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COMBINATORY DIFFERENTIAL FIELDS

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Abstract. Combinatory differential fields arise if differential fields are augmented by operations which allow functions that are programmable in the usual recursive sense to be denoted. The present paper defines this concept. It is shown that every differential field whose field of constants is ordered can be extended to a combinatory field. We generalize the basic notions of the Liouville–Ritt–Risch theory of closed-form solvability to combinatory field extensions and present some explorative examples of problems and solutions.

1. The solvability problem for algebraic, functional and differential equations

Since it is the purpose of this paper to introduce and justify some additions to the algebraic toolkit for the treatment of equations, we start by giving a brief survey of the algebraic viewpoint in problem solving. The problems considered here always have the form "find a function (on $\mathbf{R}$ or $\mathbf{C}$) which has this or that property". The algebraic point of view enters in two ways: first in the specification of what an admissible problem-formulation is, and second in specifying the search-space for the solution. Admissible problem-formulations take the form of equations (for the unknown function), using an adequate supply of operations to write out these equations, e.g. differential operators. The search space for the solution or solutions is chosen as an algebraic structure, typically as an extension of the structure in which the parameters that enter the (problem-)equation are elements.

The most straightforward examples are the classical algebraic functions. Let $\mathbf{Q}(x)$ be the field of rational functions over $\mathbf{Q}$ (with the indeterminate "x" playing the role of an independent variable), and consider a polynomial equation $P(x, y) = 0$, e.g. $y^2 - x = 0$. This is the problem equation; the solution space is an extension of $\mathbf{Q}(x)$ to the field $\mathbf{Q}(x, \sqrt{x})$ in which there are the only solutions, $\sqrt{x}$ and $-\sqrt{x}$, algebraic functions as it were. Ordinary differential equations can also be considered as algebraic equations, now of course with an additional algebraic operation $'$, differentiation with respect to x, which makes the field $\mathbf{Q}(x)$ into a differential field: for each element f of $\mathbf{Q}(x)$ there is an element $f' \in \mathbf{Q}(x)$ such that the well-known differentiation rules of calculus apply: $(f+g)' = f' + g'$, $(fg)' = f'g + fg'$, $x' = 1$, $q' = 0$ for $q \in \mathbf{Q}$. The differential equation $y' = y$, having no solution in $\mathbf{Q}(x)$, has of course one in the extension $\mathbf{Q}(x, e^x)$. Finally as an example of a functional equation, we

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use the composition operation of unary functions ($f \circ g$ is the function with $(f \circ g)(x) = f(g(x))$, to formulate the familiar functional equation $y \circ (fg) = y \circ f + y \circ g$. If $f, g \in \mathbf{Q}(x)$, we can find a solution in another extension of $\mathbf{Q}(x)$, e.g. $\mathbf{Q}(x, \log x)$, an extension, which by the way, can be encountered also as the search space for a differential equation, $y' = 1/x$.

The basic mathematical question in this area concerns the subtle and fruitful relationships starting with Galois theory between (a) methods of extending algebraic structures by adjoining solutions to particular problems, and (b) the types of equations formulated in the original algebraic structure which can be solved in such extensions. Clearly, the more liberal the extension methods are, the broader will be the set of solvable equations. The graduations in this typology are what attracted us to this study, especially because we had at our disposal some additional operations to augment the variety of possible equations. Before we enter into this, we briefly review one such relationship which has received considerable interest in recent years, not least, because of its usefulness for computer algebra, specially the closed-form integrability of elementary functions and differential equations.

Let $\mathcal{F}^\infty$ be the field of meromorphic functions in a specific region in $\mathbf{C}$. We consider differential subfields $\mathcal{F}$ of $\mathcal{F}^\infty$ and their extensions within $\mathcal{F}^\infty$. The field of constants of $\mathcal{F}$ is always $\{f \in \mathcal{F}: f' = 0$, the constant function zero$\}$, it may increase during an extension. An extension $\mathcal{F}'$ of $\mathcal{F}$ is called algebraic, if $\mathcal{F}' = \mathcal{F}(f) \neq \mathcal{F}$ and f satisfies, in $\mathcal{F}'$, a polynomial equation $p(f) = 0$, $p \in \mathcal{F}[f]$. It is called logarithmic, if there is an element $g \in \mathcal{F}$ such that $f' = g'/g$; and it is called exponential, if for some $g \in \mathcal{F}$ the element $f \in \mathcal{F}(f)$ satisfies $f' = g'f$. A field which is obtained from $\mathcal{F}$ by a finite succession of algebraic, logarithmic or exponential extensions is called an elementary extension. The "method" mentioned above, is here therefore that of elementary extensions. What are the corresponding problems? Classically it is the integration problem of elementary functions. Let $\mathcal{F}$ be an elementary extension of $\mathbf{Q}(x)$ and let $f \in \mathcal{F}$. Does there exist an elementary extension $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{F}'$ contains an element g, the integral of f as it were, such that g satisfies the (differential) equation $g' = f$? The results on this question (starting with work by Liouville in the 1830s, with important contributions by Kolchin, Ostrowski, Ritt and Risch [4]), are by now well known and in fact at least partially integrated in computer algebra packages such as MACSYMA, REDUCE, MAPLE and MATHEMATICA (see e.g. [1]).

