Categories of Mapping Filters*, **

By

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The subject of this paper is a topic in that branch of universal algebra called the theory of models. The main trait that distinguishes model theory from other approaches to algebra is the fact that in model theory the language in which theorems and definitions are to be coined is explicitly, indeed formally, specified. This gives, of course, a peculiar slant to the type of problems that are of immediate interest to model theorists. The generalities in which we are interested concern the exact extent of definability of mathematical notions and the characterizability of types of mathematical structures in various formal languages, the existence of structures with particular properties in formally characterizable classes of structures, formal descriptions of types of properties preserved under various mathematical constructions, and the like. In a nutshell, the difference between an algebraist and a model theorist is the following: To an algebraist two mathematical structures $\mathscr{A}$, $\mathscr{B}$ are "essentially the same" if they are isomorphic,

$$\mathscr{A} \cong \mathscr{B},$$

while to a model theorist they are essentially the same in case they are elementarily equivalent,

$$\mathscr{A} \equiv \mathscr{B},$$

(if $\mathscr{A}$ and $\mathscr{B}$ have the same first-order properties).

While the morphisms are well-known in the algebraic case (and are at the basis of a categorical treatment of algebra) a satisfactory notion of morphism in the model theoretic case has been missing.

1. Filtration of Categories

Recall that a filter on a set P is a family F of subsets of P such that, (a) if S, $T \in F$ then $S \cap T \in F$, (b) if $S \in F$ and $S \subseteq T \subseteq P$ then $T \in F$. A filter is *proper* if $\emptyset \notin F$. An *ultrafilter* is a proper filter which for every

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subset S of P contains either S or its complement S relative to P. If F_0 is a family of subsets of P in which no finite collection of elements has an empty intersection then the filter *generated* by F_0, denoted by $[F_0]$, consists of all sets $T \subseteq P$ such that T contains an intersection of finitely many elements of F_0. A filter F is called *principal* if there exists $p \in P$ such that $F = [\{p\}]$, (often abbreviated as $[p]$).

Let $\boldsymbol{C}$ be a category; we define the *filtration* $\boldsymbol{C}'$ of the *category* $\boldsymbol{C}$ to be the following family of filters: For every pair of objects $A, B \in \boldsymbol{C}$ consider $\boldsymbol{C}(A,B)$, the family of all maps (in $\boldsymbol{C}$) from A to B. Let $\boldsymbol{C}'$ be the family of all proper filters F on $\boldsymbol{C}(A,B)$ (for all A, B). First we define how to compose filters. Suppose F is a filter on $\boldsymbol{C}(A,B)$, G a filter on $\boldsymbol{C}(B,C)$, then the following set $G \cdot F$ will turn out to be a filter on $\boldsymbol{C}(A,C)$:

$$S \in G \cdot F \text{ iff } \{q \in \boldsymbol{C}(B,C) : \{p \in \boldsymbol{C}(A,B) : qp \in S\} \in F\} \in G.$$

Indeed: (a) $\emptyset \notin G \cdot F$ if both G, F are proper; (b) If $S_1, S_2 \in G \cdot F$ then $S_1 \cap S_2 \in G \cdot F$. Namely: from $\{q : \{p : qp \in S_1\} \in F\} \in G$ and

$$\{q : \{p : qp \in S_2\} \in F\} \in G \quad \text{it follows that}$$
$$\{q : \{p : qp \in S_1\} \in F\} \cap \{q : \{p : qp \in S_2\} \in F\} \in G,$$
$$\{q : \{p : qp \in S_1\} \text{ and } \{p : qp \in S_2\} \in F\} \in G,$$
$$\{q : \{p : q\, p \in S_1\} \text{ and } qp \in S_2\} \in F\} \in G, \quad \text{and thus}$$
$$\{q : \{p : q\, p \in S_1 \cap S_2\} \in F\} \in G, \quad \text{i.e.} \quad S_1 \cap S_2 \in G \cdot F;$$

(c) If $S \in G \cdot F$ and $S \subseteq T \subseteq \boldsymbol{C}(A,C)$ then $T \in G \cdot F$.
Namely: $\{q : \{p : qp \in S\} \in F\} \subseteq \{p : qp \in T\} \in F\}$.

