Formal Universes

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Abstract. This essay addresses the concerns of the foundations of mathematics of the early 20th century which led to the creation of formally axiomatized universes. These are confronted with contemporary developments, particularly in computational logic and neuroscience. Our approach uses computational models of mental experiments with the infinite in set-theory and symbol-manipulation systems, in particular models of combinatory logic.

1 Introduction

To guide the perplexed, philosophers used to propose their systems, physicists their Grand Unified Theories, and theologians their faith and scriptures. What about mathematicians? Formal universes.

In this essay[1] I try to tell this story as I see it, distinguishing two kinds of universes: One is centered in set theory, the other in symbol-manipulating systems. Both claim universality for mathematics. For evidence both adduce developments of vast stretches of mathematics, reducing it to the respective "foundations". I permit myself some skeptical comments on the whole enterprise.

2 Universalism I : Paradise Lost?

Mathematics creates its own universe, not out of chaos or *tohu-wabohu* as in Genesis, but out of nothingness, the empty set. Start with the empty set $V_0 = \emptyset$ and repeat taking the set of all subsets: $V_1 = P(\emptyset), V_2 = P(P(\emptyset)), \ldots$, all the while collecting what you have obtained so far. At the beginning all is finite; the first time an infinite set obtains is when collecting up after infinitely many steps. This is the set V_ω of hereditarily finite sets, a small universe of mathematical objects. In it we may recognize (or fashion) the natural numbers, the integers, the rationals (individually, not as totalities). According to the famous 19th century algebraist Kronecker this is all that God created, the rest is "Menschenwerk". Works of man are for example the real numbers which arise if we take the power-set of the hereditarily finite sets and, continuing the process, functions, function spaces, ... the whole menagerie, the universe of mathematics, called the

[1] In memory and appreciation of a quarter century of discussion on the foundation and philosophy of mathematics and computer science with José Meseguer.

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cumulative hierarchy, due to von Neumann and Zermelo [18]. All of this out of nothingness, insubstantial.

Indeed? What is the substance of mathematics; is there one? Substance is something that you have to reckon with, can reckon with. Mathematicians can manipulate, reckon with the objects of their domain, the universe of sets, named "Cantor's Paradise" by Hilbert [12].

In this view, the problem of foundations is to capture the kind of reckoning that goes on in the construction of the universe we just described. Around 1900 this meant to axiomatize: Postulate the empty set, pairing, power set, union, etc. in fact the axioms of Zermelo. These axioms arguably constitute the basis of mathematics today, they are still taken as "axiomatic" by most. Zermelo also produced an axiomatization of thermodynamics, long since forgotten. Axiomatization at that time meant mainly to describe compactly and plausibly the knowledge of a subject matter—with the intention to support logical conclusions, theorems, covering the field—*more geometrico* (Spinoza.)

Hilbert was one to recognize that axiomatization did not by itself create the universe of discourse, nor does it automatically fix it. His axiomatization of real numbers arguably fixes them,[2] but their "reality" had to be seen in the system of logical deductions from these axioms. Consequently, foundations as an axiomatic discipline became based on formal logic. The problem of foundation was reduced to proving deductive consistency by finite universally accepted means, and existence became identified with being free of formal logical contradictions. In the words of the main adversary, the founding intuitionist Brouwer, this meant that mathematics consisted only of "marks on paper".—And, as is well known, Hilbert's program was shown to fail by Gödel.—Banned from the Garden of Eden we now follow Voltaire's advice and cultivate our own gardens, outside.

This, of course, is not the whole story. In fact, the challenge was taken up by an impressive and ongoing development of proof theory (e.g. locating incompleteness of Peano arithmetic in the formal limitations of induction), and of axiomatic set theory (showing up the limitations of "platonic insight" into the transfinite by set-theoretic independence results), and much more. In the following we describe a small section of these developments from a somewhat personal perspective.

3 Universalism II : Paradise Regained?

Of course, nowadays numbers, reals, sets, functions do not only live on paper, they also live in our computers: as the data types of programming languages, supported by interpreters and compilers on sophisticated hardware. This universe of objects "to be reckoned with",[3] the types of objects available for computing is ever expanding. The definition of data types in most programing languages proceeds in a more or less pragmatic manner, guided by emerging applications, including the "internet of things". This is a valid approach to universality, and

[2] With respect to the "naive" set theory in which its structure is discussed.
[3] Cf. German "rechnen" = to compute.

important efforts are being made to provide it with systematic foundations as we shall see.

