Function spaces of generalised smoothness and pseudo-differential operators associated to a continuous negative definite function

H a b i l i t a t i o n s s c h r i f t

zur Feststellung der Lehrbefähigung für das Fach Mathematik

vorgelegt dem Rat der

Fakultät für Mathematik, Informatik und Statistik

der Ludwig-Maximilians-Universität München

von **Dr. rer. nat. Erich Walter Farkas**

München 2002

To my wife

Acknowledgement

It is a pleasure for me to thank Professor Heinz Siedentop for his constant support and encouragement.

Several fruitfull discussions with Professor Niels Jacob, with Professor Hans-Gerd Leopold, with Professor Réne L. Schilling, and with Professor Hans Triebel during the last years are gratefully acknowledged.

Contents

Preface

At least since the publication of M. Fukushima's work [Fu71] on Dirichlet forms and Markov processes, the (functional) analytic approach to stochastic processes turned into the centre of interest for researchers working in probability theory.
More precisely, the point of view based on the relation of Fourier analysis and Markov processes, subject which was first taken up by P. Lévy and by S. Bochner when discussing stochastically continuous processes with stationary and independent increments (*Lévy processes*), is at present an area of intensive development.
The key observation is that every Lévy process $(X_t)_{t\geqslant 0}$ with state space $\mathbb{R}^n$ is completely determined by one and only one function $\psi : \mathbb{R}^n \to \mathbb{C}$ which is defined by the relation $E(e^{iX_t\cdot\xi}) = e^{-t\psi(\xi)}$. The function ψ, called the characteristic exponent of $(X_t)_{t\geqslant 0}$, is a continuous negative definite function and contains all information about $(X_t)_{t\geqslant 0}$.

The main aim of this work is twofold.
First of all we want to highlight new aspects of the close relationship between various aspects concerning Markov and Lévy process (as central objects in pure and applied probability and in the theory of stochastic processes) and Fourier analysis, the theory of function spaces, and the theory of pseudo-differential operators as essential tools in the analysis and in the theory of partial differential equations.
Secondly, this work should be regarded as a contribution to the modern theory of function spaces (of generalised Sobolev and Bessel potential type) and to the theory of pseudo-differential operators (with symbols in various types of Hörmander classes).
Our contribution can be summarised in the following way.

- We investigate systematically domains of generators of L_p-sub-Markovian semigroups, determine those domains in terms of function spaces and indicate (based on introducing a capacity associated to these function spaces), how one can construct a Hunt process without exceptional set starting with a semigroup.

- We point out that if the Lévy measure of a continuous negative definite function is supported in a bounded neighbourhood of the origin (assumption which in probability corresponds to the fact that the jumps of the associated process are bounded) then the domain of definition of a generator of an L_p-sub-Markovian semigroup can be regarded as a function spaces of generalised smoothness of Triebel-Lizorkin type.

- We give a unified approach on function spaces of generalised smoothness characterising most of those known spaces in terms of modern tools such as local means and atomic decompositions.

- We apply the atomic decomposition theorem obtained for function spaces of generalised smoothness and prove that pseudo-differential operators with so-called exotic symbols (the class $S^0_{1,1}$) have nice mapping properties. This extends the results known so far.

- We propose an investigation in an L_p-setting of a symbol class which is a refinement of Hörmander's class of symbols and indicate how one can get Feller and sub-Markovian semigroups generated by pseudo-differential operators with those symbols.

Introduction

Motivation

The key notion in our work is the concept of a continuous negative definite function. At least two places where these functions naturally appear should be mentioned: in connection with generators of time-homogeneous Markov processes in $\mathbb{R}^n$ and in connection with translation invariant (symmetric) Dirichlet forms.

• It is well-known that the generator A of a time-homogeneous Markov process in $\mathbb{R}^n$ is typically given by a Lev́y-type operator

$$\begin{aligned} Au(x) &= \sum_{k,l=1}^{n} a_{kl}(x)\,\frac{\partial^2 u(x)}{\partial x_k \partial x_l} + \sum_{j=1}^{n} b_j(x)\,\frac{\partial u(x)}{\partial x_j} + c(x)u(x) \\ &\quad + \int_{\mathbb{R}^n\setminus\{0\}} \left(u(x+y) - u(x) + \sum_{j=1}^{n} \frac{y_j}{1+|y|^2}\,\frac{\partial u(x)}{\partial x_j} \right) \nu(x,dy) \qquad (0.1) \end{aligned}$$

for $u \in C_0^\infty(\mathbb{R}^n)$.

This follows immediately from the fact that the generator of a transition semigroup satisfies the positive maximum principle i.e., for any u in the domain of the generator and $x_0 \in \mathbb{R}^n$ such that

$$u(x_0) = \sup_{x\in\mathbb{R}^n} u(x) \geqslant 0 \quad \text{we have} \quad Au(x_0) \leqslant 0,$$

and from a result of Ph. Courrège [Cou66] which characterises the operators satisfying the positive maximum principle as operators of type (0.1).

But Ph. Courrège gave also another equivalent representation of this class of operators as pseudo-differential operators

$$Au(x) = -a(x,D)u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\,x\cdot\xi} a(x,\xi)\,\widehat{u}(\xi)\,d\xi \qquad (0.2)$$

where $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a measurable, locally bounded function such that for every $x \in \mathbb{R}^n$ the function $a(x,\cdot) : \mathbb{R}^n \to \mathbb{C}$ is continuous and negative definite (in the sense of I. J. Schoenberg). Those symbols a are called negative definite symbols.

Conversely, if the symbol is a continuous negative definite function for every fixed $x \in \mathbb{R}^n$ then the operator $-a(x,D)$ satisfies the positive maximum principle on $C_0^\infty(\mathbb{R}^n)$.

The relation between (0.1) and (0.2) is given by the Lévy-Khinchin formula

$$a(x,\xi) = -c(x) + i\sum_{j=1}^{n} b_j(x)\xi_j + \sum_{k,l=1}^{n} a_{kl}(x)\xi_j\xi_k + \int_{y\neq 0} \left(1 - e^{-iy\cdot\xi} - \frac{iy\cdot\xi}{1+|y|^2} \right) \nu(x,dy)$$

where $c \leqslant 0$, $(a_{kl})_{kl} \in \mathbb{R}^{n\times n}$ is a symmetric, positive semidefinite matrix, $b \in \mathbb{R}^n$, and $\nu(x,dy)$ is a kernel satisfying $\int_{y\neq 0} \min\{|y|^2, 1\}\,\nu(x,dy) < \infty$, compare Theorem 2.1.4.

Note that representation (0.2) shows immediately that *any* pseudo-differential operator $-a(x,D)$ with a negative definite symbol $a(x,\xi)$ naturally satisfies the positive maximum principle, independent of the question whether $-a(x,D)$ extends to the generator of a Feller semigroup.

In the particular case of a symbol $a(x,\xi) = a(\xi)$ which is independent of x, the operator $-a(D)$ generates a convolution semigroup and the corresponding process is a Lévy process. Moreover the negative definite symbol function ψ is nothing but the characteristic exponent of the Lévy process and in this way a complete one-to-one correspondence between negative definite functions and Lévy processes is given.
Even in this simple x-independent case the most standard example of symmetric α-stable processes show that the corresponding symbol $a(\xi) = |\xi|^{2\alpha}$, $0 < \alpha \leqslant 1$, is not differentiable unless $\alpha = 1$, i.e. in the case of Brownian motion. From this we see that it is an intrinsic property of the regarded symbol class that they are in general not differentiable with respect to ξ. Hence these symbols do not fit into any known class of pseudo-differential operators and one cannot apply pseudo-differential calculus without further considerations. For that reason many approaches to Lévy-type operators beside those which study the case of dominating diffusion term either concentrate on the representation (0.1) with certain integrability conditions on the Lévy kernel $\nu(x,dy)$ or they make some homogeneity assumptions on the symbol with respect to ξ and often consider perturbations of α-stable and so-called stable-like processes, see [Ho98a] and [Ja01] for references and further comments.
We will focus on symbols with Lévy measures supported in a bounded neighbourhood of the origin. Recall that in probability this assumption is often made from the very beginning and corresponds to the fact that the jumps of the associated process are bounded. It turns out that these symbols are differentiable with respect to ξ, see Corollary 2.1.7 and this plays a key role in our investigation.

• The main reason for introducing Dirichlet forms was to give an axiomatic approach to potential theory starting with the notion of energy. Regarding this aspect it is not surprising that within the framework of Dirichlet spaces many potential theoretical considerations can be done. Notions like capacities, energy, (equilibrium) potentials, reduced functions, and balayage are best studied. We refer to the classical monograph by M. Fukushima [Fu80] and mention also the books [BoHi91], [FOT94], [MaRö92] and [Si74].
Recall that a Dirichlet space (on $\mathbb{R}^n$ for simplicity) is a pair $(\mathcal{F},\mathcal{E})$ consisting of a space of real-valued functions $\mathcal{F} \subset L_2(\mathbb{R}^n)$ and a symmetric quadratic form $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ which is closed, densely defined, non-negative, and satisfies the following contraction condition:

$$\text{if} \quad u \in \mathcal{F} \quad \text{then} \quad v := (0 \vee u) \wedge 1 \in \mathcal{F} \quad \text{and} \quad \mathcal{E}(v,v) \leqslant \mathcal{E}(u,u).$$

All translation invariant (symmetric) Dirichlet forms (on $\mathbb{R}^n$) are given by

$$\mathcal{E}^\psi(u,v) = \int_{\mathbb{R}^n} \psi(\xi)\, \widehat{u}(\xi)\, \overline{\widehat{v}(\xi)}\, d\xi, \quad u,v \in \mathcal{S}(\mathbb{R}^n),$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function.
The domain $\mathcal{F}^\psi$ of $\mathcal{E}^\psi$ is then given by

$$\mathcal{F}^\psi := H_2^{\psi,1}(\mathbb{R}^n) := \left\{ u \in L_2(\mathbb{R}^n) \, : \, \int_{\mathbb{R}^n} (1+\psi(\xi))|\widehat{u}(\xi)|^2\, d\xi < \infty \right\}.$$

It is well known that one can associate with ψ (or with $(\mathcal{F}^\psi, \mathcal{E}^\psi)$) the operator semigroup $(T_t^{(2)})_{t\geqslant 0}$ on $L_2(\mathbb{R}^n)$ defined by

$$T_t^{(2)}u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, e^{-t\psi(\xi)}\, \widehat{u}(\xi)\, d\xi = \int_{\mathbb{R}^n} u(x-y)\, \mu_t(dy), \qquad (0.3)$$

where $(\mu_t)_{t\geqslant 0}$ is a vaguely continuous convolution semigroup of sub-probability measures on $\mathbb{R}^n$ with Fourier transform $\widehat{\mu_t}(\xi) = (2\pi)^{-n/2}\, e^{-t\psi(\xi)}$.
The generator $(A^{(2)}, D(A^{(2)}))$ of the semigroup $(T_t^{(2)})_{t\geqslant 0}$ is given by

$$A^{(2)}u(x) = -\psi(D)u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \psi(\xi)\, \widehat{u}(\xi)\, d\xi$$

with domain

$$H_2^{\psi,2}(\mathbb{R}^n) := \left\{ u \in L_2(\mathbb{R}^n) \,:\, \int_{\mathbb{R}^n} (1+\psi(\xi))^2\, |\widehat{u}(\xi)|^2\, d\xi < \infty \right\}.$$

Note that the measures μ_t are also the transition probabilities for a Lévy process $(X_t)_{t\geqslant 0}$ and therefore we have

$$E\left(e^{iX_t\xi}\right) = e^{-t\psi(\xi)}. \qquad (0.4)$$

Thus ψ is also a characteristic exponent of a Lévy process.

The function spaces of type $H_2^{\psi,1}$ and $H_2^{\psi,2}$ we are interested in, appeared in their generality for the first time in the work of A. Beurling and J. Deny [BeDe58, BeDe59], see also [Den70], on Dirichlet spaces.
In general they are contained neither in the Besov- $B^s_{p,q}$ or Triebel - Lizorkin- $F^s_{p,q}$ scales nor in the classes of anisotropic spaces considered so far.
They are so-called function spaces of generalised smoothness, because the smoothness properties are related to the function ψ.

Function spaces of generalised smoothness have been introduced and considered by several authors, in particular since the middle of the seventies up to the end of the eighties with different starting points and in different contexts.
The work of M. L. Goldman and G. A. Kalyabin and of their co-authors was the starting point for many contributions to the topic. They developed independently an approach via approximation by series of entire analytic functions and coverings, see for example [Go79], [Go80], [Ka77a] and [Ka80]. Another approach is due to M. L. Goldman, see [Go84], who gave a systematic treatment based on differences and moduli of continuity of those type of spaces. His setting has P. L. Ul'yanov (1968) and A. S. Dzhafarov (1965) as forerunners.
In both cases mentioned above, the spaces consist of functions belonging to L_p with additional smoothness properties.
Many remarkable and final results were obtained, for example results concerning embeddings in different kinds of spaces of smoothness level zero, equivalent norms, trace theorems and estimates of capacities. The survey [KaLi87], the supplement in [Tr86], or [KuNi88, Chapter 5 §4] cover in particular the literature up to the end of the eighties in this direction.

In our work we will take up some basic ideas from the above settings but now from the standpoint of a Fourier analytic characterisation. This allows us the description of the full scale of spaces, including spaces of negative smoothness and duality results.
Further, spaces of generalised smoothness defined on ideal spaces E as basic spaces, instead of L_p, were considered in [Go86], [Go92], [Net88], and in [Net89].
Moreover, at the end of the eighties C. Merucci, see [Mer86], F. Cobos and D. L. Fernandez, see [CoFe86], investigated some classes of function spaces of generalised smoothness. More precisely they characterised interpolation spaces between L_p and W_p^k which were obtained with respect to a generalised real interpolation method.
We have noticed an increasing interest in spaces of generalised smoothness in the last years. First of all, this interest is in connection with embeddings, limiting embeddings and entropy numbers. We mention here the papers [Le98], [EGO97], [EdHa99], [OpTr99] were such problems were considered.
Additionally, in connection with generalised d-sets and h-sets (special fractals) those spaces appeared in a natural way in [EdTr98], [EdTr99], [Mo99], [Mo01], and in [Bri02].
As one could see in the above discussion, function spaces of generalised smoothness appear in a natural way in the theory of Dirichlet forms and of Markov processes.

Once the function space $H_2^{\psi,1}(\mathbb{R}^n)$ is understood to be a "good" space for potential theoretic questions it is natural to extend $H_2^{\psi,1}(\mathbb{R}^n)$ to a scale of spaces in order to handle operators derived from $\psi(D)$ in an L_p- context.
Although we feel that from the mathematical point of view investigating L_p- variants of the spaces $H_2^{\psi,1}(\mathbb{R}^n)$ does not need any justification—it is interesting and non-trivial mathematics in itself—let us point out some important arguments for a systematic approach to these spaces.

Our starting point is formula (0.4) telling us that continuous negative definite functions are closely related with Lévy processes. As a matter of fact, every reasonable Feller process with state space $\mathbb{R}^n$ is characterized by a family (parametrised by $\mathbb{R}^n$) of continuous negative definite functions. More precisely, following [Ja98a], see also [Sc98b], we find for the Feller process $((X_t)_{t\geqslant 0}, P^x)_{x\in\mathbb{R}^n}$ that

$$-a(x,\xi) = \lim_{t\to 0} \frac{E^x\left(e^{i(X_t-x)\cdot\xi}\right) - 1}{t}$$

is the symbol of the generator of the semigroup

$$T_t u(x) = E^x(u(X_t))$$

associated with $((X_t)_{t\geqslant 0}, P^x)_{x\in\mathbb{R}^n}$, i.e. on $C_0^\infty(\mathbb{R}^n)$ we have

$$Au(x) = -a(x,D)u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, a(x,\xi)\,\widehat{u}(\xi)\,d\xi. \tag{0.5}$$

Moreover, $\xi \mapsto a(x,\xi)$ is for each $x \in \mathbb{R}^n$ a continuous negative definite function. Note that this result complements the theorem of Ph. Courrège [Cou66] which states that on $C_0^\infty(\mathbb{R}^n)$ the generator of a Feller semigroup has necessarily the structure (0.5).

Now, assuming for example that $a(x,\xi) \sim \psi(\xi)$ where $\psi : \mathbb{R}^n \to \mathbb{R}$ is a fixed continuous negative definite function, one should expect that the operator $a(x,D)$ behaves like a

perturbation of $\psi(D)$. Hence the scales of spaces associated with ψ should play for $a(x, D)$ the same role as Sobolev or Besov and Triebel - Lizorkin spaces do for elliptic operators in the classical situation, i.e. for operators with symbol $q(x, \xi) \sim |\xi|^{2m}$.
We can, however, also ask the converse question: when does an operator given by (0.5) (defined on $C_0^\infty(\mathbb{R}^n)$) admit an extension to a generator of a Markov process?

It is a cornerstone in the modern theory of stochastic processes that to each (regular) Dirichlet form one can associate a stochastic process. This result is originally due to M. Fukushima [Fu71]. Constructions of stochastic processes starting with $-a(x, D)$ either in the Hilbert space situation (Dirichlet space case) or in the Feller situation (in $C_\infty(\mathbb{R}^n)$) were obtained by N. Jacob in [Ja92] [Ja93], [Ja94a], [Ja94b], and subsequently extended in a series of papers by W. Hoh [Ho93]–[Ho98a], see also [HoJa92].
We refer also to more recent (and special) considerations due to F. Baldus [Bal99] and V. Kolokoltsov [Ko00]. In fact, even operators of variable order of differentiability were handled, notably in the papers [Ho00] of W. Hoh, [KiNe97] of K. Kikuchi and A. Negoro, [Neg94] of A. Negoro and [JaLe93] of N. Jacob and H. G. Leopold. In some of these cases spaces of variable order of differentiation are needed. The interested reader should consult also the survey [KaLi87] of G. A. Kaljabin an P. I. Lizorkin on function spaces of generalised smoothness.

From the probabilistic point of view there is a disadvantage in working with processes associated with Dirichlet spaces. The process is only defined up to an exceptional set, i.e. a set of capacity zero. It seems that H. Kaneko in [Kan86] was the first who proposed to use an L_p- setting to overcome this difficulty. He considered certain L_p-Bessel potential spaces associated with a sub-Markovian semigroup. With such an L_p-Bessel potential space there is always associated a capacity, but for p sufficiently large it might happen that the only exceptional set, i.e. the only set of capacity zero, is the empty set.
In this case no problems occur when constructing the associated Markov process. Thus it is clear that L_p- variants of Dirichlet spaces are of greater interest for probabilistic reasons.

Style and structure

We would like to say a few words about the organising and contents of the manuscript. Since we discuss topics not only from the theory of function spaces and pseudo-differential operators but also from potential theory and from the theory of stochastic processes, we tried to make our exposition as most as possible self contained.
We divided our work into four chapters and it can be seen in two ways: first as a collection of the four papers [FJS01a], [FJS01b], [Fa02] and [FaLe01] and of our work in progress [Fa03] or, secondly, as a streamline work which contains besides an overview of the results obtained and published in [FJS01a] and [FJS01b] in the first chapter, three other chapters which contain results unpublished yet.
The results from Chapter III are contained and announced in the preprint [FaLe01] (which is however larger than our third part).
The results from Chapter II and the last section from Chapter IV will be included in (our work in progress) [Fa03].

The results from Chapter IV contained in Sections 14-15 are announced in the preprint [Fa02].
Each chapter has its own introduction in which we give an overview on the results and on the techniques used.

An overview on the results

We want to present here, briefly, the contents of our work.
In an introductory section called **Preliminaries**, we set up terminology and notation and recall some fundamental results which will play a key role in our further considerations (the Fefferman-Stein inequality for Hardy-Littlewood maximal functions and the Michlin-Hörmander multiplier theorem).

Chapter I is entitled *Negative definite functions and L_p-domains of generators of Lévy processes* and contains a collection of results obtained in the already published papers [FJS01a] and [FJS01b].
Section 2 has a preparatory character and only known background material is collected.
We recall basic properties of continuous negative definite functions and their relation to convolution semigroups of measures and then we recall some basic facts on one-parameter semigroups.
In addition we introduce Bernstein functions and present just some basic facts on subordination in the sense of Bochner.

Section 3 contains a very brief overview on some results obtained [FJS01a] and is, in some sense, of more theoretical nature.
In **Subsection 3.1** we recall the motivation of [FJS01a] preparing the following subsections.
Subsection 3.2 is devoted to the structure of generators of L_p-sub-Markovian semigroups. The form of generators of Feller semigroups is well known. They satisfy the positive maximum principle and once the domain contains $C_0^\infty(\mathbb{R}^n)$, they are already certain differential - integrodifferential operators with negative definite symbol. Under suitable regularity assumptions on the respective domains and the mapping behaviour, see Theorem 3.2.1 for details, we infer that each L_q-generator satisfies the positive maximum principle and has the same structure as a Feller-generator. This result is quite important since it tells us something about the type of the operator one has to start in order to construct an L_p-sub-Markovian semigroup, or, if $p = 2$, a Dirichlet form.
In **Subsection 3.3** we concentrate on fractional powers of second order elliptic differential operators generating L_p-sub-Markovian (diffusion) semigroups. We need not assume to the analyticity of the original diffusion semigroup since by a result of A. Carasso and T. Kato [CarKa91] the subordinate semigroup is automatically analytic if the corresponding Bernstein function is a complete Bernstein function. Interpolation results for fractional powers of generators lead to a large class of strong L_p-sub-Markovian semigroups. Moreover we get concrete, non-trivial examples of the structure theorem for generators, see Theorem 3.2.1.
Subordination in the sense of Bochner is applied in **Subsection 3.4** to discuss the Γ-transform $(V_r^{(p)})_{r\geqslant 0}$ of an L_p-sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ which is needed to

handle refinements of that semigroup. This is, of course, closely related to the work of P. Malliavin and M. Fukushima (with coauthors) and this construction is well-known, but it seems that in [FJS01a] for the first time systematic use is made from the fact that the Γ- transform is a special case of subordination.
This enables us to determine $V_r^{(p)}$ as $(\mathrm{id} - A^{(p)})^{-r/2}$, $A^{(p)}$ being the generator of $(T_t^{(p)})_{t\geqslant 0}$, and to identify the abstract Bessel potential space $\mathcal{F}_{r,p}$ with $D(\mathrm{id} - A^{(p)})^{r/2}$, see Theorem 3.4.1 and Corollary 3.4.2. For $p = 2$ and a selfadjoint operator $A^{(2)}$ this was proved in [Fu93] and [FuKa85] using spectral theory. Our approach is based on a functional calculus for generators of semigroups and Bernstein functions, see [Sc98a].

In **Subsection 3.5** we discuss the problem of constructing refinements of L_p-sub-Markovian semigroups. In particular we are interested in L_p-sub-Markovian semigroups $(T_t^{(p)})_{t\geqslant 0}$ with the property $T_t^{(p)}\chi_A \in C_b(\mathbb{R}^n)$ for all $t > 0$ and all Borel sets A with finite Lebesgue measure. We call these semigroups *strong* L_p-sub-Markovian semigroups in analogy to strong Feller semigroups. Whenever $(T_t^{(p)})_{t\geqslant 0}$ is a strong L_p-sub-Markovian semigroup we may use $p_t(x, A) := T_t^{(p)}\chi_A(x)$ to construct an associated Hunt process without any exceptional set. Otherwise we shall try to reduce the exceptional set whenever possible by using capacities associated with $(T_t^{(p)})_{t\geqslant 0}$ and $\mathcal{F}_{r,p}$ for some suitable r.
The key observation (which seems to be new in our context) is that a combination of the regularising effects of an analytic semigroup with the concrete characterisation of the domain(s) (of powers) of the generator, and Sobolev-type embeddings will immediately give the strong L_p-sub-Markov property, see Proposition 3.5.2 and Theorem 3.5.3. A first example for this idea is provided by semigroups generated by second order elliptic differential operators. Of course, not every (analytic) L_p-sub-Markovian semigroup is a strong L_p-sub-Markovian semigroup. In this case we use the theory of (r, p)-capacities to get refinements, see [Fu92] - [Fu93] or [FuKa85] which is briefly recorded for the reader's convenience.

Section 4 contains a brief overview on [FJS01b] and deals with ψ- Bessel potential spaces, i.e. Bessel potential spaces associated with a fixed continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$.
These function spaces should be natural domains of L_p-generators. They are constructed for translation invariant operators, i.e., Lévy processes, and we recall some recent results from [FJS01b].
Since we do not dispose of Plancherel's theorem, the L_p-analysis for $p \neq 2$ is much harder than the L_2-analysis.
The first and obvious attempt to define these spaces would be to take all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\| \mathcal{F}^{-1}\left[(1+\psi(\cdot))^{r/2}\, \widehat{u} \right] \mid L_p(\mathbb{R}^n) \right\| < \infty$$

is finite. The problem is, however, that in general ψ is a continuous, but not differentiable function, hence $(1 + \psi(\cdot))^{r/2}\, \widehat{u}$ is a priori not well-defined. We overcome this difficulty in introducing the space $H_p^{\psi,2}(\mathbb{R}^n)$ by making use of the Lévy - Khinchin formula to decompose ψ. This makes it possible to identify

$$H_p^{\psi,2}(\mathbb{R}^n) := \left\{ u \in L_p(\mathbb{R}^n) \,:\, \left\| \mathcal{F}^{-1}\left[(1+\psi(\cdot))\, \widehat{u} \right] \mid L_p(\mathbb{R}^n) \right\| < \infty \right\}$$

with the domain of definition of the operator $-\psi(D)$ as generator of the L_p-sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ given by (0.3). This enables us to introduce the scale $H_p^{\psi,s}(\mathbb{R}^n)$, $1 < p < \infty$ and $s \in \mathbb{R}$, first by using the functional calculus for the operator $-\psi(D)$ and then by identifying this space with the closure of $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm $\left\|\mathcal{F}^{-1}\left[(1+\psi(\cdot))^{s/2}\,\widehat{u}\right] \mid L_p(\mathbb{R}^n)\right\|$.
Elementary properties of these spaces (including a characterisation of the dual space) are given in [FJS01b] and will not be repeated here. Note that these scales contain both the classical Bessel potential spaces, we just have to take $\psi(\xi) = |\xi|^2$, and the classical anisotropic Bessel potential spaces associated with the anisotropic distance function $\sqrt{\psi}$ where $\psi(\xi) = |\xi_1|^{2/a_1} + ... + |\xi_n|^{2/a_n}$ for $a_k \geqslant 1$, $k = 1, ..., n$.
However we want to emphasise that due to our examples of continuous negative definite functions, in particular Examples 2.1.14 and 2.1.15, the class under consideration is much larger (even than the classes studied in [KaLi87] and in [Mo99]) and contains function spaces not considered so far.
Subsection 4.2 collects the embedding results and in **Subsection 4.3** we make use of the fact that the semigroup $(T_t^{(p)})_{t\geqslant 0}$ and the operators $(\mathrm{id} - A^{(p)})^{-r/2}$, $r > 0$, are positivity preserving. Therefore we can associate a capacity $\mathrm{cap}_{r,p}^{\psi}$ with each of the spaces $H_p^{\psi,r}(\mathbb{R}^n)$, $r > 0$. This capacity enables us to consider (r,p)- quasi-continuous modifications of elements $u \in H_p^{\psi,r}(\mathbb{R}^n)$, and we show that each $u \in H_p^{\psi,r}(\mathbb{R}^n)$ has a unique quasi-continuous modification (up to (r,p)- quasi-everywhere equality). Further we obtain comparison results for $\mathrm{cap}_{r_1,p_1}^{\psi_1}$ and $\mathrm{cap}_{r_2,p_2}^{\psi_2}$ based on embedding theorems.