Of course, the problem of integration in finite terms does not stop here: others than exponential and logarithmic extensions may be added to the "methods", e.g. the error function or the logarithmic integral given by differential equations. But we may also be more radically innovative in choosing "methods": why not, for example, allow functions given explicitly as computer programs? To subsume this topical idea under the present algebraic approach presupposes however, that programmed functions themselves become available algebraically, i.e. as solutions to corresponding equations: the combinatory equations to be considered in the next section.

2. Basic definitions

Combinatory differential fields consist of elements, which the reader may profitably first visualize as rational functions, i.e. as elements of $\mathbf{Q}(x)$; later, being forced to abandon this intuitive model, a valuable intuition is supported by fields of functions defined on subsets of $\mathbb{R}$ with values in $\mathbb{R}$. But actually, the author prefers to visualize these elements as sets of properties, formulated in some formal language, which such functions might have. But, of course, visualization is nothing but a pons asinorum, and the mathematical content of what follows resides wholly in the definitions and consistency results expressed below.

The axioms for combinatory differential fields are formulated within a first-order predicate logic with equality. Individual constants 0, 1 and ι and variables $x, y, z, \ldots$ (with indices and primes if necessary) are atomic terms. Using $\sigma, \tau, \ldots$ as metavariables for terms, the set of terms of the language are built up by means of the binary operations $+, \cdot, \circ$ for which we use an infix notation, the unary operations $-$, $^{-1}$, $'$, the ternary operation *cond* (in parenthesis-free notation) and, for each n, the operations $\mu_{x_i}^{x_1,\ldots,x_n}$, $i = 1, \ldots, n$, each of them $2n$-ary. The operations $\mu_{x_i}^{x_1,\ldots,x_n}$, $i = 1, \ldots, n$, are in fact variable-binding, and we need to introduce the notion of free variables for terms; $FV(\tau)$ denotes that set for τ:

$$FV(0) = FV(1) = FV(\iota) = \emptyset, \qquad FV(\tau) = \{\tau\}, \quad \text{if } \tau \text{ is a variable;}$$

$$FV(\tau_1 + \tau_2) = FV(\tau_1\ \tau_2) = FV(\tau_1 \circ \tau_2) = FV(\tau_1) \cup FV(\tau_2);$$

$$FV(-\tau) = FV(\tau^{-1}) = FV(\tau') = FV(\tau);$$

$$FV(cond\ \tau_1\ \tau_2\ \tau_3) = FV(\tau_1) \cup FV(\tau_2) \cup FV(\tau_3);$$

$$FV(\mu_{x_i}^{x_1,\ldots,x_n}(\tau_1, \ldots, \tau_n)_{\sigma_1,\ldots,\sigma_n}) = \bigcup_{i=1}^{n} (FV(\tau_i) \cup FV(\sigma_i)) - \{x_1, \ldots, x_n\};$$

with the variable-condition $x_i \notin FV(\sigma_i)$, $i = 1, \ldots, n$.

There is one unary predicate $const(x)$, one binary predicate $\tau_1 < \tau_2$ and equality. The axioms are divided into groups according to the basic operations.

2.1. *The field axioms*

A combinatory differential field $\mathcal{F}$ is a field of characteristic 0 with neutral element 0 for addition and 1 for multiplication. The set of elements x with $const(x)$ true are called the constant elements of $\mathcal{F}$ and form an ordered subfield of $\mathcal{F}$, comprising the elements 0 and 1:

$$const(0),\ const(1),\ const(\sigma) \wedge const(\tau) \rightarrow const(\sigma + \tau) \wedge const(\sigma\tau),$$

$$const(\tau) \rightarrow const(-\tau),\ const(\tau) \wedge \tau \neq 0 \rightarrow const(\tau^{-1}).$$

2.2. Axioms of composition

These axioms express the facts which follow from interpreting the operation $\circ$ as composition of unary functions with ι the identity function:

$$\tau_1 \circ (\tau_2 \circ \tau_3) = (\tau_1 \circ \tau_2) \circ \tau_3,$$

$$\tau \circ \iota = \iota \circ \tau = \tau,$$

$$(\tau_1 + \tau_2) \circ \sigma = \tau_1 \circ \sigma + \tau_2 \circ \sigma,$$

$$(\tau_1 \tau_2) \circ \sigma = (\tau_1 \circ \sigma)(\tau_2 \circ \sigma),$$

$$(-\tau) \circ \sigma = -(\tau \circ \sigma),$$

$$(\tau^{-1}) \circ \sigma = (\tau \circ \sigma)^{-1},$$

$$\mathit{const}(\sigma) \rightarrow \sigma \circ \tau = \sigma.$$

$$\tau_1 < \tau_2 \equiv \forall x(\mathit{const}(\tau_1 \circ x) \wedge \mathit{const}(\tau_2 \circ x) \rightarrow \tau_1 \circ x < \tau_2 \circ x).$$