To illustrate this second kind of approach to universalism in mathematics, I shall use its historically first worked-out, and technically simplest, example. It is based on a development that originated in the 1930s in answer to the same "crisis of foundations" sketched above: Combinatory logic, lambda calculus, and type theories. Today these theories can also be seen as a foundational approach to computer science. Since Turing, in particular since universal Turing Machines, and also since von Neumann-Machines, inputs and programs are of the same nature and which we may simply call "data". They are perhaps marks on the Turing tape or bits in the computer memory. The basic operation is application: Programs may be applied to input data and of course result in data, which may again be programs. Programs may also be applied to programs, again resulting in data, etc. Indeed, we may admit that all combinations of applications on data result again in data, (e.g. in the model of Sect. 4 below).

Formally we have a the universe D of data which admits a binary operation on all its elements: if a and b are in D, then so is a applied to b, written $a \cdot b$. Any combination of data is results in data. For example $(x \cdot z) \cdot (y \cdot z)$ is a kind of data: If the program y is used to modify input z to a new input $y \cdot z$, and x uses the input z to obtain a new program $x \cdot z$, then $(x \cdot z) \cdot (y \cdot z)$ is the data resulting from applying the new program to the new input. Obtaining this data is in fact a program $\mathbf{S}$ which is applied to data x, y and z in sequence: $((\mathbf{S} \cdot x) \cdot y) \cdot z)$ equals $(x \cdot z) \cdot (y \cdot z)$.

Universality is expressed by the following axiom scheme: For every combination $\phi(x_1, \ldots x_n)$ of data there is a data t_ϕ such that $t_\phi x_1 \ldots x_n = \phi(x_1, \ldots x_n)$, (association to the left is understood tacitly.) Such t_ϕ are called "combinators". A set D with an application operation satisfying the axiom scheme is called a combinatory algebra. Combinatory Logic, was invented by H.B. Curry in his 1929 Göttingen thesis [3] directed by Paul Bernays, also Doktorvater of the present author as well as of Saunders MacLane and Gerhard Gentzen. It is the formal theory of equations between combinators as its objects. As Schönfinkel had already shown, two combinators suffice for expressing all combinators, namely the above $\mathbf{S}$, characterized by its equation, together with $\mathbf{K}$, characterized by $\mathbf{K}xy = x$. These equations may be understood as rewrite rules of the logic. The two combinators come out in the proof of the axiom scheme.

For Combinatory Logic the question of consistency arises again. But while Hilbert's program failed for the first formal universe (of set theory and Peano arithmetic), Curry's formal system is consistent, as proved by Church and Rosser using the same finitist proof-theoretic tools that failed in the first case. This proof considers combinatory logic and algebra as a rewrite system. As a rewrite system, with rules such as $\mathbf{S}xyz \to (xz)(yz)$, it is surprisingly rich: it admits natural numbers (as combinations of $\mathbf{S}$ and $\mathbf{K}$), partial recursive functions (and thereby a theory of computability equivalent to Turing Machines), exhibits the phenomenon of undecidability (of termination of rewriting), etc. It is for this reason that we may consider combinatory logic as a foundation for computer science. It is open to formal extensions. One important such extension results from the introduction of types (e.g. to Lambda Calculus).

As a symbol manipulation system, Combinatory Logic is the prototype of a development that has been mastery reviewed by Meseguer [14]. The extensive development of rewrite logic supports the claim that the concept of mathematics as a symbol-manipulating enterprise is sustainable.

Rewrite systems prefer to live on computers. The currently successful computational system COQ offers such an environment. We remarked above that types are introduced in the classical programming languages in a restricted manner. To be part of a mathematical, or computational universe, containing the structures on which mathematics is "really" done, it would need types. COQ introduces basic types such as natural numbers and allows definition of new types by recursion, products (of dependent types) and continues through a hierarchy of classical mathematical structures, groups, fields, etc. to contemporary structures such as homotopy groups; all of this by uniform definitional templates. Moreover, it embodies procedures which support construction, and checking, of mathematical proofs—after a process of formalization that is empowered by its structural richness. The mathematics on which this rests is Homotopy Type Theory.[4]

A quite different example of universalism akin to rewrite systems is Wolfram's proposal to understand the mathematical sciences by rewrite rules in the shape of rules for cellular automata [17]. As a remarkable single-handed effort it challenges by its suggestiveness and scope. It should not be blamed for not completely and convincingly reaching its goal.[5]

4 Reductionism

Reducing one corpus of knowledge (e.g. chemistry) to another (e.g. quantum theory) is more a kind of (successful) mind-set than a true methodology. Called "reductionism", it is a well-recognized topic in the philosophy of science and has been the subject of an extensive literature. Whether it has ever been fully successful in the natural sciences is open to doubt.[6] How about mathematics?