Since the former set belongs to G so does the latter, hence $T \in G \cdot F$. Thus, *if F, G are proper filters then so is $G \cdot F$*; moreover, as is easy to verify, if both F,G are ultrafilters then so is $G \cdot F$. This leads us to define the *ultrafiltration* $\boldsymbol{C}^u$ of a category $\boldsymbol{C}$ as the family of all ultrafilters on sets $\boldsymbol{C}(A,B)$.

Observe next that $H \cdot (G \cdot F) = (H \cdot G) \cdot F$ for all $F \in \boldsymbol{C}(A,B)$, $G \in \boldsymbol{C}(B,C), H \in \boldsymbol{C}(C,D)$. Namely: $S \in H \cdot (G \cdot F)$ iff

$$\{r : \{s : rs \in S\} \in G \cdot F\} \in H, \quad \text{iff} \quad \{r : \{q : \{p : qp \in \{s : rs \in S\}\} \in F\} \in G\} \in H,$$
$$\text{iff} \qquad \{r : \{q : \{p : rqp \in S\} \in F\} \in G\} \in H,$$

similarly, $S \in (H \cdot G) \cdot F$ can be found equivalent to this condition.

Finally consider the family of principal filters in $\boldsymbol{C}'$, let it be denoted by $\boldsymbol{C}^*$. $\boldsymbol{C}^*$ is a subcategory of $\boldsymbol{C}'$, in fact, if $F = [p_0]$, $G = [q_0]$ then $G \cdot F = [q_0 p_0]$. In particular, if e_A, e_B are the identities in $\boldsymbol{C}$, on A, B respectively, then $E_A = [e_A]$, $E_B = [e_B]$ are the identities in $\boldsymbol{C}'$ on A, B. Indeed, if $F \in \boldsymbol{C}'(A,B)$, then $E_B \cdot F = F \cdot E_A = F$ as can be easily verified. We have thus proved the following proposition: *For every category* $\boldsymbol{C}$ *its filtration* $\boldsymbol{C}'$ *and ultrafiltration* $\boldsymbol{C}^u$ *are categories with the same objects.*

Let $\boldsymbol{C}$ be a category, A, B, C objects in $\boldsymbol{C}$ and suppose that $\boldsymbol{C}(A,B)$, $\boldsymbol{C}(B,A)$, $\boldsymbol{C}(A,C)$, $\boldsymbol{C}(B,C)$ are all nonempty.

Triangle Lemma. *For any proper filters* U *on* $\boldsymbol{C}(A,B)$, V *on* $\boldsymbol{C}(B,A)$ *there exist ultrafilters* $F \in \boldsymbol{C}^u(A,C), G \in \boldsymbol{C}^u(B,C)$ *such that* $F \supseteq G \cdot U$ *and* $G \supseteq F \cdot V$; moreover, given $F_0 \in \boldsymbol{C}'(A,C)$, $G_0 \in \boldsymbol{C}'(B,C)$ such that $F_0 \supseteq G_0 \cdot U, G_0 \supseteq F_0 \cdot V$ then F, G may be found as refinements of F_0, G_0.

To prove the first part of the lemma, let $\{R_\nu\}$, $(\nu < \alpha)$, be a (transfinite) enumeration of the set of all subsets of $\boldsymbol{C}(A,C)$ possibly with repetitions; similarly, let $\{S_\nu\}$, $(\nu < \alpha)$, be an enumeration of the set of all subsets of $\boldsymbol{C}(B,C)$. By transfinite recursion define F_ν, G_ν, $(\nu < \alpha)$, simultaneously as follows:

$$F_0 = \{\boldsymbol{C}(A,C)\},\ G_0 = \{\boldsymbol{C}(B,C)\}. \text{ For } \nu > 0 \text{ we let } F_\nu = \bigcup_{\mu<\nu} H_\mu,$$

$$G_\nu = \bigcup_{\mu<\nu} K_\mu, \text{ where}$$

$$H_\mu = \begin{cases} [G_\mu \cdot U \cup \{R_\mu\}], & \text{if this filter is proper,} \\ G_\mu \cdot U & \text{otherwise;} \end{cases}$$

$$K_\mu = \begin{cases} [F_\mu \cdot V \cup \{S_\mu\}], & \text{if this filter is proper,} \\ F_\mu \cdot V & \text{otherwise.} \end{cases}$$

