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Algebras and combinators

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1. Graph algebras and a general representation theorem

Let A be a non-empty set and let h be a partial function from A to A. The *graph* H of the function h is the set of ordered pairs $H=\{(a\to b): b=F(a), a\in A\}$. We have written $(a\to b)$ instead of the ordered pair to make notation mnemonic. Viewed as an operation on sets, the *application* of H to a singular argument $\{a\}$ can be understood as

$$H*\{a\}=\{b:\exists x\in\{a\}.\ (x\to b)\in H\}.$$

The notion of a *graph algebra* arises as a generalization of the above set-operation on graphs of functions as follows.

For $A\neq\emptyset$, $n\in\mathbb{N}$, let $G_n(A)$ be defined recursively by $G_0(A)=A$, $G_{n+1}(A)=G_n(A)\cup\{(\alpha\to b):\alpha\subseteq G_n(A),\ \alpha \text{ finite},\ b\in G_n(A)\}$, and let $G(A)=\bigcup_{n\in\mathbb{N}}G_n(A)$. For $M, N\subseteq G(A)$ let

$$M*N=\{b:\exists\alpha\subseteq N.\ (\alpha\to b)\in M\}.$$

DEFINITION. A graph algebra over A is a collection of subsets of $G(A)$ which is closed under the binary operation $*$.

REPRESENTATION THEOREM. *Every algebra* $\mathbf{A}=\langle A,\cdot\rangle$ *with one binary operation* $\cdot$ *is isomorphic to a graph algebra over* A.

Proof. Construct the set $G(A)$ as above, starting with the carrier set A of the given algebraic structure $\mathbf{A}$. Then define a map f of A into the powerset of $G(A)$ recursively by

$$f(a)=\bigcup_i f_i(a),$$

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where

$$f_0(a)=\{a\},$$

$$f_{i+1}(a)=f_i(a)\cup\{\alpha\to y:\exists b\in A.\ b\in\alpha\subseteq f_i(b)\wedge y\in f_i(a\cdot b)\wedge\alpha \text{ finite}\}.$$

Note that $f(a)\cap A=\{a\}$. Hence

(1) If $f(a)=f(b)$ then $a=b$, because then $\{a\}=f(a)\cap A=f(b)\cap A=\{b\}$. Thus, it remains to prove

(2) $f(a\cdot b)=f(a)\cdot f(b)$.

For this we compute as follows:

$$\begin{aligned} f(a)\cdot f(b) &=\{y:\exists\alpha\subseteq f(b).\ \alpha\to y\in f(a)\}\\ &=\{y:\exists\alpha\subseteq f(b)\ \exists \text{ minimal } i.\ \alpha\to y\in f_{i+1}(a)\}\\ &=\{y:\exists\alpha\subseteq f(b)\ \exists i\ \exists u,\ v\in A.\ au=v\wedge u\in\alpha\subseteq f_i(u)\wedge y\in f_i(v)\}.\end{aligned}$$

Because $u\in\alpha\subseteq f(b)\cap f_i(u)$ and $u\in A$, we have $u=b$ and $v=a\cdot b$, using $f(a)\cap A=\{a\}$ again. Hence

$$\begin{aligned} f(a)\cdot f(b) &=\{y:\exists\alpha\subseteq f(b)\ \exists i.\ b\in\alpha\subseteq f_i(b)\wedge y\in f_i(a\cdot b)\}\\ &=\{y:\exists i.\ y\in f_i(a\cdot b)\}=\bigcup_i f_i(a\cdot b)=f(a\cdot b).\end{aligned}$$

Thus (2) holds, and f is an isomorphic embedding as claimed. ∎

Remark. The concept of graph algebra can easily be generalized to more than one operation, to partial operations and to relations; corresponding representation theorems hold.

2. Application: combinatory algebras

A combinatory algebra is an algebraic structure $\mathbf{A}=\langle A,\cdot\rangle$ which is "combinatorially complete", i.e.:

For every expression $\Phi(x_1,\ldots,x_n)$ built up from constants (denoting elements of A) and variables $x_1,\ldots,x_n$ by means of parentheses and the operation symbol "·" there exists an element f in A such that for all $a_1,\ldots,a_n\in A$

$$(\cdots((f\cdot a_1)\cdot a_2)\cdot\cdots\cdot a_n)=\Phi(a_1,\ldots,a_n).$$

The existence of non-trivial combinatory algebras follows either from a Church–Rosser theorem as an algebra of equivalence-classes of terms, or by constructions such as Scott's D_∞ or Plotkin–Scott's P_ω (see [1]). Our representation theorem suggests that combinatory algebras be constructed as graph algebras; indeed, all combinatory algebras are isomorphic to graph algebras.

