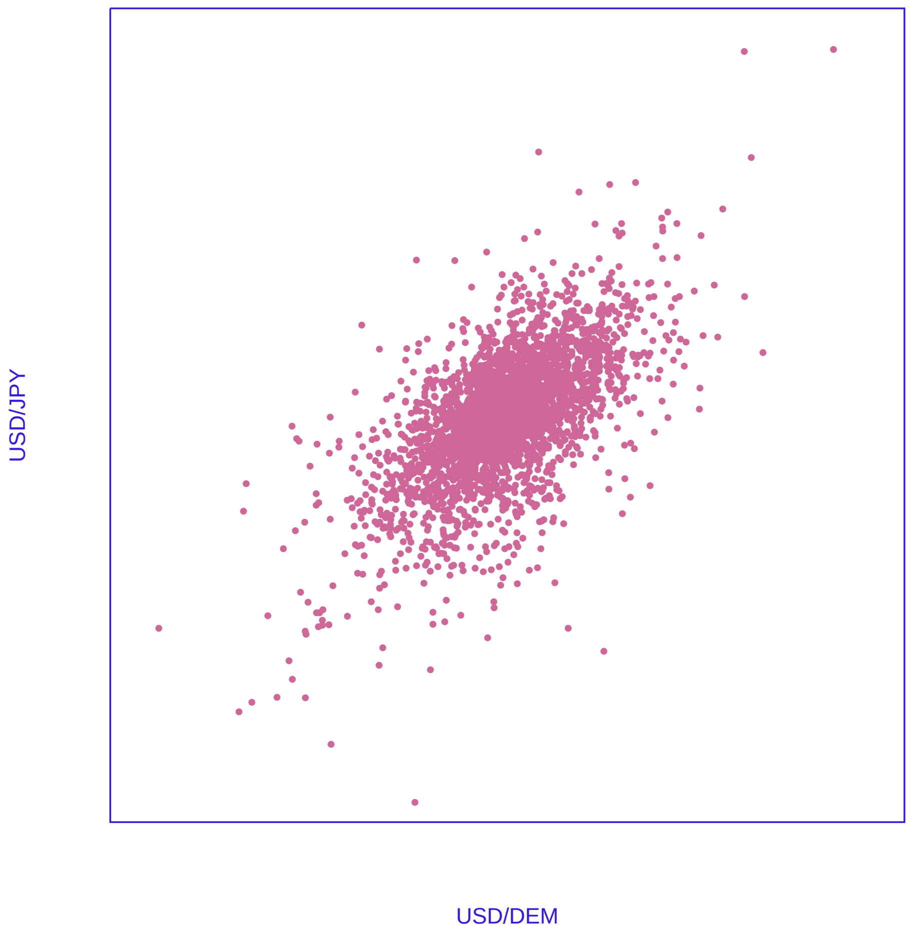
Example 1: EVT - POT

- Modeling of Operational Risk (Basel II)

- Data example:



Example 2: MEVT



Bivariate daily returns of DEM and JPY FX rates quoted against US Dollar from February 1986 up to the end of December 1998

Example 2: MEVT

- Suppose that the d-dimensional random vector $\mathbf{X}$ has a **regularly varying tail distribution**, i.e., the tail behaviour of $\mathbf{X}$ is characterised by a tail index α and the limit

$$\frac{P(\| \mathbf{X} \| > tx, \mathbf{X}/ \| \mathbf{X} \| \in \cdot)}{P(\| \mathbf{X} \| > t)} \quad \xrightarrow{v} \quad x^{-\alpha} P(\Theta \in \cdot),$$