Chapter II is entitled *Admissible continuous negative definite functions and associated Bessel potential spaces* and the material contained inside is not yet published but will be included in [Fa03] (together with an extended version of Section 16 from the last chapter).
This second chapter has a transitory character and it may be interpreted as a bridge between Chapter I and Chapter III since our aim in this part is on the one hand to complement the results obtained in [FJS01b] (and presented in Section 4) and, on the other hand to motivate (partly) what is done in Chapter III.
To be more precise, our investigation is devoted to the study of Bessel potential spaces associated with a given real-valued continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ if the function ψ has properties similar to those one of the functions of type $\xi \mapsto f(|\xi|^2)$ where f is an appropriate Bernstein function.
We call these functions admissible continuous negative definite functions and the precise formulation is given in **Section 6**, in particular see Definition 6.1.1. They are not only smooth but they have additional properties like a hypoellipticity type property.
To each admissible function ψ we associate a sequence of non-negative numbers by the formula $N_j = \sup\{\langle\xi\rangle : \psi(\xi) \leqslant 2^{2j}\}$ for any $j \in \mathbb{N}_0$. It turns out that there exists a $\lambda_0 > 1$ such that this sequence satisfies $\lambda_0\, N_j \leqslant N_{j+1}$ for any $j \in \mathbb{N}_0$, compare Lemma 6.2.2.
The associated sequence $N = (N_j)_{j\in\mathbb{N}_0}$ to an admissible continuous negative definite function allows us in **Section 7** to show that for any $u \in H_p^{\psi,s}$ its norm is equivalent to $\left\|(2^{js}\,\varphi_j^N(D)u)_{j\in\mathbb{N}_0} \mid L_p(l_2)\right\|$ where $(\varphi_j^N)_{j\in\mathbb{N}_0}$ is a smooth partition of unity associated in a "canonical way" to the sequence N. The precise formulation is given Corollary 7.1.4.
Consequently, dealing with ψ-Bessel potential in which ψ is of the form $f(|\cdot|^2)$, f a Bernstein function, we may identify these spaces with spaces of generalised smoothness

of type $F^{\sigma,N}_{p,2}$. This motivates partly our investigations in the next chapter.
We conclude this second chapter discussing Sobolev type embeddings.
We use a result of G. A. Kalyabin from [Ka81] obtain in Corollary 7.2.5 a Sobolev type embedding for $H^{\psi,s}$. This result can be stated as below.
If $\psi(\xi) = f(|\xi|^2)$ where f is a Bernstein function with $\lim_{t\to\infty} f(t) = \infty$ such that there exists a number $r_0 \in (0,1]$ such that $t \mapsto f(t)\, t^{-r_0}$ is increasing in t and if $1 < p < \infty$ and if $s > \frac{1}{r_0}\frac{n}{p}$ then $H^{\psi,s}_p \hookrightarrow C_\infty$, compare Corollary 7.2.5 and Corollary 7.2.6.
This is the natural extension of the embedding result for classical spaces H^s_p but is sharper than that one from [Ja02, Corollary 3.3.32] where the restriction was $s > \frac{1}{r_0}\frac{n(p+1)}{p}$.

Chapter III is entitled *Function spaces of generalised smoothness* and is more or less self-contained. A great part of the results in this third chapter were announced in the preprint [FaLe01].
The aim of Chapter III is twofold. The first one is to give a unified approach on function spaces of generalised smoothness and the second one is to characterise these spaces in terms of new tools such as local means and atoms.
In obtaining this unified approach we are additionally motivated by what has been done in the previous chapter where we have shown that for reasonable negative definite functions, the associated Bessel potential space is also a function space of generalised smoothness.

Our approach has as background the Fourier-analytic characterisation of function spaces based on a suitable resolution of unity on the Fourier side and a suitable weighted summation of the resulting parts.
Let us recall the classical construction of $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ spaces. These contain as special cases many well-known spaces as Hölder-Zygmund spaces,

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty,\infty} \quad \text{if} \quad s > 0,$$

Lebesgue spaces

$$L_p(\mathbb{R}^n) = F^0_{p,2} \quad \text{if} \quad 1 < p < \infty,$$

Sobolev spaces,

$$W^m_p(\mathbb{R}^n) = F^m_{p,2} \quad \text{if} \quad m \in \mathbb{N} \quad \text{and} \quad 1 < p < \infty,$$

Bessel potential spaces (fractional Sobolev spaces),

$$H^s_p(\mathbb{R}^n) = F^s_{p,2} \quad \text{if} \quad s \in \mathbb{R} \quad \text{and} \quad 1 < p < \infty,$$

local Hardy spaces

$$h_p = F^0_{p,2} \quad \text{if} \quad 0 < p < 1,$$

and functions of bounded mean oscillation

$$bmo = F^0_{\infty,2} \quad ,$$

see and compare [Tr92, Chapter 1] for historical references.
In what follows we will assume $p < \infty$ in the case of F spaces.

In introducing these spaces, any temperate distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is decomposed in a sum of entire analytic functions $(\varphi_j \widehat{f})^\vee$. This decomposition in the Fourier-image is in the classical case usually related to the symbol of the Laplacian and to the sequence 2^j. Then this sequence of entire analytic functions $(\varphi_j \widehat{f})^\vee$ is considered in L_p and afterwards in a weighted l_q space with weight sequence 2^{sj} in the case of $B^s_{p,q}(\mathbb{R}^n)$, and vice-versa for $F^s_{p,q}(\mathbb{R}^n)$.

To extend this classical construction to the case of generalised smoothness we replace the sequences 2^j and 2^{sj} by two sequences N and σ. The first one is strongly increasing and determines the decomposition on the Fourier side. The second one is the weight sequence for l_q and is, together with its inverse, of bounded growth.
We show that such a construction is suitable and covers many classes of function spaces of generalised smoothness known so far in the literature. Furthermore we give a comprehensive study of those spaces including Littlewood - Paley theorems, existence of a lift operator and duality.

In the eighties and nineties new far-reaching tools for classical spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$, have been developed. The key words are maximal functions, local means, atomic and, most recently, quarkonial decompositions.
First, under some mild restrictions on the sequence N determining the decomposition on the Fourier-side, we prove a general characterisation of these spaces in terms of maximal functions and local means, which essentially generalises the characterisation from [BPT96] and [BPT97] of H.-Q. Bui, M. Paluszyński, and M. Taibleson (which complemented some earlier results of J. Peetre, see [Pe75], and H. Triebel, see [Tr88] and [Tr92]).
This result, see the precise formulation in Theorem 11.3.4, is of independent interest but it played the key role in proving the central result of this chapter, the atomic decomposition theorem.

Entire analytic functions may be considered as building blocks for the spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ in the sense described above or in the sense of approximation theory.
However there is a well-known other type of decomposition in simple building blocks, the so-called atoms.
Historically, atomic decompositions of functions appeared in the 70's in connection with Hardy spaces. Later the fundamental works of M. Frazier and B. Jawerth, [FrJa85] and [FrJa90], posed the cornerstone to a systematically developed theory, embracing the whole scale of (homogeneous) Besov and Triebel-Lizorkin spaces.
The central idea of this approach was to characterise a function space, say $F^s_{p,q}$, as the collection of all elements F which can be decomposed as $f = \sum_{k\in I} \lambda_k \rho_k$ where the (countably many) coefficients λ_k belong to a suitable sequence space and the functions ρ_k are particularly "simple" functions with compact support (atoms). This has nothing to do with Hilbert-type decompositions: for a function f the coefficients are not even uniquely determined, and the atoms represent neither a base nor a so-called frame.
Nonetheless, there are some analogies, for example the knowledge of the coefficients still gives the possibility to reconstruct the norm of f. We do not go into further details since we will be more specific in the sequel.
We only want to mention that the (smooth) atoms in $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ spaces as they were defined by in [FrJa85], [FrJa90] (cf. also [FJW91]), proved to be a powerful tool in the theory of function spaces. We also wish to emphasise that there exist many

other types of atomic decompositions in spaces but we will not discuss this point here. More information about this subject is given in [FrJa90], [Tr92] and [AdHe96] where one can find many modifications and applications as well as comprehensive references extending the subject.
We conclude our third chapter obtaining a decomposition theorem which extends the atomic decomposition theorem of M. Frazier and B. Jawerth, see [FrJa85] and [FrJa90], to the function spaces $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$ and $F^{\sigma,N}_{p,q}(\mathbb{R}^n)$. The precise formulation is given in Theorem 12.2.1.
Consequently, the study of function spaces of generalised smoothness can be done with the help of some sequence spaces in an analogous way as it is done in the classical (isotropic) case. In particular this characterisation may be applied to ψ-Bessel potential spaces with admissible ψ.

Chapter IV is entitled *Pseudo-differential operators related to an admissible continuous negative definite function* and the material contained is partly announced in the preprint [Fa02] (Section 14 and Section 15). The last section will be included (together with the contents of the second chapter) in [Fa03].

This last chapter has two objectives. The first one is to treat mapping properties of exotic pseudo-differential operators in spaces of generalised smoothness; we will do this using the atomic decomposition theorem and this study of mapping properties will show (as a by-product) how powerful this decomposition could be. The second objective is to indicate how one can use the contents of all previous chapters for developing an L_p-theory for pseudo-differential operators and discussing conditions under which these are generators of L_p-sub-Markovian semigroups.

To be more precise, let us mention that in **Section 14** we just collect some fundamental concepts and results from the theory of pseudo-differential operators. We recall the definition of the Hörmander symbol class $S^{\mu}_{\rho,\delta}$, give examples and discuss some mapping properties also from the historical point of view.

In **Section 15** our interest is focused on the so-called exotic symbols (and on the associated operators). A function $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ of class C^∞ such that for any multi-index β and for any multi-index α there exists $c_{\beta\alpha} > 0$ with

$$|D^\beta_x D^\alpha_\xi a(x,\xi)| \leqslant c_{\beta\alpha} \langle\xi\rangle^{-|\alpha|+|\beta|} \quad \text{for any} \quad x \in \mathbb{R}^n,\, \xi \in \mathbb{R}^n$$

is a symbol from $S^0_{1,1}$.
There is a striking difference between mapping properties for non-exotic pseudo-differential operators, i.e. operators with symbols in $S^\mu_{1,\delta}$ with $\delta < 1$ on the one hand and mapping properties for exotic pseudo-differential operators.
It was observed by C.-H. Ching in [Ch72] that exotic pseudo-differential operators do not necessarily map L_2 into L_2. More precisely, he showed that there exist pseudo-differential operators which do not map L_2 into L_2. We recall his example in Example 14.1.4.
An important step was done at the beginning of the eighties, when Y. Meyer proved that pseudo-differential operators with exotic symbols map $H^s_p = F^s_{p,2}$ ($s > 0$ and $1 < p < \infty$) into itself.

Afterwards exotic pseudo-differential operators attracted more and more attention, in particular in connection with Bony's application of exotic pseudo-differential operators

to non-linear problems. Corresponding investigations covering general spaces of $B^s_{p,q}$ and $F^s_{p,q}$ type were given (among other) by T. Runst using paramultiplication in [Ru85], and by R. H. Torres using so-called molecular decompositions in [To90].
Our main result of this section is Theorem 15.1.1 in which we show that if $a \in S^0_{1,1}$ is an exotic symbol and if s satisfies some reasonable assumptions then $a(\cdot, D)$ maps (function space of generalised smoothness) $F^{\sigma^s,N}_{p,q}$ linear and bounded into itself.
The proof is highly technical but in some sense canonical once the philosophy of the atoms is well understood. Actually it requires besides atomic decompositions also the Fefferman-Stein inequality for maximal functions extremely useful in proving the crucial estimate contained in Lemma 15.2.7.
In Subsection 15.3 we point out that this covers the results of T. Runst and R. H. Torres and as corollary, if $1 < p < \infty$ and if $s > 0$ one gets that the pseudo-differential operator $a(\cdot, D)$ with exotic symbol a maps the classical Bessel potential space $H^s_p = F^s_{p,2}$ linear and bounded into itself and this is the famous result of Y. Meyer, see [Mey80].
In particular our mapping result stated in Theorem 15.1.1 can be applied to ψ-Bessel potential spaces, where ψ is admissible in the sense of Chapter II. We think also it might be useful treating several classes of (exotic) partial differential equations.

We conclude the last chapter with **Section 16** in which we start with an admissible continuous negative ψ and introduce a class of symbols (related to ψ) which is a refinement of the classical symbols classes $S^\mu_{\rho,\delta}$, see the precise formulation in 16.2.1. Of course our class is related to some earlier works of R. Beals from [Be75] and H.-G. Leopold from [Leo89a], [Leo89b] and [Le91].
We think that the class of symbols we introduced and (partly) studied in this last section fits good in fulfilling our aim of developing an L_p-theory for generators of sub-Markovian semigroups, compare the discussion in Subsection 16.4.

Complements and an outlook

As one can see from the presentation above, we tried to treat in a unified manner some aspects concerning function spaces and pseudo-differential in connection to problems arising from the theory of sub-Markovian semigroups.
In spite of the fact that this work is still a qualification work (in the german system), having the intention to treat the subject described above more or less unitarily, but also having in mind to keep a reasonable length for this work, we decided not to include here the recent preprint (with co-authors) [BFHHS02].
In [BFHHS02] a slightly different topic was treated. We study the energy of relativistic electrons and positrons interacting through the second quantised Coulomb interaction, in the field of a nucleus of charge Z, within the Hartree-Fock approximation. This is a work related to some previous investigations of V. Bach, J.-M. Barbaroux, B. Helffer and H. Siedentop done in [BBHS98] and in [BBHS99].
This topic comes from mathematical physics and maybe here is the right place to point out that pseudo-differential operators as treated in the current work (mainly motivated by the problem of generating Markov processes) play directly, or indirectly, an important role also in other mathematical fields.

For example, in some parts of mathematical physics the operator

$$\begin{aligned} Hu(x) &= H_0(x) + V(x)\,u(x) \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} \left(\sqrt{|\xi|^2+m^2} - m\right) \widehat{u}(\xi)\,d\xi + V(x)\,u(x) \end{aligned} \tag{0.6}$$

is considered as a certain type of the relativistic Hamiltonian. In particular in problems concerning the stability of matter this Hamiltonian is very often used, see for example R. Carmona [Ca89], R. Carmona, W. C. Masters and B. Simon [CMS90], I. Daubechies [Da83] and [Da84], I. Daubechies and E. Lieb [DaLi83], C. Fefferman [Fe86], E. Lieb [Li90], E. Lieb and H.-T. Yau [LiYa87] and [LiYa88], and the references given there. In fact (0.6) enters, indirectly, already in the considerations of Chandrasekhar concerning the evolution of stars, see [Li90] for comments. However the function $\psi(\xi) = \sqrt{|\xi|^2+m^2} - m$ is a continuous negative definite function and so H_0 is also a generator of Lévy process.

In quantum theory the spectral theory of (0.6) and related operators is also of interest. Since the work of M. Kac, see [Kac51], it is known that the stochastic process generated by the free Hamiltonian H_0 is very useful in the spectral analysis of the operator H. This is essentially due to Feynman-Kac formula which reads as follows:

$$e^{-tH} = E^x\left(f(X_t)\,e^{-\int_0^t V(X_s)\,ds}\right),$$

where we do not want to go into details when the formula above makes sense. We want to emphasise here that one has to define admissible potential classes (Kato classes etc.) relative to the operator H_0 or equivalently to the corresponding processes. Hence the fact that we can associate a Feller process with H_0 gives a further tool in spectral analysis. For example, it was shown in [CMS90] that the behaviour of the eigenfunctions is related to the transience or recurrence of the process generated by the free Hamiltonian.

The Hamiltonian mentioned in (0.6) does not include a magnetic potential A. But also this case has been intensively studied with the help of pseudo-differential operators generating a Feller process. We would like to mention the papers of T. Ichinose [Ic89], [Ic90], and of T. Ichinose and H. Tamura [IcTa86] where operators corresponding to the relativistic energy

$$\sqrt{(p-A(x))^2+m^2} + V(x)$$

are considered and obtained by the Weyl quantisation.

The above mentioned Hamiltonian, including a magnetic potential, was used by E. Lieb, M. Loss and H. Siedentop in constructing a Thomas-Fermi-Weizsäcker type theory by means of which they were able to give a relatively simple proof of the stability of relativistic matter.

Furthermore, stability of relativistic matter with magnetic fields was investigated by E. Lieb, H. Siedentop and J.-P. Solovej in [LiSiSo97].

We will discuss in our first chapter the correspondence of continuous negative definite functions and convolution semigroups of measures. Let us return to the important example that for $m \geqslant 0$ the function $\psi(\xi) = \sqrt{|\xi|^2+m^2} - m$ is a continuous negative definite function.

The corresponding convolution semigroup $(\rho_t)_{t\geqslant 0}$ consists of measures each having density h_t with respect to the n-dimensional Lebesgue measure:

$$h_t(x) = 2 \cdot (2\pi)^{-\frac{n+1}{2}} m^{\frac{n+1}{2}} e^{mt} t(|x|^2 + t^2)^{-\frac{n+1}{4}} \cdot K_{\frac{n+1}{2}} \left(m(|x|^2 + t^2)^{1/2} \right) \tag{0.7}$$

where K_j is the modified Bessel function of third kind of order j.
A related example is given by $\widetilde{\psi}(\xi) = \sqrt{|\xi|^2 + m^2}$ which leads to a convolution semigroup $(\widetilde{\rho}_t)_{t\geqslant 0}$ with densities

$$\widetilde{h}_t(x) = 2 \cdot (2\pi)^{-\frac{n+1}{2}} m^{\frac{n+1}{2}} (|x|^2 + t^2)^{-\frac{n+1}{4}} \cdot K_{\frac{n+1}{2}} \left(m(|x|^2 + t^2)^{1/2} \right).$$

Formula (0.7) goes back to T. Ichinose [Ic89], the modification for $\widetilde{h}_t$ is obvious.
For $n = 3$ the formula for $\widetilde{h}_t$ was given in [LiYa88] by E. Lieb and H. T. Yau, we refer also to I. Herbst and A. Sloan, see [HeSl78].

In an other context let us mention that the function

$$\mathbb{R} \ni \xi \mapsto -\log \left(\frac{1}{K_1(1)} \frac{K_1(1 + \sqrt{1 + |\xi|^2})}{\sqrt{1 + |\xi|^2}} \right),$$

where K_1 is the modified Bessel function of the third kind with index 1, is also a (one-dimensional) continuous negative definite function. The associated semigroup is the hyperbolic convolution semigroup of measures and E. Eberlein and U. Keller used this semigroup for modeling some problems in financial mathematics, see [EbKe95].

The classical examples of generators of diffusion semigroups and the examples of non-local generators of Feller semigroups led M. Demuth and J. van Casteren, see [DeCa89] - [DeCa94c], to build up a theory they called stochastic *spectral analysis*. For a general background concerning unbounded operators and spectral theory in Hilbert spaces we refer the reader to the book [BiSo87].
The idea is to study the spectral properties of operators $H = H_0 + V$ where H_0 is a selfadjoint generator of a "nice" Feller semigroup and V is a certain potential. More precisely they assumed that H_0 is the generator of a symmetric Feller semigroup and their basic assumptions are expressed in terms of the transition function of this semigroup. For further details the reader is referred to the above mentioned works.

0 Preliminaries

0.1 Notation

In this work we adopt a unique progressive numbering system for theorems, propositions, remarks, etc. of the form $m.n.k$ where m stands for the number of the section, n stands for the subsection and k is a progressive index within the considered subsection. Equations are numbered and then referenced as $(m.k)$ where m is the number of the section and k is a progressive index within the considered section.

Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, where $n \in \mathbb{N}$; as usual $\mathbb{R} = \mathbb{R}^1$. For $x \in \mathbb{R}^n$ let $\langle x \rangle = (1 + |x|^2)^{1/2}$.
If $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ is a multi-index its length is $|\alpha| = \alpha_1 + ... + \alpha_n$, the derivatives D^α have the usual meaning and if $x = (x_1, ..., x_n) \in \mathbb{R}^n$ then $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.
If X is a Banach space we denote its norm by $\|\cdot \,|\, X\|$.

We denote $C(\mathbb{R}^n)$ respectively $C_b(\mathbb{R}^n)$, the collection of all complex-valued continuous, respectively continuous and bounded, functions defined on $\mathbb{R}^n$.
We say that $u : \mathbb{R}^n \to \mathbb{C}$ vanishes at infinity if for any $\varepsilon > 0$ there exists a compact set K in $\mathbb{R}^n$ such that $|u(x)| < \varepsilon$ if $x \in K^c$. Let

$$C_\infty(\mathbb{R}^n) = \{u \in C(\mathbb{R}^n) \,:\, u \text{ vanishes at infinity}\}.$$

The spaces $C_b(\mathbb{R}^n)$ and $C_\infty(\mathbb{R}^n)$ are normed in the usual way, for example

$$\|u \,|\, C_\infty(\mathbb{R}^n)\| = \sup_{x \in \mathbb{R}^n} |u(x)|, \quad \text{if} \quad u \in C_\infty(\mathbb{R}^n).$$

The space $C_0^\infty(\mathbb{R}^n)$ denotes, as usual, the collection of all complex-valued infinitely differentiable functions having compact support and it is clear that $C_0^\infty(\mathbb{R}^n)$ is dense in $C_\infty(\mathbb{R}^n)$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ equipped with the usual topology. By $\mathcal{S}'(\mathbb{R}^n)$ we denote its topological dual, the space of all tempered distributions on $\mathbb{R}^n$. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then

$$\widehat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x)\, dx$$

and $\check{\varphi} = \mathcal{F}^{-1}\varphi$ are respectively the Fourier and inverse Fourier transform of φ. One extends $\mathcal{F}$ and $\mathcal{F}^{-1}$ in the usual way from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ we will use the notation $\varphi(D)f\,(x) = [\mathcal{F}^{-1}(\varphi \mathcal{F} f)](x)$ where this is the extension of

$$\varphi(D)\psi\,(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} \varphi(\xi)\psi(y)\, dy\, d\xi \quad , \quad \psi \in \mathcal{S}(\mathbb{R}^n)$$

to elements $f \in \mathcal{S}'(\mathbb{R}^n)$.
Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \leqslant \infty$, is the standard quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f \,|\, L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}$$

with the obvious modification if $p=\infty$.
We adopt here and in the sequel the following convention: if there is no danger of confusion we omit $\mathbb{R}^n$ in $\mathcal{S}(\mathbb{R}^n)$ and in the other spaces below.
If $0<q\leqslant\infty$, then l_q is the set of all sequences $(a_k)_{k\in\mathbb{N}_0}$ of complex numbers such that

$$\|(a_k)_{k\in\mathbb{N}_0}\,|\,l_q\|=\left(\sum_{k=0}^{\infty}|a_k|^q\right)^{1/q}<\infty,$$

with the obvious modification if $q=\infty$.
Let $0<p\leqslant\infty$ and $0<q\leqslant\infty$. If $(f_k)_{k\in\mathbb{N}_0}$ is a sequence of complex-valued Lebesgue measurable functions on $\mathbb{R}^n$, then

$$\|(f_k)_{k\in\mathbb{N}_0}\,|\,l_q(L_p)\|=\left(\sum_{k=0}^{\infty}\left(\int_{\mathbb{R}^n}|f_k(x)|^p dx\right)^{q/p}\right)^{1/q}$$

and

$$\|(f_k)_{k\in\mathbb{N}_0}\,|\,L_p(l_q)\|=\left(\int_{\mathbb{R}^n}\left(\sum_{k=0}^{\infty}|f_k(x)|^q\right)^{p/q}dx\right)^{1/p},$$

again with obvious modifications if $p=\infty$ and/or $q=\infty$.

The equivalence $a_k\sim b_k$ or $\varphi(x)\sim\psi(x)$ means that there are two positive constants c_1 and c_2 such that $c_1a_k\leqslant b_k\leqslant c_2a_k$ or $c_1\varphi(x)\leqslant\psi(x)\leqslant c_2\varphi(x)$ for all admissible values of the discrete variable k or of the continuous variable x.
All unimportant positive constants are denoted with c, occasionally with additional subscripts within the same formulas.

0.2 Some fundamental results

0.2.1 Maximal inequalities

If f is a complex-valued locally Lebesgue integrable function on $\mathbb{R}^n$, then

$$(\mathcal{M}f)(x)=\sup\frac{1}{|B|}\int_B|f(y)|\,dy$$

is the Hardy-Littlewood maximal function, where the supremum is taken over all balls B centred at x.
If $1<p<\infty$ and $1<q<\infty$ then there exists a constant $c>0$ such that

$$\left\|(\mathcal{M}f_k)_{k\in\mathbb{N}_0}\,|\,L_p(l_q)\right\|\leqslant c\left\|(f_k)_{k\in\mathbb{N}_0}\,|\,L_p(l_q)\right\| \tag{0.8}$$

for any sequence $(f_k)_{k\in\mathbb{N}_0}$ of complex-valued locally Lebesgue-integrable functions on $\mathbb{R}^n$. This fundamental inequality is due to C. Fefferman and E. M. Stein, see [FeSt71].

0.2.2 A Fourier multiplier theorem

For a system $(m_{k,j})_{k,j\in\mathbb{N}_0} \subset L_\infty(\mathbb{R}^n)$ let

$$M = \sup \left(R^{2|\alpha|-n} \int\limits_{\frac{R}{2}\leqslant|\xi|\leqslant 2R} \sum_{k,j=0}^{\infty} |D^\alpha m_{k,j}(\xi)|^2 \, d\xi \right)^{1/2} \tag{0.9}$$

where the supremum is taken over all $R > 0$ and all multi-indices α with $0 \leqslant |\alpha| \leqslant 1 + \left[\frac{n}{2}\right]$.

Proposition 0.2.1 *Let $1 < p < \infty$ and $1 < q < \infty$. Let $f = (f_j)_{j\in\mathbb{N}_0}$ be a system of measurable functions in $\mathbb{R}^n$.*
(i) *(General case) There exists a positive constant c such that*

$$\left\| \left(\sum_{j=0}^{\infty} m_{k,j}(D) f_j \right)_{k\in\mathbb{N}_0} \,|\, L_p(l_2) \right\| \leqslant c\, M \cdot \|(f_j)_{j\in\mathbb{N}_0} \,|\, L_p(l_2)\| \tag{0.10}$$

for all systems $(m_{k,j})_{k,j\in\mathbb{N}_0} \subset L_\infty(\mathbb{R}^n)$, the number M being given in (0.9)*.*

(ii) *(Diagonal case) There exist a positive constant c such that*

$$\left\| (m_{j,j}(D) f_j)_{j\in\mathbb{N}_0} \,|\, L_p(l_q) \right\| \leqslant c\, M \cdot \|(f_j)_{j\in\mathbb{N}_0} \,|\, L_p(l_q)\| \tag{0.11}$$

for all systems $(m_{k,j})_{k,j\in\mathbb{N}_0} \subset L_\infty(\mathbb{R}^n)$ with $m_{k,j} \equiv 0$ if $k \neq j$, the number M being given in (0.9)*.*

A proof of this classical Fourier-multiplier theorem of Michlin-Hörmander type can be found in [Tr78, Theorem 2.2.4]. The first part is contained also in [Tr83, Equation 2.5.6/(1)].

Chapter I.
Negative definite functions and L_p-domains of generators of Lévy processes

1 Introduction to Chapter I

This chapter contains besides some basic facts on continuous negative definite functions, semigroups of operators, subordination in the sense of Bochner, also an overview of the results obtained in [FJS01a] and [FJS01b]. We give no proofs but we will point out the role of Sobolev type embeddings for domains of generators of L_p-sub-Markovian semigroups in constructing Markov processes with no exceptional sets.

Section 2 has a preparatory character and only known background material is collected here.
We recall basic properties of continuous negative definite functions and their relation to convolution semigroups of measures and then we recall some basic facts on one-parameter semigroups. Then we introduce Bernstein functions and recall some basic facts on subordination, concept which goes back to S. Bochner. This construction which allows to construct Markov processes by starting with a given one, works on the level of paths of the process by changing the time, but works also on the level of the semigroup and of its generator in form of a certain functional calculus.

Section 3 contains a very brief overview on some results obtained [FJS01a].
After a short subsection with preparatory character (**Subsection 3.1**) we discuss in **Subsection 3.2** the structure of generators of L_p-sub-Markovian semigroups.
After that in **Subsection 3.3** we concentrate on fractional powers of second order elliptic differential operators generating L_p-sub-Markovian (diffusion) semigroups.
Subordination in the sense of Bochner is applied in **Subsection 3.4** to discuss the Γ-transform $(V_r^{(p)})_{r\geqslant 0}$ of an L_p-sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ which is needed to handle refinements of that semigroup.
In **Subsection 3.5** we discuss the problem of constructing refinements of L_p-sub-Markovian semigroups. In particular we are interested in L_p-sub-Markovian semigroups $(T_t^{(p)})_{t\geqslant 0}$ with the property $T_t^{(p)}\chi_A \in C_b(\mathbb{R}^n)$ for all $t > 0$ and all Borel sets A with finite Lebesgue measure.

Section 4 contains an overview on [FJS01b] and deals with ψ- Bessel potential spaces, i.e. Bessel potential spaces associated with a fixed continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$.
These function spaces should be natural domains of L_p-generators. They are constructed for translation invariant operators, i.e., Lévy processes. We will point out that these scales of spaces contain both the classical Bessel potential spaces and the classical anisotropic Bessel potential spaces.
However we want to emphasise that due to our examples of continuous negative definite functions in particular Examples 2.1.14 and 2.1.15, the class under consideration is

much larger (even than the classes studied in [KaLi87] and in [Mo99]) and contains function spaces not considered so far.
Subsection 4.2 collects the embedding results and in **Subsection 4.3** we associate a capacity $\mathrm{cap}^{\psi}_{r,p}$ with each of the spaces $H^{\psi,r}_p(\mathbb{R}^n)$, $r > 0$. This capacity enables us to consider (r,p)- quasi-continuous modifications of elements $u \in H^{\psi,r}_p(\mathbb{R}^n)$, and we show that each $u \in H^{\psi,r}_p(\mathbb{R}^n)$ has a unique quasi-continuous modification (up to (r,p)-quasi-everywhere equality).

