Measuring Risk for Function of Dependent Risks

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INTRODUCTION

Suppose we measure risk through VaR and that we are concerned with the following problem:

- one-period risks X_1, X_2 are given
- marginal Value-at-Risks are known

At the integrated level the bank has to measure the risk of the joint position $X_1 + X_2$

The intuitive statement "The worst case VaR for a portfolio X_1+X_2 occurs when the linear correlation function is maximal" is in general (non-elliptical portfolios) wrong

The techniques summarized enable risk managers to tackle the following problems:

- How can one bound the Value-at-Risk of a global, aggregated position, if one only has information on the marginal distributions (VaR's)?

- How do these bounds change when specific dependence information is assumed?

Generalized inverses and VaR

Definition 1 (Generalized inverses) The generalized left and right continuous inverses of an increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ are:

$$\varphi^{-1}(y) := \inf\{x \in \mathbb{R} \,|\, \varphi(x) \geq y\}$$
$$\varphi^{\wedge}(y) := \sup\{x \in \mathbb{R} \,|\, \varphi(x) \leq y\}$$

Definition 2 (VaR) For $0 \leq \alpha \leq 1$ the *Value-at-Risk at probability level* α of a r.v. X with d.f. F_X is its α-quantile

$$\mathrm{VaR}_\alpha(X) := F_X^{-1}(\alpha)$$

Setup

- risks X_1, X_2 with d.f. F_1, F_2
- $\psi : \mathbb{R}^2 \to \mathbb{R}$ increasing and continuous

We are interested in bounding VaR of the joint position $\psi(X_1, X_2)$

Examples

a) portfolio position:
$\psi(x_1, x_2) = a_1 x_1 + a_2 x_2$

b) excess-of-loss reinsurance:
$\psi(x_1, x_2) = (x_1 - k)^+ + (x_2 - k)^+, \ k \geq 0$

c) stop-loss reinsurance:
$\psi(x_1, x_2) = (x_1 + x_2 - k)^+, \ k \geq 0$

d) basket options, credit derivatives

Remarks

- results extend to aribitrary dimensions
- for notational reasons losses are in the right tails of the F_i's
- more general risk measures can also be treated

CONTENT

1. Copulae

2. Distributional bounds

3. Comonotonicity and independence

4. Computational aspects

5. Examples

1 COPULAE

1.1 Definitions, properties and examples

Definition 3 A (2 -dimensional) copula C is a (2-dimensional) d.f. on $[0,1]^2$ with uniform-$(0,1)$ marginals

Properties

a) C is continuous and increasing in each argument

b) $C(1,u) = C(u,1) = u$ for any $0 \leq u \leq 1$

Examples

a) Independence: $C_{\mathrm{I}}(u,v) = uv$

b) Gumbel: $0 < \beta \leq 1$

$$C_\beta^{\mathrm{Gu}}(u,v) = \exp\left[-\left\{(-\log u)^{1/\beta} + (-\log v)^{1/\beta}\right\}^\beta\right]$$

c) Comonotonicity: $C_{\mathrm{U}}(u,v) = \min\{u,v\}$

d) Countermonotonicity:

$$C_{\mathrm{L}}(u,v) := (u+v-1)^+$$

Remark

- $C_1^{\mathrm{Gu}} = C_{\mathrm{I}}$ and $\lim_{\beta\downarrow 0} C_\beta^{\mathrm{Gu}} = C_{\mathrm{U}}$

1.2 Copulae as dependence structures

Consider a 2-dimensional d.f. F and one-dimensional d.f. F_1, F_2

Idea: separate the dependence structure in F from the marginal behaviour

Theorem 1 (Sklar)
F has marginals F_1, F_2 if and only if there is a copula C such that

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$$

Remarks

- if F_1, F_2 are continuous, then C is unique
- C couples the marginals F_1, F_2 to form the joint d.f. F and is therefore referred to as dependence structure

1.3 Fréchet bounds

Any copula C satisfies

$$C_{\mathsf{L}} \leq C \leq C_{\mathsf{U}}$$

and the expressions C_{L} and C_{U} are called lower- and upper-Fréchet bound

1.4 Comonotonicity and Quadrant dependence

Definition 4 X_1, X_2 with a C_{U}-dependence structure are called comonotonic

Comonotonicity is a strong dependence concept.