2.3. Axioms of branching

These axioms describe properties of the branching operation *cond*, whose intuitive interpretation is this. If f, g, h are (partial) functions, then $\mathit{cond}\, f\, g\, h$ is defined where all three are defined and the value is $g(x)$ where $f(x) > 0$ and $h(x)$ where $f(x) \leq 0$.

$$\mathit{const}(\sigma) \wedge \sigma > 0 \;\rightarrow\; \mathit{cond}\, \sigma\, \tau_1\, \tau_2 = \tau_1,$$

$$\mathit{const}(\sigma) \wedge \sigma \leq 0 \;\rightarrow\; \mathit{cond}\, \sigma\, \tau_1\, \tau_2 = \tau_2,$$

$$(\mathit{cond}\, \tau_1\, \tau_2\, \tau_3) \circ \sigma = \mathit{cond}\, \tau_1 \circ \sigma\, \tau_2 \circ \sigma\, \tau_3 \circ \sigma.$$

2.4. Axioms of recursion

With the axioms below we formulate that the $\mu_{x_i}^{x_1,\ldots,x_n}$, $i = 1, \ldots, n$, intended to define a solution to a simultaneous set of fixpoint equations; these axioms are incomplete, the intuitive interpretation being that we obtain the least fixpoints "above" the approximations $\sigma_1, \ldots, \sigma_n$. For each formula ϕ of the language, the following is an axiom:

$$(\phi(\sigma_1, \ldots, \sigma_n) \wedge \forall x_1, \ldots, x_n(\phi(x_1, \ldots, x_n)$$
$$\rightarrow \phi(\tau_1(x_1, \ldots, x_n), \ldots, \tau_n(x_1, \ldots, x_n))))$$
$$\rightarrow \phi(\mu_{x_1}^{x_1,\ldots,x_n}(\tau_1, \ldots, \tau_n)_{\sigma_1,\ldots,\sigma_n}, \ldots, \mu_{x_n}^{x_1,\ldots,x_n}(\tau_1, \ldots, \tau_n)_{\sigma_1,\ldots,\sigma_n}),$$

with $x_1, \ldots, x_n$ not free in $\sigma_1, \ldots, \sigma_n$.

2.5. Axioms of differentiation

These axioms provide the laws for formally taking derivatives of all terms that can be formed in the language. If we restrict attention to the field operations we obtain, of course, the notion of a differential field. The novel axioms are the ones that concern the "combinatory" operations $\circ$, *cond* and μ.

$$(\sigma+\tau)'=\sigma'+\tau',$$

$$(\sigma\tau)'=\sigma'\tau+\sigma\tau',$$

$$const(\tau)\rightarrow\tau'=0,$$

$$\iota'=1,$$

$$(\sigma\circ\tau)'=\tau'(\sigma'\circ\tau),$$

$$(cond\,\sigma\,\tau_1\,\tau_2)'=cond\,\sigma\,\tau_1'\,\tau_2'.$$

For variable x, the axiom for differentiation is $x'=x'$ where (by abuse of notation), the second "$'$" is a prime, i.e. part of the term, and not the differentiation operator. Finally, the derivative of a recursively defined function is given by

$$(\mu_{x_i}^{x_1,\ldots,x_n}(\tau_1,\ldots,\tau_n)_{\sigma_1,\ldots,\sigma_n})'$$
$$=\mu_{x_i'}^{x_1,\ldots,x_n,x_1',\ldots,x_n'}(\tau_1,\ldots,\tau_n,\tau_1',\ldots,\tau_n')_{\sigma_1,\ldots,\sigma_n,\sigma_1',\ldots,\sigma_n'}.$$

3. The combinatory interpretation of combinatory differential fields

The technical interpretation of combinatory differential fields presented here is based on the sketch presented in [2] and elaborated abstractly in [3]. The basic idea is this. Given any algebraic (indeed, relational) structure, it is possible to embed this structure inside a much richer structure, one which provides for elements that correspond to exactly those operations on the original structure that are programmable (using the basic operations and relations of the original structure as building blocks for recursive procedures). In this sense, the richer structure is the correct algorithmic environment of the originally given structure. In fact, parenthetically, such an embedding can be obtained uniformly for axiomatically described classes of structures (e.g. for varieties, [3], but with little effort also for much larger types of classes; a study of the axiomatic questions raised in this context is contained in [5]).