Universalism and reductionism go hand in hand. This is the case of the set-theoretic universe. Indeed it is the tacit understanding of most mathematicians that all mathematics can "in principle" be reduced to set theory and logic. But this comes with steep costs: Mathematics, when fully reduced to its set-theoretic basic components may become quite opaque to mere humans. But not always.

Let us return to the second kind of universe, combinatory logic, which we introduced as a specimen of a symbol-manipulating systems and whose "existence" relies on a formal consistency proof. If the first universe, of set theory, is "good enough", then it should be possible to reduce the second universe to it, that is: to create a pocket universe for combinatory logic in set theory.

[4] An ongoing project accessible online at HomotopyTypeTheory.org.
[5] For a critique cf. [11].
[6] Among others by my late colleague, the quantum chemist H. Primas, in [1].

This would be an algebraic structure with elements corresponding to the combinators and their laws. My favorite construction[7] starts with an arbitrary non-empty set A, the basic data (e.g. marks on the Turing tape or bits in computer memory), and builds upon it a set $G(A)$ of formal expressions. It is recursively defined by $G(A) = A \cup \{\alpha \longrightarrow x : \alpha \subseteq G(A), x \in G(A)\}$, α finite. It thus consist of expressions $\alpha \longrightarrow x$ obtained by starting with basic data by iterating replacement of components x and $y \in \alpha$.—From this we define the combinatory algebra $\mathcal{D}_{\mathcal{A}}$: Its set D of elements is the set of all subsets of $G(A)$, which stands as the set of data, $D = P(G(A))$. The application operation is defined by $X \cdot Y = \{x : \exists \alpha \subseteq Y, \alpha \longrightarrow x \in X\}$ for arbitrary data X and Y in D.[8]

This author's Combinatory Program [8] shows in detail that much of mathematics can be developed by suitably enriching $\mathcal{D}_{\mathcal{A}}$ with additional operations and admitting specific basic sets A.

The extension to the theory of dependent types as in Homotopy Type Theory, mentioned above, also calls for a reduction to the set-theoretic universe. This is accomplished by Voevodsky in the current univalent foundation project,[9] which we cannot discuss here.

5 Criticism

Universalist tendencies are present in many fields and for many reasons; they are obviously attractive intellectually. But universality claims can also be the source of fundamentalism. In religion and politics unspeakable extremes have resulted throughout history. If such fundamentalism is applied to the natural sciences it may be strange (geology: disputing the age of the world; biology: disputing evolution). But occasionally, and frighteningly, there are even cases concerning mathematics (symbolic logic, banned as "bourgeois idealism" under Stalin; some "decadent" mathematical developments to be prosecuted, banned and replaced by "Deutsche Mathematik" by the Nazi).

So there are reasons to be skeptical towards universalism, even if it is esthetically attractive such as in the search for a Grand Unified Theory (for physics) and in the formal universes for mathematics and computer science of which we talked above. There are many others, equally attractive such as Feferman and Jäger's explicit mathematics and their universes [13].

How could one be skeptical about mathematics? Using the Zermelo-Fraenkel axioms of set theory *ZFC*, are we really sure that we fully understand infinite sets and their properties? Isn't set theory a kind of story that mathematicians tell among themselves, somewhat akin to fairy tales: Infinite sets are like the Easter Bunny which we have never actually seen but about some of whose properties

[7] The Plotkin-Scott-Engeler model.

[8] Understanding the model may be helped by considering sets of expressions $\alpha \longrightarrow x$ as partial and many-valued function from $G(A)$ to $G(A)$, namely as sets of pairs of arguments and values in $G(A)$. (If we so wish, we may also see these expressions as lists with head x and tail α).

[9] Reviewed in a recent survey [15].

we all agree; set theory also hides its precious gifts for us to search for them. Now, of course this is a doubtful kind of doubting. But still, on what basis rests the undeniable consent of mathematicians on these axioms? Some even doubt the power-set axiom for infinite sets, for example Gödel at one time, in his 1933 talk "The present situation in the foundations of mathematics" [9].