Lemma 1 *Equivalent are:*

(i) X_1, X_2 *are comonotonic*
(ii) there exist increasing f_1, f_2 *and a r.v.* Z *so that* $(X_1, X_2) \stackrel{d}{=} (f_1(Z), f_2(Z))$

Remark

- Lemma 1 (ii) motivates the use of the concept of comonotonicity in financial applications, the r.v. Z can beseen as a common underlying factor

- for X_1 and X_2 comonotonic, correlation between X_1 and X_2 is maximal

Question:

- How to compare two or more dependence structures?
- Which copula leads to a strong or to a weak kind of dependence?

Possible approach: consider stochastic orders for probability distributions and define the degree of dependence through this partial order

For $\mathbf{X} = (X_1, X_2)$ with joint d.f. $F_{\mathbf{X}}$ consider the joint survival function

$$\overline{F}_{\mathbf{X}}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$$

Definition 5 (correlation order) For $\mathbf{X}$ and $\mathbf{Y}$ with pairwise equal marginals

$$\mathbf{X} \leq_{\mathrm{c}} \mathbf{Y} \Leftrightarrow F_{\mathbf{X}} \leq F_{\mathbf{Y}} \Leftrightarrow \overline{F}_{\mathbf{X}} \leq \overline{F}_{\mathbf{Y}}$$

Remarks

- $\mathbf{X} \leq_{\mathrm{c}} \mathbf{Y}$ means that Y_1, Y_2 are more likely to take simultaneously small (large) values compared to X_1, X_2
- $(X_1, X_2) \leq_{\mathrm{c}} (Y_1, Y_2)$ is equivalent with:

$$\mathrm{Cor}(f(X_1), g(X_2)) \leq \mathrm{Cor}(f(Y_1), g(Y_2)), \qquad \text{for all increasing } f, g$$

- C_{U} is a maximal element w.r.t. $\leq_{\mathrm{c}}$, hence, for this order, comonotonicity correspond to the strongest possible dependence

Let $\tilde{X}_1, \tilde{X}_2$ be independent copies of X_1, X_2

Definition 6 If $(\tilde{X}_1, \tilde{X}_2) \leq_c (X_1, X_2)$, then X_1, X_2 are positive quadrant dependent (PQD)

Remarks

- for X_1, X_2 PQD the correlations between by increasing transformed X_1, X_2 are nonnegative

- PQD assumption is quite natural when modelling positive dependence:

  association $\Rightarrow$ PQD

  cond. increas. in sequence (CIS) $\Rightarrow$ PQD

2 DISTRIBUTIONAL BOUNDS

2.1 Notation

Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be increasing and continuous and C a two-dimensional copula

- $\psi_x(\cdot) = \psi(x, \cdot)$

- the dual copula of C is

$$C^d(u_1, u_2) = u_1 + u_2 - C(u_1, u_2)$$

Remark

- for (X_1, X_2) with copula C and marginals F_1, F_2:

$$C^d(F_1(x_1), F_2(x_2)) = P(\{X_1 \leq x_1\} \cup \{X_2 \leq x_2\})$$

Define

$$\tau_{C,\psi}(F_1, F_2)(s) := \sup_{x\in\mathbb{R}} C(F_1(x), F_2(\psi_x^\wedge(s)))$$

$$\sigma_{C,\psi}(F_1, F_2)(s) := \int_{\{\psi\leq s\}} dC(F_1(u), F_2(v))$$

$$\rho_{C,\psi}(F_1, F_2)(s) := \inf_{x\in\mathbb{R}} C^d(F_1(x), F_2(\psi_x^\wedge(s)))$$