We conclude this first chapter with a summary of our approach and an outlook.

2 Preliminaries

2.1 Continuous negative definite functions and semigroups of operators

2.1.1 Basic facts on continuous negative definite functions

The concept and definition of negative definite functions goes back to I. J. Schoenberg, see [Sch38], who introduced it in connection with isometric embeddings of metric spaces into Hilbert spaces. For $\mathbb{R}^n$ his result may be stated as below:
Let d be a metric on $\mathbb{R}^n$. In order that the metric space $(\mathbb{R}^n, d)$ is isometric to a Hilbert space $(\mathbb{R}^n, (\cdot,\cdot))$ it is necessary and sufficient that for all $m \in \mathbb{N}$, all points ξ^0, ξ^1, ..., $\xi^m \in \mathbb{R}^n$, and all $c_j \in \mathbb{R}$, $1 \leqslant j \leqslant m$, the inequality

$$\sum_{k,l=1}^{m} \left(d^2(\xi^0,\xi^k) + d^2(\xi^0,\xi^l) - d^2(\xi^k,\xi^l)\right) c_k c_l \geqslant 0$$

is satisfied. Setting $\psi(\xi - \eta) := d^2(\xi,\eta)$ we find

$$\sum_{k,l=1}^{m} \left(\psi(\xi^0 - \xi^k) + \psi(\xi^0 - \xi^l) - \psi(\xi^k - \xi^l)\right) c_k c_l \geqslant 0.$$

In particular, using the symmetry of d and replacing $\xi^0 - \xi^k$ by ξ^k, we have

$$\sum_{k,l=1}^{m} \left(\psi(\xi^k) + \psi(\xi^l) - \psi(\xi^l - \xi^k)\right) c_k c_l \geqslant 0. \tag{2.1}$$

Hence, given an even function $\psi : \mathbb{R}^n \to \mathbb{R}$ satisfying for all $m \in \mathbb{N}_0$, $\xi^k \in \mathbb{R}^n$, $1 \leqslant k \leqslant m$, and $c_k \in \mathbb{R}$, $1 \leqslant k \leqslant m$, the inequality (2.1), one can expect that $(\xi,\eta) \mapsto \psi^{1/2}(\xi - \eta)$ behaves like a metric.

The following presentation is based on the monograph [BeFo75] by C. Berg and G. Forst, see also [Ja01].
Recall that a function $g : \mathbb{R}^n \to \mathbb{C}$ is called positive definite if for any $k \in \mathbb{N}$ and any vectors ξ^1,..., $\xi^k \in \mathbb{R}^n$ the matrix $(g(\xi^j - \xi^l))_{j,l=1,\ldots,k}$ is positive Hermitian, that is

$$\sum_{j,l=1}^{k} g(\xi^j - \xi^l)\, z_j \overline{z_l} \geqslant 0$$

for any numbers $z_1, ..., z_k \in \mathbb{C}$.
It is not hard to see that the Fourier transform $\widehat{\mu}$ of a bounded Borel measure μ on $\mathbb{R}^n$ is a positive definite function.
Bochner's theorem states the converse: given a continuous positive definite function g on $\mathbb{R}^n$ there exists a bounded Borel measure μ with Fourier transform $\widehat{\mu} = g$. Note that for a fixed $x \in \mathbb{R}^n$ the function $\xi \mapsto e^{-ix\cdot\xi}$ is a continuous positive definite function.
We are now able to give the definition of a negative definite function.

Definition 2.1.1 *A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is called* negative definite *if for any $k \in \mathbb{N}$ and all vectors ξ^1,..., $\xi^k \in \mathbb{R}^n$ the matrix*

$$\left(\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l)\right)_{j,l=1,\dots,k}$$

is positive Hermitian.

The following result is known as Schoenberg's theorem.

Theorem 2.1.2 *A continuous function $\psi : \mathbb{R}^n \to \mathbb{C}$ is negative definite if, and only if, $\psi(0) \geqslant 0$ and for all $t > 0$ the function $\xi \mapsto e^{-t\psi(\xi)}$ is continuous and positive definite.*

In the following we state some elementary properties of continuous negative definite functions.
Clearly, the set of all continuous negative definite functions is a convex cone which is closed under locally uniform convergence.

Lemma 2.1.3 **(i)** *If $\psi : \mathbb{R}^n \to \mathbb{C}$ is a continuous negative definite function then for all ξ, $\eta \in \mathbb{R}^n$*

$$\sqrt{|\psi(\xi+\eta)|} \leqslant \sqrt{|\psi(\xi)|} + \sqrt{|\psi(\eta)|}.$$

(ii) *If $\psi : \mathbb{R}^n \to \mathbb{C}$ is a continuous negative definite function then*

$$|\psi(\xi)| \leqslant c_\psi\,(1 + |\xi|^2) \quad \textit{for all} \quad \xi \in \mathbb{R}^n,$$

where $c_\psi = 2 \sup\limits_{|\eta|\leqslant 1} |\psi(\eta)|$.

Every continuous negative definite function admits a Lévy-Khinchin representation:

Theorem 2.1.4 *If $\psi : \mathbb{R}^n \to \mathbb{C}$ is a continuous negative definite function then there exist: a constant $c > 0$, a vector $d \in \mathbb{R}^n$, a symmetric positive semidefinite quadratic form q on $\mathbb{R}^n$ and a finite measure μ on $\mathbb{R}^n \backslash \{0\}$ such that*

$$\psi(\xi) = c + i(d\cdot\xi) + q(\xi) + \int_{\mathbb{R}^n\backslash\{0\}} \left(1 - e^{-ix\cdot\xi} - \frac{ix\cdot\xi}{1+|x|^2}\right) \frac{1+|x|^2}{|x|^2}\,\mu(dx). \qquad (2.2)$$

The quadruple (c, d, q, μ) is uniquely determined by ψ.
Conversely, given (c, d, q, μ) as above, the function ψ defined by (2.2) is continuous and negative definite.

Let us state the Lévy-Khinchin formula for real-valued continuous negative definite functions explicitly.

Corollary 2.1.5 *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a real-valued continuous negative definite function. Then we have the representation*

$$\psi(\xi) = c + q(\xi) + \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(x \cdot \xi)) \frac{1 + |x|^2}{|x|^2} \mu(dx) \tag{2.3}$$

with c, q, and μ as in Theorem 2.1.4. *In addition μ is a symmetric measure.*

Instead of the measure μ it is often convenient to use the Lévy measure associated with ψ, i.e. the measure

$$\nu(dx) := \frac{1 + |x|^2}{|x|^2} \mu(dx).$$

Thus ν is a Radon measure on $\mathbb{R}^n\setminus\{0\}$ satisfying the integrability condition

$$\int_{\mathbb{R}^n\setminus\{0\}} (|x|^2 \wedge 1) \, \nu(dx) < \infty.$$

The following result is due to W. Hoh, [Ho98a, Proposition 2.1] and it relates the smoothness of ψ to integrability properties of ν.

Theorem 2.1.6 *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function with Lévy-Khinchin representation* (2.3). *Suppose that for $2 \leqslant l \leqslant m$ all absolute moments of the Lévy measure ν exist, i.e.*

$$M_l := \int_{\mathbb{R}^n\setminus\{0\}} |x|^l \, \nu(dx) < \infty, \quad 2 \leqslant l \leqslant m.$$

Then ψ is of class $C^m(\mathbb{R}^n)$ and for $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leqslant m$, we have the estimate

$$|\partial^\alpha \psi(\xi)| \leqslant c_{|\alpha|} \cdot \begin{cases} \psi(\xi) & , \quad \alpha = 0 \\ \psi^{1/2}(\xi) & , \quad |\alpha| = 1 \\ 1 & , \quad |\alpha| \geqslant 2 \end{cases} ,$$

with $c_0 = 1$, $c_1 = (2M_2)^{1/2} + 2\lambda^{1/2}$, $c_2 = M_2 + 2\lambda$ and $c_l = M_l$ for $3 \leqslant l \leqslant m$, where λ is the maximal eigenvalue of the quadratic form q in (2.3).

For our purposes we will deal mostly with continuous negative definite functions $\psi : \mathbb{R}^n \to \mathbb{R}$ of the form

$$\psi(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(x \cdot \xi)) \, \nu(dx), \tag{2.4}$$

where ν is a Lévy measure on $\mathbb{R}^n\setminus\{0\}$.

Corollary 2.1.7 *Suppose that the Lévy measure ν associated with the continuous negative definite function ψ from* (2.4) *has its support in a bounded set, i.e.* $\operatorname{supp} \nu \subset \overline{B(0, R)}$ *for some $R > 0$. Then the function ψ is arbitrarily often differentiable and the function itself, as well as all its partial derivatives, are of at most quadratic growth.*

Moreover, we have the obvious

Lemma 2.1.8 *Let ψ be a continuous negative definite function given by* (2.4)*. If the support of the Lévy measure ν satisfies* $\operatorname{supp}\nu \subset B^c(0,R)$ *for some $R>0$, then ψ is a bounded continuous function.*

From Corollary 2.1.7 and Lemma 2.1.8 we find that every continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ with representation (2.4) has a decomposition $\psi = \psi_R + \widetilde{\psi}_R$, $R>0$, into continuous negative definite functions ψ_R and $\widetilde{\psi}_R$ such that ψ_R is arbitrarily often differentiable and ψ_R as well as its partial derivatives are at most of quadratic growth, and $\widetilde{\psi}_R$ is bounded and continuous. In fact we just have to define

$$\psi_R(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(x\cdot\xi))\,\chi_{B(0,R)}(x)\,\nu(dx)$$

and

$$\widetilde{\psi}_R(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(x\cdot\xi))\,\chi_{B^c(0,R)}(x)\,\nu(dx).$$

We want to point out that there is a one-to-one correspondence between convolution semigroups of sub-probability measures on $\mathbb{R}^n$ and continuous negative definite functions.

Definition 2.1.9 *A family $(\mu_t)_{t\geqslant 0}$ of sub-probability measures on $\mathbb{R}^n$ is called a* convolution semigroup *if the following conditions are fulfilled*

(i) $\mu_t * \mu_s = \mu_{t+s}$ *for any $s,t>0$ and $\mu_0 = \varepsilon_0$ (Dirac measure);*

(ii) $\mu_s \longrightarrow \varepsilon_0$ *vaguely for $s\to 0$.*

Note that some authors do not require the normalization $\mu_0=\varepsilon_0$.

Theorem 2.1.10 *For every convolution semigroup $(\mu_t)_{t\geqslant 0}$ of sub-probability measures on $\mathbb{R}^n$ there exists a uniquely determined continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ such that*

$$\widehat{\mu_t}(\xi) = (2\pi)^{-n/2}\,e^{-t\,\psi(\xi)} \quad , \quad s\geqslant 0 \quad \textit{and} \quad \xi\in\mathbb{R}^n. \tag{2.5}$$

Conversely, given a continuous negative definite function $\psi : \mathbb{R}^n\to\mathbb{C}$ there exists a unique convolution semigroup $(\mu_t)_{t\geqslant 0}$ on $\mathbb{R}^n$ such that (2.5) *holds.*

Example 2.1.11 For any $\alpha,\beta\in(0,1]$ the functions $\xi\mapsto|\xi|^{2\alpha}$ and $\xi\mapsto|\xi|^{2\alpha}+|\xi|^{2\beta}$, $\xi\in\mathbb{R}^n$, are continuous and negative definite.

Example 2.1.12 Let $a_1,...,a_n$ be real numbers such that $a_k\geqslant 1$ for $k=1,...,n$. The function $\psi:\mathbb{R}^n\to\mathbb{R}$ defined by

$$\psi(\xi) = |\xi_1|^{2/a_1} + ... + |\xi_n|^{2/a_n} \tag{2.6}$$

is a continuous negative definite function. This is a simple consequence of the previous example and elementary properties of continuous negative functions.
The function $\sqrt{\psi}$, where ψ is given by (2.6), is a so-called anisotropic distance function, see for example H.-J. Schmeisser and H. Triebel [ScTr87, Subsection 4.2.1] and M. Yamazaki [Yam86a].

Example 2.1.13 From the Lévy-Khinchin formula we deduce immediately that

(i) any symmetric positive semidefinite quadratic form q on $\mathbb{R}^n$,

(ii) any function of the form $\xi \mapsto i\ell \cdot \xi$, $\ell \in \mathbb{R}^n$,

(iii) the functions $\xi \mapsto 1 - e^{-ih\cdot\xi}$ and $\xi \mapsto 1 - \cos(h \cdot \xi)$ with $h \in \mathbb{R}^n$,

(iv) any combination of **(i)**–**(iii)**

are continuous and negative definite functions.

Example 2.1.14 Fix any $\lambda \in (0, 2)$ and choose $M = M(\lambda) \in \mathbb{N}$ such that $M > \frac{2}{2-\lambda}$. Then the following measure

$$\nu(dx) := \sum_{j=1}^{\infty} 2^{\lambda M^j - j}\, \varepsilon_{2^{-M^j}}(dx)$$

is easily seen to be a Lévy measure. Therefore the function $\psi : \mathbb{R} \to \mathbb{R}$,

$$\psi(\xi) := \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(x\xi))\nu(dx) = \sum_{j=1}^{\infty} 2^{\lambda M^j - j} \left(1 - \cos\left(2^{-M^j}\xi\right)\right)$$

is a continuous negative definite function. This function enjoys the following properties:

$$\begin{aligned}
\liminf_{|\xi|\to\infty} \psi(\xi) &= 0; \\
\limsup_{|\xi|\to\infty} \frac{\psi(\xi)}{|\xi|^{\lambda-\rho}} &= \infty \qquad \text{for} \quad \rho > 0; \\
\lim_{|\xi|\to\infty} \frac{\psi(\xi)}{|\xi|^{\lambda+\rho}} &= 0 \qquad \text{for} \quad \rho > 0.
\end{aligned}$$

The proof is given in [FJS01b, Example 1.1.15].

Example 2.1.15 Pick $0 < \kappa < \lambda < 2$ and denote by $\psi_\lambda(\xi)$ the function constructed in Example 2.1.14. Then

$$\psi(\xi) := \psi_\lambda(\xi) + |\xi|^\kappa$$

is a continuous negative function that oscillates for $|\xi| \to \infty$ between the curves $\xi \mapsto |\xi|^\kappa$ and $\xi \mapsto 2|\xi|^\lambda$. Moreover, $\psi(\xi) = \mathcal{O}(|\xi|^\lambda)$ as $|\xi| \to \infty$.

Further examples will be constructed with the help of Bernstein functions, see the next section and in particular Theorem 2.2.4 and Example 2.2.5.

2.1.2 One-parameter operator semigroups

Let $(\mu_t)_{t\geqslant 0}$ be a convolution semigroup of sub-probability measures on $\mathbb{R}^n$. For $t \geqslant 0$ we define the operator

$$T_t u(x) = \int_{\mathbb{R}^n} u(x-y)\,\mu_t(dy) = \mu_t * u(x).$$

Obviously, T_t is defined for all $u \in \mathcal{S}(\mathbb{R}^n)$. We find using the convolution theorem

$$(T_t u)^\wedge(\xi) = (2\pi)^{n/2}\,\widehat{u}(\xi)\,\widehat{\mu_t}(\xi) = \widehat{u}(\xi)\,e^{-t\psi(\xi)}, \tag{2.7}$$

where $\psi : \mathbb{R}^n \to \mathbb{C}$ is the continuous negative definite function associated with $(\mu_t)_{t\geqslant 0}$. It is easy to prove that T_t extends from $\mathcal{S}(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$, $1 < p < \infty$, as well as to $C_\infty(\mathbb{R}^n)$. These extensions will also be denoted by T_t. For the moment we will write $(X, \|\cdot\,|\,X\|)$ for any of the above Banach spaces.
The (extended) operators T_t, $t \geqslant 0$, have the following properties on $(X, \|\cdot\,|\,X\|)$:

(i) $T_{t+s} = T_t \circ T_s$ and $T_0 = \mathrm{id}$;

(ii) $\lim\limits_{t\to 0} \|T_t u - u\,|\,X\| = 0$;

(iii) $\|T_t u\,|\,X\| \leqslant \|u\,|\,X\|$.

Furthermore we have in the case of the spaces $L_p(\mathbb{R}^n)$

$$0 \leqslant u \leqslant 1 \quad \text{a.e. implies} \quad 0 \leqslant T_t u \leqslant 1 \quad \text{a.e.}, \tag{2.8}$$

and in the context of $C_\infty(\mathbb{R}^n)$

$$0 \leqslant u \leqslant 1 \quad \text{implies} \quad 0 \leqslant T_t u \leqslant 1. \tag{2.9}$$

Definition 2.1.16 *A family of linear operators $(T_t)_{t\geqslant 0}$ on a Banach space $(X, \|\cdot\,|\,X\|)$ is called a* strongly continuous contraction semigroup *if the conditions* (i)–(iii) *are satisfied.*

Since we are only considering either convolution semigroups of sub-probability measures or strongly continuous contraction semigroups we will sometimes use semigroups for short.

Definition 2.1.17 **(i)** *A strongly continuous contraction semigroup on $L_p(\mathbb{R}^n)$, $1 \leqslant p < \infty$, is called an* L_p-sub-Markovian semigroup *if* (2.8) *is satisfied.*

(ii) *A strongly continuous contraction semigroup on $C_\infty(\mathbb{R}^n)$ satisfying* (2.9) *is called a* Feller semigroup.

From our introductory considerations we conclude that a family $(T_t)_{t\geqslant 0}$ of operators defined on a small space can be extended to different Banach spaces as a strongly continuous contraction semigroup. In particular, extensions to the spaces $L_p(\mathbb{R}^n)$, $1 \leqslant p < \infty$, will be denoted by $(T_t^{(p)})_{t\geqslant 0}$ and the extension to $C_\infty(\mathbb{R}^n)$ will be denoted by $(T_t^{(\infty)})_{t\geqslant 0}$.

Let us recall the definition of the generator of a semigroup.

Definition 2.1.18 *Let* $(T_t)_{t\geqslant 0}$ *be strongly continuous contraction semigroup on a Banach space* $(X, \|\cdot\,|\,X\|)$*. Its* (infinitesimal) generator *is the operator*

$$Au := \lim_{t\to 0} \frac{1}{t}(T_t u - u) \quad \textit{(strong limit)}$$

with domain

$$D(A) := \left\{ u \in X \,:\, \lim_{t\to 0} \frac{1}{t}(T_t u - u) \quad \textit{exists strongly in} \quad X \right\}.$$

The generator is always a densely defined closed operator which is *dissipative*, i.e. the inequality

$$\lambda\,\|u\,|\,X\| \leqslant \|(\lambda - A)u\,|\,X\|$$

is satisfied for all $\lambda > 0$ and all $u \in D(A)$.

A major problem is to determine the domain $D(A)$ of A. In particular for L_p-sub-Markovian semigroups it is interesting to characterise $D(A)$ in terms of function spaces.

A strongly continuous semigroup $(T_t)_{t\geqslant 0}$ on $L_2(\mathbb{R}^n)$ is called *symmetric* if

$$(T_t u, v)_{L_2(\mathbb{R}^n)} = (u, T_t v)_{L_2(\mathbb{R}^n)} \quad \text{for all} \quad u, v \in L_2(\mathbb{R}^n).$$

Theorem 2.1.19 *Let* $(T_t^{(2)})_{t\geqslant 0}$ *be a symmetric sub-Markovian semigroup on* $L_2(\mathbb{R}^n)$*. Then it extends from* $L_2(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$ *to a sub-Markovian semigroup* $(T_t^{(p)})_{t\geqslant 0}$ *on* $L_p(\mathbb{R}^n)$*,* $p \in [1, \infty)$*.*

For a proof of this result see for example E. B. Davies [Dav80].

A semigroup is called *analytic* if $t \mapsto T_t u$ admits an analytic extension $z \mapsto T_z u$ to some sector $S_{\theta,d_0} := \{z \in \mathbb{C} \,:\, \arg(z - d_0) < \theta\}$.
A result of E. M. Stein [St70a] says that in the case of a symmetric sub-Markovian semigroup $(T_t^{(2)})_{t\geqslant 0}$ on $L_2(\mathbb{R}^n)$ this semigroup as well as its extensions to $L_p(\mathbb{R}^n)$, $1 < p < \infty$, are analytic.
For every analytic semigroup $(T_t)_{t\geqslant 0}$ on a Banach space the following regularisation result holds:

$$T_t u \in \bigcap_{k\geqslant 0} D(A^k), \quad u \in X.$$

Here $D(A^k)$ is the domain of the k-th power of the generator of $(A, D(A))$.

We want to characterise the generators of Feller semigroups and L_p-sub-Markovian semigroups.
For the sake of completeness we recall a version of the classical Hille - Yosida theorem, see S. Ethier and Th. Kurtz [EtKu86, p. 16].

Theorem 2.1.20 *A linear operator on a Banach space* $(X, \|\cdot\,|\,X\|)$ *is closable and its closure* $\overline{A}$ *is the generator of a strongly continuous contraction semigroup on* X *if, and only if, the following three conditions are satisfied:*

(i) $D(A) \subset X$ *is dense;*

(ii) *A is dissipative;*

(iii) *for some $\lambda > 0$ the range $R(\lambda - A)$ of $\lambda - A$ is dense in X.*

For Feller semigroups we have the following characterisation, see [EtKu86, p. 165], often called the Hille - Yosida - Ray theorem.

Theorem 2.1.21 *A linear operator $(A^{(\infty)}, D(A^{(\infty)}))$ on $C_\infty(\mathbb{R}^n)$ is closable and its closure is the generator of a Feller semigroup if, and only if, the following three conditions are satisfied:*

(i) *$D(A^{(\infty)}) \subset C_\infty(\mathbb{R}^n)$ is dense;*

(ii) *$A^{(\infty)}$ satisfies the positive maximum principle, i.e.*

$$u(x_0) = \sup_{x \in \mathbb{R}^n} u(x) \geqslant 0 \quad \textit{implies} \quad A^{(\infty)} u(x_0) \leqslant 0;$$

(iii) *for some $\lambda > 0$ the range $R(\lambda - A^{(\infty)})$ of $\lambda - A^{(\infty)}$ is dense in $C_\infty(\mathbb{R}^n)$.*

2.2 Bernstein functions and subordination in the sense of Bochner

2.2.1 Bernstein functions

Convolution semigroups of measures $(\eta_t)_{t \geqslant 0}$ supported in $[0, \infty)$, i.e. $\operatorname{supp} \eta_t \subset [0, \infty)$, are of particular interest. It turns out that they are better described by their (one-sided) Laplace transforms $\mathcal{L}(\eta_t)$ than by their Fourier transforms.
We need some preparation. Again we refer to the monograph [BeFo75] of C. Berg and G. Forst as a standard reference.

Definition 2.2.1 *An arbitrarily often differentiable function $f : (0, \infty) \to \mathbb{R}$ with continuous extension to $[0, \infty)$ is called a* Bernstein function *if*

$$f \geqslant 0 \quad \textit{and} \quad (-1)^k f^{(k)} \leqslant 0 \quad \textit{for all} \quad k \in \mathbb{N}.$$

Bernstein functions have a representation formula which is analogous to the Lévy - Khinchin formula.

Theorem 2.2.2 *Let f be a Bernstein function. Then there exist constants $a, b \geqslant 0$ and a measure μ on $(0, \infty)$ satisfying*

$$\int_{0+}^{\infty} \frac{r}{1+r} \, \mu(dr) < \infty \tag{2.10}$$

such that

$$f(t) = a + bt + \int_{0+}^{\infty} (1 - e^{-tr}) \, \mu(dr), \quad t > 0. \tag{2.11}$$

The triple (a, b, μ) is uniquely determined by f.
Conversely, given $a, b \geqslant 0$ and a measure μ satisfying (2.10), then (2.11) defines a Bernstein function.

There is a one-to-one correspondence between Bernstein functions and convolution semigroups on $[0, \infty)$.

Theorem 2.2.3 *Let $f : (0, \infty) \to \mathbb{R}$ be a Bernstein function. Then there exists a unique convolution semigroup $(\eta_s)_{s\geqslant 0}$ supported in $[0, \infty)$ such that*

$$\mathcal{L}(\eta_s)(t) = \int_0^\infty e^{-rt}\eta_s(dr) = e^{-sf(t)}, \quad t > 0 \quad \textit{and} \quad s > 0. \tag{2.12}$$

Conversely, for any convolution semigroup $(\eta_s)_{s\geqslant 0}$ supported in $[0, \infty)$ there exists a unique Bernstein function f such that (2.12) *holds.*

It is not difficult to see based on (2.11) every Bernstein function f extends to the half plane $\operatorname{Re} z \geqslant 0$, details are given for example in [Ja01, Section 3.9]. From this one may deduce one of the most important properties of Bernstein functions: they operate on negative definite functions.

Proposition 2.2.4 *For any Bernstein function f and any continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ the function $f \circ \psi$ is again continuous and negative definite.*

In particular we obtain the following important example.

Example 2.2.5 For any Bernstein function f the function $\psi : \mathbb{R}^n \to \mathbb{R}$, $\psi(\xi) = f(|\xi|^2)$ is continuous and negative definite.

Now let ψ and f be as in Proposition 2.2.4. Since $f \circ \psi$ is a continuous negative definite function on $\mathbb{R}^n$, there exists a convolution semigroup $(\mu_s^f)_{s\geqslant 0}$ associated with $f \circ \psi$.

Theorem 2.2.6 *Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous negative definite function with associated convolution semigroup $(\mu_s)_{s\geqslant 0}$ on $\mathbb{R}^n$. Let f be a Bernstein function with associated semigroup $(\eta_s)_{s\geqslant 0}$ supported in $[0, \infty)$.*
The convolution semigroup $(\mu_s^f)_{s\geqslant 0}$ on $\mathbb{R}^n$ associated with the continuous negative definite function $f \circ \psi$ is given by

$$\int_{\mathbb{R}^n} \varphi(x)\, \mu_s^f(dx) = \int_0^\infty \int_{\mathbb{R}^n} \varphi(x)\, \mu_r(dx)\, \eta_s(dr), \quad \varphi \in C_0^\infty(\mathbb{R}^n). \tag{2.13}$$

Remark 2.2.7 Instead of (2.13) one can write

$$\mu_s^f = \int_0^\infty \mu_r\, \eta_s(dr) \quad \text{vaguely.} \tag{2.14}$$

Definition 2.2.8 *In the situation of Theorem* 2.2.6 *we call the convolution semigroup $(\mu_s^f)_{s\geqslant 0}$ the semigroup* subordinate (in the sense of Bochner) *to $(\mu_s)_{s\geqslant 0}$ (with respect to $(\eta_s)_{s\geqslant 0}$). The convolution semigroup $(\eta_s)_{s\geqslant 0}$ is sometimes called a* subordinator.

Example 2.2.9 (i) The function $t \mapsto a$, $a \geqslant 0$, is a Bernstein function as well as the function $t \mapsto bt$, $b \geqslant 0$. The associated semigroups are $(e^{-at}\varepsilon_0)_{t\geqslant 0}$ and $(\varepsilon_{bt})_{t\geqslant 0}$, respectively.

(ii) If $r \geqslant 0$ the function $f(t) = 1 - e^{-rt}$ is a Bernstein function. It corresponds to the *Poisson semigroup* with jumps of size r, i.e.

$$\eta_s = \sum_{k=0}^{\infty} e^{-s} \frac{s^k}{k!} \varepsilon_{rk}, \quad s \geqslant 0.$$

(iii) The function $f(t) = \log(1+t)$ is a Bernstein function. Note that

$$\log(1+t) = \int_0^\infty (1 - e^{-tr}) r^{-1} e^{-r} \, dr, \quad t > 0.$$

The semigroup associated with this Bernstein function is called the Γ- *semigroup* which is given by $\eta_s = g_s(\cdot)\lambda^{(1)}$ where

$$g_s(x) = \chi_{(0,\infty)}(x) \frac{1}{\Gamma(s)} x^{s-1} e^{-x}.$$

Clearly $t \mapsto \frac{1}{2} \log(1+t)$ is also a Bernstein function with corresponding convolution semigroup

$$\eta_s(dr) = \chi_{(0,\infty)}(r) \frac{1}{\Gamma(\frac{s}{2})} r^{\frac{s}{2}-1} e^{-r} \lambda^{(1)}(dr).$$

We call this semigroup the *modified* Γ- *semigroup*. It will become of greater importance later on.