Induction on ν shows that each F_ν, G_ν is proper. Hence so are $F = \bigcup_{\nu<\alpha} F_\nu$, $G = \bigcup_{\nu<\alpha} G_\nu$. F and G are maximal filters by construction, hence F and G are ultrafilters. To prove $G \cdot U \subseteq F$ and $F \cdot V \subseteq G$ we need the following: whenever G_ν, $(\nu < \alpha)$, is an increasing chain of proper filters and U is a proper filter then $\left(\bigcup_{\nu<\alpha} G_\nu\right) \cdot U = \bigcup_{\nu<\alpha} (G_\nu \cdot U)$. Namely: $R \in \left(\bigcup_{\nu<\alpha} G_\nu\right) \cdot U$ iff $\{q : \{p : qp\} \in R\} \in U\} \in \bigcup_{\nu<\alpha} G_\nu$, iff $R \in G_\nu \cdot U$ for some $\nu < \alpha$, iff $R \in \bigcup_{\nu<\alpha} (G_\nu \cdot U)$. Thus we may compute as follows:

$$G \cdot U = \left(\bigcup_{\nu<\alpha} G_\nu\right) \cdot U = \bigcup_{\nu<\alpha} (G_\nu \cdot U) = \bigcup_{\nu<\alpha} \left(\bigcup_{\mu<\nu} G_\mu \cdot U\right) \subseteq \bigcup_{\mu<\alpha} F_\nu = F,$$

and similarly for $F \cdot V \subseteq G$.

The proof of the second part of the lemma is an obvious modification of the above proof.

2. Relational Categories, their Filtration, and Model Theory

Subsets of the set A^n of all n-tuples of elements of a non-empty set A are called n-ary *relations* on A, $n = 1, 2, \ldots$. Every function p from a set B into a set A induces a *relational map* $\hat{p}$ from the set of all relations on A into the set of all relations on B as follows: for every $R \subseteq A^n$ and $\langle b_1, \ldots, b_n\rangle \in B^n$ let $\langle b_1, \ldots, b_n\rangle \in \hat{p}R$ iff $\langle p\, b_1, \ldots, pb_n\rangle \in R$. A family of sets together with a family of induced relational maps between them that form a category in the usual sense will be called a *relational category.*

The filtration $\boldsymbol{C}'$ of a relational category is again a category of relational maps. We have to indicate how a filter F on $\boldsymbol{C}(A,B)$ may be regarded as a relational map of A into B: Let R be an n-ary relation on A, then FR is the n-ary relation on B such that $\langle b_1, \ldots, b_n\rangle \in FR$ iff $\{\hat{p} \in \boldsymbol{C}(A,B) : \langle b_1, \ldots, b_n\rangle \in \hat{p}R\} \in F$. This definition of mapping relations by means of filters agrees with our definition of composition of filters, indeed

$$G(F\,R) = (G \cdot F)\,R,$$

as can be easily verified.

Let A, B be any objects in a relational category $\boldsymbol{C}$, let $F \in \boldsymbol{C}'(A,B)$. An $n+1$-ary relation R on A is called a *function* on A if for any $a_1, \ldots, a_n \in A$ there exists a unique $a_{n+1} \in A$ such that $\langle a_1, \ldots, a_n, a_{n+1}\rangle \in R$. Observe that we do not necessarily have that $F(R)$ is a function if R is one. Observe also that the F-image of the identity relation on A is not necessarily the identity relation on B, if it is we shall call F a *normal* filter. Let Φ be a set of functions on A, if F is a normal filter and $F(f)$ is a function for all $f \in \Phi$ then we call F a *Φ-complete* filter. A filter which is Φ-complete for every Φ is called *functionally complete*, (see our paper [*1*] for characterizations and various properties of functionally complete filters). In the present paper we shall be interested in Φ-complete filters for particular Φ's, introduced in the theory of models by Skolem.

A *relational structure* $\mathscr{A} = \langle A, A_i\rangle_{i\in I}$ *of type* t in the sense of model theory consists of a set A and $t(i)$-ary relations A_i on A for all $i \in I$. The cardinality of $\mathscr{A}$ is that of A and the cardinality of the type t is that of I. If $K \supset I$ and s is an extension of t to K and S_k is an $s(k)$-ary relation on A such that $S_i = A_i$ for all $i \in I$ we shall denote by $(\mathscr{A}, S_k)_{k\in K-I}$ the relational structure $\langle A, S_k\rangle_{k\in K}$. $\mathscr{A}$ is said to be he restriction of $(\mathscr{A}, S_k)_{k\in K-I}$ to type t.