The structures which we called "rich" above, are in fact combinatory models, that is, algebraic structures

$$\mathbb{D}=\langle D,\cdot\rangle$$

which contain elements $\boldsymbol{K}$, $\boldsymbol{S}$, $\boldsymbol{L}$ with the following properties:

$$\boldsymbol{K}XY = X,$$

$$\boldsymbol{S}XYZ = XY(XZ),$$

$$(\boldsymbol{L}XY = XY \text{ and } \forall Z(XZ = YZ)) \rightarrow \boldsymbol{L}X = \boldsymbol{L}Y.$$

We are using capital X, Y, Z etc. for variables ranging over D and suppress the (application-)symbol "$\cdot$", associating multiple applications to the left. In D we have particular elements with corresponding properties as follows:

$$\boldsymbol{B}XYZ = X(YZ) \quad \text{(composition)};$$

$$\boldsymbol{Y}X = X(\boldsymbol{Y}X) \quad \text{(fixpoint)};$$

$$X_i(\boldsymbol{Y}_i^n X_1, \dots, X_n) = \boldsymbol{Y}_i^n X_1, \dots, X_n, \quad i = 1, \dots, n$$
(simultaneous fixpoints).

Let now $\mathscr{F}$ be an arbitrary differential field of characterisic 0, whose set of constants form an ordered field $\mathscr{F}$ in a suitable combinatory model. Let @ be a new individual constant (of the language of terms) and consider the set T of terms formed from @, 0, 1, ι and all $f \in \mathscr{F}$ by use of the field operations $+$, $\cdot$, $-$, $^{-1}$, $'$. Let A be the set of atomic formulas $\sigma < \tau$, $\sigma = \tau$ and their negations, with σ, $\tau \in T$, and construct $\mathbf{D}_A$ as in [2]. Thus, let

$$G_0(A) = A,$$

$$G_{n+1}(A) = G_n(A) \cup \{\alpha \rightarrow a \colon \alpha \text{ finite}, \alpha \subseteq G_n(A), a \in G_n(A)\},$$

$$G(A) = \bigcup_{n=0}^{\infty} G_n(A),$$

D_A = set of all subsets of $G(A)$, and $\mathbf{D}_A = \langle D_A, \cdot \rangle$, with

$$M \cdot N = \{a \colon \exists \alpha \subseteq N, \alpha \rightarrow a \in M\},$$

for all M, $N \subseteq G(A)$. Then $\boldsymbol{K}$, $\boldsymbol{S}$, $\boldsymbol{L}$, $\boldsymbol{B}$, $\boldsymbol{Y}$, $\boldsymbol{Y}_i^n$ can all be explicitly given (as subsets of $G(A)$) and satisfy the laws above. Indeed, $\boldsymbol{Y}$ yields the least fixpoint, that is $X \cdot Z = Z$ implies $\boldsymbol{Y}Z \subseteq Z$; similarly for the $\boldsymbol{Y}_i^n$.

To embed $\mathscr{F}$ in D_A we start by associating a subset of A to each element of $\mathscr{F}$. Denoting substitution of σ for τ in ϕ by $\phi|_\tau^\sigma$, this is:

$$f \mapsto \boldsymbol{f} = \{\phi(f)|_f^{@} \colon \phi(f) \in A \text{ and } \phi(f) \text{ holds in } \mathscr{F}\}^{\vdash} \cup \{@ = f\},$$

where $M^{\vdash}$ is the deductive closure of $M \subseteq A$ under the axioms of differential fields together with the atomic formulas holding in $\mathscr{F}$. Because we want to characterize the image of $\mathscr{F}$ in $\mathbf{D}_A$ by means appropriate to that structure, we first extend $\mathscr{F}$ by two elements, $\top$ and $\bot$, whose behaviour under the basic operations of the differential field is defined by extending such an op to $\bar{op}$:

$$\bar{op}(x,y)=\begin{cases} op(x,y) & \text{if } x, y \in \mathscr{F}, \\ \bot & \text{if } x=\bot \text{ or } y=\bot, \\ \top & \text{otherwise.} \end{cases}$$

The enlarged structure $\bar{\mathscr{F}}$ is then mapped into $\mathbf{D}_A$ by augmenting the above embedding as follows:

$$f \mapsto f, \qquad \bot \mapsto \emptyset, \qquad \top \mapsto G(A).$$

It is now easy to construct an element $F \in \mathbf{D}_A$ which characterizes the image of $\bar{\mathscr{F}}$ in $\mathbf{D}_A$. Consider

$$\begin{aligned} F := &\{\{@=f, \phi\} \to \phi : f \in \mathscr{F}, \phi \in A\} \\ &\cup \{\{u\} \to v : u, v \in G(A), u \notin A\} \\ &\cup \{\{\phi_1, \ldots, \phi_k\} \to \phi_{k+1} : \phi_1, \ldots, \phi_k \vdash \phi_{k+1}, \phi_1, \ldots, \phi_{k+1} \in A\} \\ &\cup \{\{0=1\} \to u ; u \in G(A)\}. \end{aligned}$$

Then F is a retraction, i.e. $F(FX)=FX$ for all X, and $X \in \mathbf{D}_A$ is an image of an element of $\bar{\mathscr{F}}$ iff $FX=X$.