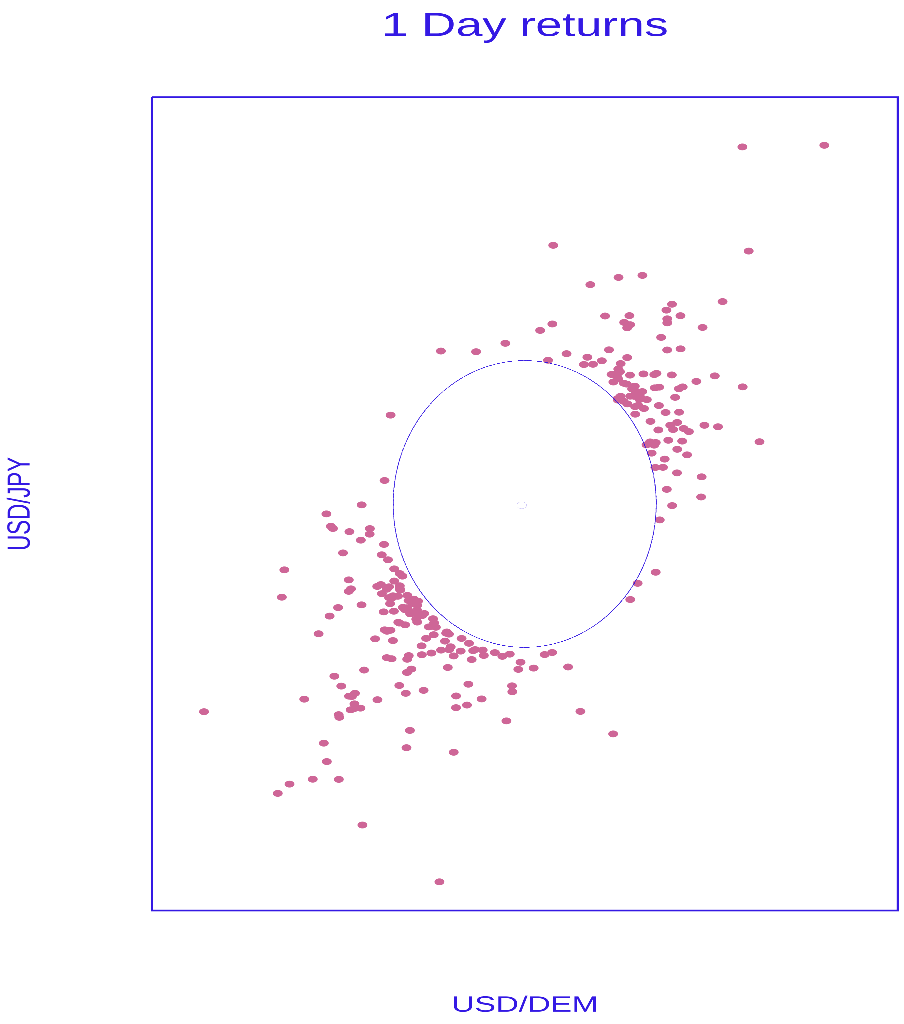
where $x > 0$, $t \to \infty$, exists. The distribution function of Θ is the **spectral distribution** of $\mathbf{X}$