Remark

- for (X_1, X_2) with copula C and marginals F_1, F_2:

$$\sigma_{C,\psi}(F_1, F_2) = F_{\psi(X_1,X_2)}$$

2.2 Existence

Theorem 2 *Let* (X_1, X_2) *have marginal distribution functions* F_1, F_2 *and let* $\psi : \mathbb{R}^2 \to \mathbb{R}$ *be increasing and continuous. If a copula* C *for* (X_1, X_2) *satisfies* $C \geq C_0$ *for some given copula* C_0*, then*

$$\tau_{C_0,\psi}(F_1, F_2) \leq \sigma_{C,\psi}(F_1, F_2) \leq \rho_{C_0,\psi}(F_1, F_2)$$

Remarks

- $\tau_{C_0,\psi}(F_1, F_2)$ and $\rho_{C_0,\psi}(F_1, F_2)$ are d.f.
- $\tau_{C_0,\psi}(F_1, F_2)$ and $\rho_{C_0,\psi}(F_1, F_2)$ are in general not d.f. of r.v. $\psi(Y_1, Y_2)$ and $\psi(Z_1, Z_2)$ with $Y_i, Z_i \sim F_i$
- the bounds obtained when increasing C_0 become tighter

Scenarios the condition $C \geq C_0$ leads to different dependence scenarios:

Examples

(Sc1) $C \geq C_L$: no dependence restriction

(Sc2) $C \geq C_I$: PQD dependence

(Sc3) $C \geq C_{0.2}^{\text{Gu}}$: at least Gumbel dependence

2.3 Optimality

The distributional bounds $\tau_{C_0,\psi}(F_1, F_2)$ and $\rho_{C_0,\psi}(F_1, F_2)$ are pointwise best-possible

Theorem 3 *Let $s \in \mathbb{R}$ be fixed. For a copula C_0, marginals F_1, F_2 and $\psi : \mathbb{R}^2 \to \mathbb{R}$ increasing and continuous let*

$$\alpha := \tau_{C_0,\psi}(F_1, F_2)(s)$$
$$\beta := \rho_{C_0,\psi}(F_1, F_2)(s)$$

There is a family of copulae $\{C^\gamma\}_{0\leq\gamma\leq 1}$ such that

$$\sigma_{C^\alpha,\psi}(F_1, F_2)(s) = \alpha$$
$$\sigma_{C^\beta,\psi}(F_1, F_2)(s) = \beta$$

Notation: Under the assumptions of Theorem 2 we write:

$$F_{\text{min}} := \tau_{C_0,\psi}(F_1, F_2)$$
$$F_{\text{max}} := \rho_{C_0,\psi}(F_1, F_2)$$

Theorems 2 and 3 rewritten in quantile versions:

- $F^{-1}_{\text{max}}(\alpha) \leq \text{VaR}_\alpha(\psi(X_1, X_2)) \leq F^{-1}_{\text{min}}(\alpha)$
- $F^{-1}_{\text{max}}(\cdot)$, $F^{-1}_{\text{min}}(\cdot)$ are best-possible

Remark

- Theorems 2 and 3 can be modified for functionals ψ which are decreasing in both arguments or increasing in one argument and decreasing in the other

3 COMONOTONICITY AND INDEPENDENCE

3.1 Comonotonicity

VaR calculations for comonotonic risks can be transported through ψ

Proposition 1 *Let* $\psi : \mathbb{R}^2 \to \mathbb{R}$ *be increasing and left continuous in each argument. Then, for any* $0 \leq \alpha \leq 1$ *such that the* VaR*'s are finite and comonotonic* X_1, X_2*, we have that*

$$\mathrm{VaR}_\alpha(\psi(X_1, X_2)) = \psi(\mathrm{VaR}_\alpha(X_1), \mathrm{VaR}_\alpha(X_2))$$

3.2 Independence

For independent risks $F_{\psi(X_1,X_2)}$ can be explicitly calculated

Proposition 2 *Let* $\psi : \mathbb{R}^2 \to \mathbb{R}$ *be increasing and left continuous in each argument. Then for independent* X_1, X_2 *with d.f.* F_1, F_2*, we have*

$$F_{\psi(X_1,X_2)}(s) = \int F_2(\psi_x^{\wedge}(s))\, dF_1(x)$$

4 COMPUTATIONAL ASPECTS

In most cases, the bounds F_{min} and F_{max} do not allow for a closed form expression and one has to resort to numerical approximations. The numerical procedure is based on the following steps:

- discretization of F_{min} and F_{max}
- alternative representation of F_{min} and F_{max}
- duality principle

4.1 Discretization

Approximate an arbitrary d.f. F by step functions $\underline{F}_N, \overline{F}_N$, $N \in \mathbb{N}$

$$\underline{F}_N(s) := \frac{1}{N} \sum_{r=1}^{N} 1_{[q_r,\infty)}(s)$$

$$\overline{F}_N(s) := \frac{1}{N} \sum_{r=0}^{N-1} 1_{[q_r,\infty)}(s)$$

The jump points $q_0, \ldots, q_N$ are

$$\begin{aligned} q_0 &:= \inf \operatorname{supp}(F) \\ q_r &:= F^{-1}(r/N) \qquad r = 1, \ldots, N-1 \\ q_N &:= \sup \operatorname{supp}(F) \end{aligned}$$

Remarks

- $\underline{F}_N \le F \le \overline{F}_N$
- $\lim_{N\to\infty} \underline{F}_N = \lim_{N\to\infty} \overline{F}_N = F$

4.2 Alternative representations and duality

Recall:

$$F_{\text{min}}(s) = \sup_{x \in \mathbb{R}} C_0(F_1(x), F_2(\psi_x^{\wedge}(s)))$$
$$F_{\text{max}}(s) = \inf_{x \in \mathbb{R}} C_0^d(F_1(x), F_2(\psi_x^{\wedge}(s)))$$

- $\psi(t_1, t_2) = s \Leftrightarrow \psi_{t_1}^{\wedge}(s-) \leq t_2 \leq \psi_{t_1}^{\wedge}(s)$
- $C_0(F_1(t_1), F_2(t_2))$ is increasing in both t_1 and t_2

Alternative representation for F_{min}:

$$F_{\text{min}}(s) = \sup_{\psi(t_1, t_2) = s} C_0(F_1(t_1), F_2(t_2)) \tag{1}$$

Similar arguments for $F_{\max}$? Consider

$$\inf_{\psi(t_1,t_2)=s} C_0^d(F_1(t_1), F_2(t_2)) \tag{2}$$

Remarks
There are examples where (2)

- is at some points strictly smaller than $F_{\max}$
- is not even an upper bound for $F_{\psi(X_1,X_2)}$

However, (2) is the left continuous version of $F_{\max}$ and hence it leads to the same quantiles

Conclusion: both (1) and (2) can be used to obtain the quantile functions $F^{-1}_{\min}$ and $F^{-1}_{\max}$

Theorem 4 (Duality) *For an increasing continuous function ψ, a copula C_0 and marginals F_1, F_2 and any $0 \le \alpha < 1$ one obtains*

$$F_{\min}^{-1}(\alpha) = \inf_{C_0(u,v)=\alpha} \psi(F_1^{-1}(u), F_2^{-1}(v))$$
$$F_{\max}^{-1}(\alpha) = \sup_{C_0^d(u,v)=\alpha} \psi(F_1^{-1}(u), F_2^{-1}(v))$$

In practice

- discretize $[0,1]$ as $\{l/N \,|\, l \in \{0,\dots,N\}\}$
- take $\alpha = r/N$, $r \in \{1,\dots,N-1\}$, and solve for $\nu_{r,l}$ and $\nu_{r,l}^*$ in

$$C_0(l/N, \nu_{r,l}) = r/N, \quad C_0^d(l/N, \nu_{r,l}^*) = r/N$$

- take the minimum over all $l \in \{0,\dots,N\}$

Formally

$$q_{\text{min}}(r/N) := \min_{r \le l \le N} \psi(F_1^{-1}(l/N), F_2^{-1}(\nu_{r,l}))$$
$$q_{\text{max}}(r/N) := \max_{0 \le l \le r} \psi(F_1^{-1}(l/N), F_2^{-1}(\nu^*_{r,l}))$$

and (we suppress N)

$$\underline{F_{\text{min}}}(s) = \frac{1}{N} \sum_{r=1}^{N} 1_{[q_{\text{min}}(r/N),\infty)}(s)$$
$$\overline{F_{\text{max}}}(s) = \frac{1}{N} \sum_{r=0}^{N-1} 1_{[q_{\text{max}}(r/N),\infty)}(s)$$