Mathematics is what all sufficiently patient learners and inventive practitioners of mathematics agree on. Let's fix on that and make it into a concrete model. Imagine mathematicians making thought-experiments on that part of the universe of sets that is immediately available to finite minds: the set of hereditarily finite sets (the first limit in the cumulative hierarchy, cf. Sect. 2). The goal of experiments is to verify the axioms of *ZFC* as expressed in first-order logic. Experiments are performed on the hereditarily finite sets by mentally executing programs based on the basic operations and basic predicates used in the axioms. The patience of such a mathematician may be measured by her or his willingness to test a universal quantifier by running through all sets up to rank n in the cumulative hierarchy. The inventiveness may be measured by the complexity n of programs that this mathematician creates to test an existential quantifier (by some measurement of the complexity of programs, only finitely many programs having smaller complexity than n). A mathematician of strength n in patience and inventiveness admits his individual theory ZFC_n of sets. Now consider ZFC_∞, the statements accepted by all sufficiently strong mathematicians. It turns out to contain *ZFC*.[10] Unfortunately (for logic?—for the model?), the set ZFC_∞, while it does not contain any theorem together with its negation, is not closed under classical logical deduction [16]. As were the intuitionists, we may be lead to doubt the reliability of classical logic when we approach infinite objects. How confident can we be?

Here is an extreme viewpoint: Some years ago van Dantzig [4] asked the question whether $10^{10^{10}}$ is a finite number, doubting that our mind is able to conceive this number as being built up by continued attaching individual marks to a given object (the intuitive basis of the number concept as proposed by Peano, Whitehead and Russell, Hilbert), or by a series of mental acts. Still, we seem to be content that $10^{10^{10}} + 10^{10^{20}} = 10^{10^{20}} + 10^{10^{10}}$, trusting a proof by induction that $a + b = b + a$ for all natural numbers. But does not such a proof beg the question in that it presumes in the induction step that local laws, verified in the small, persist in the large?

Again, this doubt is rather doubtful. But it points in another direction, that of the limits of the mind, of mental imagination.

In view of the exciting advanced of neurology, e.g. in explaining mechanisms of numerical abilities[11] it may be of interest to use mathematical models to approach the question on how and why the human brain can deal with natural numbers and conceive of them as a totality.

Let me present this in a somewhat idiosyncratic fashion; let us talk about thinking. Thinking means to apply thoughts to thoughts, thoughts being things

[10] Worked out in a 1971/78 paper by the author, reprinted in his collection Algorithmic Properties of Structures, World Publ.Co., 1990, pp. 87–95.

[11] Cf. the review by an originator of the idea of a number sense [7].

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like concepts, impressions, memories, activities, projects—anything that you can think about, including mathematics. And, of course, the results of applying thoughts to thoughts. For example, applying the thought ⌜this object is even⌝ to the thought ⌜the number 14⌝ results in the thought ⌜14 is an even number⌝, a thought that happens to be true, (but this is of no concern.) We can also apply the thought ⌜this object is green⌝ to ⌜the number 14⌝, resulting in the possible, but rather unusual synesthetic thought ⌜the number 14 is green⌝. Thinking is free, all combinations of thoughts are admitted into the universe of thoughts. We arrive again at combinators, now talking about "thoughts" instead of "data".

The totality of thoughts may thus be understood as a combinatory algebra. In the following we let neurology suggest a model for the algebra of thoughts. Consider the brain as a set of connected neurons, firing at discrete times. The Neural Algebra sketched below describes the total activity of this neural net. Recall that for the graph model $\mathcal{D}_{\mathcal{A}}$ (Sect. 4 above), we were free to choose the base set A. Let A therefore consist here of statements $a(t)$ expressing the fact that neuron a fires at time t. Assume we have knowledge of the firing history of some brain and of the underlying firing laws. These laws fix the causality of the activation of, say, neuron a_n at time t_n by neurons $a_0, \ldots, a_{n-1}$ connected to it and firing at (earlier) times $t_0, \ldots, t_{n-1}$ respectively. Let the expression $\{a_0(t_0), \ldots, a_{n-1}(t_{n-1})\} \longrightarrow a_n(t_n)$ denote this fact.

The construction analog to that of $\mathcal{D}_{\mathcal{A}}$ above, now performed on these givens, produces a combinatory algebra $\mathcal{N}_{\mathcal{A}}$, our Neural Algebra. Its elements are sets X each of which describes a time segment of the distributed activation history in respective parts of the brain. The sets X by construction consist of expressions denoting causal cascades of firings.[12]

By Crick's neurological hypothesis [2] all mental activities, perceptions, concepts, etc., "thoughts" for short, correspond to such activation histories and are therefore modeled here by elements in the algebra $\mathcal{N}_{\mathcal{A}}$. Moreover, the operation of application $X \cdot Y$ describes the causal functioning of the brain: the processing X of an input Y, e.g. a visual input, produces $X \cdot Y$, the perception of an observed object. Remark: This is a much simplified version of the model used in the author's ongoing project on Neural Algebra. The full model ties more closely to the basic neural net, and allows to pass from objects in the algebra to underlying structures of the net.[13]