(iv) For $\varrho \in [0,1]$ the function $f_\varrho(t) = t^\varrho$ is a Bernstein function. For $\varrho = 0$ or $\varrho = 1$ this is obvious, for $\varrho \in (0,1)$ we note

$$t^\varrho = \frac{\varrho}{\Gamma(1-\varrho)} \int_0^\infty (1 - e^{-tr}) r^{-\varrho-1} \, dr, \quad t \geqslant 0.$$

The corresponding semigroup is called the *one-sided stable semigroup of order* ϱ and is denoted by $(\sigma_s^\varrho)_{s \geqslant 0}$. Only for $\varrho = \frac{1}{2}$ a closed expression for σ_s^ϱ is known,

$$\sigma_s^{\frac{1}{2}}(dr) = \chi_{(0,\infty)}(r) \frac{1}{\sqrt{4\pi}} s \, r^{-\frac{3}{2}} e^{-\frac{s^2}{2r}} \lambda^{(1)}(dr).$$

(v) For $m > 0$ the function $f(t) = \sqrt{t + m^2} - m$ is a Bernstein function.

Often people are interested in a subclass of Bernstein function, the so-called complete Bernstein functions.

A function $f : (0,\infty) \to \mathbb{R}$ is called a *complete Bernstein function* if f has the representation

$$f(t) = a + bt + \int_{0+}^\infty \frac{t}{r+t} \rho(dr)$$

with a measure ρ satisfying $\int_{0+}^\infty \frac{1}{1+r} \rho(dr) < \infty$.

A complete Bernstein function is itself a Bernstein function.

Example 2.2.10 The following functions are complete Bernstein functions:

$$\begin{aligned}
t^\alpha &= \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \frac{t}{t+r}\, r^{\alpha-1}\, dr, \quad 0<\alpha<1;\\
\frac{t}{t+\lambda} &= \int_0^\infty \frac{t}{t+r}\, \varepsilon_\lambda(dr), \quad \lambda>0;\\
\log(1+t) &= \int_0^\infty \frac{t}{t+r}\, \chi_{[1,\infty)}(r)\, \frac{dr}{r}.\\
\sqrt{t}\,\log(1+\sqrt{t}) &= \frac{1}{2\pi}\int_0^\infty \frac{t}{t+r}\, r^{-1/2}\, \log(1+r)\, dr\\
\sqrt{t}\,(1-\exp(-4\sqrt{t})) &= \frac{2}{\pi}\int_0^\infty \frac{t}{t+r}\, r^{-1/2}(\sin 2r^{1/2})^2\, dr\\
\sqrt{t}\,\log(1+\coth\sqrt{t}) &= \frac{1}{2\pi}\int_0^\infty \frac{t}{t+r}\, r^{1/2}\log(1+\coth r^{1/2})\, dr.
\end{aligned}$$

As a corollary we obtain the following examples.

Example 2.2.11 The functions $f_1(t)=\sqrt{t}\,\log(1+\sqrt{t})$, $f_2(t)=\sqrt{t}\,(1-\exp(-4\sqrt{t}))$, $f_3(t)=\sqrt{t}\,\log(1+\coth\sqrt{t})$, $f_4(t)=\frac{t}{t+\lambda}$ with $\lambda>0$ are further examples of Bernstein functions.

2.2.2 Subordination in the sense of Bochner

We have seen that Bernstein functions can be used in order to obtain new negative definite functions, and thus new convolution semigroups, from a given one.
Let us take now formula (2.14) as our starting point in order to treat subordination of contraction semigroups.
Denote by $(X, \|\cdot\,|\,X\|)$ some Banach space of functions, which will be in later sections $L_p(\mathbb{R}^n)$, $1\leqslant p<\infty$, or $C_\infty(\mathbb{R}^n)$ and let $(T_t)_{t\geqslant 0}$ be a strongly continuous contraction semigroup on X with generator $(A, D(A))$. By $(\eta_t)_{t\geqslant 0}$ we denote a convolution semigroup of sub-probability measures supported in $[0,\infty)$ which is associated with the Bernstein function f of the form (2.11). Define

$$T_t^f u := \int_0^\infty T_s u\, \eta_t(ds), \qquad t\geqslant 0, \quad u\in X, \tag{2.15}$$

where the right-hand side is given by a Bochner integral. Since $\|T_t u\,|\,X\| \leqslant \|u\,|\,X\|$ and since η_t is a sub-probability measure, (2.15) is well-defined. Moreover, it is not hard to see that $(T_t^f)_{t\geqslant 0}$ is a strongly continuous contraction semigroup and that T_t^f is again a sub-Markovian or Feller operator whenever $(T_t)_{t\geqslant 0}$ is.

Definition 2.2.12 *Let $(T_t)_{t\geqslant 0}$ be a strongly continuous contraction semigroup on the Banach space $(X, \|\cdot|, X\|)$ and let $(\eta_t)_{t\geqslant 0}$ be a subordinator with associated Bernstein function f.*
Then the semigroup $(T_t^f)_{t\geqslant 0}$ given by (2.15) *is called* subordinate to $(T_t)_{t\geqslant 0}$ *(with respect to the Bernstein function f). Its generator will be denoted by $(A^f, D(A^f))$.*

The notion of subordination goes essentially back to S. Bochner, see [Bo49] and [Bo55]. Little is known, in general, about $D(A^f)$. Clearly, $D(A^f) = X$ if either A or f is bounded (in this case A^f is bounded). Besides this trivial case,

$$D(A^f) = D(A) \quad \text{if, and only if,} \quad b = \lim_{x\to\infty} \frac{f(x)}{x} \neq 0,$$

cf. [Sc96] and [Sc98a]. More information is available if we restrict ourselves to the class of complete Bernstein functions but we do not go here into details and refer to [FJS01a] and [FJS01b] and the references therein.
In fact subordination of generators generalises the fractional powers $(-A)^\gamma$, $\gamma \geqslant 0$, for generators of strongly continuous contraction semigroups in which we are interested in. For fractional powers we refer also to the book by M. Birman and M. Solomyak, [BiSo87, p.231]. It is known that in this case the Bernstein functional calculus is in accordance with the classical Dunford-Taylor integral but we do not go into details and refer to [Sc96].
Important results in this field are due to U. Westphal [We70a, We70b]. In particular, it should be noted that for $\alpha, \gamma > 0$ we have (as identity for closed operators)

$$(-A)^\alpha \circ (-A)^\gamma = (-A)^{\alpha+\gamma},$$

and this equality has an appropriate extension to $\alpha, \gamma \in \mathbb{R}$. Moreover, there is a generalisation of A. V. Balakrishnan's formula, see [Ba60] or U. Westphal [We70a, We70b], which is stated below.

Theorem 2.2.13 *Let $(A, D(A))$ be the generator of a strongly continuous contraction semigroup $(T_t)_{t\geqslant 0}$ with resolvent $(R_\lambda^A)_{\lambda>0}$ on the Banach space X. For $m < \gamma < m+1$, $m \in \mathbb{N}$, we have for $u \in D(A^{m+1})$*

$$(-A)^\gamma u = \frac{\sin \pi(\gamma - m)}{\pi} \int_0^\infty \lambda^{\gamma-m-1} R_\lambda^A\, A^{m+1} u \, d\lambda.$$

3 Sub-Markovian semigroups and Bessel potential type spaces

3.1 A motivation

Probably the most important equation connecting the theory of Markov processes with functional analysis is given by

$$p_t(x, A) = T_t\, \chi_A(x) = E^x(\chi_A(X_t)). \tag{3.1}$$

Here $(T_t)_{t\geqslant 0}$ is a semigroup of operators on some function space over $\mathbb{R}^n$ (for simplicity), $((X_t)_{t\geqslant 0}, P^x)_{x\in\mathbb{R}^n}$ is a Markov process with state space $\mathbb{R}^n$ and transition function $p_t(x, A)$; χ_A is the characteristic function of the set A. In order to construct a Markov process using the Kolmogorov theorem we have to know the family $p_t(x, A)$ of (sub-)Markovian kernels. One way to construct $p_t(x, A)$ is to start with a given operator semigroup $(T_t)_{t\geqslant 0}$ and to define $p_t(x, A)$ through (3.1). In this case it is natural to use the theory of strongly continuous contraction semigroups on Banach spaces. The direct approach is, of course, a pointwise construction working with continuous functions. This means that we start with a Feller semigroup $(T_t^{(\infty)})_{t\geqslant 0}$, and we have the nice structure theorem for the generator of $(T_t^{(\infty)})_{t\geqslant 0}$ due to Ph. Courrège. However, there are two major drawbacks: in order to obtain non-trivial examples of Feller semigroups, one uses the Hille-Yosida-Ray theorem. This means that one has to solve equations in the Banach space $C_\infty(\mathbb{R}^n)$ which can be quite difficult. Moreover, operators with non-smooth coefficients cannot be treated in general.

M. Fukushima proposed to start with a symmetric L_2-sub-Markovian semigroup $(T_t^{(2)})_{t\geqslant 0}$, i.e., a strongly continuous L_2-contraction semigroup satisfying the sub-Markov property

$$0 \leqslant u \leqslant 1 \quad \text{(a.e.)} \quad \text{implies} \quad 0 \leqslant T_t^{(2)} u \leqslant 1 \quad \text{(a.e.)}.$$

Using the potential theory of the associated quadratic form, the Dirichlet form, it is possible to construct the transition function up to an exceptional set, i.e., a set of capacity zero. This method has the advantage that $L_2(\mathbb{R}^n)$ is a Hilbert space where it is easier to solve equations and thus to construct semigroups using the Hille-Yosida-Ray theorem; moreover, one can treat operators with non-smooth coefficients. A major problem is, of course, the presence of exceptional sets which implies that the constructed process effectively lives on $\mathbb{R}^n$ *less an exceptional set* and that all considerations have to be done modulo this set. This problem can be overcome if we consider L_2-sub-Markovian semigroups $(T_t^{(2)})_{t\geqslant 0}$ with the property that for all bounded and measurable sets A the functions

$$x \mapsto T_t\, \chi_A(x) \tag{3.2}$$

are continuous. Recall the result of E. M. Stein that symmetric sub-Markovian semigroups are analytic, hence

$$T_t\, \chi_A \in \bigcap_{k\geqslant 0} D\left((A^{(2)})^k\right)$$

holds, where $D\left((A^{(2)})^k\right)$ is the domain of the k-th power of the generator $(A^{(2)}, D(A^{(2)}))$ of $(T_t^{(2)})_{t\geqslant 0}$. We may, therefore, establish the continuity of (3.2) for those cases where we can embed the intersection (of some finite number) of domains of powers of $A^{(2)}$ into $C(\mathbb{R}^n)$. Usually, it is quite hard to obtain precise information on $D\left((A^{(2)})^k\right)$ for $k \geqslant 2$ and this requires (in general) higher regularity of the coefficients.

With the Sobolev embedding theorem and the theory of (second order) elliptic differential operators in mind, it might be helpful to pass from the L_2-theory to an L_p-setting, $p > 2$, and to consider operators with domains in some L_p-space such that we may embed these domains into $C(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$.
Thus it would be very natural to start with $-a(x, D)$ as in (0.2) and to prove that it extends under certain conditions on $a(x, \xi)$ to a generator $A^{(p)}$ of an analytic L_p-sub-Markovian semigroup with some nice function space containing $D((-A^{(p)})^r)$ for some r.
In the case $p = 2$ and symmetric operators many concrete examples are known. In the general case several problems arise. First of all there are non-analytic sub-Markovian semigroups. More important however is the fact that a general continuous negative definite function $\xi \mapsto \psi(\xi)$ is neither smooth nor homogeneous implying that standard

L_p-analysis tools such as the Calderon - Zygmund theory of singular integrals or multiplier theorems of Michlin - Hörmander or Lizorkin type do not apply. For this reason we are interested in a theory of L_p-generators for $p \neq 2$.

3.2 Generators of L_p-sub-Markovian semigroups

If $(T_t^{(\infty)})_{t\geqslant 0}$ is a Feller semigroup with generator $(A^{(\infty)}, D(A^{(\infty)}))$ such that $C_0^\infty \subset D(A^{(\infty)}) \subset C_\infty$ then it is well-known that $A^{(\infty)}$ satisfies the *positive maximum principle* From the result of Ph. Courrège we know that on C_0^∞ the operator $A^{(\infty)}$ is a pseudo-differential operator with negative symbol of the form (0.2) or (0.1). Thus the structure of the generators of Feller semigroups is (essentially) known.

In case of a (symmetric) sub-Markovian semigroup $(T_t^{(2)})_{t\geqslant 0}$ on L_2, N. Bouleau and F. Hirsch [BoHi86] showed that its generator $(A^{(2)}, D(A^{(2)}))$ is a Dirichlet operator in the sense that

$$\int_{\mathbb{R}^n} (A^{(2)}u)((u-1)^+)\,dx \leqslant 0$$

holds for all $u \in D(A^{(2)})$.
For non-symmetric sub-Markovian semigroups on L_2 this result is shown in the monograph [MaRö92] by Z.-M. Ma and M. Röckner. However, from the above form we cannot deduce a structure theorem like Courrège's result. The notion of a Dirichlet operator in the context of L_p-sub-Markovian semigroups was introduced by N. Jacob, see [Ja98b, Ja01], where also related and independent results of A. Eberle [Eb98], V. Liskevich and Yu. Semenov [LiSe96], and E. M. Ouhabaz [Ou92, Ou98] are discussed. An operator $A^{(p)}$, defined on $D(A^{(p)}) \subset L_p$, is called an L_p-*Dirichlet operator* if

$$\int_{\mathbb{R}^n} (A^{(p)}u)((u-1)^+)^{p-1}\,dx \leqslant 0$$

holds for all $u \in D(A^{(p)})$.
In [Ja98b] it was proved that if $(A^{(p)}, D(A^{(p)}))$ is an L_p-Dirichlet operator which generates a strongly continuous contraction semigroup $(T_t^{(p)})_{t\geqslant 0}$ on L_p then $(T_t^{(p)})_{t\geqslant 0}$ is sub-Markovian.
But also the converse assertion is true: if $(A^{(p)}, D(A^{(p)}))$ is the generator of a sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ on L_p, then $(A^{(p)}, D(A^{(p)}))$ is an L_p-Dirichlet operator compare [Ja98b].
In [Ja98b], see also [Ja01], it was proved that if an operator $(A^{(\infty)}, D(A^{(\infty)}))$ generates a Feller semigroup and extends to a generator $(A^{(p)}, D(A^{(p)}))$ of a strongly continuous contraction semigroup on L_p, then $A^{(p)}$ is an L_p-Dirichlet operator.

Using an extension result from [FJS01a, Section 1] we conclude that any L_p-Dirichlet operator extends to L_q-Dirichlet operators for all $p < q < \infty$.
Moreover, in [FJS01a, Theorem 2.6] it was proved the following result.

Theorem 3.2.1 *Let $(T_t^{(p)})_{t\geqslant 0}$ be an L_p-sub-Markovian semigroup and denote its L_q-extensions by $(T_t^{(q)})_{t\geqslant 0}$, $p < q < \infty$. Suppose that each of the generators $(A^{(q)}, D(A^{(q)}))$, maps C_0^∞ into C_b. Then $A^{(q)}|_{C_0^\infty}$ satisfies the positive maximum principle and hence, by the theorem of Ph. Courrège it has the structure* (0.2) *or* (0.1)*, respectively.*

Remark 3.2.2 Having Theorem 3.2.1 in mind, it is clear that for constructing L_p-sub-Markovian semigroups one should start with operators defined on C_0^∞ having the structure (0.2) or (0.1) respectively.

For later purposes let us introduce the notion of strong L_p-sub-Markovian semigroups.

Definition 3.2.3 *An L_p-sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ is called a* strong L_p-sub-Markovian semigroup *if each of the operators $T_t^{(p)}$ maps L_p into $L_p \cap C$.*

Suppose that $(T_t^{(p)})_{t\geqslant 0}$ is a strong L_p-sub-Markovian semigroup. In this case for any bounded Borel set $A \subset \mathbb{R}^n$ we find $T_t^{(p)}\chi_A \in C_b$. This observation will be used later on to avoid exceptional sets when constructing Markov processes starting with L_p-semigroups.

3.3 Subordination of second order elliptic differential operators

We want to present here a nice result on subordination which follows from the estimates for elliptic differential operators given by F. Browder in [Bro61]. A detailed treatment can be found in [Ja02], in particular Theorem 6.1.44.

Theorem 3.3.1 *Let*

$$L(x,D) = \sum_{k,l=1}^n a_{kl}(x)\,\frac{\partial^2}{\partial x_k \partial x_l} + \sum_{j=1}^n b_j(x)\,\frac{\partial}{\partial x_j} + c(x)$$

be a uniformly elliptic operator of second order with coefficients $a_{kl} = a_{lk} \in C_b^2$, $b_j \in C_b^1$ and $c \in C_b$. In addition suppose that $c(x) \leqslant 0$ and

$$\sum_{j=1}^n \frac{\partial}{\partial x_j}\left(b_j(x) - \sum_{k=1}^n \frac{\partial}{\partial x_j}\, a_{kj}(x)\right) \geqslant 0.$$

Then $(L(x,D), W_p^2)$ generates an L_p-sub-Markovian semigroup for $1 < p < \infty$.

For convenience we denote the L_p-generator $(L(x,D), W_p^2)$ by $(A^{(p)}, W_p^2)$ and the corresponding L_p-sub-Markovian semigroup by $(T_t^{(p)})_{t\geqslant 0}$. In particular, the semigroup $(T_t^{(p)})_{t\geqslant 0}$ is a strongly continuous contraction semigroup on L_p.
According to H. Triebel, [Tr78, page 91], the operator $-(A^{(p)} - \lambda\,\mathrm{id})$ is for $\lambda > 0$ *positive* in the sense that

$$\|(\kappa\,\mathrm{id} - (A^{(p)} - \lambda\,\mathrm{id}))^{-1}\| \leqslant \frac{c_\lambda}{1+\kappa}, \qquad \kappa > 0,$$

holds. Moreover, $(A^{(p)} - \lambda\,\mathrm{id}, W_p^2)$ is also the generator of an L_p-sub-Markovian semigroup, namely the semigroup $(T_{\lambda,t}^{(p)})_{t\geqslant 0}$ where $T_{\lambda,t}^{(p)} = e^{-\lambda t}\,T_t^{(p)}$. Consequently, $A_\lambda^{(p)} := A^{(p)} - \lambda\,\mathrm{id}$ has a positivity preserving resolvent and $(e^{-\lambda t}T_t^{(p)})_{t\geqslant 0}$ is of negative type since

$$\|e^{-\lambda t}T_t^{(p)}\|_{L_p \to L_p} \leqslant e^{-\lambda t}, \quad t \geqslant 0.$$

From Example 4.7.3.(c) in H. Amann [Am95] we deduce that $A_\lambda^{(p)}$, $1 < p < \infty$, has bounded imaginary powers,

$$\left\|(A_\lambda^{(p)})^{i\kappa}\right\|_{L_p\to L_p} \leqslant c\,(1+\kappa^2)\,e^{\pi|\kappa|/2}, \quad \kappa\in\mathbb{R}.$$

Now we may apply Theorem 1.15.3 from [Tr78] to deduce

Theorem 3.3.2 *Suppose $L(x,D)$ fulfils the assumptions of Theorem 3.3.1 and let $\lambda > 0$ be fixed. Each of the operators $(A_\lambda^{(p)}, W_p^2)$ generates an L_p-sub-Markovian semigroup and the same holds for the fractional powers $-(-A_\lambda^{(p)})^\alpha$, $0<\alpha<1$. The domains of these operators are determined by complex interpolation leading to*

$$D\left((-A_\lambda^{(p)})^\alpha\right) = \left[L_p, W_p^2\right]_\alpha = H_p^{2\alpha}.$$

Moreover, since $0\in\rho(-A_\lambda^{(p)})$ the operator $-(-A_\lambda^{(p)})^{-\beta}$, $0<\beta<1$, maps L_p into $H_p^{2\beta}$.

Denote $\left(T_{\lambda,t}^{(p),\alpha}\right)_{t\geqslant 0}$ the L_p-sub-Markovian semigroup generated by $(-(-A_\lambda^{(p)})^\alpha, H_p^{2\alpha})$. From the results of A. Carasso and T. Kato [CarKa91] it follows that these semigroups are analytic.

Corollary 3.3.3 *Let $L(x,D)$ and $\lambda>0$ be as in Theorem 3.3.2 and $\left(T_{\lambda,t}^{(p),\alpha}\right)_{t\geqslant 0}$ be the L_p-sub-Markovian semigroup generated by $(-(-A_\lambda^{(p)})^\alpha, H_p^{2\alpha})$. Then for all $u\in L_p$ we have*

$$T_{\lambda,t}^{(p),\alpha}u\in H_p^{2\alpha}.$$

In particular, for $p>\frac{n}{2\alpha}$ the semigroup $\left(T_{\lambda,t}^{(p),\alpha}\right)_{t\geqslant 0}$ is strongly L_p-sub-Markovian.

Clearly, $C_0^\infty\subset H_p^{2\alpha}$ for all $1<p<\infty$ and therefore

$$-(-A_\lambda^{(p)})^\alpha u\in H_p^{2(1-\alpha)} \quad \text{for any} \quad u\in C_0^\infty.$$

Thus for $p>\frac{n}{2(1-\alpha)}$ each of the operators $-(-A_\lambda^{(p)})^\alpha$ maps C_0^∞ into C_∞. Therefore we may apply Theorem 3.2.1 implying that for $0<\alpha<1$ the operator $-(-A_\lambda^{(p)})^\alpha|_{C_0^\infty}$ is indeed a pseudo-differential operator

$$a_{\lambda,\alpha}(x,D)u(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\,a_{\lambda,\alpha}(x,\xi)\,\widehat{u}(\xi)\,d\xi$$

with negative definite symbol $a_{\lambda,\alpha}(x,\xi)$. Of course for $a(x,\xi)=\psi(\xi)$ we have $a_{\lambda,\alpha}(x,\xi)=(\psi(\xi)+\lambda)^\alpha$.

3.4 The Γ-transform of L_p-sub-Markovian semigroups

As before, $(T_t^{(p)})_{t\geqslant 0}$ denotes an L_p-sub-Markovian semigroup, $1 < p < \infty$. For $u \in L_p$ and $r > 0$ we define the *gamma-transform* of $(T_t^{(p)})_{t\geqslant 0}$ by

$$V_r^{(p)} u := \frac{1}{\Gamma(\frac{r}{2})} \int_0^\infty t^{\frac{r}{2}-1}\, e^{-t}\, T_t^{(p)} u\, dt.$$

Then the semigroup $(V_r^{(p)})_{r\geqslant 0}$ is an L_p-sub-Markovian semigroup obtained from $(T_t^{(p)})_{t\geqslant 0}$ by subordination in the sense of Bochner. The corresponding Bernstein function is given by $f(s) = \frac{1}{2}\log(1+s)$, and the corresponding convolution semigroup $(\eta_t)_{t\geqslant 0}$ is given in Example 2.2.9.
Thus we have $\|V_r^{(p)} u \,|\, L_p\| \leqslant \|u \,|\, L_p\|$ and $V_{r_1}^{(p)} V_{r_2}^{(p)} = V_{r_1+r_2}^{(p)}$. Moreover, according to a result of A. Carasso and T. Kato, see [CarKa91], the semigroup $(V_r^{(p)})_{r\geqslant 0}$ is always analytic.

Theorem 3.4.1 *Let $(A^{(p)}, D(A^{(p)}))$ be the generator of the L_p-sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$. For all $r > 0$ and all $u \in L_p$ we have*

$$V_r^{(p)} u = (\mathrm{id} - A^{(p)})^{-r/2} u.$$

In particular, each of the operators $V_r^{(p)}$ is injective.

The proof of the above result was given in [FJS01a, Theorem 4.1].
Since each of the operators $V_r^{(p)}$ is injective we may define for $1 < p < \infty$ the spaces

$$\mathcal{F}_{r,p} := V_r^{(p)}(L_p) \quad \text{and} \quad \|u \,|\, \mathcal{F}_{r,p}\| := \|v \,|\, L_p\| \quad \text{for} \quad u = V_r v.$$

Clearly $(\mathcal{F}_{r,p}, \|\cdot\,|\,\mathcal{F}_{r,p}\|)$ is a separable Banach space.
The next result can be found in [FJS01a, Corollary 4.2].

Corollary 3.4.2 *In the situation of Theorem* 3.4.1 *we have* $\mathcal{F}_{r,p} = D((\mathrm{id} - A^{(p)})^{r/2})$.

Remark 3.4.3 In the Hilbert space case, i.e. $p = 2$, and if $(A^{(2)}, D(A^{(2)}))$ is a selfadjoint generator, the results of Theorem 3.4.1 and its corollary are well known (see also comments in the next section) and are proved by the spectral theorem for selfadjoint operators.

3.5 Refinements for analytic L_p-sub-Markovian semigroups

Given an L_p-sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$, $1 < p < \infty$, it is of course possible to define for any Borel set $A \subset \mathbb{R}^n$ with finite Lebesgue measure $\lambda^{(n)}(A) < \infty$ the function $p_t(x, A) = T_t^{(p)} \chi_A(x)$. As an element in L_p, the function $x \mapsto p_t(x, A)$ is only *almost everywhere* determined; it is therefore not possible to use the family $p_t(x, A)$, $t \geqslant 0$, $x \in \mathbb{R}^n$, $A \in \mathcal{B}^n$, in order to construct a Markov process. However, if it would be possible to find for each $A \in \mathcal{B}^n$, with $\lambda^{(n)}(A) < \infty$, and all $t > 0$ a *unique* representative $\widetilde{p}_t(\cdot, A)$ of $x \mapsto p_t(x, A)$ such that the Chapman-Kolmogorov equations

$$\widetilde{p}_{t+s}(x, A) = \int_{\mathbb{R}^n} \widetilde{p}_t(y, A)\, \widetilde{p}_s(x, dy)$$

hold, we could construct a Markov process starting in *every* point $x \in \mathbb{R}^n$. Clearly, if $T_t^{(p)}$ maps for all $t > 0$ the space L_p into $C \cap L_p$ (in the sense that $T_t^{(p)} u$ is considered to be a uniquely determined continuous function), i.e., if $(T_t^{(p)})_{t \geqslant 0}$ is strongly L_p-sub-Markovian in the sense of Definition 3.2.3.B, then we are done.

Another approach is to use capacities and to define the process only up to a set $\mathcal{N}$ of capacity zero in the state space. The drawback of this method is that—unless the set $\mathcal{N}$ is the empty set—the process is only defined on $\mathbb{R} \setminus \mathcal{N}$, i.e., it can only start at points *outside* $\mathcal{N}$. This was the idea of M. Fukushima in [Fu71] where he used Dirichlet forms to construct Hunt processes.

In this section we will first discuss conditions for an L_p-sub-Markovian semigroup to be strongly L_p-sub-Markovian, and then the theory of (r,p)-capacities and its application to analytic L_p-sub-Markovian semigroups.
The concept of (r,p)-capacities was introduced by P. Malliavin in [Mal84], see also [Mal97], and many investigations have been done in the context of sub-Markovian semigroups by M. Fukushima and H. Kaneko, see [Fu92, Fu93], [FuKa85] and [Kan86]. In order to study capacities we need further properties of the spaces $\mathcal{F}_{r,p}$ defined in Section 3.4. We start mentioning that for all $s, r \geqslant 0$ we have $\mathcal{F}_{r+s,p} \subset \mathcal{F}_{r,p}$. For $k \in \mathbb{N}$ the space $\mathcal{F}_{k+2,p}$ is dense in $\mathcal{F}_{k,p}$ as proved in [FJS01a, Lemma 5.1].

Definition 3.5.1 *Let $(T_t^{(p)})_{t \geqslant 0}$ be an L_p-sub-Markovian semigroup and $\mathcal{F}_{r,p}$ as above. We call $\mathcal{F}_{r,p}$* regular *if $\mathcal{F}_{r,p} \cap C$ is dense in $(\mathcal{F}_{r,p}, \| \cdot \mid \mathcal{F}_{r,p}\|)$.*

Proposition 3.5.2 *Let $k \in \mathbb{N}$ and suppose that the set $C \cap D([A^{(p)}]^k)$ is an operator core for $[A^{(p)}]^k$. Then $\mathcal{F}_{2k,p}$ is regular.*

The proof of the above result was given in [FJS01a, Proposition 5.3].

Thus the regularity problem for $\mathcal{F}_{r,p}$ can be reduced to find a good operator core for $A^{(p)}$ or $[A^{(p)}]^k$. Our next theorem gives a first answer when one can find a good version of $p_t(x, A)$.

Theorem 3.5.3 *Let $(T_t^{(p)})_{t \geqslant 0}$ be an analytic L_p-sub-Markovian semigroup with generator $(A^{(p)}, D(A^{(p)}))$. If for some $k_0 \in \mathbb{N}$ the space $D([A^{(p)}]^{k_0})$ is contained in $C \cap L_p$, then all the spaces $\mathcal{F}_{r,p}$, $r > 0$, are regular, and $(T_t^{(p)})_{t \geqslant 0}$ is a strong L_p-sub-Markovian semigroup, i.e., maps L_p into $L_p \cap C$.*

The proof of the above result was given in [FJS01a, Theorem 5.4].