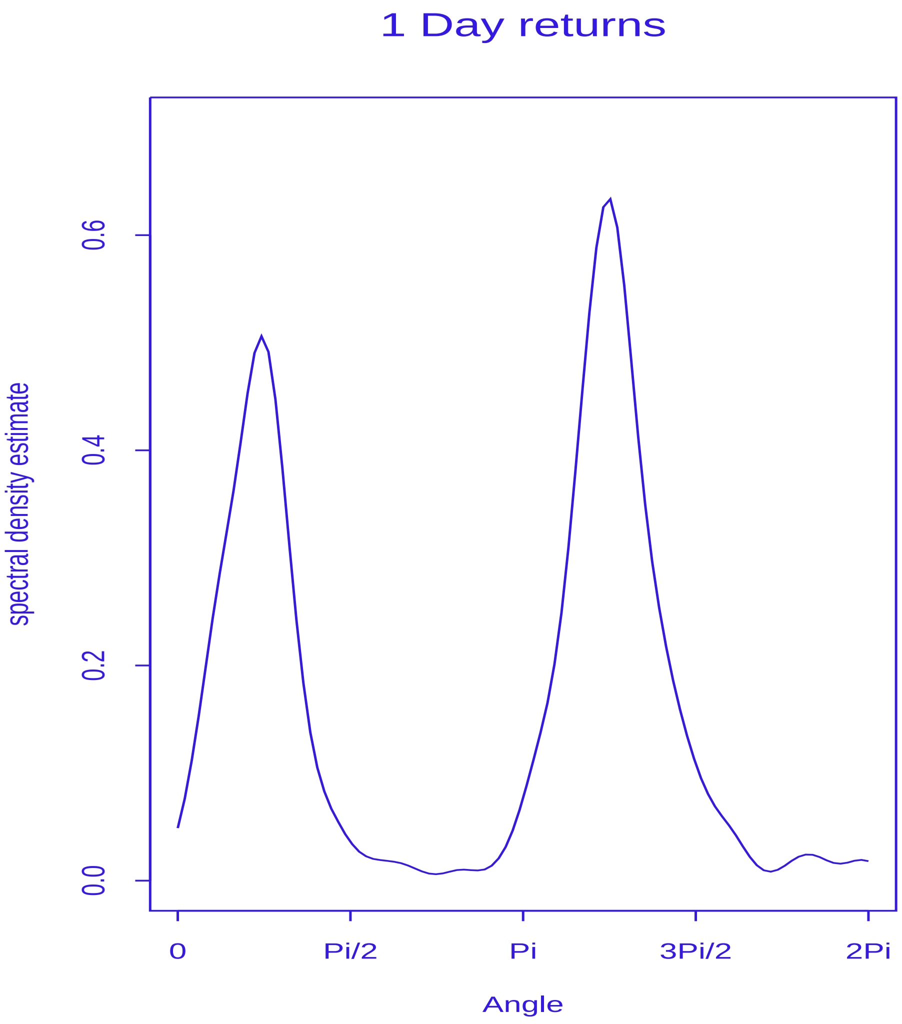
- **Estimator:**

$$\widehat{P}(\Theta \in S) = \frac{1}{k_n} \sum_{i=1}^{n} \epsilon_{\mathbf{X}_i/\|\mathbf{X}\|_{k_n,n}}(V(S))$$

where $V(S) = \{\mathbf{x} \in \mathbb{S}_+^{d-1} : \mathbf{x}/ \| \mathbf{x} \| \in S\}$

Example 2: MEVT





Example 3: COPULAE

d risks $X_1, \ldots, X_d$ with **continuous** marginal distribution functions $F_1, \ldots, F_d$

- Given the joint law $F_{\mathbf{X}}(\mathbf{x}) = P(X_1 \le x_1, \ldots, X_d \le x_d)$, there exists a **unique** function C on $[0,1]^d$ so that

  $$F_{\mathbf{X}}(\mathbf{x}) = C(F_1(x_1), \ldots, F_d(x_d))$$

  (a **copula** C is a df on $[0,1]^d$ with uniform margins)

- Given **only** $F_1, \ldots, F_d$ and **any** copula C,

  $$C(F_1(x_1), \ldots, F_d(x_d))$$

  yields **a** joint model with the prescribed margins

WHY ARE COPULAE USEFUL

- pedagogical: "Thinking beyond linear correlation"

- stress testing dependence: joint extremes, spillover, contagion, ...

- worst case analysis under incomplete information:

  given: $X_i \sim F_i$, $i = 1, \ldots, d$, marginal 1-period risks
  $\Psi(\mathbf{X})$: a financial position
  Δ: a 1-period risk or pricing measure

  task: find $\min \Delta(\Psi(\mathbf{X}))$ and $\max \Delta(\Psi(\mathbf{X}))$ under the above constraints

MANY QUESTIONS REMAIN TO BE STUDIED

- statistical fitting
- high-dimensional copulae
- dynamic modeling
- stylized facts on copulae