5 EXAMPLES

Range for $VaR(\psi(X_1, X_2))$ under different dependence scenarios

(Sc1) $C \geq C_L$: no dependence restriction

(Sc2) $C \geq C_I$: PQD dependence

(Sc3) $C \geq C_{0.2}^{\text{Gu}}$: at least a Gumbel dependence

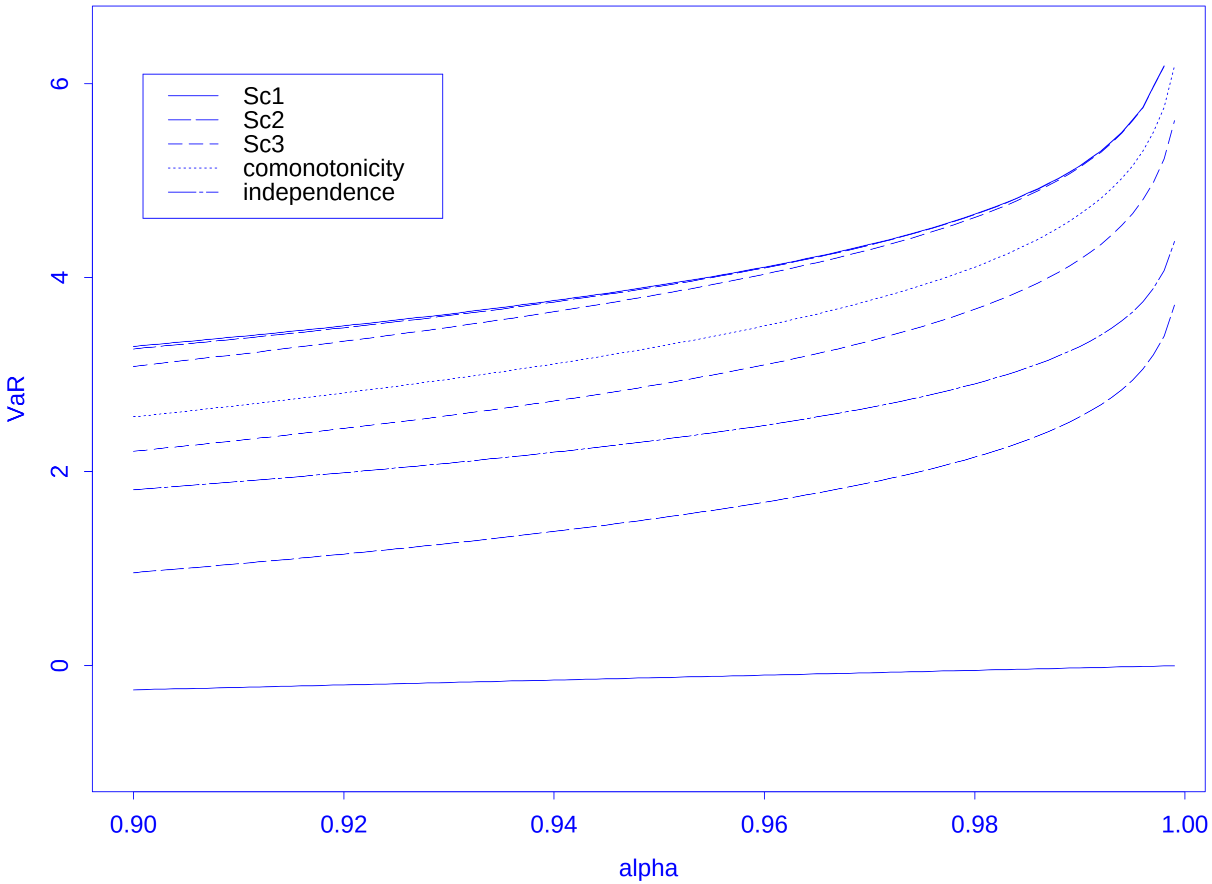
5.1 The plain vanilla case

- $X_i \sim N(0,1), i = 1,2$
- $\psi(x_1, x_2) = x_1 + x_2$

Note that:

- for X_1, X_2 comonotonic $VaR(X_1 + X_2) = VaR(X_1) + VaR(X_2)$
- there is a non-coherence gap, i.e. copulae for which $VaR(X_1 + X_2) > VaR(X_1) + VaR(X_2)$
- the worst case for $VaR(X_1 + X_2)$ under $X_i \sim N(0,1)$ conditions is obtained (correlation is not maximal)

Figure 1: Range for $VaR(X_1 + X_2)$ **with** $X_i \sim N(0,1)$



	$\alpha = 0.95$			$\alpha = 0.99$		
scenarios	exact	min	max	exact	min	max
(Sc1)		-0.13	3.92		-0.03	5.15
(Sc2)		1.52	3.91		2.56	5.15
(Sc3)		2.90	3.83		4.19	5.14
$C = C_I$	2.33			3.29		
$C = C_U$	3.29			4.65		

Table 1: Range for $\mathrm{VaR}_{0.95}(X_1 + X_2)$ and $\mathrm{VaR}_{0.99}(X_1 + X_2)$ for a standard normal portfolio

5.2 Further examples

Consider the functionals

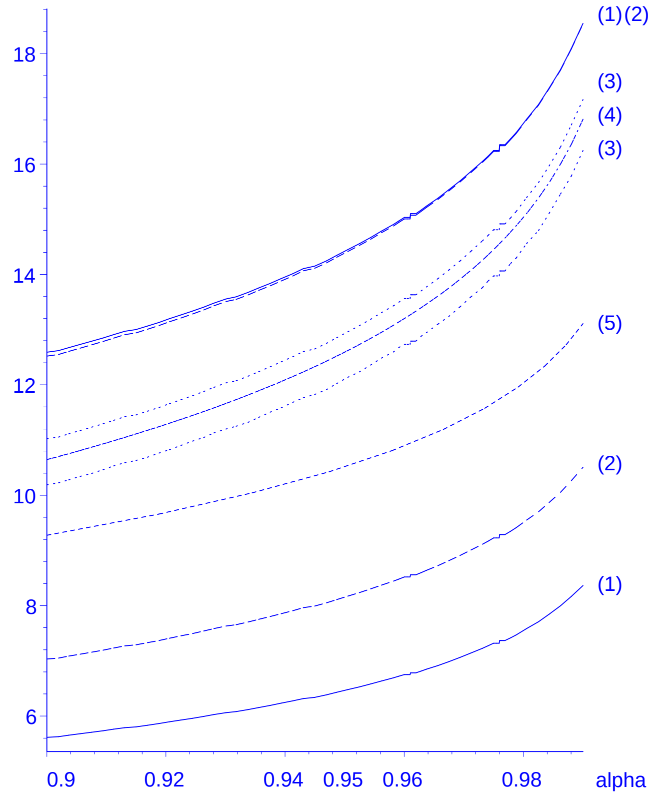
1. $\psi(x_1, x_2) = x_1 + x_2$

2. $\psi_1(x_1, x_2) = (\max(x_1, x_2) - \text{const})^+$

3. $\psi_2(x_1, x_2) = x_1 \cdot I_{\{x_2 > F_2^{-1}(0.9)\}}$

for a $\gamma(3, 1)$ portfolio

Figure 2: $VaR(X_1 + X_2)$ for a $\gamma(3, 1)$ portfolio



Lines (1), respectively (2) and (3) are the upper, and lower bounds under scenarios (Sc1), respectively (Sc2) and (Sc3). Lines (4) and (5) correspond to comonotonicity and independence for X_1 and X_2