Example 3.5.4 We want to present an application of Theorem 3.5.3. It is well known that many second order elliptic differential operators

$$L(x, D)u(x) = \sum_{k,l=1}^{n} a_{kl}(x) \frac{\partial^2 u(x)}{\partial x_k \partial x_l} + \sum_{j=1}^{n} b_j(x) \frac{\partial u(x)}{\partial x_j} + c(x)u(x)$$

are L_p-Dirichlet operators, see [Ja01], and extend to generators $A^{(p)}$ of analytic L_p-contraction semigroups, see [Lu95]. Under mild regularity assumptions on the coefficients one can prove that the domain $D(A^{(p)})$ of the generator is the Sobolev space W_p^2, see again [Lu95]. By the Sobolev embedding theorem,

$$W_p^2 \hookrightarrow C_\infty \quad \text{for} \quad p > \frac{n}{2},$$

and the analyticity of $(T_t^{(p)})_{t\geqslant 0}$ for $p > \frac{n}{2}$ we have

$$T_t^{(p)} u \in \bigcap_{k\geqslant 0} D((-A^{(p)})^k) \subset W_p^2 \subset C_\infty .$$

This, however, means that $(T_t^{(p)})_{t\geqslant 0}$ is already a strong L_p-sub-Markovian semigroup.

Example 3.5.4 suggests a strategy to find strong L_p-sub-Markovian semigroups: Determine the domain of its generator in terms of function spaces and prove good embedding results for these function spaces.

Clearly one cannot expect every L_p-sub-Markovian semigroup to be strongly L_p-sub-Markov. Therefore we aim to find good representatives of $T_t^{(p)}\chi_A(\cdot)$ on a subset $\mathbb{R}^n \setminus \mathcal{N}$ where $\mathcal{N}$ is negligible in an appropriate sense. This can be achieved by introducing a capacity $\operatorname{cap}_{r,p}$ in each of the spaces $\mathcal{F}_{r,p}$.

Let us recall some results due to M. Fukushima and H. Kaneko. For an open set $G \subset \mathbb{R}^n$ we define the (r,p)-capacity by

$$\operatorname{cap}_{r,p}(G) := \inf \{ \|u \,|\, \mathcal{F}_{r,p}\|^p \,:\, u \in \mathcal{F}_{r,p} \quad \text{and} \quad u \geqslant 1 \text{ a.e. on } G \} .$$

Defining for an arbitrary set $E \subset \mathbb{R}^n$

$$\operatorname{cap}_{r,p}(E) = \inf \left\{ \operatorname{cap}_{r,p}(G) \,:\, E \subset G \quad \text{and} \quad G \quad \text{open} \right\},$$

$\operatorname{cap}_{r,p}$ extends to an outer capacity. The following lemma can be found in [FuKa85].

Lemma 3.5.5 *Let $(T_t^{(p)})_{t\geqslant 0}$ be an L_p-sub-Markovian semigroup.*
A. *For any measurable set $E \subset \mathbb{R}^n$ we have: $\lambda^{(n)}(E) \leqslant \operatorname{cap}_{r,p}(E)$.*
B. *Whenever $E \subseteq F \subset \mathbb{R}^n$, $r \leqslant r'$, or $p \leqslant p'$ then $\operatorname{cap}_{r,p}(E) \leqslant \operatorname{cap}_{r',p'}(F)$.*
C. *For any sequence $(E_j)_{j\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$ we have $\operatorname{cap}_{r,p}\left(\bigcup_{j=1}^{\infty} E_j \right) \leqslant \sum_{j=1}^{\infty} \operatorname{cap}_{r,p}(E_j)$.*

Using $\operatorname{cap}_{r,p}$ we may introduce the concepts of exceptional sets and quasi-continuous functions.

Definition 3.5.6 *Let $(T_t^{(p)})_{t\geqslant 0}$ be an L_p-sub-Markovian semigroup.*
A. *A set $\mathcal{N} \subset \mathbb{R}^n$ satisfying $\operatorname{cap}_{r,p}(\mathcal{N}) = 0$ is called* (r,p)-exceptional *(w.r.t. $(T_t^{(p)})_{t\geqslant 0}$).*
B. *A statement is said to hold* (r,p)-quasi-everywhere *(w.r.t. $(T_t^{(p)})_{t\geqslant 0}$) if there exists an (r,p)-exceptional set $\mathcal{N}$ such that the statement holds on $\mathbb{R}^n \setminus \mathcal{N}$. We will use the abbreviation (r,p)-q.e. for (r,p)-quasi-everywhere.*
C. *A real valued function u defined (r,p)-quasi-everywhere on $\mathbb{R}^n$ is called* (r,p)-quasi-continuous *(w.r.t. $(T_t^{(p)})_{t\geqslant 0}$) if for any $\varepsilon > 0$ there exists an open set $G \subset \mathbb{R}^n$ such that $\operatorname{cap}_{r,p}(G) < \varepsilon$ and $u|_{G^c}$ is continuous.*

The following theorem is again taken from [FuKa85].

Theorem 3.5.7 *Let $(T_t^{(p)})_{t\geqslant 0}$ be an L_p-sub-Markovian semigroup and assume $\mathcal{F}_{r,p}$ is regular.*
A. *If u is (r,p)-quasi-continuous and $u \geqslant 0$ a.e. on an open set G, then $u \geqslant 0$ (r,p)-q.e. on G.*
B. *Each $u \in \mathcal{F}_{r,p}$ admits an (r,p)-quasi-continuous modification denoted by $\widetilde{u}$, and we have*

$$\operatorname{cap}_{r,p}(\{|\widetilde{u}| > \varrho\}) \leqslant \frac{1}{\varrho^p} \|u \,|\, \mathcal{F}_{r,p}\|^p , \quad \varrho > 0.$$

Further we have, see [FuKa85],

Proposition 3.5.8 *For any $A \subset \mathbb{R}^n$ with finite (r,p)-capacity there exists a unique function $e_A \in \{u \in \mathcal{F}_{r,p} : \widetilde{u} \geqslant 1\ (r,p)\text{-q.e. on } A\}$ minimising the norm $\| \cdot \,|\, \mathcal{F}_{r,p}\|$. The function e_A is non-negative and satisfies*

$$\operatorname{cap}_{r,p}(A) = \|e_A \,|\, \mathcal{F}_{r,p}\|^p.$$

For the next results of this section one should note that the semigroup $(T_t^{(p)})_{t\geqslant 0}$ has to be *symmetric* and *analytic*. The next proposition is due to H. Kaneko, see [Kan86].

Proposition 3.5.9 *Let $(T_t^{(p)})_{t\geqslant 0}$ be a symmetric, analytic L_p-sub-Markovian semigroup and suppose that $\mathcal{F}_{r,p}$ is regular. For each $u \in L_p$ we can choose a function $\widetilde{T_t^{(p)}u}$ such that the function $(x,t) \mapsto \widetilde{T_t^{(p)}u}(x)$ has the following properties:*
(i) *For each $t > 0$ the function $x \mapsto \widetilde{T_t^{(p)}u}(x)$ is an (r,p)-quasi-continuous version of $T_t^{(p)}u$. Moreover, for any $\varepsilon > 0$ there exists an open set G independent of t such that $\operatorname{cap}_{r,p}(G) < \varepsilon$ and the functions $x \mapsto \widetilde{T_t^{(p)}u}(x)$ are continuous on $\mathbb{R}^n \backslash G$ for all $t > 0$.*
(ii) *For (r,p)-quasi-every $x \in \mathbb{R}^n$ the function $t \mapsto \widetilde{T_t^{(p)}u}(x)$ is analytic.*

For our purposes it is important to note that Proposition 3.5.9 allows to select a nice representative for the function $p_t(x,B) = T_t^{(p)}\chi_B(x)$. In particular, suppose that we can find a real number r_0 such that $\operatorname{cap}_{r_0,p}(A) = 0$ implies $A = \emptyset$. Then it follows that we have even a *continuous* representative for $x \mapsto T_t^{(p)}u(x)$, $u \in L_p$, and $(T_t^{(p)})_{t\geqslant 0}$ is strongly L_p-sub-Markovian. This proves

Theorem 3.5.10 *Let $(T_t^{(p)})_{t\geqslant 0}$ be a symmetric, analytic L_p-sub-Markovian semigroup and suppose that for some $r_0 > 0$ the space $\mathcal{F}_{r_0,p}$ is regular and that for every $A \subset \mathbb{R}^n$ such that $\operatorname{cap}_{r_0,p}(A) = 0$ it follows that $A = \emptyset$. Then $(T_t^{(p)})_{t\geqslant 0}$ is a strong L_p-sub-Markovian semigroup, i.e., each $T_t^{(p)}$ maps L_p into $L_p \cap C$.*

Remark 3.5.11 We have already remarked that the regularity problem for $\mathcal{F}_{r,p}$ can be solved by characterising these spaces or the spaces $D((-A^{(p)})^k)$ in terms of function spaces. A criterion for the condition

$$\operatorname{cap}_{r,p}(A) = 0 \quad \text{implies} \quad A = \emptyset. \tag{3.3}$$

can also be derived when characterising the spaces $\mathcal{F}_{r,p}$ or the spaces $D((-A^{(p)})^k)$ in terms of function spaces. Thus if $\mathcal{F}_{r,p} \hookrightarrow C_\infty$ for some values of r and p then we have

$$\inf\left\{\operatorname{cap}_{r,p}(E) \,:\, \emptyset \neq E \subset \mathbb{R}^n\right\} > 0,$$

i.e., every statement which holds (r,p)-quasi-everywhere reduces already to a statement which holds everywhere. Therefore we are looking for Sobolev - type embeddings for the spaces $\mathcal{F}_{r,p}$. This will be discussed in the next Subsection.
Conversely, (3.3) already implies the inclusion $\mathcal{F}_{r,p} \subset C_\infty$.

Remark 3.5.12 Combining Proposition 3.5.9 and Corollary 3.4.2 with H. Kaneko's construction of Hunt processes associated to a symmetric, analytic L_p-sub-Markovian semigroup to find that if $D(A^{(p)})$ is regular, then the process is determined up to a $\operatorname{cap}_{2,p}$-exceptional set. In particular, for $p = 2$, i.e., the Dirichlet form situation, it follows that the process is always determined up to an exceptional set determined by the regular domain of the generator, not only up to an exceptional set determined by the domain of the Dirichlet form.

4 L_p-domains of generators of Lévy processes

In this section we recall some results from [FJS01b] and outline possible applications.

4.1 Preliminaries

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a fixed continuous negative definite function with representation

$$\psi(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(x\cdot\xi))\,\nu(dx) \tag{4.1}$$

where ν is a Lévy measure integrating the function $x \mapsto 1 \wedge |x|^2$. For any $R > 0$ we decompose ψ according to $\psi(\xi) = \psi_R(\xi) + \widetilde{\psi}_R(\xi)$ where

$$\psi_R(\xi) := \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos(x\cdot\xi))\,\chi_{B(0,R)}(x)\,\nu(dx) \qquad \text{and} \qquad \widetilde{\psi}_R(\xi) := \psi(\xi) - \psi_R(\xi).$$

Both ψ_R and $\widetilde{\psi}_R$ are continuous and negative definite. Moreover, ψ_R is smooth and polynomially bounded, whereas $\widetilde{\psi}_R$ is just bounded. We define on $\mathcal{S}$ the norm

$$\|u\|_{\psi,R,p} := \|(\mathrm{id} + \psi_R(D))u \mid L_p\|, \quad R > 0, \quad 1 < p < \infty.$$

The following result can be found in [FJS01b].

Theorem 4.1.1 *For $R > 0$ and $S > 0$ the norms $\|\cdot\|_{\psi,R,p}$ and $\|\cdot\|_{\psi,S,p}$ are equivalent and each of these norms is equivalent to the norm $\|(\mathrm{id} + \psi(D))u \mid L_p\|$.*

Further, let us define the space

$$H_p^{\psi,2} := \left\{u \in L_p \,:\, \left\|u \mid H_p^{\psi,2}\right\| < \infty\right\}$$

where

$$\left\|u \mid H_p^{\psi,2}\right\| := \|(\mathrm{id} + \psi(D))u \mid L_p\|.$$

Then we find that $\mathcal{S}$ is dense in $H_p^{\psi,2}$ and, in addition, the next theorem holds.

Theorem 4.1.2 *The generator of the L_p-sub-Markovian semigroup associated with the continuous negative definite function* (4.1) *has as its domain the space $H_p^{\psi,2}$.*

Using Theorem 4.1.2 we may extend $H_p^{\psi,2}$ now to the scale

$$H_p^{\psi,s} := \mathcal{F}_{s,p,\psi}, \quad 0 \leqslant s < \infty,$$

where $\mathcal{F}_{s,p,\psi}$ is the (abstract) Bessel potential space associated with the L_p-generator $-\psi(D)$. Again it is possible to prove that $\mathcal{S}$ is dense in $H_p^{\psi,s}$.
If now $s < 0$ then the space $H_p^{\psi,s}(\mathbb{R}^n)$ is the closure of $\mathcal{S}(\mathbb{R}^n)$ in the norm

$$\big\| u \mid H_p^{\psi,s}(\mathbb{R}^n) \big\| = \big\| \mathcal{F}^{-1}[(1+\psi)^{s/2}\, \widehat{u}] \mid L_p(\mathbb{R}^n) \big\|. \tag{4.2}$$

Thus we have a scale of Bessel potential spaces $H_p^{\psi,s}(\mathbb{R}^n)$, $s \in \mathbb{R}$, for which $\mathcal{S}(\mathbb{R}^n)$ is a dense subset with respect to the norm (4.2).

4.2 Embeddings

For the purposes of this work we have the following important embedding results.

Theorem 4.2.1 *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function as in* (4.1), *$1 < p < \infty$, and $s > 0$. Then $H_p^{\psi,s} \hookrightarrow C_\infty$ if, and only if,*

$$\mathcal{F}^{-1}\left[(1+\psi(\cdot))^{-s/2}\right] \in L_{p'}, \qquad \frac{1}{p} + \frac{1}{p'} = 1. \tag{4.3}$$

Note that (4.3) means that $(1+\psi(\cdot))^{-s/2}$ must be a Fourier multiplier of type (p,∞). Here is a sufficient criterion for (4.3) to hold:

Theorem 4.2.2 *Suppose that $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function with representation* (4.1) *and such that*

$$1 + \psi(\xi) \geqslant c_0\,(1+|\xi|^2)^{r_0}, \qquad \xi \in \mathbb{R}^n, \tag{4.4}$$

holds for some constant $c_0 > 0$ and some $0 < r_0 \leqslant 1$. For $0 < \varepsilon < 1$ and $\frac{1}{2-\varepsilon} < \theta < 1$ we have the following continuous embedding

$$H_p^{\psi,\theta n/r_0} \hookrightarrow C_\infty \qquad \textit{if} \qquad p = p_{\varepsilon,\theta} := \frac{1+\theta\varepsilon}{1+(\varepsilon-1)\theta}.$$

Let us point out by a simple but concrete example that Theorem 4.2.2 gives indeed non-trivial results. Choose $\varepsilon = \frac{1}{2}$, $\theta = \frac{3}{4}$.
If $r_0 = \frac{3}{8}$ and $n = 1$ then $H^{\psi,2}_{\frac{11}{5}}$ is embedded into C_∞.
If $r_0 = \frac{3}{4}$ and $n = 2$ then $H^{\psi,2}_{\frac{11}{5}}$ is embedded into C_∞.

One should note that the representation (4.1) excludes the continuous negative definite function $\xi \mapsto |\xi|^2$. However, this function leads to the classical Bessel potential spaces H_p^s, see e.g. [St70b], and these spaces are well understood. Observe that in this case the homogeneity of $\xi \mapsto |\xi|^2$ leads to sharper results. For example, we have the embedding

$$H_p^s \hookrightarrow C_\infty \quad \text{for} \quad s > \frac{n}{p}.$$

Remark 4.2.3 **(i)** Note that if $\psi(\xi) = |\xi|^2$ or $\psi(\xi) = q(\xi)$ is a non-degenerate, positive definite, symmetric quadratic form, then $H_p^{\psi,2}(\mathbb{R}^n)$ is the classical Sobolev space $W_p^2(\mathbb{R}^n) = H_p^2(\mathbb{R}^n)$.
Moreover, for the continuous negative definite function $\xi \mapsto (1+|\cdot|^2)^{r/2}$, $0 < r < 2$, we obtain the classical Bessel potential space $H_p^r(\mathbb{R}^n)$.
(ii) Note also that if $\psi(\xi) = |\xi|^{2\varrho}$, with $0 < \varrho \leqslant 1$ then the ψ-Bessel potential space $H_p^{\psi,s}$ is the classical one $H_p^{s\varrho}$.
Moreover, if $\psi(\xi) = \sqrt{|\xi|^2 + m^2} - m$, with $m > 0$, then $H_p^{\psi,s}$ is a again a classical space, namely $H_p^{s/2}$.
(iii) If $\psi(\xi) = |\xi_1|^{2/a_1} + ... + |\xi_n|^{2/a_n}$ where a_1,...,$a_n \geqslant 1$, cf. Example 2.1.12, then $H_p^{\psi,2}$ is the (classical) anisotropic Bessel potential space (of order 2), see [Ni77] and [ScTr87, Section 4.2.2].
(iv) For $p = 2$ the spaces $H_2^{\psi,2}(\mathbb{R}^n)$ are Hilbert spaces and are denoted $H^{\psi,2}(\mathbb{R}^n)$. The spaces $H^{\psi,2}(\mathbb{R}^n)$ coincide with the analogue of the Hörmander - $B_{k,p}$ - spaces; these spaces are denoted by $B^2_{\psi,2}(\mathbb{R}^n)$ and are discussed in detail in [Ja01, Section 4.10].

Remark 4.2.4 Let ψ be a continuous negative definite function of the form (4.1) and let $q(\xi)$ be a non-degenerate, positive definite, symmetric quadratic form. Then $\Psi := q+\psi$ is again continuous negative definite, and we have $H_p^{\Psi,2}(\mathbb{R}^n) = H_p^2(\mathbb{R}^n) = W_p^2(\mathbb{R}^n)$ as proved in [FJS01b, Remark 2.1.7].

Because of this remark we can without loss of generality restrict ourselves to negative definite functions without quadratic part. We will do so if this helps to avoid cumbersome notation.

4.3 Capacities and quasi-continuous modifications

Let us, finally, consider very briefly the question of comparability of (r,p)-capacities associated with different continuous negative definite functions $\psi_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2$. For the case $p = 2$ and $r = 1$, i.e. the symmetric Dirichlet space situation, such a comparison result is due to J. Hawkes [Ha79]. His theorem reads as follows (stated in our context).

Theorem 4.3.1 *Let $\psi_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2$, be two continuous negative definite functions such that for all $\xi \in \mathbb{R}^n$*

$$\frac{1}{M}\,\frac{1}{\lambda + \psi_1(\xi)} \leqslant \frac{1}{\lambda + \psi_2(\xi)}$$

holds for some $M > 0$ and some $\lambda > 0$. Then we have

$$\frac{1}{4}\,M\ \mathrm{cap}_{1,2}^{\psi_1}(A) \leqslant \mathrm{cap}_{1,2}^{\psi_2}(A)$$

for all analytic sets $A \subset \mathbb{R}^n$.

In [FJS01b, Theorem 2.5.12] we obtained a comparison result for capacities from embedding results for the spaces $H_{p_1}^{\psi_1,r_1}(\mathbb{R}^n)$ and $H_{p_2}^{\psi_2,r_2}(\mathbb{R}^n)$. We state it below.

Corollary 4.3.2 *Let $\psi_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2$, be two continuous negative definite functions and $0 < r_1, r_2 < \infty$, $1 < p_1, p_2 < \infty$. If the space $H_{p_2}^{\psi_2,r_2}(\mathbb{R}^n)$ is continuously embedded in the space $H_{p_1}^{\psi_1,r_1}(\mathbb{R}^n)$ and the estimate*

$$\left\|u \mid H_{p_1}^{\psi_1,r_1}(\mathbb{R}^n)\right\| \leqslant c \left\|u \mid H_{p_2}^{\psi_2,r_2}(\mathbb{R}^n)\right\| \tag{4.5}$$

holds, then for all analytic sets $A \subset \mathbb{R}^n$:

$$\operatorname{cap}_{r_1,p_1}^{\psi_1}(A) \leqslant c \operatorname{cap}_{r_2,p_2}^{\psi_2}(A).$$

Note that in [FJS01b, Section 2.3] we gave some conditions in order that (4.5) holds. An immediate consequence of Theorem 4.2.1 is

Corollary 4.3.3 *If $\mathcal{F}^{-1}\left[(1+\psi)^{-r/2}\right] \in L_{p'}(\mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{p'} = 1$, then $\operatorname{cap}_{r,p}(A) = 0$ implies $A = \emptyset$.*

The above Corollary is useful in the context described in Remark 3.5.11.

Some conclusions and an outlook

Our approach described in this chapter, which is in fact that one from [FJS01a] and [FJS01b], can be summed up in the following way:

Let $(T_t^{(p)})_{t\geqslant 0}$ be a given L_p-sub-Markovian semigroup with L_q-extensions $(T_t^{(q)})_{t\geqslant 0}$, $p < q < \infty$, and assume that $C_0^\infty(\mathbb{R}^n) \subset \bigcap_{q\geqslant p} D(A^{(q)})$. If each operator $A^{(q)}$ maps $C_0^\infty(\mathbb{R}^n)$ into $L_q(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, then $A^{(p)}$ (and each $A^{(q)}$) restricted to $C_0^\infty(\mathbb{R}^n)$ is a pseudo-differential operator with negative definite symbol, i.e.,

$$A^{(p)}|_{C_0^\infty(\mathbb{R}^n)} u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\, \widehat{u}(\xi)\, d\xi = -a(x,D)u(x), \tag{4.6}$$

where $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a continuous and negative definite in ξ. If, in addition, the semigroup $(T_t^{(p)})_{t\geqslant 0}$ is analytic and if for some $r > 0$ the space $D(-(-A^{(p)})^r)$ is contained in a space of continuous functions, then $(T_t^{(p)})_{t\geqslant 0}$ is a strong L_p-sub-Markovian semigroup.

Despite the lack (up to now) of a general L_p-theory for the pseudo-differential operators (4.6) our discussion in the previous sections shows the following:

- there is a good applicability of the L_p-theory to operators obtained from given generators by standard constructions such as subordination or perturbation;
- there is a natural limit in the (r,p)-capacity refinements of L_p-sub-Markovian semigroups, namely the strong L_p-sub-Markovian semigroups;
- the determination of domains in terms of concrete function spaces is the key to get concrete refinement results.

Let us conclude this chapter indicating how one can use the above mentioned results provided a good L_p-theory for pseudo-differential operators $a(x,D)$ does exist. For this, suppose that $-a(x,D)$ extends to a generator of an L_p-sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ with domain $H_p^{\psi,2}$ where ψ is a fixed continuous negative definite function.

If, in addition, ψ satisfies the assumptions of Theorem 4.2.1, or the more concrete assumptions of Theorem 4.2.2, and if the semigroup $(T_t^{(p)})_{t\geqslant 0}$ is analytic, then $(T_t^{(p)})_{t\geqslant 0}$ is a strong L_p-sub-Markovian semigroup and we may associate with $-a(x,D)$ a Hunt process *without any exceptional set*.

We want however to mention that it has been done much work for parabolic pseudo-differential operators with symbols in some *classical* symbol classes, for example for operators of type $\frac{\partial}{\partial t} - a(x,D)$, with $a \in S^m_{\rho,\delta}$ (we will return to the Hörmander symbol class $S^m_{\rho,\delta}$ later on).
Clearly, under the assumption that $a(x,\cdot) : \mathbb{R}^n \to \mathbb{C}$ is a continuous negative definite function and $q \in S^m_{\rho,\delta}$, we can apply the existing parabolic theory of pseudo-differential operators to construct sub-Markovian or Feller semigroups and processes. For this we refer to the monographs of H. Kumano-go [Ku74] and that of G. Grubb [Gr86] as well as to the survey of S. D. Ejdel'man [Ej94]. The simplest example is the operator $(1-\Delta)^t$, $0 < t < 1$, which is an L_p-Dirichlet operator with domain H_p^{2t}; obviously, $\xi \mapsto (1+|\xi|^2)^t$ is a continuous negative definite function.

Chapter II. Admissible continuous negative definite functions and associated Bessel potential spaces

5 Introduction to Chapter II

In the previous chapter we discussed ψ-Bessel potential spaces as domains of generators for L_p-sub-Markovian semigroups and in particular we have seen that it is important to have Sobolev type embeddings for the ψ-Bessel potential spaces in order to construct strong L_p-sub-Markovian semigroups.

Due to the complicated structure of a general continuous negative definite functions (which is, in general, non-smooth, non-isotropic and non-homogeneous) it is quite difficult to handle ψ-Bessel potential spaces and in particular it is difficult to obtain necessary and sufficient conditions for those Sobolev embeddings, compare Theorem 4.2.1 and Theorem 4.2.2.

Recall formula (2.3) for a a general real-valued continuous negative definite function. In this chapter we pay a special attention on continuous negative definite functions of the form $\xi \mapsto f(|\xi|^2)$ where f is a Bernstein function. These functions are not only smooth outside the origin but they have additional properties like a hypoellipticity type property.

In **Section 6** we introduce and consider a subclass of the real valued continuous negative definite functions. We call these functions, modeled after functions of type $f(1+|\cdot|^2)$ (f Bernstein), admissible continuous negative definite functions. They are C^∞ functions but have some additional properties, see Definition 6.1.1 for the precise formulation. These additional properties turn out to be very useful in the context of identification of the associated ψ-Bessel potential spaces with spaces of generalised smoothness of type $F^{\sigma,N}_{p,2}$ and in the context of obtaining Sobolev type embeddings.

To each admissible function ψ we will associate a sequence of non-negative numbers by the formula $N_j = \sup\{\langle\xi\rangle \,:\, \psi(\xi) \leqslant 2^{2j}\}$ for any $j \in \mathbb{N}_0$. It turns out that there exists a $\lambda_0 > 1$ such that this sequence satisfies $\lambda_0\, N_j \leqslant N_{j+1}$ for any $j \in \mathbb{N}_0$, compare Lemma 6.2.2. This is a special case of a so-called strongly increasing sequence (compare the next chapter, and in particular Section 9.1).

The associated sequence $N = (N_j)_{j\in\mathbb{N}_0}$ to an admissible continuous negative definite function allows us in **Section 7** to show that for any $u \in H^{\psi,s}_p$ its norm is equivalent to $\big\|(2^{js}\,\varphi^N_j(D)u)_{j\in\mathbb{N}_0} \,|\, L_p(l_2)\big\|$ where $(\varphi^N_j)_{j\in\mathbb{N}_0}$ is a smooth partition of unity associated in a "canonical way" to the sequence N. The precise formulation is given in Corollary 7.1.4.

This equivalence allows us to use and embedding result of G. A. Kalyabin from [Ka81] and to obtain in Corollary 7.2.5 a Sobolev type embedding for $H^{\psi,s}$ with ψ admissible.

6 Admissible continuous negative definite functions and associated sequences

6.1 The class Ψ: Definition and examples

Recall that for $\xi \in \mathbb{R}^n$ we use the notation $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

Definition 6.1.1 *Let Ψ be the class of all continuous negative definite functions $\psi : \mathbb{R}^n \to \mathbb{R}$ of class C^∞ with the following properties:*
(i) *ψ is radial symmetric and $\lim_{|\xi|\to\infty} \psi(\xi) = \infty$;*
(ii) *ψ is increasing in $|\xi|$, i.e. if $|\xi| \leqslant |\eta|$ then $\psi(\xi) \leqslant \psi(\eta)$;*
(iii) *there exists a number $w > 0$ such that $\xi \mapsto \psi(\xi)\langle \xi \rangle^{-2w}$ is decreasing in $|\xi|$, i.e.*

$$\textit{if} \quad |\xi| \leqslant |\eta| \quad \textit{then} \quad \frac{\psi(\xi)}{\langle \xi \rangle^{2w}} \geqslant \frac{\psi(\eta)}{\langle \eta \rangle^{2w}} \quad ;$$

(iv) *for every multi-index $\alpha \in \mathbb{N}_0^n$ there exists some $c_\alpha > 0$ such that*

$$|D^\alpha \psi(\xi)| \leqslant c_\alpha \, \psi(\xi) \, \langle \xi \rangle^{-|\alpha|} \quad \textit{if} \quad \xi \in \mathbb{R}^n. \tag{6.1}$$

We call the functions ψ from Ψ admissible continuous negative definite functions.