Let $A \neq \emptyset$ and let $G(A)$ be constructed as in Section one. Then the graph algebra of all subsets of $G(A)$ already forms a combinatory algebra. Following Schönfinkel–Curry, combinatorial completeness follows from two of its instances: it suffices to isolate two different subsets K and S of $G(A)$ such that for all $M, N, L \subseteq G(A)$ we have

$$KMN = M \quad \text{and}$$
$$SMNL = ML(NL).$$

The following definitions accomplish this, where we write B for $G(A)$.

DEFINITION.

$$K := \{\sigma \to (\rho \to s) : \sigma,\ \rho \subseteq B,\ s \in \sigma\}$$
$$S := \{\{\tau \to (\{r_1, \ldots, r_n\} \to s)\} \to (\{\sigma_1 \to r_1, \ldots, \sigma_n \to r_n\} \to (\sigma \to s)) : n \geqq 0,\ r_1, \ldots, r_n \in B,\ \tau \cup \bigcup \sigma_i = \sigma \subseteq B\}.$$

THEOREM. *The graph algebra of all subsets of $B = G(A)$ is a combinatory algebra.*

Proof. Clearly $K \neq S$, since $(\{a\} \to (\{a\} \to a)) \in K$, $(\{a\} \to (\{a\} \to a)) \notin S$. The combinatorial laws follow by straightforward verification:

$$\begin{aligned} KMN &= \{s : \exists \alpha \subseteq N\ \exists \beta \subseteq M.\ \beta \to (\alpha \to s) \in K\} \\ &= \{s : \exists \beta \subseteq M.\ s \in \beta\} = M. \end{aligned}$$

$$\begin{aligned} ML(NL) &= \{s : \exists \rho \subseteq NL.\ \rho \to s \in ML\} \\ &= \{s : \exists n \geqq 0\ \exists r_1, \ldots, r_n \in B\ \exists \sigma_1, \ldots, \sigma_n \subseteq L. \\ &\qquad \{r_1, \ldots, r_n\} \to s \in ML \wedge \sigma_1 \to r_1, \ldots, \sigma_n \to r_n \in N\} \\ &= \{s : \exists n \geqq 0\ \exists r_1, \ldots, r_n \in B\ \exists \sigma_1, \ldots, \sigma_n \subseteq L\ \exists \tau \subseteq L. \\ &\qquad \tau \to (\{r_1, \ldots, r_n\} \to s) \in M \wedge \sigma_1 \to r_1, \ldots, \sigma_n \to r_n \in N\}. \end{aligned}$$

$$SMNL = \{s : \exists \sigma \subseteq L\ \exists \eta \subseteq N\ \exists \varepsilon \subseteq M.\ (\varepsilon \to (\eta \to (\sigma \to s))) \in S\}$$

$$= \{s : \exists \sigma \subseteq L\ \exists n \geqq 1\ \exists r_1, \ldots, r_n \in B\ \exists \tau,\ \sigma_1, \ldots, \delta_n \subseteq B.$$

$$\tau \to (\{r_1, \ldots, r_n\} \to s) \in M \wedge \sigma_1 \to r_1, \ldots, \sigma_n \to r_n \in N \wedge \sigma = \tau \cup \bigcup \sigma_i\}$$

$$= \{s : \exists n \geqq 0\ \exists r_1, \ldots, r_n \in B\ \exists \tau,\ \sigma_1, \ldots, \sigma_n \subseteq L.$$

$$\tau \to (\{r_1, \ldots, r_n\} \to s) \in M \wedge \sigma_1 \to r_1, \ldots, \sigma_n \to r_n \in N\}$$

$$= ML(NL). \quad \blacksquare$$

Remark. The graph algebra of all subsets of $G(A)$ is obviously not extensional; for example there are various K which satisfy $KMN = M$ for all M and N. But this algebra does have extensional combinatory algebras as well as all lambda calculus models (of restricted size) as subalgebras. The problem of associating a subset of $G(A)$ to a closed λ-term can be solved in various effective ways. Structures similar to graph algebras have been considered by Plotkin, Scott in connection with models of lambda calculus and in recursion theory in connection with functionals, see [2].

REFERENCES

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[2] D. S. SCOTT, *Lambda Calculus: Some Models, some Philosophy*, 33 pp., to appear.

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