Having defined a "combinatory" embedding of $\bar{\mathscr{F}}$ in $\mathbf{D}_A$ for the elements, we now extend it to the operations. For images of elements of $\mathscr{F}$ this is quite straightforward:

$$f+g=\{\phi(z)|_z^a : f|_a^x, g|_a^y \vdash \phi(x+y)\}$$

where the turnstile symbol "$\vdash$" stands for provability (in the first order theory of differential fields). The operations $f \cdot g$, $-f$, $(f)^{-1}$ and $(f)'$ are defined similarly, e.g. for the last one:

$$(f)'=\{\phi(z)|_z^a : f|_a^x \vdash \phi(x')\}.$$

If the operations involve the elements $\emptyset$, $G(A)$, the definition is extended (for example in the case of addition), to

$$X+Y=\begin{cases} f+g & \text{if } X=f,\ Y=g,\ f, g \in \mathscr{F}, \\ \emptyset & \text{if } X=\emptyset \text{ or } Y=\emptyset, \\ G(A) & \text{otherwise} \end{cases}$$

This definition allows such operations to be represented combinatorially, by left multiplication with an appropriate element of $\mathbf{D}_A$, whose construction is quite straightforward. Thus for addition, there is $Sum \in \mathbf{D}_A$ with

$$Sum\ XY=Z \text{ iff } X+Y=Z$$

for all X, Y, Z in the image, i.e. with $FX=X$, $FY=Y$, $FZ=Z$. We then have the following lemma.

Lemma 3.1. *Every differential field can be combinatorially embedded in some combinatory model.*

The proof rests on straightforward verifications which we do not give here.

Through this isomorphic embedding, differential fields are now provided with an algebraic environment in which recursive programs are represented by combinators (in the manner known since early denotational semantics), and we could rest the case here. However, our goal is to provide a model for an extension of the differential field $\mathcal{F}$ to a combinatory differential field $\mathcal{F}^*$. Thus in addition to what we have done so far, we need to locate elements and operators that correspond to the additional entities and operations postulated by the axioms. And, if for nothing else but "Purity of methods", we would wish to construct these combinatorially.

Let T^C be as T above, but with the additional operations *cond*, $\circ$ *and* $\mu_{x_i}^{x_1,\ldots,x_n}$. Let A^C be the set of boolean combinations of the corresponding atomic formulas. Let $\mathbf{D}_{A^C}$ be constructed as before. Then the same embedding as before maps $\bar{\mathcal{F}}$ isomorphically into $\mathbf{D}_{A^C}$, and indeed the image of $\bar{\mathcal{F}}$ is again characterized by a retraction F^C:

$$\begin{aligned}F^C := &\{\{@ = f, \phi\} \to \phi : f \in \mathcal{F}, \phi \in A^C\}\\ &\cup \{\{u\} \to v : u, v \in G(A^C), u \notin A^C\}\\ &\cup \{\{\phi_1, \ldots, \phi_k\} \to \phi_{k+1} : \phi_1, \ldots, \phi_k \vdash^C \phi_{k+1}, \phi_1, \ldots, \phi_{k+1} \in A^C\}\\ &\cup \{\{0 = 1\} \to u : u \in G(A^C)\}.\end{aligned}$$

The elements X of the retract are $\emptyset$, $G(A^C)$ and consistent subsets X of A^C which contain exactly one element of the form $@ = f$, $f \in \mathcal{F}$ and are deductively closed under $\vdash^C$, which here means provability from the axioms of combinatory differential fields together with the formulas (of A^C) defined and holding in $\mathcal{F}$. We denote by f^C the element corresponding to $f \in \mathcal{F}$ in this manner.

Let $\bar{\mathcal{F}}^*$ denote the combinatory differential field (augmented by $\top$ and $\bot$) which we are about to construct. Its set of elements are characterized again by a retraction, denoted F^*, which is a liberalization of the retraction F^C and defined by

$$\begin{aligned}F^* := &\{\{@ = \tau, \phi\} \to \phi : \tau \in T^C, \phi \in A^C\}\\ &\cup \{\{u\} \to v : u, v \in G(A^C), u \notin A^C\}\\ &\cup \{\{\phi_1, \ldots, \phi_k\} \to \phi_{k+1} : \phi_1, \ldots, \phi_k \vdash^C \phi_{k+1}, \phi_1, \ldots, \phi_{k+1} \in A^C\}\\ &\cup \{\{0 = 1\} \to u : u \in G(A^C)\}.\end{aligned}$$

Now, if $F^*X = X$ then X is either $\emptyset$, $G(A^C)$, denoted by $\bot$ and $\top$, respectively, or a consistent, deductively closed subset of A^C containing at least one element of the form $@ = \tau$, $\tau \in T^C$. Clearly, $\bar{\mathcal{F}}^*$ extends (the isomorphic image of) $\bar{\mathcal{F}}$, because $F^C X = X$ implies $F^* X = X$.