Remark 6.1.2 Condition (iii) from Definition 6.1.1 implies that $\psi(\xi) \leqslant c \, \langle \xi \rangle^{2w}$ for large ξ and $\psi(\xi) \leqslant C \, (1 + |\xi|^2)^w$ for some $C > 0$ and for any $\xi \in \mathbb{R}^n$.
Moreover, since any continuous negative definite function has at most quadratic growth, it follows that the number w from the above definition satisfies $w \leqslant 1$.

Clearly the function $\xi \mapsto 1 + |\xi|^2$ is admissible, the number w in the above Definition can be taken 1.

Remark 6.1.3 Recall formula (2.3) for a general real-valued continuous negative definite function

$$\psi(\xi) = c + q(\xi) + \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(x \cdot \xi)) \, \nu(dx)$$

with c, q, and ν as in Theorem 2.1.4.
One might be tempted to consider the requirement that any admissible function ψ is of class C^∞ too restrictive. This is not the case: based on Corollary 2.1.7 one can obtain continuous negative definite functions which are infinitely differentiable if the Lévy measure ν is supported in a bounded neighbourhood of the origin. Note that this is a reasonable assumption at least because in probability this assumption is often made from the very beginning and corresponds to the fact that the jumps of the associated process are bounded.

In introducing the class Ψ, we had in mind of course negative definite functions of the form $\xi \mapsto f(|\xi|^2)$ where f is a Bernstein function with $\lim_{t\to\infty} f(t) = \infty$. This especially due to a result of W. Hoh proved in [Ho98a] where he showed that if f is a Bernstein function then for any $j \in \mathbb{N}$

$$|f^{(j)}(t)| \leqslant \frac{j!}{t^j} f(t) \quad , \quad t > 0. \tag{6.2}$$

In particular, for $j=1$ we have

$$0 \leqslant f'(t) \leqslant \frac{f(t)}{t} \quad \text{for} \quad t > 0. \tag{6.3}$$

and this implies the fact that $t \mapsto \frac{f(t)}{t}$ is decreasing.
Consequently, we get the important

Lemma 6.1.4 *If f is a Bernstein function and $\psi : \mathbb{R}^n \to (0,\infty)$ is a continuous negative definite function satisfying* (6.1) *then $b(\xi) = f(\psi(\xi))$ satisfies* (6.1) *too.*

Proof. To show (6.1) for the function b, let us recall that for the arbitrarily often differentiable functions $f : (0,\infty) \to (0,\infty)$ and $\psi : \mathbb{R}^n \to (0,\infty)$ and for any $\alpha \in \mathbb{N}_0^n$ one has

$$\partial^\alpha (f \circ \psi) = \sum_{j=1}^{|\alpha|} f^{(j)}(\psi(\cdot)) \sum \frac{\alpha!}{\delta_\beta! \delta_\gamma! \cdot \ldots \cdot \delta_\omega!} \Big(\frac{\partial^\beta \psi(\cdot)}{\beta!}\Big)^{\delta_\beta} \cdot \ldots \cdot \Big(\frac{\partial^\omega \psi(\cdot)}{\omega!}\Big)^{\delta_\omega} \tag{6.4}$$

the second sum being taken over all pairwise different multi-indices $0 \neq \beta, \gamma, \ldots, \omega \in \mathbb{N}_0^n$ and all $\delta_\beta, \delta_\gamma, \ldots, \delta_\omega \in \mathbb{N}$ such that $\delta_\beta \beta + \delta_\gamma \gamma + \ldots + \delta_\omega \omega = \alpha$ and $\delta_\beta + \delta_\gamma + \ldots + \delta_\omega = j$. Using (6.2) and the fact that ψ satisfies (6.1) we get for any $\alpha \in \mathbb{N}_0^n$ and any ξ

$$\begin{aligned} |D^\alpha b(\xi)| &\leqslant \sum_{j=1}^{|\alpha|} \frac{j!}{\psi(\xi)^j} f(\psi(\xi)) \sum \frac{\alpha!}{\delta_\beta! \delta_\gamma! \cdot \ldots \cdot \delta_\omega!} \Big(\frac{\partial^\beta \psi(\xi)}{\beta!}\Big)^{\delta_\beta} \cdot \ldots \cdot \Big(\frac{\partial^\omega \psi(\xi)}{\omega!}\Big)^{\delta_\omega} \\ &\leqslant c_\alpha \sum_{j=1}^{|\alpha|} \frac{j!}{\psi(\xi)^j} f(\psi(\xi)) \prod_\beta \psi(\xi)^{\delta_\beta} \langle \xi \rangle^{-\delta_\beta |\beta|} \\ &\leqslant c'_\alpha \, f(\psi(\xi)) \, (1+|\xi|^2)^{-\frac{|\alpha|}{2}} \end{aligned}$$

and this completes the proof. ■

From the above Lemma, using the monotonicity of Bernstein functions, and property (6.3) we get:

Corollary 6.1.5 *For any Bernstein function f with $\lim_{t\to\infty} f(t) = \infty$ the function $b(\xi) = f(1+|\xi|^2)$ is admissible in the sense of Definition* 6.1.1*; in particular one has $w = 1$ in* (iii)*.*

Remark 6.1.6 Given a Bernstein function with $\lim_{t\to\infty} f(t) = \infty$ then $\xi \mapsto f(|\xi|^2)$ is a C^∞ function outside of the origin. Moreover, inequality (6.1) is satisfied for $\xi \neq 0$. To avoid the "battle" with the differentiability in the origin we prefer to study functions $f(1 + |\cdot|^2)$.

As a simple consequence we obtain, compare Example 2.2.9 and Example 2.2.11, the following examples.

Example 6.1.7 The functions

$$\begin{aligned}
\psi(\xi) &= (1+|\xi|^2)^{\varrho} \quad , \quad \varrho\in[0,1],\\
\psi(\xi) &= \log(2+|\xi|^2),\\
\psi(\xi) &= \sqrt{1+|\xi|^2+m^2}-m \quad , \quad \text{for some} \quad m\geqslant 0,\\
\psi(\xi) &= \langle\xi\rangle \log(1+\langle\xi\rangle),\\
\psi(\xi) &= \langle\xi\rangle\,(1-\exp(-4\,\langle\xi\rangle)),\\
\psi(\xi) &= \langle\xi\rangle \log(1+\coth\langle\xi\rangle),
\end{aligned}$$

are admissible continuous negative definite functions.

Note that if f, g are two Bernstein functions then obviously $f\circ g$ is also a Bernstein function.
Consequently, using Corollary 6.1.5 we obtain many non-trivial examples of admissible ψ of the form $f(1+|\xi|^2)$, with f a Bernstein function satisfying $\lim\limits_{t\to\infty} f(t)=\infty$.

6.2 Associated sequences and a Littlewood-Paley type theorem

Definition 6.2.1 *For a function $\psi\in\Psi$ and for an $r>0$ let*

$$N_j^{\psi,r} = \sup\{\langle\xi\rangle \,:\, \psi(\xi)\leqslant 2^{jr}\} \quad , \quad \textit{for any} \quad j\in\mathbb{N}_0. \tag{6.5}$$

We will call $N^{\psi,r}=(N_j^{\psi,r})_{j\in\mathbb{N}_0}$ the sequence associated to ψ and to r.

It is clear from (6.5) that $N^{\psi,r}=(N_j^{\psi,r})_{j\in\mathbb{N}_0}$ is an increasing sequence. The following lemma gives additional information on the sequence $N^{\psi,r}$.

Lemma 6.2.2 *For a function $\psi\in\Psi$ and for an $r>0$ let $N^{\psi,r}=(N_j^{\psi,r})_{j\in\mathbb{N}_0}$ the sequence associated to ψ and to r. Then*
(i) *for any $j\in\mathbb{N}_0$ there exists a $\xi^j\in\mathbb{R}^n$ such that $\langle\xi^j\rangle = N_j^{\psi,r}$ and $\psi(\xi^j)=2^{jr}$.*
(ii) *there exists a $\lambda_0>1$ such that $\lambda_0\, N_j^{\psi,r}\leqslant N_{j+1}^{\psi,r}$ for any $j\in\mathbb{N}_0$.*

Proof. For simplicity let us denote $N_j^{\psi,r}=N_j$ for any $j\in\mathbb{N}_0$.
(i) Due to the continuity of ψ, it is clear that the supremum in (6.5) is attained. So there exists a $\xi^j\in\mathbb{R}^n$ such that $\langle\xi^j\rangle=N_j$.
If we would have $\psi(\xi^j)<2^{jr}$, then we will arrive at a contradiction. Indeed, due to the properties of the function ψ it is clear that the mapping $t\mapsto\psi(t\xi^j)$ is a one-dimensional continuous function with $\lim\limits_{t\to\infty}\psi(t\xi^j)=\infty$. Consequently, there exists a $t_0>1$ with

$$\psi(t_0\xi^j)=2^{jr}.$$

Then taking $\eta^j = t_0\xi^j$ one has on the one hand $|\eta^j| = t_0|\xi^j| > |\xi^j|$, $\langle\eta^j\rangle > \langle\xi^j\rangle$, $\psi(\eta^j)=2^{jr}$ and on the other hand

$$\langle\eta^j\rangle \leqslant N_j = \sup\{\langle\eta\rangle \,:\, \psi(\eta)\leqslant 2^{jr}\} = \langle\xi^j\rangle$$

which is impossible. Hence $\langle \xi^j \rangle = N_j$ and $\psi(\xi^j) = 2^{jr}$.

(ii) According to part (i) let ξ^j and $\xi^{j+1} \in \mathbb{R}^n$ such that $\langle \xi^j \rangle = N_j$, $\psi(\xi^j) = 2^{jr}$ and $\langle \xi^{j+1} \rangle = N_{j+1}$, $\psi(\xi^{j+1}) = 2^{(j+1)r}$.
Applying now property (iii) from Definition 6.1.1 we have

$$\frac{2^{jr}}{N_j^{2w}} = \frac{\psi(\xi^j)}{\langle \xi^j \rangle^{2w}} \geqslant \frac{\psi(\xi^{j+1})}{\langle \xi^{j+1} \rangle^{2w}} = \frac{2^{(j+1)r}}{N_{j+1}^{2w}}.$$

Consequently, taking $\lambda_0 = 2^{\frac{r}{2w}}$ we get $N_{j+1} \geqslant \lambda_0 \, N_j, \quad j \in \mathbb{N}_0$. ■

A sequence $(N_j)_{j \in \mathbb{N}_0}$ of non-negative real numbers for which there exists a $\lambda_0 > 1$ such that $\lambda_0 \, N_j \leqslant N_{j+1}$ for any $j \in \mathbb{N}_0$ is a special case of a so-called strongly increasing sequence.
Extended considerations on strongly increasing sequences will be made in Section 9.

For a fixed admissible continuous negative definite function ψ and for an $r > 0$ consider the strongly increasing sequence $(N_j^{\psi,r})_{j \in \mathbb{N}_0}$ associated to ψ and r as in (6.5). For simplicity denote $N_j^{\psi,r} = N_j$ for $j \in \mathbb{N}_0$.

Let $g \in C_0^\infty(\mathbb{R})$ with $g(t) = 1$ if $|t| \leqslant 1$ and $\operatorname{supp} g \subset \{t \in \mathbb{R} : |t| \leqslant 2\}$. Let

$$\varphi_0^N(\xi) = g(N_0^{-1}|\xi|)$$

and

$$\varphi_j^N(\xi) = g(N_j^{-1}|\xi|) - g(N_{j-1}^{-1}|\xi|) \quad \text{for any} \quad j \geqslant 1.$$

Then it is easy to see that the system $\varphi^N = (\varphi_j^N)_{j \in \mathbb{N}_0}$ satisfies the following four properties

$$\varphi_j^N \in C_0^\infty(\mathbb{R}^n) \quad \text{and} \quad \varphi_j^N(\xi) \geqslant 0 \quad \text{if} \quad \xi \in \mathbb{R}^n \quad \text{for any} \quad j \in \mathbb{N}_0\,; \tag{6.6}$$

$$\begin{cases} \operatorname{supp} \varphi_0^N \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 2\,N_0\} \\ \operatorname{supp} \varphi_j^N \subset \{\xi \in \mathbb{R}^n : N_{j-1} \leqslant |\xi| \leqslant N_{j+1}\} \quad \text{if} \quad j \geqslant 1; \end{cases} \tag{6.7}$$

for any $\gamma \in \mathbb{N}_0^n$ there exists a constant $c_\gamma > 0$ such that for any $j \in \mathbb{N}_0$

$$|D^\gamma \varphi_j^N(\xi)| \leqslant c_\gamma \, \langle \xi \rangle^{-\gamma} \quad \text{for any} \quad \xi \in \mathbb{R}^n; \tag{6.8}$$

and

$$\sum_{j=0}^{\infty} \varphi_j^N(\xi) = 1 \quad \text{for any} \quad \xi \in \mathbb{R}^n. \tag{6.9}$$

The following result is a Littlewood-Paley type theorem.
Let $N = (N_j)_{j \in \mathbb{N}_0}$ be a sequence such that for some $\lambda_0 > 1$ one has $\lambda_0 \, N_j \leqslant N_{j+1}$ for any $j \in \mathbb{N}_0$. Moreover, let $(\varphi_j^N)_{j \in \mathbb{N}_0}$ a sequence of functions fulfilling (6.6)-(6.9).
Let $F_{p,2}^{\sigma^0,N}$ be the collection of all tempered distributions u such that

$$\left\| u \,|\, F_{p,2}^{\sigma^0,N} \right\| = \left\| (\varphi_j^N(D)u)_{j \in \mathbb{N}_0} \,|\, L_p(l_2) \right\| < \infty.$$

Theorem 6.2.3 *Let* $1 < p < \infty$*. Then there exist two constants* c_1*,* $c_2 > 0$ *such that*

$$c_1 \, \|u \,|\, L_p\| \leqslant \left\| u \,|\, F_{p,2}^{\sigma^0,N} \right\| \leqslant c_2 \, \|u \,|\, L_p\|$$

for any $u \in L_p$*.*

Proof. The proof is similar to that from [Tr83, Theorem 2.5.6] and it is based on the first part of Proposition 0.2.1 so that we will only sketch it.
If $u \in L_p$ we take for any $k \in \mathbb{N}_0$ the function $m_{k,0} = \varphi_k^N$ and $m_{k,j} = 0$ if $j \geqslant 1$. We apply (0.10) with $f_0 = u$ and $f_j = 0$ if $j \geqslant 1$ and get $\|u \,|\, F_{p,2}^{\sigma^0,N}\| \leqslant c \, \|u \,|\, L_p\|$.
To prove the second inequality, let for any $k \in \mathbb{N}_0$

$$\psi_k^N(\xi) = \sum_{r=-3}^{3} \varphi_{k+r}^N(\xi) \quad \text{with} \quad \varphi_{-3}^N = \varphi_{-2}^N = \varphi_{-1}^N = 0.$$

Clearly $\psi_k^N(\xi) = 1$ if $\xi \in \operatorname{supp} \varphi_k^N$. Taking for any $j \in \mathbb{N}_0$ the function $m_{0,j} = \psi_j^N$ and $m_{k,j} = 0$ if $k \geqslant 1$ we apply (0.10) with $f_j = \varphi_j^N(D)u$ and get

$$\begin{aligned} \|u \,| L_p\| &= \|(\delta_{k,0} \sum_{j=0}^{\infty} \psi_j^N(D) \, \varphi_j^N(D) \, u)_{k \in \mathbb{N}_0} \,|\, L_p(l_2)\| \\ &\leqslant c \, \|(\varphi_j^N(D)u)_{j \in \mathbb{N}_0} \,|\, L_p(l_2)\| \end{aligned}$$

and consequently $\|f \,|\, L_p\| \leqslant c \, \|f \,|\, F_{p,2}^{\sigma^0,N}\|$ which proves the second inequality. ■

7 Special properties of ψ-Bessel potential spaces for admissible ψ

7.1 An equivalent norm

In what follows we will consider ψ again an admissible continuous negative function.

Remark 7.1.1 It is easy to see that if $\varphi^N = (\varphi_j^N)_{j \in \mathbb{N}_0}$ is a system of functions that fulfills (6.6)-(6.8) then for any multi-index α there is a constant $c_\alpha > 0$ such that

$$\sum_{j=0}^{\infty} |D^\alpha \varphi_j^N(\xi)| \leqslant c_\alpha \, \langle \xi \rangle^{-|\alpha|} \quad , \quad \text{for any} \quad \xi \in \mathbb{R}^n.$$

In particular, the last inequality implies

$$\sup \left(R^{2|\alpha|-n} \int\limits_{\frac{R}{2} \leqslant |\xi| \leqslant 2R} \sum_{j=0}^{\infty} |D^\alpha \varphi_j^N(\xi)|^2 \, d\xi \right)^{1/2} < \infty$$

where the supremum is taken over all $R > 0$ and all $\alpha \in \mathbb{N}_0^n$ with $0 \leqslant |\alpha| \leqslant 1 + \left[\frac{n}{2}\right]$.

Lemma 7.1.2 *If $\psi : \mathbb{R}^n \to [0,\infty)$ is a function which satisfies* (6.1) *then for any real number μ the function $b(\xi) = (1+\psi(\xi))^\mu$ satisfies also* (6.1).

Proof. Applying (6.4) with $f(t) = t^\mu$ and $1+\psi(\cdot)$ instead of $\psi(\cdot)$, and using the assumption on ψ, we get for any $\alpha \in \mathbb{N}_0^n$ and any ξ

$$\begin{aligned}
|D^\alpha b(\xi)| &\leqslant \sum_{j=1}^{|\alpha|} c_j\,(1+\psi(\xi))^{\mu-j} \\
&\quad \times \sum \frac{\alpha!}{\delta_\beta!\delta_\gamma!\cdot\ldots\cdot\delta_\omega!}\Big(\frac{\partial^\beta(1+\psi(\xi))}{\beta!}\Big)^{\delta_\beta}\cdot\ldots\cdot\Big(\frac{\partial^\omega(1+\psi(\xi))}{\omega!}\Big)^{\delta_\omega} \\
&\leqslant \sum_{j=1}^{|\alpha|} c_j\,(1+\psi(\xi))^{\mu-j}\,c_\alpha \prod_\beta (1+\psi(\xi))^{\delta_\beta}\,\langle\xi\rangle^{-\delta_\beta|\beta|} \\
&\leqslant c'_\alpha\,(1+\psi(\xi))^\mu\,\langle\xi\rangle^{-|\alpha|}
\end{aligned}$$

and this shows that b satisfies also (6.1). ∎

Consider a sequence $N = (N_j)_{j\in\mathbb{N}_0}$ such that there exists a $\lambda_0 > 1$ with $\lambda_0\,N_j \leqslant N_{j+1}$ for any $j \in \mathbb{N}_0$ and let $s \in \mathbb{R}$. Then we denote $F_{p,2}^{\sigma^s,N}$ the collection of all tempered distributions u such that

$$\|u\,|\,F_{p,2}^{\sigma^s,N}\| = \|(2^{js}\,\varphi_j^N(D)u)_{j\in\mathbb{N}_0}\,|\,L_p(l_2)\| < \infty.$$

With this preparation we are able to prove the main result of this chapter.

Theorem 7.1.3 *Let ψ an admissible continuous negative definite function, let $r > 0$, and let $N^{\psi,r}$ the sequence associated to ψ and r, see* (6.5). *Let $1 < p < \infty$ and $s \in \mathbb{R}$. Then*

$$\|(\mathrm{id}+\psi(D))^{s/r}\,u\,|\,L_p\| \sim \|u\,|\,F_{p,2}^{\sigma^s,N^{\psi,r}}\|.$$

Proof. For simplicity let us denote $N = N^{\psi,r}$ and $N_j = N_j^{\psi,r}$. Using Lemma 7.1.2 and the construction of the strongly increasing sequence $N = (N_j)_{j\in\mathbb{N}_0}$, we get for any multi-index α

$$\begin{aligned}
D^\alpha\Big(2^{-js}(1+\psi(\xi))^{s/r}\,\chi_{\mathrm{supp}\,\varphi_j^N}(\xi)\Big) &\leqslant c_\alpha\,2^{-js}(1+\psi(\xi))^{s/r}\langle\xi\rangle^{-\alpha}\chi_{\mathrm{supp}\,\varphi_j^N}(\xi) \\
&\leqslant c'_\alpha\,\langle\xi\rangle^{-\alpha}
\end{aligned}$$

since $(1+\psi(\xi))^{s/r} \sim 2^{js}$ on $\mathrm{supp}\,\varphi_j^N \subset \{\xi \in \mathbb{R}^n : N_{j-1} \leqslant |\xi| \leqslant N_{j+1}\}$. Consequently we may apply Proposition 0.2.1, diagonal case - see (0.11), and Theorem 6.2.3 and get

$$\begin{aligned}
\big\|(\mathrm{id}+\psi(D))^{s/r}u\,|\,L_p\big\| &\leqslant c_1\,\|(\mathrm{id}+\psi(D))^{s/r}u\,|\,F_{p,2}^{\sigma^0,N}\| \\
&= \|\mathcal{F}^{-1}[\varphi_j^N(\xi)(1+\psi(\xi))^{s/r}\mathcal{F}u]\,|\,L_p(l_2)\| \\
&= \|\mathcal{F}^{-1}[2^{-js}\,(1+\psi(\xi))^{s/r}\,2^{js}\,\varphi_j^N(\xi)\mathcal{F}u]\,|\,L_p(l_2)\| \\
&\leqslant c\,\|2^{js}\,\varphi_j^N(D)u\,|\,L_p(l_2)\| \\
&= c\,\|u\,|\,F_{p,2}^{\sigma^s,N}\|.
\end{aligned}$$

For the second inequality note that

$$\begin{aligned} D^\alpha\left(2^{js}(1+\psi(\xi))^{-s/r}\,\chi_{\operatorname{supp}\varphi_j^N}(\xi)\right) &\leqslant c_\alpha\,2^{js}(1+\psi(\xi))^{-s/r}\langle\xi\rangle^{-\alpha}\chi_{\operatorname{supp}\varphi_j^N}(\xi)\\ &\leqslant c'_\alpha\,\langle\xi\rangle^{-\alpha}\end{aligned}$$

since $(1+\psi(\xi))^{-s/r}\sim 2^{-js}$ on $\operatorname{supp}\varphi_j^N\subset\{\xi\in\mathbb{R}^n\,:\,N_{j-1}\leqslant|\xi|\leqslant N_{j+1}\}$. Consequently we may apply again Proposition 0.2.1, diagonal case - see (0.11), and Theorem 6.2.3, and get

$$\begin{aligned}\|u\,|\,F_{p,2}^{\sigma^s,N}\| &= \|(2^{js}\,\varphi_j^N(D)u)_{j\in\mathbb{N}_0}\,|\,L_p(l_2)\|\\ &= \|\mathcal{F}^{-1}[2^{js}\,(1+\psi(\xi))^{-s/r}\,\varphi_j^N(\xi)\,(1+\psi(\xi))^{s/r}\mathcal{F}u]\,|\,L_p(l_2)\|\\ &\leqslant c\,\|\mathcal{F}^{-1}[\varphi_j^N(\xi)(1+\psi(\xi))^{s/r}\mathcal{F}u]\,|\,L_p(l_2)\|\\ &= c\,\|(\mathrm{id}+\psi(D))^{s/r}u\,|\,F_{p,2}^{\sigma^0,N}\|\leqslant c_2\left\|(\mathrm{id}+\psi(D))^{s/r}u\,|\,L_p\right\|\end{aligned}$$

and the proof is complete taking into account Theorem 6.2.3. ■

As a simple consequence of the above theorem using $\left\|u\,|\,H_p^{\psi,s}\right\|=\left\|(\mathrm{id}+\psi(D))^{s/2}u\,|\,L_p\right\|$ we get

Corollary 7.1.4 *Let* ψ *an admissible continuous negative definite function, and let* $N^{\psi,2}$ *the strongly increasing sequence associated to* ψ *and to* $r=2$*, see* (6.5)*. Let* $1<p<\infty$ *and* $s\in\mathbb{R}$*. Then there exist two constants* $c_1,c_2>0$ *such that*

$$c_1\left\|u\,|\,H_p^{\psi,s}\right\|\leqslant\|u\,|\,F_{p,2}^{\sigma^s,N^{\psi,2}}\|\leqslant c_2\left\|u\,|\,H_p^{\psi,s}\right\|$$

for any $u\in H_p^{\psi,s}$.

Spaces of type $F_{p,2}^{\sigma^s,N}$ will be extensively studied in the next chapter.

7.2 A Sobolev type embedding

If $N=(N_j)_{j\in\mathbb{N}_0}$ is a sequence such that $\lambda_0\,N_j\leqslant N_{j+1}$ for any $j\in\mathbb{N}_0$ then for spaces $F_{p,2}^{\sigma^s,N}$ many different results are already known from the works of G. A. Kalyabin and M. L. Goldman.
We mention here only one embedding result, proved first (in a more general context) in [Ka81]. We will return to this aspect in Theorem 10.1.4.

Proposition 7.2.1 *Let* $1<p<\infty$ *and* $\frac{1}{p}+\frac{1}{p'}=1$*. Let also* $N=(N_j)_{j\in\mathbb{N}_0}$ *be a strongly increasing sequence and let* $s>0$*. Then* $F_{p,2}^{\sigma^s,N}$ *is embedded into* $C_\infty(\mathbb{R}^n)$ *if, and only if,* $(2^{-js}\,N_j^{n/p})_{j\in\mathbb{N}_0}\in l_{p'}$.

We want to point out that due to Corollary 7.1.4, the above embedding result leads to embeddings of Sobolev type for $H_p^{\psi,s}$ if ψ is an admissible continuous negative definite function according to Definition 6.1.1.

Corollary 7.2.2 *Let ψ be an admissible continuous negative definite function and let $N^{\psi,2} = (N_j^{\psi,2})_{j\in\mathbb{N}_0}$ the sequence associated to ψ and to $r = 2$ according to* (6.5).
If $1 < p < \infty$ and $s > 0$ then $H_p^{\psi,s}$ is embedded in $C_\infty(\mathbb{R}^n)$ if, and only if,

$$(2^{-js}\,(N_j^{\psi,2})^{n/p})_{j\in\mathbb{N}_0} \in l_{p'} \quad . \tag{7.1}$$

Obviously the embedding result stated in the above corollary looks nicer than Theorem 4.2.1. But the price one has to pay is to restrict to the class of admissible functions ψ.

It is clear that if there would exist a number $\lambda_1 > 1$ such that $N_{j+1}^{\psi,2} \leqslant \lambda_1\, N_j^{\psi,2}$ for any $j \in \mathbb{N}_0$ (and consequently $N_j^{\psi,2} \leqslant \lambda_1^j\, N_0^{\psi,2}$) then condition (7.1) would be implied by the restriction $s > (\log_2 \lambda_1)\,\frac{n}{p}$.

However, if ψ is an admissible continuous negative definite function and if $N^{\psi,2} = (N_j^{\psi,2})_{j\in\mathbb{N}_0}$ is the sequence associated to ψ and to $r = 2$ according to (6.5) then it is not necessary (one can check this directly considering $\psi(\xi) = \log(2 + |\xi|^2)$) that there exists a $\lambda_1 > 1$ such that $N_{j+1}^{\psi,2} \leqslant \lambda_1\, N_j^{\psi,2}$ for any $j \in \mathbb{N}_0$.
A sufficient condition for this is indicated in the next Lemma.

Lemma 7.2.3 *Let ψ be a continuous negative definite function and let $N^{\psi,2} = (N_j^{\psi,2})_{j\in\mathbb{N}_0}$ the sequence associated to ψ and to $r = 2$ according to* (6.5).
If there exists an $0 < r_0 \leqslant 1$ such that $\psi(\xi)\langle\xi\rangle^{-2r_0}$ is increasing in $|\xi|$, i.e.

$$\text{if} \quad |\xi| \leqslant |\eta| \quad \text{then} \quad \frac{\psi(\xi)}{\langle\xi\rangle^{2r_0}} \leqslant \frac{\psi(\eta)}{\langle\eta\rangle^{2r_0}} \quad , \tag{7.2}$$

then there exists a $\lambda_1 > 1$ such that $N_{j+1}^{\psi,2} \leqslant \lambda_1\, N_j^{\psi,2}$, $j \in \mathbb{N}_0$.