RISK MANAGEMENT CONSEQUENCES

- **First Fundamental Theorem of Integrated Risk Management (1FTIRM):** For elliptically distributed risk vectors, classical IRM tools like VaR, Markowitz portfolio approach, work fine

  Recall:

  – $\mathbf{Y}$ in $\mathbb{R}^d$ is spherical if $\mathbf{Y} \stackrel{d}{=} U\mathbf{Y}$ for all **orthogonal** matrices U

  – $\mathbf{X} = A\mathbf{Y} + \mathbf{b}$, $A \in \mathbb{R}^{d \times d}$, $\mathbf{b} \in \mathbb{R}^d$ is called elliptical

  – Let $\mathbf{Z} \sim \mathrm{N}_d(\mathbf{0}, \mathbf{\Sigma})$, $W \geq 0$, **independent** of $\mathbf{Z}$, then

  $$\mathbf{X} = \boldsymbol{\mu} + W\mathbf{Z}$$

  is elliptical (multivariate Normal variance-mixtures)

RISK MANAGEMENT CONSEQUENCES

– If one takes

$W = \sqrt{\nu/V}$, $V \sim \chi^2_\nu$, then $\mathbf{X}$ is multivariate t_ν

W normal inverse Gaussian, then $\mathbf{X}$ is generalized hyperbolic

- **2FTIRM:** (much more important!)
  For non-elliptically distributed risk vectors, 1FTIRM breaks down:

  - VaR is typically non-subadditive
  - risk capital allocation is non-consistent
  - portfolio optimization is risk-measure dependent
  - correlation based methods are insufficient

- **A**(n early) **stylized fact:**
  In practice, portfolio risk factors typycally are **non-elliptical**
  Question: are these deviations relevant, important?

SOME COMMON FALLACIES (often appearing in disguise!)

- **Fallacy 1:** marginal distributions and their correlation matrix uniquely determine the joint distribution

  True for elliptical families, **wrong** in general (copulae)

- **Fallacy 2:** given two one-period risks X_1, X_2, VaR$(X_1 + X_2)$ is maximal for the case where the correlation $\rho(X_1, X_2)$ is maximal

  True for elliptical families, **wrong** in general (non-coherence of VaR)

- **Fallacy 3:** small correlation $\rho(X_1, X_2)$ implies that X_1 and X_2 are close to being independent

AN EXAMPLE CONCERNING FALLACY 3

- Two country risks X_1 and X_2
  - $Z \sim \mathrm{N}(0,1)$, $U \sim \mathrm{UNIF}(\{-1,+1\})$, $P(U=-1) = 1/2 = P(U=1)$
    U stands for an economic stress generator, **independent** of Z
  - As a consequence: $X_1 = Z \sim \mathrm{N}(0,1)$ and $X_2 = UZ \sim \mathrm{N}(0,1)$
    Moreover: $Cov(X_1, X_2) = E(X_1X_2) = E(UZ^2) = E(U)E(Z^2) = 0$
    hence $\rho(X_1, X_2) = 0$
    However, X_1 and X_2 are **strongly dependent**: with 50% probability comonotone, with 50% countermonotone
  - Also note that $X_1 + X_2 = Z(1+U)$ is **not** normally distributed

- This Example can be made much more realistic

CONCLUSION

- EVT and copula techniques are no doubt **most useful tools** in quantitative finance and insurance

- More work is needed

- Reader Guidelines to **P. Embrechts**, **C. Klüppelberg** and **T. Mikosch** (1997) **Modelling of Extremal Events for Insurance and Finance**, Springer, Berlin:
  "Though not providing a risk manager in a bank with the final product he or she can use for monitoring financial risk on a global scale, we will provide that manager with stochastic methodology needed for the construction of various components of such a global tool"

REFERENCES

- See various papers on:

  `www.math.ethz.ch/~embrechts`

- Forthcoming book:

  P. Embrechts, **R. Frey** and **A. McNeil** (2003/4)
  Stochastic Methods for Quantitative Risk Management