On $\bar{\mathscr{F}}^*$ we explain the basic relations and operations of combinatory differential fields as follows for $X, Y, \ldots \in \bar{\mathscr{F}}^*$:

$$X+Y=\begin{cases}\{\phi(z)|_z^{@}: X|_{@}^{x}, Y|_{@}^{y} \vdash \phi(x+y)\} & \text{if } X, Y \neq \bot, \top,\\ \bot & \text{if } X=\bot \text{ or } Y=\bot,\\ \top & \text{otherwise.}\end{cases}$$

Similarly for the other operations of differential fields. The composition operator is also defined in this way:

$$X \circ Y=\begin{cases}\{\phi(z)|_z^{@}: X|_{@}^{x}, Y|_{@}^{y} \vdash \phi(x \circ y)\} & \text{if } X, Y \neq \bot, \top,\\ \bot & \text{if } X=\bot \text{ or } Y=\bot,\\ \top & \text{otherwise.}\end{cases}$$

while the *cond* operator has to take care of constants:

$$\textit{cond}\, XYZ=\begin{cases} Y & \text{if } @>0 \in X,\\ Z & \text{if } @\leqslant 0 \in X,\\ \bot & \text{if } X=\bot \text{ or } Y=\bot \text{ or } Z=\bot,\\ \top & \text{otherwise.}\end{cases}$$

Each one of these operators can be represented by a combinator which accomplishes the same effect by left application to elements X, Y of $\bar{\mathscr{F}}^*$ (see [3] for analogous details).

There remain the fixpoint operations $\mu_{x_i}^{x_1,\ldots,x_n}$. In the simplest case, $\mu_x^x(\tau(x))_\sigma$, we let the value be $\bigcup_{m=0}^{\infty} X_m$, where X_0 is the largest consistent subset of σ for which

$$\{@=\mu_x^x\} \subseteq X_0 \subseteq \tau(X_0) \quad \text{and} \quad X_{m+1}=\tau(X_m), \quad m=0,1,\ldots .$$

To represent $\mu_x^x(\tau(x))_\sigma$ combinatorially, given a representation T for τ, that is $TZ=\tau(Z)$, means to construct combinators C and Y_{X_0} with

$$T(Y_{X_0}T)=Y_{X_0}T \supseteq CT\sigma = X_0.$$

The details of this construction are not difficult to extend from the known construction of the least-fixpoint operator Y in $\mathbf{D}_A$. Thus, the assumption on the representability of τ is always fulfilled and μ_x^x is taken care of. This approach extends easily, mutatis mutandis, to all $\mu_{x_i}^{x_1,\ldots,x_n}$, and we may state the following.

Theorem 3.2. *Every differential field of characteristic* 0 *and with ordered field of constants can be extended to a combinatory differential field.*

Remark. We cannot reasonably hope for uniqueness of this extension. This is a consequence of the fact, that there cannot be a complete recursive axiomatization of the quantifier-free theory of combinatory fields. Such a theory would then be decidable. The impossibility of this follows, if we can construct a quantifier-free interpretation of the natural numbers (with addition and multiplication) within such a theory. But it is easy to write a formula $\phi(x)$ characterizing N in any sufficiently strong theory of combinatory differential fields:

$$\phi(x) :\equiv const(x) \wedge x \geqslant 0 \wedge 0 = \mu_x^x(cond\, x - y\, y + 1\, y)_0.$$

The above expression uses a simultaneous recursion for a function which yields 0 if $x \in \mathrm{N}$ and is undefined otherwise. If the theory is strong enough, it would prove this fact (in all fields considered here).

4. Equations, extensions and solutions

Let now $\mathscr{F}^*$ be a combinatory differential field extending $\mathscr{F}$, with (partial) operations $+$, $\cdot$, $-$, $^{-1}$, $'$, *cond*, $\circ$, and $\mu_{x_i}^{x_1,\ldots,x_n}$, predicates *const*, $<$, equality, and individual constants 0, 1, and ι. For concreteness, assume that $\mathscr{F}^* = \bar{\mathscr{F}}^* - \{\bot, \top\}$. The elements of $\bar{\mathscr{F}}^*$ are sets of quantifier-free formulae of the above language, augmented by the additional individual constant @. The field operations are defined as total operations on sets, in particular $\mu_{x_i}^{x_1,\ldots,x_n}$ are using least fixpoint operators.