Figure 3: $VaR(\psi_1(X_1, X_2))$ for a $\gamma(3, 1)$ portfolio

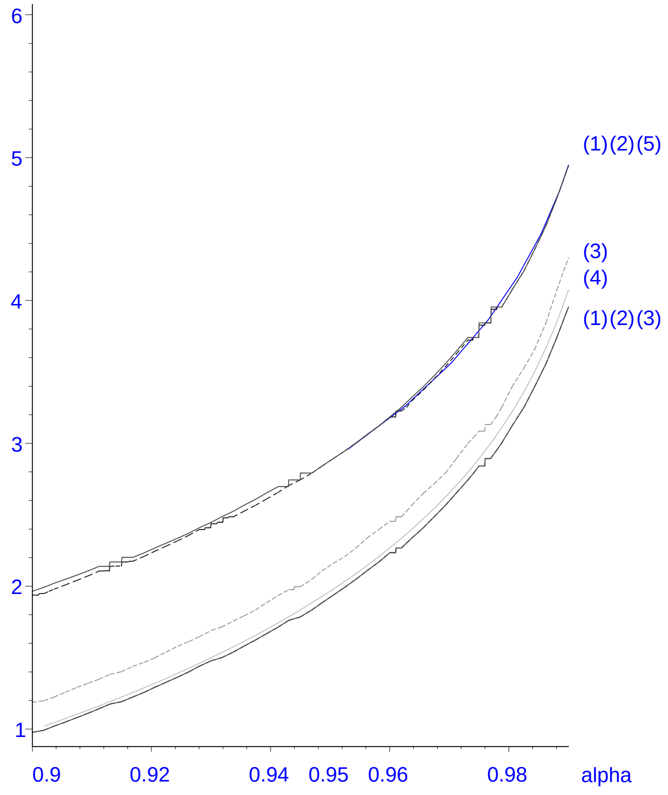
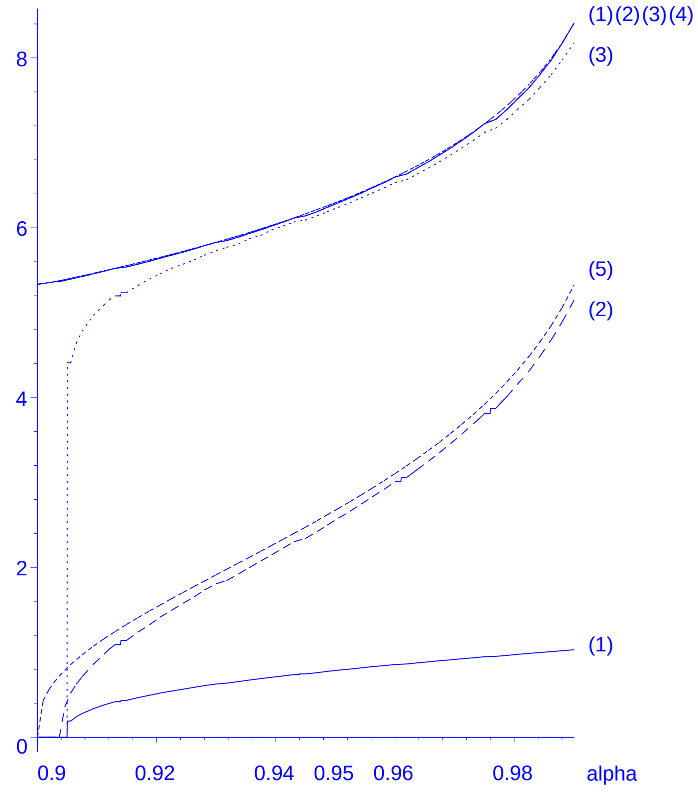


Figure 4: $VaR(\psi_2(X_1, X_2))$ for a $\gamma(3, 1)$ portfolio



Bibliography

For details and further references, see:

Embrechts, P., Höing, A. and Juri, A. (2001). *Using Copulae to bound the Value-at-Risk for functions of dependent risks. Finance and Stochastics, to appear.*