The brain model $\mathcal{N}_{\mathcal{A}}$ allows us to speculate about the presence of the natural numbers in the human brain. First proposed by Dedekind [6], the infinity of natural numbers may be constructed "psychologically" as the following thought: Taking any thought object, e.g. the thought ⌜I am thinking of a number⌝. Reflecting on this thought is again a thought; and so on, yielding an infinity of thoughts.[14] In the Neural Algebra context there is no problem: We may assume that the element N_0

[12] To interpret expression in X as cascades, transform subexpressions such as $\{a\} \longrightarrow (\{b, c\} \longrightarrow d)$ successively by absorption into $\{a, b, c\} \longrightarrow d$.

[13] Diverse papers available online from the author's website, cf. "Neural Algebra".

[14] Proof of theorem 66; dismissed by the critical comment of Emmy Noether, one of the editors.

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of $\mathcal{N}_{\mathcal{A}}$ represents thinking of the number zero, and the element R represents the mental operation of ⌜reflect on a thought⌝. Then Dedekind's construction goes like this: $N_{i+1} = N_i \cup R \cdot N_i, i = 0, 1, \ldots$. Collecting up, this gives the thought N of the totality, namely $N = \bigcup_i N_i$ which solves the recursion equation $N = R \cdot N$ with initial condition N_0. This N is a legitimate object of $\mathcal{N}_{\mathcal{A}}$ and stands for ⌜I am thinking of the totality of numbers⌝. It is a recursively defined object; not an easy subject, for many people it is too close to a vicious circle. The difficulty is that N is an infinite set and can therefore not be totally present in a finite brain restricted to finite time; the brain would be "lost in thought". At best N is present as an approximate thought, an illusion. Jumping a few steps to include other concepts and operations of arithmetic as objects in $\mathcal{N}_{\mathcal{A}}$, what we then are "really" able to conceive about arithmetic remains a sort of illusion, strangely convincing.

Shall we be content with that? Will we have to accept the judgement of the Erdgeist in Goethe's Faust?[15]

The Spirit of Nature, called up by the polymath Doctor Faustus by powerful formulas, spoke thus:

> "You grasp the mind
> you understand,
> not mine."[16]

With all respect for (Goethe's) Nature: certainly not!

6 Rejoinder

It is a good policy, unfortunately rarely followed, to be skeptical of one's own models.

In view of the missing consistency proof, is set theory and indeed most of the mathematical enterprise, just dogmata implanted in the brains of those students that we did not fail in our courses? Are we truly only guided by experiences in the strictly finite and reachable? This goes counter to all experience. The thriving industry of mathematics, pure, applied and expanding into all sciences is evidence enough. What seems to be at work here is what Martin Davis calls "Pragmatic Platonism" [5], which guides the mathematician's "Anschauung" into uncharted realms.

Mathematicians, like all scientists, are legitimized to use whatever technical means are available to aid the naked eye and brain. Otherwise they would be like savages, frightened by the telescope. Of course there are limits. Proof-finding algorithms (first satirized by Swift in Gulliver's Travels) are largely chimeric, and checking out otherwise inaccessible structures and highly complex and long proofs on their computer may leave some colleagues wondering about the reliability of coding and supporting software. But it pays to push these limits.

I refuse to believe that numbers are some sort of evanescent ghosts in our biological brains. Such beliefs were called "the most ridiculous" by Gödel.[17] The

[15] J.W.v.Goethe, Faust, Der Tragödie erster Teil, Nacht.

[16] Free translation by this author. Original: "Du gleichst dem Geist den Du begreifst, nicht mir".

[17] In a letter to A. Robinson, [10].

interpretation of N proposed above as the view we have of the natural numbers (as an object in $\mathcal{N}_{\mathcal{A}}$) is simplistic: By combining thought objects we may form the thought object of composing this N. This is an example of generating an abstract entity as a manipulable object in $\mathcal{N}_{\mathcal{A}}$. Such objects are present in our minds in a vein similar to the thoughts about the thoughts of another person: objects that are clearly present in our mind but not completed there. This is what I think the Pragmatic Platonist experiences as the "strangely convincing" presence of N mentioned above.

To conclude: Mathematics is a wonderful cultural treasure, shared and expanding by many and pursued pragmatically, without prejudice and mindfully. Universalist tendencies, with moderation and caution, help us to appreciate its coherence and beauty.

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