The discussion of the relation between solutions of equations and field extensions takes place in the lattice of fields between $\mathscr{F}$ and $\mathscr{F}^*$. If such a field is closed under $'$ it is a *differential field*, if it is closed under $\circ$, *cond* and the $\mu_{x_i}^{x_1,\ldots,x_n}$ it is a *combinatory field*, and if it is closed under all of them, it is a *combinatory differential field*. Of course, closing a field under such an operation may result in an enormous extension which overrides much of the fine-structure of field extensions which are so useful in solvability discussion. For example, the closure of $\mathbf{Q}(x, \mathrm{e}^x)$ under $\circ$ admits e^{x^2} which is transcendental over the original field. To retain the fine-structure, we therefore proceed in the classical manner, extending a field $\mathscr{F}_1$ to $\mathscr{F}_2$ by adjoining solutions of some equation, formulated in terms of available operations. Mostly, we restrict our attention to the sublattice of differential fields.

Let $\mathscr{F}_1$ be given and let $\sigma(y) = \tau(y)$ be an equation between terms formed by using parameters from $\mathscr{F}_1$, the combinatory differential field operations and the variable y. An element X of $\mathscr{F}_2$ is called a *formal solution* of this equation if $X = \{\sigma(@) = \tau(@)\}^{\vdash C}$. It is called a *closed-form* solution, if there is a closed term ρ (without @), such that the equation $@ = \rho$ belongs to X. We then say that ρ denotes X in $\mathscr{F}_2$. Finally, we define the notion of an approximate solution. Let Λ be the set of formulas that describe upper and lower bounds of elements of $\mathscr{F}^*$ as follows. Λ consists of all boolean combinations of formulas of the form

$$cond\, \rho\ @\ 0 < cond\, \rho\, \kappa_1\, 1 \text{ and } cond\, \rho\, \kappa_2\, 0 < cond\, \rho\ @\ 1$$

where ρ, κ_1 and κ_2 are closed terms without @. Then $X \in \mathscr{F}_2$ is an *approximate solution* of $\sigma(y) = \tau(y)$ if

$$\sigma(X) \cap \Lambda = \tau(X) \cap \Lambda.$$

Our interest may be to obtain approximate solutions X that are denoted by a suitable term, say ξ. We shall then write $\sigma(\xi) \approx \tau(\xi)$ and say that ξ is a ***closed-form approximate solution.***

Let us now turn to some simple illustrations.

Example 4.1. Algebraic extensions for algebraic equations: While $y^2 - \iota\iota = 0$ has the obvious closed-form solutions ι and $-\iota$, the equation $y^2 - \iota = 0$ defines in general an algebraic extension (by the square-root function). In the spirit of algebra, we immediately have of course the formal solution $\{@^2 = \iota\}^{\vdash C}$. This latter set (of properties of the square-root function) reasonably only consists of the consequences of its definition. Closed-form approximate solutions also exist.

Example 4.2. Rational recursion extension for algebraic differential equations and closed-form solutions: One of the simplest algebraic differential equations is $y' = yf'$. It defines in general an extension of the differential field to which f belongs, a so-called exponential extension. Together with logarithmic extensions (using $fy' = f'$) and algebraic extensions, these constitute the elementary extensions which are studied in the Liouville–Risch theory of closed-form integrability. There, the goal of the algorithmic approach is to find effective procedures for deciding (under additional assumptions, e.g. about constants in the field extensions) whether a logarithmic or exponential extension is algebraic or transcendental, to lift integration algorithms from a field to its extension and discuss closed-form solvability of differential equations, in such extensions. While this set of goals clearly belongs to the future aims of our approach, we use the example here only to illustrate another type of extension, that by simultaneous *rational recursion*, and show that it provides a solution.

Consider the set of simultaneous equations

$$y_{k+1} := y_k + u_k, \qquad u_{k+1} := u_k\iota / n_k, \qquad n_{k+1} := n_k + 1$$

with starting values 1, ι, 1 respectively for y_0, u_0, n_0. Of course, this recursion is but a "disguise" of the usual power-series expansion of the exponential function. The point here is that it translates into a term, namely

$$\varepsilon = \mu_y^{y\,u\,n}(y+u,\ u\iota/n,\ n+1)_{1,\iota,1}.$$

In fact ε is a closed-form solution to $y' = y$, that is, we can show $\varepsilon' = \varepsilon$ from the theory of combinatory differential fields. By the rules of differentiation, we have

$$\varepsilon' = \mu_{y'}^{y\,u\,n\,y'\,u'\,n'}(y+u,\ u\iota/n,\ n+1,\ y'+u',\ ((u'\iota + v) - u\iota n')/n^2,\ 0)_{1,\iota,1,0,1,0}.$$

If X_m and X'_m represent the mth approximation (of the least-fixpoint computation) to the values of ε, respectively ε', then we see that $X_m = X'_{m+1}$, by inspection of the power-series expansions which the terms ε and ε' describe. Therefore $\bigcup X_m = \bigcup X'_m$ and therefore, as claimed, $\varepsilon = \varepsilon'$.