Proof. For simplicity let again $N_j = N_j^{\psi,2}$ if $j \in \mathbb{N}_0$. We proceed as in Lemma 6.2.2. Let ξ^j and $\xi^{j+1} \in \mathbb{R}^n$ such that $\langle\xi^j\rangle = N_j$, $\psi(\xi^j) = 2^{2j}$ and $\langle\xi^{j+1}\rangle = N_{j+1}$, $\psi(\xi^{j+1}) = 2^{2(j+1)}$.
Using now our assumption on ψ we have

$$\frac{2^{2j}}{N_j^{2r_0}} = \frac{\psi(\xi^j)}{\langle\xi^j\rangle^{2r_0}} \leqslant \frac{\psi(\xi^{j+1})}{\langle\xi^{j+1}\rangle^{2r_0}} = \frac{2^{2(j+1)}}{N_{j+1}^{2r_0}}.$$

Consequently, taking $\lambda_1 = 2^{\frac{1}{r_0}}$ we have $N_{j+1} \leqslant \lambda_1\, N_j, \quad j \in \mathbb{N}_0$. ■

Remark 7.2.4 The restriction on ψ in the above lemma is not very surprising since it implies $\psi(\xi) \geqslant \langle\xi\rangle^{2r_0}$ for large ξ or

$$\psi(\xi) \geqslant c\,(1 + |\xi|^2)^{r_0} \quad \text{for any} \quad \xi \in \mathbb{R}^n. \tag{7.3}$$

This is a very often used assumption in connection with Sobolev type embeddings for function spaces associated to a continuous negative definite function, see for example [Ho98a] and the references therein and compare (4.4).
Moreover, note that we obviously have $r_0 \leqslant w$, where w is the number from Definition 6.1.1.

The next two embedding results are trivial consequences of the above considerations.

Corollary 7.2.5 *Let ψ be an admissible continuous negative definite function and assume that there exists an $0 < r_0 \leqslant 1$ such that $\psi(\xi)\langle\xi\rangle^{-2r_0}$ is increasing in $|\xi|$.*
If $1 < p < \infty$ and

$$\text{if} \quad s > \frac{1}{r_0}\frac{n}{p} \quad \text{then} \quad H_p^{\psi,s} \hookrightarrow C_\infty.$$

Of course the assumption of s becomes better and better when r_0 approaches 1. If $r_0 \to 1$ then the restriction on s becomes $s > \frac{n}{p}$ and this is the restriction in the Sobolev embedding since if $r_0 = 1$ (and consequently $w = 1$) then $\psi(\xi) = 1 + |\xi|^2$ and we have the standard Bessel potential spaces H_p^s.

Corollary 7.2.6 *Let $f : (0, \infty) \to (0, \infty)$ a Bernstein function with $\lim_{t\to\infty} f(t) = \infty$ and such that*

$$\text{there exists an} \quad r_0 \in (0, 1] \quad \text{such that} \quad t \mapsto \frac{f(t)}{t^{r_0}} \quad \text{is increasing.} \tag{7.4}$$

If $1 < p < \infty$ and

$$\text{if} \quad s > \frac{1}{r_0}\frac{n}{p} \quad \text{then} \quad H_p^{f(1+|\cdot|^2),s} \hookrightarrow C_\infty.$$

Remark 7.2.7 Note that the Sobolev embedding results stated in Corollary 7.2.5 and in Corollary 7.2.6 are sharper than those one obtained in [Ja02, Corollary 3.3.32] where the restriction was $s > \frac{1}{r_0}\frac{n(p+1)}{p}$.

Example 7.2.8 **(i)** The function $f(t) = \log(1 + t)$ does not satisfy condition (7.4) whereas the functions $g(t) = \sqrt{t}\,\log(1 + \sqrt{t})$ and $h(t) = \sqrt{t}\,(1 - \exp(-4\sqrt{t}))$ satisfy condition (7.4) with $r_0 = 1/2$.
(ii) It is obvious that the function $f(t) = t^\varrho$ with $0 < \varrho \leqslant 1$ satisfies (7.4). However due to Remark 4.2.3.(ii) the associated Bessel potential spaces are classical and the above embedding is known.

Corollary 7.2.9 *The functions $\psi_1 : \mathbb{R}^n \to \mathbb{R}$, $\psi_1(\xi) = \langle\xi\rangle \log(1 + \langle\xi\rangle)$ and $\psi_2 : \mathbb{R}^n \to \mathbb{R}$, $\psi_2(\xi) = \langle\xi\rangle\,(1 - \exp(-4\,\langle\xi\rangle))$ satisfy the assumptions of Corollary 7.2.5 with $r_0 = 1/2$.*

We conclude this section with a remark not directly related to embeddings.

Remark 7.2.10 Let ψ be an admissible continuous negative definite function and assume that there exists an $0 < r_0 \leqslant 1$ such that $\psi(\xi)\langle\xi\rangle^{-2r_0}$ is increasing in $|\xi|$. Then one has (7.3). As a corollary of [Ja02, 3.3.34] one gets that if $s \geqslant 0$ then C_0^∞ is dense in $H_p^{\psi,s}$.

Chapter III.
Function spaces of generalised smoothness

8 Introduction to Chapter III

Motivated partly by the considerations in the previous chapter we will give below a unified approach on function spaces of generalised smoothness and we will characterise these spaces in terms of modern tools such as local means and atoms.
Our results cover the results on classical Besov and Triebel-Lizorkin spaces and on spaces of type $B^{(s,\Psi)}_{p,q}$ and $F^{(s,\Psi)}_{p,q}$ as treated by D. E. Edmunds, H. Triebel and S. Moura in [EdTr98], [EdTr99], [Mo99], [Mo01], spaces which appear in our context as particular cases.
Furthermore, we will obtain new characterisations of ψ-Bessel potential spaces if the continuous negative definite function ψ is admissible.
The results of this chapter will allow us to treat pseudo-differential operators on function spaces of generalised smoothness, in particular on ψ-Bessel potential spaces.

Briefly about the organising and contents of this chapter.
We tried to make our exposition in Chapter III as much as possible self-contained so that **Section 9** has a preparatory character. We set up notation and introduce the sequences determining the generalised smoothness.
The first sequence is a so-called strongly increasing sequence $N = (N_j)_{j\in\mathbb{N}_0}$ (an almost increasing sequence such that, additionally, there exists a natural number κ_0 with $2N_j \leqslant N_k$ for all j and all k with $j+\kappa_0 \leqslant k$, see the precise formulation in Definition 9.1.1) - which generalises the sequence $(2^j)_{j\in\mathbb{N}_0}$ and induces a decomposition in $\mathbb{R}^n$ in the sets $\Omega^N_j = \{\xi \in \mathbb{R}^n : |\xi| \leqslant N_{j+\kappa_0}\}$ for $j = 0, 1, ..., \kappa_0 - 1$ and $\Omega^N_j = \{\xi \in \mathbb{R}^n : N_{j-\kappa_0} \leqslant |\xi| \leqslant N_{j+\kappa_0}\}$ for $j \geqslant \kappa_0$. To this decomposition of $\mathbb{R}^n$ it is associated a family $(\varphi^N_j)_{j\in\mathbb{N}_0}$ of compactly supported smooth functions which extends the classical partition of unity. We then have a decomposition of any tempered distribution f into a series of entire analytic functions $f_j = (\varphi^N_j \widehat{f})^\vee$ like in the classical case.
Secondly, we will consider a so-called admissible sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ (i.e. it satisfies $d_0\,\sigma_j \leqslant \sigma_{j+1} \leqslant d_1\,\sigma_j$ for any $j \in \mathbb{N}_0$) which generalises the sequence $(2^{js})_{j\in\mathbb{N}_0}$ and which is a smoothness weight for the different functions f_j. We want to point out that an admissible sequence σ is considerably *more general* than $(2^{js})_{j\in\mathbb{N}_0}$ or than $(2^{js}\Psi(2^{-j}))_{j\in\mathbb{N}_0}$ (for monotone functions Ψ on $(0,1]$ with $\Psi(2^{-j}) \sim \Psi(2^{-2j})$), see Example 9.1.9.

In **Section 10**, for given sequences N and σ and for $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, Besov, respectively Triebel - Lizorkin spaces of generalised smoothness are defined as the collection of all tempered distributions f such that $\|f \,|\, B^{\sigma,N}_{p,q}\| = \|\sigma_j(\varphi^N_j \widehat{f})^\vee \,|\, l_q(L_p)\|$, respectively $\|f \,|\, F^{\sigma,N}_{p,q}\| = \|\sigma_j(\varphi^N_j \widehat{f})^\vee \,|\, L_p(l_q)\|$, is finite, see Definition 10.1.1.
To show that the definition of the spaces is consistent, one has to use the classical Fourier-multiplier theorem of Michlin - Hörmander type (for convenience we recall it in Proposition 0.2.1). It is easy to show that standard properties in the classical situation,

such as the density of test functions $\mathcal{S}$ (for appropriate values of the parameters) are still true.
Then we prove a theorem of Littlewood - Paley type: $F^{\sigma^0,N}_{p,2} = L_p$ for any strongly increasing sequence N (here σ^0 denotes the sequence with all terms equal 1), we mention embeddings on level of zero-smoothness, the existence of a lift operator between spaces $B^{\sigma,N}_{p,q}$ and $B^{\beta,N}_{p,q}$ (and also for F-spaces) and finally a duality result.
In particular in Subsection 10.1.2 we study some special classes of function spaces of generalised smoothness, those in which the strongly increasing sequence $N = (N_j)_{j\in\mathbb{N}_0}$ is obtained from a smooth given so-called admissible function (see Definition 10.1.11) in a canonical way. In particular the considerations in this subsection cover the topic on ψ-Bessel potential spaces (with admissible ψ) as discussed in the previous chapter.

To extend the definition of the spaces of generalised smoothness to $p = \infty$, $p = 1$ and $0 < p < 1$, an additional assumption on the sequence N is necessary, namely the sequence N has to be not only strongly increasing but also of bounded growth. The reason is, that we have to use in these cases another Fourier-multiplier theorem than before. A brief discussion is contained in Subsection 10.2.
Subsection 10.3 illustrates how our approach covers many classes of function spaces of generalised smoothness known up to now in the literature.
Simultaneously, due to the flexibility of the admissible sequence σ, this covering is a strict one. For the sake of completeness we decided to include also examples of representatives in function spaces of generalised smoothness.

Section 11 is the core of this chapter. Under the assumption that the sequence $N = (N_j)_{j\in\mathbb{N}_0}$ satisfies $\lambda_0 \, N_j \leqslant N_{j+1} \leqslant \lambda_1 \, N_{j+1}$ for any $j \in \mathbb{N}_0$, with some constants $1 < \lambda_0 \leqslant \lambda_1$, we prove an useful characterisation of function spaces of generalised smoothness based on the so-called local means.
[Tr92, Section 1.8] might be considered the "philosophical" background to local means: step by step, from Cauchy-Poisson and Gauss-Weierstrass semigroups of operators and quasi-norms in classical spaces $B^s_{p,q}$ and $F^s_{p,q}$ related to these semigroups, one arrives finally at [Tr92, Section 1.8.4] where the idea of local means as a generalisation of these approaches comes in.
The theorem on local means that we proved for function spaces of generalised smoothness (Theorem 11.3.4) is highly technical and its proof required (compared with the classical situation) new techniques and ideas. We summarised the basic ideas of the proof in Subsection 11.4.

Section 12 contains the atomic decomposition theorem in function spaces of generalised smoothness, see Theorem 12.2.1. It is on the one hand a direct application of the theorem on local means and on the other hand it paves the way to discuss in the next chapter mapping properties for pseudo-differential operators.
Roughly speaking, the atomic decomposition theorem states that for any $u \in F^{\sigma,N}_{p,q}(\mathbb{R}^n)$ it is possible to find a decomposition (convergence in $\mathcal{S}'(\mathbb{R}^n)$)

$$u = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} \, \rho_{\nu m} \quad ,$$

where $\rho_{\nu m}$ are the N-atoms and $\lambda = \{\lambda_{\nu m} \, : \, \nu \in \mathbb{N}_0 \, , m \in \mathbb{Z}^n\}$ belongs to an appropriate sequence space $f^N_{p,q}$, such that

$$\|u \,|\, F^{\sigma,N}_{p,q}(\mathbb{R}^n)\| \sim \inf \ \|\lambda \,|\, f^N_{p,q}\|$$

where the infimum is taken over all admissible representations of u, and a corresponding assertion for $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$ spaces (with an appropriate different sequence space). The precise formulation is given, as mentioned, in Theorem 12.2.1.
We conclude the section with some comments and examples.

We would like to mention that around 1996/97 H. Triebel has given an important contribution to the theory of "nice building blocks" of function spaces with the introduction of *quarks*. They are a sort of refined atoms and (in contrast to atoms) can be taken as a frame in the desired function space. Moreover, the coefficients can be chosen linearly dependent on u in a canonical way. The price to pay is a bit more involved approach to the subject. We also have contributed to this subject in some anisotropic situations in [Fa00] (paper which is not included here).
This work contains no subatomic (quarkonial) decomposition in function spaces of generalised smoothness. In a particular case ($N_j = 2^j$ for any $j \in \mathbb{N}_0$) they were recently obtained by M. Bricchi in [Bri02] who used our results on atomic decompositions from this chapter as announced in the preprint [FaLe01]. It has to be expected that one can get a subatomic decomposition for the general case but no attempt in this direction has been done here.

Finally we would like to mention that it has to be expected that our main results from this chapter have an anisotropic counterpart but, due to the technical complications, we decided to shift this approach to a later work.

9 Preliminaries

9.1 Sequences

Definition 9.1.1 *A sequence $\gamma = (\gamma_j)_{j\in\mathbb{N}_0}$ of positive real numbers is called:*
(i) almost increasing *if there is a positive constant d_0 such that*

$$d_0\gamma_j \leqslant \gamma_k \quad \textit{for all} \quad j \quad \textit{and} \quad k \quad \textit{with} \quad 0 \leqslant j \leqslant k.$$

(ii) strongly increasing *if it is almost increasing and, in addition, there is a natural number κ_0 such that*

$$2\gamma_j \leqslant \gamma_k \quad \textit{for all} \quad j \quad \textit{and} \quad k \quad \textit{with} \quad j + \kappa_0 \leqslant k.$$

(iii) of bounded growth *if there are a positive constant d_1 and a number $J_0 \in \mathbb{N}_0$ such that*

$$\gamma_{j+1} \leqslant d_1\gamma_j \quad \textit{for any} \quad j \geqslant J_0.$$

Remark 9.1.2 It is easy to see that each sequence $\gamma = (\gamma_j)_{j\in\mathbb{N}_0}$ with the property that there is a constant $\lambda_0 > 1$ such that

$$\lambda_0\gamma_j \leqslant \gamma_{j+1} \quad \text{for all} \quad j \in \mathbb{N}_0, \tag{9.1}$$

is strongly increasing in the sense of the above definition. However not every strongly increasing sequence satisfies property (9.1).

Example 9.1.3 **(i)** The sequence $\gamma = (\gamma_j)_{j \in \mathbb{N}_0}$ with $\gamma_j = 2^{j\delta}(1+j)^b$, where $\delta > 0$, $b \in \mathbb{R}$ is strongly increasing and of bounded growth.
(ii) The sequence $\gamma = (\gamma_j)_{j \in \mathbb{N}_0}$ with $\gamma_j = j!$ is strongly increasing but not of bounded growth.
(iii) The sequence $\gamma = (\gamma_j)_{j \in \mathbb{N}_0}$ with $\gamma_j = j$ is not strongly increasing, but of bounded growth.

Example 9.1.4 For an admissible continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ let $N_j^{\psi,2} = \sup\{\langle\xi\rangle : \psi(\xi) \leqslant 2^{2j}\}$ if $j \in \mathbb{N}_0$, compare (6.5). Then $2^{1/w}\, N_j^{\psi,2} \leqslant N_{j+1}^{\psi,2}$ as proved in Lemma 6.2.2 so $(N_j^{\psi,2})_{j \in \mathbb{N}_0}$ is a strongly increasing sequence.
If $\psi(\xi) = \log(2 + |\xi|^2)$ then $(N_j^{\psi,2})_{j \in \mathbb{N}_0}$ is not of bounded growth.
If $\psi(\xi) = \langle\xi\rangle\, \log(1+\langle\xi\rangle)$ then $(N_j^{\psi,2})_{j \in \mathbb{N}_0}$ is of bounded growth, compare Corollary 7.2.5.

In the function spaces we will consider below we will have two parameters determining the generalised smoothness.
First we will deal with a sequence $N = (N_j)_{j \in \mathbb{N}_0}$ which will be strongly increasing in the next section and additionally of bounded growth in the main theorems of this chapter (local means and atomic decompositions).
Secondly, we will consider a sequence $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ which acts as a smoothness weight on the different functions f_j, which are the result of the decomposition on the Fourier side. This sequence will fulfill

$$d_0\, \sigma_j \leqslant \sigma_{j+1} \leqslant d_1\, \sigma_j \quad \text{for all} \quad j \in \mathbb{N} \tag{9.2}$$

with two positive constants d_0 and d_1. In other words, both $(\sigma_j)_{j \in \mathbb{N}_0}$ and $(\sigma_j^{-1})_{j \in \mathbb{N}_0}$ are of bounded growth.
Sequences σ satisfying (9.2) will be called *admissible sequences.*
To illustrate the flexibility of the last condition we give some examples:

Example 9.1.5 The sequence $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$,

$$\sigma_j = 2^{js}(1+j)^b(1+\log(1+j))^c \tag{9.3}$$

with arbitrary fixed real numbers s, b and c is the standard example of an admissible sequence and it can be considered in some sense as a model sequence. However, as it is shown in the next examples, the general definition includes also other sequences, which cannot be reduced to the one above or to a similar one.

Notation 9.1.6 For any $s \in \mathbb{R}$ we will denote

$$\sigma^s = (2^{js})_{j \in \mathbb{N}_0} \quad . \tag{9.4}$$

Of course σ^s is a special case of (9.3) with $b = 0$ and $c = 0$. In particular σ^0 will denote the sequence with all terms equal with 1.

Example 9.1.7 Let $s \in \mathbb{R}$ be fixed and

$$\sigma_j = 2^{js}\Psi(2^{-j}) \quad , \quad j \in \mathbb{N}_0 \quad ,$$

where Ψ is a positive monotone function on $(0,1]$ and there are positive constants b_0^* and b_1^* such that for all $j \in \mathbb{N}_0$

$$b_0^* \, \Psi(2^{-j}) \leqslant \Psi(2^{-2j}) \leqslant b_1^* \, \Psi(2^{-j}).$$

Then it is easy to see that σ is an admissible sequence: we can take $d_0 = b_0 \, 2^s$ and $d_1 = b_1 \, 2^s$ in (9.2) where $b_0 = \min(b_0^*, 1, \Psi(2^{-1})\Psi(1)^{-1})$ and $b_1 = \max(b_1^*, 1, \Psi(2^{-1})\Psi(1)^{-1})$. This example goes back to [EdTr99] and gives a qualitative description of the model case in Example 9.1.5 with fixed main order 2^{js}.

Example 9.1.8 Let $(j_l)_{l\in\mathbb{N}_0}$ be a strongly increasing sequence of natural numbers, defined recursively by

$$j_0 = 0 \quad , \quad j_1 = 1 \quad , \quad j_{2l} = 2 \, j_{2l-1} - j_{2l-2} \quad , \quad j_{2l+1} = 2^{j_{2l}} \quad , \quad l \in \mathbb{N} \quad .$$

The sequence $(\sigma_j)_{j\in\mathbb{N}_0}$ is defined by

$$\sigma_j = \begin{cases} 2^{j_{2l}} & \text{if} \quad j_{2l} \leqslant j < j_{2l+1} \\ 2^{j_{2l}} \; 4^{(j-j_{2l+1})} & \text{if} \quad j_{2l+1} \leqslant j < j_{2l+2} \end{cases} .$$

Then σ is an admissible sequence. Moreover, the sequence $(\sigma_j)_{j\in\mathbb{N}_0}$ oscillates between $(j)_{j\in\mathbb{N}_0}$ and $(2^j)_{j\in\mathbb{N}_0}$, i.e.

$$j \leqslant \sigma_j \leqslant 2^j \quad ,$$

and there exist infinitely many j' and j'' such that $\sigma_{j'} = j'$ and $\sigma_{j''} = 2^{j''}$, respectively.

Example 9.1.9 Let $s \in \mathbb{R}$ be fixed and $(\sigma_j)_{j\in\mathbb{N}_0}$ be the above sequence. If

$$\tau_j = 2^{js} \; \sigma_j$$

then τ is admissible. $(\tau_j)_{j\in\mathbb{N}_0}$ oscillates between $(j \, 2^{js})_{j\in\mathbb{N}_0}$ and $(2^{j(s+1)})_{j\in\mathbb{N}_0}$, i.e.

$$j \, 2^{js} \leqslant \tau_j \leqslant 2^{j(s+1)}$$

and, again, infinitely many τ_j equal the left-hand side or the right-hand side of the above double inequality.

Remark 9.1.10 We would like to point out that the last two examples, which are due to G. A. Kalyabin, show that an admissible sequence do not have necessarily a fixed main order. Consequently the class of admissible sequences is larger than the class described in Example 9.1.7.

Remark 9.1.11 **(i)** Of course for given admissible sequences σ the numbers d_0 and d_1 are not uniquely determined. In [Bri02, Definition 2.2.11] M. Bricchi introduced a lower and an upper index associated to the sequence σ

$$\underline{\mathfrak{s}}(\sigma) = \liminf_{j\to\infty} \log\left(\frac{\sigma_{j+1}}{\sigma_j}\right) \quad \text{and} \quad \bar{\mathfrak{s}}(\sigma) = \limsup_{j\to\infty} \log\left(\frac{\sigma_{j+1}}{\sigma_j}\right)$$

and formulated his results in terms of these indices. One has to pay however attention that different admissible sequences generate the same function spaces of generalised smoothness.

(ii) Very recently, after a first version of this work was finished, M. Bricchi and S. Moura introduced in [BriMo02] the lower respectively the upper Boyd index of a given admissible sequence as follows. First let

$$\underline{\sigma}_j := \inf_{k \geqslant 0} \frac{\sigma_{j+k}}{\sigma_k} \quad \text{and} \quad \overline{\sigma}_j := \sup_{k \geqslant 0} \frac{\sigma_{j+k}}{\sigma_k} \quad j \in \mathbb{N}_0.$$

Then let

$$b^\sigma := \lim_{j\to\infty} \frac{\log \overline{\sigma}_j}{j} \quad \text{and} \quad b_\sigma := \lim_{j\to\infty} \frac{\log \underline{\sigma}_j}{j}$$

be the upper and respectively the lower Boyd index of the sequence σ.
It is not difficult to reformulate the main results of this chapter, the theorem of local means and the atomic decomposition theorem, in terms of these two Boyd indices.

9.2 Decompositions

In what follows we generalise the considerations in Section 6.
For a fixed strongly increasing sequence $N = (N_j)_{j\in\mathbb{N}_0}$ and a fixed $J \in \mathbb{N}$ we define the associate covering $\Omega^{N,J} = (\Omega_j^{N,J})_{j\in\mathbb{N}_0}$ of $\mathbb{R}^n$ by

$$\Omega_j^{N,J} = \{\xi \in \mathbb{R}^n \,:\, |\xi| \leqslant N_{j+J\kappa_0}\} \quad \text{if} \quad j = 0, 1, ..., J\kappa_0 - 1, \tag{9.5}$$

and

$$\Omega_j^{N,J} = \{\xi \in \mathbb{R}^n \,:\, N_{j-J\kappa_0} \leqslant |\xi| \leqslant N_{j+J\kappa_0}\} \quad \text{if} \quad j = J\kappa_0,\, J\kappa_0 + 1, ... \quad . \tag{9.6}$$

From the above definition it is obvious that each $\Omega_j^{N,J}$ has a non-empty intersection with at most $2(J + L + 1)\kappa_0$ different sets $\Omega_k^{N,L}$ from a covering associated to the same sequence $N = (N_j)_{j\in\mathbb{N}_0}$.

Definition 9.2.1 *For a fixed strongly increasing sequence $N = (N_j)_{j\in\mathbb{N}_0}$, a fixed $J \in \mathbb{N}$, and for the associated covering $\Omega^{N,J} = (\Omega_j^{N,J})_{j\in\mathbb{N}_0}$ of $\mathbb{R}^n$, let $\Phi^{N,J}$ be the collection of all function systems $\varphi^{N,J} = (\varphi_j^{N,J})_{j\in\mathbb{N}_0}$ such that:*
(i)

$$\varphi_j^{N,J} \in C_0^\infty(\mathbb{R}^n) \quad \textit{and} \quad \varphi_j^{N,J}(\xi) \geqslant 0 \quad \textit{if} \quad \xi \in \mathbb{R}^n \quad \textit{for any} \quad j \in \mathbb{N}_0\,; \tag{9.7}$$

(ii)

$$\operatorname{supp} \varphi_j^{N,J} \subset \Omega_j^{N,J}\,; \tag{9.8}$$

(iii) *for any $\gamma \in \mathbb{N}_0^n$ there exists a constant $c_\gamma > 0$ such that for any $j \in \mathbb{N}_0$*

$$|D^\gamma \varphi_j^{N,J}(\xi)| \leqslant c_\gamma \, \langle\xi\rangle^{-\gamma} \quad \textit{for any} \quad \xi \in \mathbb{R}^n; \tag{9.9}$$

(iv) *there exists a constant $c_\varphi > 0$ such that*

$$0 < \sum_{j=0}^{\infty} \varphi_j^{N,J}(\xi) = c_\varphi < \infty \quad \textit{for any} \quad \xi \in \mathbb{R}^n. \tag{9.10}$$

Without loss of generality we may take $\varphi_j^{N,J} \equiv 0$ if $j = 0, 1, ..., J\kappa_0 - 2$.
By the relatively free choice of the sequence $(N_j)_{j\in\mathbb{N}_0}$ the construction of function systems $(\varphi_j)_{j\in\mathbb{N}_0}$ satisfying properties (9.7) - (9.10) is a little bit complicate compared with the classical case. This is illustrated in the following examples.

Example 9.2.2 Let $\rho \in C_0^\infty(\mathbb{R})$ with

$$\rho(t) = 1 \quad \text{if} \quad |t| \leqslant 1 \quad \text{and} \quad \operatorname{supp}\rho \subset \{t \in \mathbb{R} \,:\, |t| \leqslant 2\}$$

A. Let

$$\varphi_j^{N,J}(\xi) = \rho(N_j^{-1}|\xi|) \qquad j = 0,\ 1,\ \ldots, J\kappa_0 - 1$$

or

$$\varphi_{J\kappa_0-1}^{N,J}(\xi) = \sum_{k=0}^{J\kappa_0-1} \rho(N_k^{-1}|\xi|) \quad \text{and} \quad \varphi_j^{N,J} \equiv 0 \ \text{ if } \ j = 0, 1, ..., J\kappa_0 - 2.$$

Let also

$$\varphi_j^{N,J}(\xi) = \rho(N_j^{-1}|\xi|) - \rho(N_{j-J\kappa_0}^{-1}|\xi|) \quad \text{for any} \quad j \geqslant J\kappa_0.$$

Then it is easy to see that the system $\varphi^{N,J} = (\varphi_j^{N,J})_{j\in\mathbb{N}_0}$ satisfies (9.7) - (9.10) with $c_\varphi = \kappa_0 J$.

B. Let also

$$\psi_k^N(\xi) = \sum_{r=-(2J+1)\kappa_0}^{(2J+1)\kappa_0} \varphi_{k+r}^{N,J}(\xi) \quad \text{with} \quad \varphi_{-(2J+1)\kappa_0} = \ldots = \varphi_{-1} = 0.$$

Then $(\psi_k^N)_{k\in\mathbb{N}_0}$ is a function system which satisfies properties (i) - (iii) from above with respect to the covering $\Omega^{N,3J+2}$. This system has the useful property

$$\psi_k^N(\xi) = c_\varphi \quad \text{on} \quad \operatorname{supp}\varphi_k^{N,J}.$$

Moreover, if we define

$$\widetilde{\psi}_0^N(\xi) = \psi_0^N(\xi) + \sum_{r=0}^{(2J+1)\kappa_0-1} ((2J+1)\kappa_0 - r)\ \varphi_r^{N,J}(\xi)$$

then we have

$$\widetilde{\psi}_0^N(\xi) + \sum_{k=1}^{\infty} \psi_k^N(\xi) = c_\psi = [(4J+2)\kappa_0 + 1]c_\varphi.$$

Remark 9.2.3 It is easy to see that if $(\varphi_j^{N,J})_{j\in\mathbb{N}_0}$ fulfills (9.7)-(9.9) then for any multi-index α there is a constant $c_\alpha > 0$ such that

$$\sum_{j=0}^{\infty} |D^\alpha \varphi_j^{N,J}(\xi)| \leqslant [(2J+1)\kappa_0]\, c_\alpha\, \langle\xi\rangle^{-|\alpha|} \quad , \quad \text{for any} \quad \xi \in \mathbb{R}^n.$$

In particular, the last inequality implies

$$\sup \left(R^{2|\alpha|-n} \int\limits_{\frac{R}{2}\leqslant|\xi|\leqslant 2R} \sum_{j=0}^{\infty} |D^\alpha \varphi_j^{N,J}(\xi)|^2 \, d\xi \right)^{1/2} < \infty \tag{9.11}$$

where the supremum is taken over all $R > 0$ and all $\alpha \in \mathbb{N}_0^n$ with $0 \leqslant |\alpha| \leqslant 1 + \left[\frac{n}{2}\right]$.