We use the applied first-order language associated with the type t to express properties of relational structures of type t, assuming familiarity with the basic notions of first-order logic, in particular with the concept of an assignment p (of elements of A to variables occurring free in φ) to *satisfy* a formula φ. In case $x_1, \ldots, x_n$ are all of the free variables of $\varphi(x_1, \ldots, x_n)$ we signal satisfaction by writing $\mathscr{A} \models \varphi(p(x_1), \ldots, p(x_n))$. Two relational structures $\mathscr{A}, \mathscr{B}$ of the same type are said to be *elementarily equivalent* (in symbols $\mathscr{A} \equiv \mathscr{B}$) if $\mathscr{A} \models \varphi$ exactly when $\mathscr{B} \models \varphi$ for all closed formulas of the language. If e is a function from A to B then e is said to be an *elementary embedding* of $\mathscr{A} = \langle A, A_i\rangle_{i\in I}$ into $\mathscr{B}$

$= \langle B, B_i \rangle_{i \in I}$ iff for every formula $\varphi(x_1, \ldots, x_n)$ (with all free variables displayed) and every $a_1, \ldots, a_n \in A$ we have $\mathscr{A} \models \varphi(a_1, \ldots, a_n)$ iff $\mathscr{B} \models \varphi(e(a_1), \ldots, e(a_n))$.

For every type t let $S(t)$ be the category of all relational structures of type t with induced relational maps $\hat{p}$ between them defined as above. We shall be interested in the filtration $S(t)'$ and ultrafiltration $S(t)^u$ of $S(t)$. Let $\langle a_1, \ldots, a_n \rangle \in R_\varphi^{\mathscr{A}}$ iff $\mathscr{A} \models \varphi(a_1, \ldots, a_n)$.

Theorem. *If $\mathscr{A}$ and $\mathscr{B}$ are elementarily equivalent then there exists a normal filtermap $F \in S'(t)(\mathscr{A}, \mathscr{B})$ such that $F(R_\varphi^{\mathscr{A}}) = R_\varphi^{\mathscr{B}}$ for all formulas* φ (and vice versa).

Namely, consider the filter F on $S(t)(\mathscr{A}, \mathscr{B})$ generated by the sets

$$Q_\varphi(b_1, \ldots, b_n) = \{\hat{p} : \langle b_1, \ldots, b_n \rangle \in \hat{p}(R_\varphi^{\mathscr{A}})\}$$

for all φ and $b_1, \ldots, b_n$ for which $\mathscr{B} \models \varphi(b_1, \ldots, b_n)$, and φ of the form $\varphi_1 \wedge \bigwedge_{i \neq j} x_i \neq x_j$. F has the desired properties as can be easily verified.

In general we call a filtermap *elementary* (of type t) if $F(R_\varphi^{\mathscr{A}}) = R_\varphi^{\mathscr{B}}$ for all formulas φ of type t. The above theorem thus indicates, in effect, that (elementary) filtermaps are the morphisms appropriate for model theory.

Let $\mathscr{A}$ be a relational structure of type t. For every formula of the form

$$(\exists x_{n+1})\, \varphi(x_1, \ldots, x_n, x_{n+1})$$

(with all free variables displayed), we introduce a corresponding n-ary function f_φ on A by selecting for any $a_1, \ldots, a_n \in A$ an element a_{n+1} such that

$$\mathscr{A} \models \varphi(a_1, \ldots, a_n, a_{n+1}) \quad \text{if} \quad \mathscr{A} \models (\exists x)\, \varphi(a_1, \ldots, a_n, x),$$

arbitrary otherwise. Let $\Phi_{\mathscr{A}}$ be the family of all such functions f_φ (called Skolem-functions).