Example 4.2 (*continued*) Closed-form approximate solutions: It is obvious that ε is also a closed-form approximate solution. A direct proof consists of translating the usual calculus proof of the convergence of the power-series expansion of e^{ι}. Let $\nu(m, \iota)$ denote the mth remainder term of that series. Because the series converges everywhere, the following inequalities belong to the set denoted by ε

$$cond\,\rho\,@\,0 < cond\,\rho\,\sigma_m\,1 \quad \text{and} \quad cond\,\rho\,\tau_m\,0 < cond\,\rho\,@\,1$$

for all ρ. The terms σ_m, τ_m represent the upper and lower bounds of ε, obtained after taking m terms in the power series

$$\sigma_m := \mu_y^{y\,u\,n}(cond\; m - n\;(y+u)\;y,$$

$$cond\; m - n\; u\iota/n\; u\iota/n + \nu(m, \iota),$$

$$cond\; m - n\; n+1\; m)_{1,\iota,1},$$

$$\tau_m := \text{like } \sigma_m, \quad \text{with } \nu(m, \iota) \text{ replaced by } -\nu(m, \iota).$$

Assuming that the theory of combinatory differential fields is strong enough to prove the above inequalities, and that all other relevant inequalities that are true for ε and ε' follow equally, we indeed would conclude $\varepsilon \approx \varepsilon'$. Thus this theory would have to include a fair amount of elementary calculus (including theorems about remainder terms in power series), which we are not about to list here explicitly.

Example 4.3. Combinatory equations and closed-form approximation: We again take a simple example, namely the functional equation for the inverse function. Thus, let $\tau \in \mathcal{F}$, a differential field, and pose the problem $\tau \circ y = \iota$, a combinatory equation of the simplest kind. It is in general necessary to extend $\mathcal{F}$ to find a solution to this equation; e.g. while $\iota\iota \in \mathbf{Q}(x)$, its inverse, the square-root function is not in $\mathbf{Q}(x)$. Here, we propose simultaneous *combinatory recursion*, namely the equations

$$y_{k+1} := y_k - \frac{\tau \circ y_k - \iota}{\tau' \circ y_k}$$

which represent Newton's method, starting with an appropriate seed function σ_0 for the approximation of the inverse of τ. If σ_0 is chosen well then $\tau \circ \bar{\tau} \approx \iota$ for

$$\bar{\tau} := \mu_y^y\left(y - \frac{\tau \circ y - \iota}{\tau' \circ y}\right)_{\sigma_0},$$

again, of course, assuming a strong enough theory of combinatory differential fields, including some elementary knowledge of numerical analysis.

Example 4.4. Other combinatory equations: Consider the rational recursion

$$y_{k+1} := a y_k (1 - y_k)$$

which arises from a first-order difference equation employed in population dynamics. The problem is to describe the dependence of fixpoints on the parameter value a. It has of course the formal solution

$$\alpha := \mu_y^\cdot(ay(1-y))_{\alpha_0},$$

an element in suitable combinatory extensions. The value of α would contain descriptions of the successive bifurcations and chaotic behaviour of α; this set would therefore contain formulas like

$$c_0 < a < d_0 \rightarrow @ \circ a = \beta \circ a$$

$$c_1 < a < d_1 \rightarrow @ \circ a = \beta_0 \circ a \vee @ \circ a = \beta_1 \circ a$$

$$c_2 < a < d_2 \rightarrow @ \circ a = \beta_{00} \circ a \vee @ \circ a = \beta_{01} \circ a \vee @ \circ a$$

$$= \beta_{10} \circ a \vee @ \circ a = \beta_{11} \circ a$$

$$\vdots$$

for suitable $c_0, d_0, \ldots \beta, \beta_0, \beta_1, \ldots$.

The same mathematical area is the source of more examples appropriate for our machinery. Another biological example is

$$y_{k+1} := y_k e^{r(1-y_k)}$$

with solution

$$\mu_y^\cdot y(\varepsilon \circ (r(1-y)))_\beta,$$

which is combinatory recursive over a field containing the exponential function ε.

Another interesting example which is extensively treated in the literature is

$$y_{k+1} = \begin{cases} ay_k & \text{if } y_k < \frac{1}{2}, \\ a(1-y_k) & \text{if } y_k > \frac{1}{2}. \end{cases}$$

The combinatory recursion that solves this is easy to write down and left as an exercise to the reader.

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