Consider maps $F \in S'(t)(\mathscr{A}, \mathscr{B})$ such that F is $\Phi_{\mathscr{A}}$-complete and $F(A_i) = B_i$ for all $i \in I$. To show that the *filtermaps F preserve all elementary properties*, we prove by induction on the structure of formulas φ: Whenever $R_\varphi^{\mathscr{A}}$, $R_\varphi^{\mathscr{B}}$ are the relations defined by the formula φ in $\mathscr{A}$, $\mathscr{B}$ respectively, then $F(R_\varphi^{\mathscr{A}}) = R_\varphi^{\mathscr{B}}$. For atomic formulas this is true by assumption. If φ is of the form $\varphi_1 \wedge \varphi_2$ we have $\langle b_1, \ldots, b_n \rangle \in F(R_\varphi^{\mathscr{A}})$ iff

$$\{\hat{p} : \langle b_1, \ldots, b_n \rangle \in \hat{p}(R_\varphi^{\mathscr{A}})\} \in F, \quad \text{iff} \quad \{\hat{p} : \langle p(b_1), \ldots, p(b_n) \rangle \in R_\varphi^{\mathscr{A}}\}$$

$$= \{\hat{p} : \mathscr{A} \models \varphi_1(p(b_1), \ldots, p(b_n)) \wedge \varphi_2(p(b_1), \ldots, p(b_n))\}$$

$$= \{\hat{p} : \mathscr{A} \models \varphi_1(p(b_1), \ldots, p(b_n)) \text{ and } \mathscr{A} \models \varphi_2(p(b_1), \ldots, p(b_n))\}$$

$$= \{\hat{p} : \mathscr{A} \models \varphi_1(p(b_1), \ldots, p(b_n))\} \cap \{\hat{p} : \mathscr{A} \models \varphi_2(p(b_1), \ldots, p(b_n))\} \in F,$$

iff both sets in the intersection belong to F, i.e. iff

$$\langle b_1, \ldots, b_n \rangle \in F(R^{\mathscr{A}}_{\varphi_1}) = R^{\mathscr{B}}_{\varphi_1} \text{ and } \langle b_1, \ldots, b_n \rangle \in F(R^{\mathscr{A}}_{\varphi_2}) = R^{\mathscr{B}}_{\varphi_2}$$

(induction assumption), iff $\langle b_1, \ldots, b_n \rangle \in R^{\mathscr{B}}_{\varphi}$. If φ is of the form $\neg\varphi_1$ then $\langle b_1, \ldots, b_n \rangle \in F(R^{\mathscr{A}}_{\varphi})$ iff $\{\hat{p}: \mathscr{A} \models \neg\varphi_1(p(b_1), \ldots, p(b_n))\} \in F$, iff $\{\hat{p}: \mathscr{A} \models \varphi_1(p(b_1), \ldots, p(b_n))\} \notin F$, iff $\langle b_1, \ldots, b_n \rangle \notin F(R^{\mathscr{A}}_{\varphi_1}) = R^{\mathscr{B}}_{\varphi_1}$, iff $\langle b_1, \ldots, b_n \rangle \in R^{\mathscr{B}}_{\varphi}$. Finally, if φ is of the form $(\exists x)\varphi_1(x, x_1, \ldots, x_n)$ let $g = F(f_{\varphi_1})$ and let $Q = \{\hat{p}: pg(b_1, \ldots, b_n) = f_{\varphi_1}(p(b_1), \ldots, p(b_n))\}$. Since F is $\Phi_{\mathscr{A}}$-complete $Q \in F$ and we may argue as follows: $\langle b_1, \ldots, b_n \rangle \in F(R^{\mathscr{A}}_{\varphi})$

iff $\{\hat{p}: \mathscr{A} \models (\exists x)\varphi_1(x, p(b_1), \ldots, p(b_n))\} \in F$,

iff $\{\hat{p}: \mathscr{A} \models \varphi_1(f_{\varphi_1}(p(b_1), \ldots, p(b_n)), p(b_1), \ldots, p(b_n))\} \in F$,

iff $\{\hat{p} \in Q: \mathscr{A} \models \varphi_1(p(g(b_1, \ldots, b_n)), p(b_1), \ldots, p(b_n))\} \in F$,

iff $\langle g(b_1, \ldots, b_n), b_1, \ldots, b_n \rangle \in F(R^{\mathscr{A}}_{\varphi_1}) = R^{\mathscr{B}}_{\varphi_1}$, iff $\langle b_1, \ldots, b_n \rangle \in R^{\mathscr{B}}_{\varphi}$.

Consider now two types t_1, t_2 with common restriction t_0 and some common extension t_3. Let $\mathscr{A}$ be an infinite relational structure of type t_1, $\mathscr{B}$ an infinite structure of type t_2, and let $\mathscr{A}_0, \mathscr{B}_0$ denote the restrictions of $\mathscr{A}, \mathscr{B}$ to type t_0. If $\mathscr{C}$ is a structure of type t_3 denote by $\mathscr{C}_1, \mathscr{C}_2$ its restrictions to t_1, t_2. The following theorem is due to ROBINSON [3]: *$\mathscr{A}_0 \equiv \mathscr{B}_0$ iff there exists a relational structure $\mathscr{C}$ of type t_3 such that* $\mathscr{C}_1 \equiv \mathscr{A}$ and $\mathscr{C}_2 \equiv \mathscr{B}$; by choosing appropriate additional relations in $\mathscr{A}, \mathscr{B}$ we may moreover obtain $\mathscr{C}_1, \mathscr{C}_2$ as elementary extensions of $\mathscr{A}, \mathscr{B}$.

Our proof of this result is immediate from the Triangle Lemma: Let S be a set of distinct symbols, let there be given an n-ary functional symbol f_i for each formula $(\exists x_{n+1})\varphi_i(x_1, \ldots, x_{n+1})$ of the first-order language of type t_3, extended to contain the individual symbols from S and the functional symbols f_i. Let P be the set of all relational maps $\hat{p}$ for functions p from S to A. Extend P to T in the obvious fashion by associating to the functional symbol f_i the function f_{φ_i} on A in case φ_i is of type t_1, otherwise chose an arbitrary function on A (with the same number of variables). Let Q be found analogously (for $\mathscr{B}$ instead of $\mathscr{A}$). By our previous result there exist filtermaps $U \in \boldsymbol{S}(t_0)'(\mathscr{A}_0, \mathscr{B}_0)$ and $V \in \boldsymbol{S}(t_0)'(\mathscr{B}_0, \mathscr{A}_0)$ which preserve the relations $R^{\mathscr{A}}_{\varphi}$, $R^{\mathscr{B}}_{\varphi}$ for all of φ type t_0. U and V may be assumed ultrafilters. Now we apply the Triangle Lemma to find ultrafilters F_0, G_0 on P, Q respectively such that

$$F_0 \cdot V \subseteq G_0, \quad G_0 \cdot U \subseteq F_0.$$

F_0, G_0 fail to be $\Phi_{\mathscr{A}}$, $\Phi_{\mathscr{B}}$-complete in general since they may not be normal. Normality is achieved as follows: It is straightforward to verify that for any relations $R^{\mathscr{A}}$, $R^{\mathscr{B}}$ for which $U(R^{\mathscr{A}}) = R^{\mathscr{B}}$ and $V(R^{\mathscr{B}}) = R^{\mathscr{A}}$ we have $F_0(R^{\mathscr{A}}) = G_0(R^{\mathscr{B}})$. In particular, since U, V are normal, the images under F_0, G_0 of the identity relations on A, B respectively are the same, say $\approx$; $\approx$ is an equivalence relation. Let C be the set of all equivalence-classes of T modulo $\approx$. For every function p from T into A we now select a function p^* from C into A such that $p^*(c) = p(\tau)$ for some $\tau \in c$.

Let F consist of those sets M for which $\{\hat{p} : \widehat{p^*} \in M\} \in F_0$. By construction F is a $\Phi_{\mathscr{A}}$-complete ultrafilter. Let G be defined analoguously. Observe that we still have $F \cdot V \subseteq G$, $G \cdot U \subseteq F$. Hence, if we define the relations on C according to type t_1 by the filtermap F and the relations according to type t_2 by G we know that these definitions agree on the relations of the common type t_0. Let the resulting relational structure of type t_3 be denoted by $\mathscr{C}$. Clearly $\mathscr{C}_1 \equiv \mathscr{A}$, $\mathscr{C}_2 \equiv \mathscr{B}$ since F, G are elementary filtermaps (of types t_1, t_2 respectively).

Notice that F,G do not necessarily belong to $\boldsymbol{S}(t_1)^u$, $\boldsymbol{S}(t_2)^u$; this may be remedied, if we wish, by using the second part of the triangle lemma.

In the above proof we used the artifice of Skolem-functions to assure elementary equivalence by the way of assuring $\Phi_{\mathscr{A}}$ and $\Phi_{\mathscr{B}}$-completeness of the mapping filters. The *direct power* construction (see [*2*]) is another such way, it has the advantage of leading to functional completeness, (to every function on a set A there corresponds a natural image-function in the direct power A^I of A).

References

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