The same is true for the system $(\psi_k^{N,3J+2})_{k\in\mathbb{N}_0}$ in the previous example.

10 The spaces: Basic facts

10.1 The case $1 < p < \infty$

10.1.1 Definition and fundamental properties

The main tool in defining function spaces of generalised smoothness of Besov and Triebel - Lizorkin type for $1 < p < \infty$ is the classical Fourier-multiplier theorem of Michlin-Hörmander type stated in Subsection 0.2.2. In analogy to the classical case we introduce function spaces of generalised smoothness of Besov and Triebel - Lizorkin type.

Definition 10.1.1 *Let $N = (N_j)_{j\in\mathbb{N}_0}$ be a strongly increasing sequence, not necessarily of bounded growth, let $J \in \mathbb{N}$, and let $(\varphi_j^{N,J})_{j\in\mathbb{N}_0} \in \Phi^{N,J}$. Let $(\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence.*
(i) *Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. Then the Besov space of generalised smoothness is*

$$B_{p,q}^{\sigma,N} = \left\{ f \in \mathcal{S}' \,:\, \|f \,|\, B_{p,q}^{\sigma,N}\| = \|(\sigma_j\, \varphi_j^{N,J}(D) f)_{j\in\mathbb{N}_0} \,|\, l_q(L_p)\| < \infty \right\}.$$

(ii) *Let $1 < p < \infty$, $1 < q < \infty$. Then the Triebel - Lizorkin space of generalised smoothness is*

$$F_{p,q}^{\sigma,N} = \left\{ f \in \mathcal{S}' \,:\, \|f \,|\, F_{p,q}^{\sigma,N}\| = \|(\sigma_j\, \varphi_j^{N,J}(D) f(\cdot))_{j\in\mathbb{N}_0} \,|\, L_p(l_q)\| < \infty \right\}.$$

Note that if $N_j = 2^j$ and $\sigma = \sigma^s = (2^{js})_{j\in\mathbb{N}_0}$ (recall notation (9.4) with s real), then the above spaces coincide with the usual function spaces $B_{p,q}^s$ and $F_{p,q}^s$ on $\mathbb{R}^n$, respectively. These scales of spaces, which include many many well-known spaces as Hölder-Zygmund spaces, Sobolev spaces, fractional Sobolev spaces, Besov spaces, inhomogeneous Hardy spaces and spaces of BMO-type, see the Introduction, were systematically treated in the books of H. Triebel, see [Tr78], [Tr83], [Tr92] and [Tr01] and the references therein. Further background material can be found in the books of D. E. Edmunds and H. Triebel, see [EdTr96] and of T. Runst and W. Sickel, see [RuSi96], books in which the theory is complemented by several other aspects (entropy numbers, nonlinear partial differential equations etc.)

For sequences $(\sigma_j)_{j\in\mathbb{N}_0}$ with $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in l_{q'}$ where $q' = q/(q-1)$ G. A. Kalyabin gave in [Ka80] a similar characterisation for such spaces, defined a-priori by approximation, for more details see Section 10.3.

Remark 10.1.2 Both $B^{\sigma,N}_{p,q}$ and $F^{\sigma,N}_{p,q}$ are Banach spaces which are independent of the choice of the system $(\varphi^{N,J}_j)_{j\in\mathbb{N}_0}$, in the sense of equivalent norms (and this is the reason why we may omit in our notation the subscript $(\varphi^{N,J}_j)_{j\in\mathbb{N}_0}$).
This can be shown in the standard way, compare for example [Tr78, Theorem 2.3.2] or [Tr83, Proposition 2.3.2/1].
Let us consider two different function systems $(\varphi^{N,J}_j)_{j\in\mathbb{N}_0}$ and $(\widetilde{\varphi}^{N,L}_j)_{j\in\mathbb{N}_0}$ related to the same strongly increasing sequence N.
Clearly for a fixed $j_0 \in \mathbb{N}_0$ the intersection $\operatorname{supp}\varphi^{N,J}_{j_0} \cap \operatorname{supp}\widetilde{\varphi}^{N,L}_k$ is non-empty at most for k in-between $j_0 - (L+J+1)\kappa_0$ and $j_0 + (L+J+1)\kappa_0$.
The desired equivalence result is a simple consequence of the second part of Proposition 0.2.1 (diagonal case - $m_{k,j} = 0$ if $k \neq j$), which is based on (9.11). In the case of Besov spaces, we use a scalar version - the classical Michlin-Hörmander Fourier-multiplier theorem for L_p spaces.

Remark 10.1.3 As in the classical case, compare [Tr78, Theorem 2.3.2] or [Tr83, Proposition 2.3.3], the embeddings $\mathcal{S} \hookrightarrow B^{\sigma,N}_{p,q} \hookrightarrow \mathcal{S}'$ and $\mathcal{S} \hookrightarrow F^{\sigma,N}_{p,q} \hookrightarrow \mathcal{S}'$ hold true for all admissible values of the parameters and sequences. If $q < \infty$ then $\mathcal{S}$ is dense in $B^{\sigma,N}_{p,q}$ and in $F^{\sigma,N}_{p,q}$.
Moreover, it is clear that $B^{\sigma,N}_{p,p} = F^{\sigma,N}_{p,p}$.

If the sequences $(\sigma_j)_{j\in\mathbb{N}_0}$ have additionally the property $(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in l_{q'}$, then all elements of $B^{\sigma,N}_{p,q}$ and of $F^{\sigma,N}_{p,q}$ are at least functions in L_p.
In this case many different results are already known from the works of G. A. Kalyabin and M. L. Goldman. We mention here only one embedding result, proved first in [Ka81].

Theorem 10.1.4 *Let $1 < p < \infty$ and $1 < q < \infty$. Let also $N = (N_j)_{j\in\mathbb{N}_0}$ be a strongly increasing sequence and let $(\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence with*

$$(\sigma_j^{-1})_{j\in\mathbb{N}_0} \in l_{q'}.$$

Then the following assertions are equivalent:

(i)	$F^{\sigma,N}_{p,q}$ *is embedded into* $C(\mathbb{R}^n)$,	$(B^{\sigma,N}_{p,q}$ *is embedded into* $C(\mathbb{R}^n)\,)$,
(ii)	$F^{\sigma,N}_{p,q}$ *is a multiplication algebra,*	$(B^{\sigma,N}_{p,q}$ *is a multiplication algebra* $)$,
(iii)	$(\sigma_j^{-1} N_j^{n/p})_{j\in\mathbb{N}_0} \in l_{p'}$,	$((\sigma_j^{-1} N_j^{n/p})_{j\in\mathbb{N}_0} \in l_{q'}\,)$,

respectively.

We used a weaker form of the above result in Section 6 when discussing embeddings of Sobolev type for ψ-Bessel potential spaces.

For embeddings into L_q-spaces, but also into Lorentz and Orlicz spaces we refer to [Go84], [Go84b], [Go85], or to [Go92] for embeddings in a more complicated context.
Because the Fourier analytic approach allows to consider also spaces of non-positive smoothness, we can obtain, and this is done in the rest of the section, results similar to those in the classical case.

• *A Littlewood - Paley type theorem.*
The following result generalises Theorem 6.2.3.

Theorem 10.1.5 *Let $1 < p < \infty$ and $N = (N_j)_{j\in\mathbb{N}_0}$ be a strongly increasing sequence. Recall σ^0 denotes the sequence with all terms equal 1. Then*

$$F^{\sigma^0,N}_{p,2} = L_p.$$

The proof is similar to that from [Tr83, Theorem 2.5.6] and to that given in Theorem 6.2.3 and it is based on the first part of Proposition 0.2.1 so that we omit it here.

Corollary 10.1.6 *If $N = (N_j)_{j\in\mathbb{N}_0}$ is a strongly increasing sequence then $B^{\sigma^0,N}_{2,2} = L_2$.*

This can be proved also directly using the definition of the space $B^{\sigma^0,N}_{2,2}$.

• *Embeddings on the level of zero-smoothness*

On the level of zero-smoothness we have the following embeddings with respect to the usual Besov spaces $B^0_{p,1}$ and $B^0_{p,\infty}$.

Theorem 10.1.7 *Let $N = (N_j)_{j\in\mathbb{N}_0}$ be a strongly increasing sequence and $1 < p < \infty$.*
(i) *Then*

$$L_p \hookrightarrow B^{\sigma^0,N}_{p,\infty} \hookrightarrow B^0_{p,\infty} \quad \text{and} \quad B^0_{p,1} \hookrightarrow B^{\sigma^0,N}_{p,1} \hookrightarrow L_p.$$

(ii) *If, in addition, the sequence $N = (N_j)_{j\in\mathbb{N}_0}$ is of bounded growth then for any $1 \leqslant q \leqslant \infty$*

$$B^{\sigma^0,N}_{p,q} = B^0_{p,q}.$$

The proof is given in [FaLe01, Theorem 3.1.7].

Remark 10.1.8 H.-G. Leopold communicated us that the results stated in the above theorem are sharp. To see this, one can show that if the sequence N is $N_j = j!$ (for any j) then we have, even in the case $p = 2$,

$$B^{\sigma^0,N}_{2,\infty} \hookrightarrow B^0_{2,\infty} \quad \text{and} \quad B^{\sigma^0,N}_{2,\infty} \neq B^0_{2,\infty}$$

but we do not go into details here.

• *Existence of a lift operator.*

The next theorem shows the existence of a lift operator between spaces of $B^{\sigma,N}_{p,q}$ and $F^{\sigma,N}_{p,q}$ type.

Theorem 10.1.9 *Let $(\sigma_j)_{j\in\mathbb{N}_0}$ and $(\beta_j)_{j\in\mathbb{N}_0}$ two admissible sequences and let $(\varphi^{N,J}_j)_{j\in\mathbb{N}_0}$ a function system associated to the strongly increasing sequence $(N_j)_{j\in\mathbb{N}_0}$.*
Then the operator $\mu(D)$ defined by the symbol

$$\mu(\xi) = \sum_{j=0}^{\infty} \sigma_j\, \beta_j^{-1}\, \varphi^{N,J}_j(\xi)$$

defines for all parameters $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$ an isomorphism between $B^{\sigma,N}_{p,q}$ and $B^{\beta,N}_{p,q}$ respectively between $F^{\sigma,N}_{p,q}$ and $F^{\beta,N}_{p,q}$.

The proof of the above result is given in [FaLe01, Theorem 3.1.8].

• *Duality*

In the following theorem the dual spaces of $B^{\sigma,N}_{p,q}$ and $F^{\sigma,N}_{p,q}$ are determined.
The previous results - see the end of Remark 10.1.2 - give the possibility to interpret the dual spaces $(B^{\sigma,N}_{p,q})'$ and $(F^{\sigma,N}_{p,q})'$ as subspaces of $\mathcal{S}'$. Furthermore, because $\mathcal{S}$ is dense in these spaces if $q < \infty$, f belongs to $(F^{\sigma,N}_{p,q})' \hookrightarrow \mathcal{S}'$ (similar for $(B^{\sigma,N}_{p,q})' \hookrightarrow \mathcal{S}'$), if, and only if, there is a number c such that for all $\psi \in \mathcal{S}$

$$| < f, \psi > | \leqslant c \, \|\psi \, | \, F^{\sigma,N}_{p,q}\| \quad . \tag{10.1}$$

For an admissible sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ we denote $1/\sigma = (1/\sigma_j)_{j\in\mathbb{N}_0}$. Clearly $1/\sigma$ is also admissible.

Theorem 10.1.10 *Let $N = (N_j)_{j\in\mathbb{N}_0}$ be a strongly increasing sequence and $(\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence. Furthermore, let $1 < p < \infty$, $1 \leqslant q < \infty$ and let p' and q' denote their conjugates.*
Then

$$(B^{\sigma,N}_{p,q})' = B^{1/\sigma,N}_{p',q'} \quad \textit{and} \quad (F^{\sigma,N}_{p,q})' = F^{1/\sigma,N}_{p',q'} \quad .$$

The proof follows essentially that from [Tr83, Theorem 2.11.2] and is given in [FaLe01, Theorem 3.1.9].

10.1.2 Special classes: function spaces of generalised smoothness associated to an admissible symbol

In the previous subsection we have introduced and considered function spaces of generalised smoothness associated to an arbitrary strongly increasing sequence N and to an admissible sequence σ.
In the last years there was an increasing interest in investigating function spaces of general smoothness for which the strongly increasing sequence N is associated (in a canonical way) to a fixed smooth function satisfying some reasonable conditions; we will call those smooth functions admissible symbols. In particular all considerations in Section 6 are special cases of what follows.

Definition 10.1.11 *Let $\mathcal{A}$ be the class of all non-negative functions $a : \mathbb{R}^n \to \mathbb{R}$ of class C^∞, with the following properties:*
(i) $\lim_{|\xi|\to\infty} a(\xi) = \infty$.
(ii) *a is* almost increasing in $|\xi|$, *i.e. there exists a constant $\delta_0 \geqslant 1$, and an $R > 0$ such that $a(\xi) \leqslant \delta_0 \, a(\eta)$ if $R \leqslant |\xi| \leqslant |\eta|$,*
(iii) *there exists an $m > 0$ such that $\xi \mapsto a(\xi)\langle\xi\rangle^{-m}$ is* almost decreasing in $|\xi|$, *i.e. there exists a constant δ_m, $0 < \delta_m \leqslant 1$, and an $R > 0$ such that*

$$a(\xi) \, \langle\xi\rangle^{-m} \geqslant \delta_m \, a(\eta) \, \langle\eta\rangle^{-m} \quad \textit{if} \quad R \leqslant |\xi| \leqslant |\eta|.$$

(iv) *for every multi-index $\alpha \in \mathbb{N}^n_0$ there exists some $c_\alpha > 0$ and an $R > 0$ such that*

$$|D^\alpha a(\xi)| \leqslant c_\alpha \, a(\xi) \, \langle\xi\rangle^{-|\alpha|} \quad \textit{if} \quad |\xi| \geqslant R. \tag{10.2}$$

The functions a from $\mathcal{A}$ are called admissible symbols.

Remark 10.1.12 Any admissible continuous negative definite function (compare Section 6) is an admissible symbol, the number m in the above definition has a similar role as the number w in Definition 6.1.1.

We want to point out that the class $\mathcal{A}$ is larger than the class $S^*(m, m', 0)$ considered in [Le90]. For $S^*(m, m'; 0)$ it was additionally required that there exists an $m' > 0$, such that $a(\xi)\,\langle\xi\rangle^{-m'}$ is almost increasing in $|\xi|$. Now the case $m' = 0$ and examples as $a(\xi) = (\log\langle\xi\rangle)^b$, with some positive $b > 0$, are also included.
However, the restriction that there exists an $m' > 0$, such that $a(\xi)\,\langle\xi\rangle^{-m'}$ is almost increasing in $|\xi|$ is a quite natural one if we take into account Corollary 7.2.5.

As in the case of admissible continuous negative definite functions we can associate a strongly increasing sequence to an admissible symbol.

Lemma 10.1.13 *For a function $a \in \mathcal{A}$ let*

$$N_j^a = \sup\{\langle\xi\rangle \,:\, a(\xi) \leqslant 2^j\} \quad , \quad \textit{for any} \quad j \in \mathbb{N}_0. \tag{10.3}$$

The sequence $N^a = (N_j^a)_{j\in\mathbb{N}_0}$ is a strongly increasing sequence in the sense of Definition 9.1.1.

Proof. It is clear from (10.3) that $N^a = (N_j^a)_{j\in\mathbb{N}_0}$ is increasing. Let us sketch the proof of the existence of a constant $\kappa_0 \in \mathbb{N}$ such that $2N_j^a \leqslant N_k^a$ for any j and k such that $j + \kappa_0 \leqslant k$.
For simplicity let us denote $N_j^a = N_j$ for any $j \in \mathbb{N}$. From the definition of the numbers N_j it follows that there exists an ξ_0 with

$$\frac{N_j}{2} \leqslant \langle\xi_0\rangle \leqslant N_j \quad \text{and} \quad a(\xi_0) \leqslant 2^j.$$

Due to the properties of the function a it is clear that the function $t \mapsto a(t\xi_0)$ is a one-dimensional continuous function with $\lim_{t\to\infty} a(t\xi_0) = \infty$. Consequently, for $\kappa_0 \in \mathbb{N}$, there exists a $t_0 > 1$ with

$$a(t_0\xi_0) = 2^{j-\frac{1}{2}+\kappa_0}.$$

Then taking $\eta_0 = t_0\xi_0$ one has $|\eta_0| = t_0|\xi_0| > |\xi_0|$ which of course means $\langle\eta_0\rangle > \langle\xi_0\rangle$ and

$$\langle\eta_0\rangle \leqslant N_{j+\kappa_0} = \sup\{\langle\eta\rangle \,:\, a(\eta) \leqslant 2^{j+\kappa_0}\}.$$

Applying now property (iii) from Definition 10.1.11 we have

$$\frac{2^j}{(N_j/2)^m} \geqslant \frac{a(\xi_0)}{\langle\xi_0\rangle^m} \geqslant \delta_m\,\frac{a(\eta_0)}{\langle\eta_0\rangle^m} \geqslant \delta_m\,\frac{2^{j-\frac{1}{2}+\kappa_0}}{N_{j+\kappa_0}^m}$$

for sufficiently large j (depending on R in property (iii)) and arbitrary $\kappa_0 \in \mathbb{N}$.
Consequently,

$$N_{j+\kappa_0} \geqslant \frac{1}{2}\left(\delta_m\,2^{\kappa_0-\frac{1}{2}}\right)^{1/m} N_j$$

and using the fact that $(N_j)_{j\in\mathbb{N}}$ is increasing we have $N_k \geqslant N_{j+\kappa_0} \geqslant 2\,N_j$ if $k \geqslant j + \kappa_0$ for a fixed large enough κ_0.
This completes the proof that $(N_j)_{j\in\mathbb{N}_0}$ is strongly increasing. ∎

Remark 10.1.14 Given an admissible function $a \in \mathcal{A}$ we can define for any $r > 0$

$$N_j^{a,r} = \sup\{\langle\xi\rangle \,:\, a(\xi) \leqslant 2^{jr}\} \quad , \quad \text{for any} \quad j \in \mathbb{N}_0. \tag{10.4}$$

Using the same technique as above it is easy to see that $N^{a,r} = (N_j^{a,r})_{j\in\mathbb{N}_0}$ is again a strongly increasing sequence.

The next result generalises Theorem 7.1.3. Recall the notation $\sigma^s = (2^{js})_{j\in\mathbb{N}_0}$.

Theorem 10.1.15 *Let $a \in \mathcal{A}$ an admissible symbol, let $r > 0$ and let $N = N^{a,r}$ the strongly increasing sequence associated to a and r, see (10.4). Let $1 < p < \infty$ and $1 < q < \infty$.*
Then for any real number s we have

$$\|(\mathrm{id} + a(D))^{s/r}\, u \,|\, F_{p,q}^{\sigma^0,N^{a,r}}\| \sim \|u \,|\, F_{p,q}^{\sigma^s,N^{a,r}}\|$$

and the corresponding assertion for B-spaces.

The proof of the above result is very close to that one given in Theorem 7.1.3 and it is given with all details in [FaLe01, Theorem 3.1.19] so we will omit it here.

As a simple consequence of the above theorem and of Theorem 10.1.5 we get

Corollary 10.1.16 *Let $a \in \mathcal{A}$ an admissible symbol, and let $N = N^{a,2}$ the strongly increasing sequence associated to a and to $r = 2$, see (10.4). Let $1 < p < \infty$.*
Then for any real number s we have

$$\|(\mathrm{id} + a(D))^{s/2} u] \,|\, L_p\| \sim \|u \,|\, F_{p,2}^{\sigma^s,N^{a,2}}\|.$$

Remark 10.1.17 Note that if $s > 0$ a similar result as stated in Corollary 10.1.16 was mentioned in [Ka79].
As an immediate corollary we get also Corollary 7.1.4.

10.2 The cases $0 < p \leqslant 1$ and $p = \infty$

To extend the definition of the spaces of generalised smoothness to $p = \infty$, $p = 1$, and to $0 < p < 1$, an additional assumption on the sequence N is necessary. The reason is, that we can not use in these cases the previous Fourier-multiplier theorem (Proposition 0.2.1).
A substitute of it is a Fourier-multiplier theorem which was proved in spaces of entire analytic functions with the help of maximal functions.

Proposition 10.2.1 *Let $0 < p < \infty$, $0 < q \leqslant \infty$. For every $j \in \mathbb{N}_0$ let $R_j > 0$ be a given number, let $\Omega_j = \{\xi \in \mathbb{R}^n \,:\, |\xi| \leqslant R_j\}$ and let $\Omega = (\Omega_j)_{j\in\mathbb{N}_0}$.*
If $0 < t < min\,(p,q)$ then there exists a constant $c > 0$ such that

$$\left\| \left(\sup_{z\in\mathbb{R}^n} \frac{|f_j(\cdot - z)|}{1 + |R_j z|^{n/t}} \right)_{j\in\mathbb{N}_0} |\, L_p(l_q) \right\| \leqslant c\, \|f \,|\, L_p(l_q)\|$$

for all $f = (f_j)_{j\in\mathbb{N}_0} \in L_p(l_q)$ such that $\operatorname{supp} \mathcal{F} f_j \subset \Omega_j$ for all $j \in \mathbb{N}_0$.

This result was proved in [Tr83, Theorem 1.6.2] and was the main tool in the proof of the following Fourier-multiplier theorem - [Tr83, Theorem 1.6.3].

Proposition 10.2.2 *Let $0 < p < \infty, 0 < q \leqslant \infty$. Let $(\Omega_j)_{j\in\mathbb{N}_0}$ be a sequence of compact subsets of $\mathbb{R}^n$ and $d_j > 0$ be the diameter of Ω_j.*
If $t > n/2 + n/\min(p,q)$, then there exists a constant $c > 0$ such that

$$\|(M_j(D)f_j)_{j\in\mathbb{N}_0} \,|\, L_p(l_q)\| \leqslant c \sup_{j\in\mathbb{N}_0} \|M_j(d_j\,\cdot\,) \,|\, H_2^t\| \;\cdot \|(f_j)_{j\in\mathbb{N}_0} \,|\, L_p(l_q)\|$$

holds for all systems $(f_j)_{j\in\mathbb{N}_0} \in L_p(l_q)$ with $\operatorname{supp}\mathcal{F}f_j \subset \Omega_j$ for all j, and all sequences $(M_j)_{j\in\mathbb{N}_0} \subset H_2^t$, where H_2^t is the standard Bessel potential space of smoothness t.

Let $(\varphi_j^{N,J})_{j\in\mathbb{N}_0}$ be a usual system associated to a strongly increasing sequence $N = (N_j)_{j\in\mathbb{N}_0}$. Then an easy computation shows that for an integer L we have

$$\|\varphi_j^{N,J}(2N_{j+J\kappa_0}\,\cdot\,) \,|\, W_2^L\| \leqslant c\,(2N_{j+J\kappa_0}\,N_{j-J\kappa_0}^{-1})^L \tag{10.5}$$

but unfortunately the right-hand side is - in general - not uniformly bounded with respect to j. This happens only if the sequence $(N_j)_{j\in\mathbb{N}_0}$ is additionally of bounded growth.
Assuming N is of bounded growth with $N_{j+1} \leqslant \lambda_1 N_j$ the right-hand side of (10.5) can be estimated for arbitrary j by $c\,\lambda_1^{2J\kappa_0}$.

With this preparation we extend the definition of the spaces $B_{p,q}^{\sigma,N}$ and $F_{p,q}^{\sigma,N}$ to all $0 < p \leqslant \infty$ and $0 < p < \infty$, respectively.

Definition 10.2.3 *Let $(N_j)_{j\in\mathbb{N}_0}$ be a strongly increasing sequence and of bounded growth and let $\varphi^{N,J} \in \Phi^{N,J}$. Let $(\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence.*
(i) *Let $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$. The Besov space of generalised smoothness is*

$$B_{p,q}^{\sigma,N} = \left\{ f \in \mathcal{S}' \,:\, \|f \,|\, B_{p,q}^{\sigma,N}\| = \|(\sigma_j\,\varphi_j^{N,J}(D)f)_{j\in\mathbb{N}_0} \,|\, l_q(L_p)\| < \infty \right\}.$$

(ii) *Let $0 < p < \infty$ and $0 < q \leqslant \infty$. The Triebel - Lizorkin space of generalised smoothness is*

$$F_{p,q}^{\sigma,N} = \left\{ f \in \mathcal{S}' \,:\, \|f \,|\, F_{p,q}^{\sigma,N}\| = \left\|(\sigma_j\,\varphi_j^{N,J}(D)f)_{j\in\mathbb{N}_0} \,|\, L_p(l_q)\right\| < \infty \right\}.$$

Of course it can be easily shown that all standard results (independence of the system $(\varphi_j)_{j\in\mathbb{N}_0}$, density of $\mathcal{S}$, embeddings, lift-operator, etc.) extend to the whole scale of spaces considered in the above definition.
In Section 11 we will consider strongly increasing sequences N which are of bounded growth so we will be able to deal with all admissible parameters in Definition 10.2.3 of the spaces $B_{p,q}^{\sigma,N}$ and $F_{p,q}^{\sigma,N}$.

10.3 Examples and comparison with other classes

10.3.1 Further classes of function spaces of generalised smoothness

As we have have already mentioned, if $N_j = 2^j$ and $\sigma = (2^{js})_{j\in\mathbb{N}_0}$ then $B^{\sigma,N}_{p,q}$ and $F^{\sigma,N}_{p,q}$ are the classical Besov and Triebel - Lizorkin spaces $B^s_{p,q}$ and $F^s_{p,q}$.

Due to Corollary 7.1.4 if ψ is a real valued admissible negative definite function and $(N_j)_{j\in\mathbb{N}_0}$ is the strongly increasing sequence associated to ψ and to $r = 2$ then $H^{\psi,s}_p = F^{\sigma^s,N}_{p,2}$ for $s \in \mathbb{R}$ and $1 < p < \infty$.

It is the aim of this subsection to show that the function spaces considered so far in this work cover (besides the classical Besov and Triebel - Lizorkin spaces $B^s_{p,q}$ and $F^s_{p,q}$ and the ψ-Bessel potential spaces for admissible ψ) many other classes of function spaces of generalised smoothness known in the literature.
For simplicity we will restrict ourselves in this part to function spaces of Besov type. The scale of F-spaces is usually defined in most of the cases in a natural way but we will not go into details.

• In the middle of the seventies M. L. Goldman and G. A. Kalyabin introduced and investigated independently function spaces of generalised smoothness. These spaces are defined on the basis of expansions in series of entire functions, and are connected with a general covering method - see [Go79], [Ka79], [Ka80], [Go80] or [Go89].
Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, let $(N_j)_{j\in\mathbb{N}_0}$ be strongly increasing, let $(\alpha_j)_{j\in\mathbb{N}_0}$ of bounded growth and $(\alpha_j^{-1})_{j\in\mathbb{N}_0} \in l_{q'}$. Then let $\mathbf{B}^{\alpha,N}_{p,q}(\mathbb{R}^n)$ be the collection of all $f \in L_p$ such that

$$f = \sum_{j=1}^{\infty} f_j \quad \text{in} \quad L_p$$

with $\operatorname{supp}(\mathcal{F} f_j) \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant N_j\}$ and $\|(\alpha_j\, f_j)_{j\in\mathbb{N}_0} \,|\, l_q(L_p)\| < \infty$.
By a standardisation result see [Ka77b], $\mathbf{B}^{\alpha,N}_{p,q}(\mathbb{R}^n)$ can be identified with a space $\mathbf{B}^{\beta,M}_{p,q}(\mathbb{R}^n)$, where $\beta = (\beta_j)_{j\in\mathbb{N}_0}$ almost strongly increasing and of bounded growth (and therefore an admissible sequence), and where the sequence $M = (M_j)_{j\in\mathbb{N}_0}$ is determined by the sequences β, α and N via

$$M_k = N_{\kappa(k)} \qquad \text{with} \qquad \kappa(k) = \min\left\{ m : \sum_{j=m}^{\infty} \alpha_j^{-q'} \leqslant \beta_k^{-q'} \right\}.$$

A simple calculation shows that both $\|f \,|\, \mathbf{B}^{\alpha,N}_{p,q}\|$ and $\|f \,|\, \mathbf{B}^{\beta,M}_{p,q}\|$ are equivalent to

$$\|f \,|\, B^{\beta,M}_{p,q}\| = \|(\beta_k\, \varphi_k^{M,K}(D) f)_{k\in\mathbb{N}_0} \,|\, l_q(L_p)\|$$

where $(\varphi_k^{M,K})_{k\in\mathbb{N}_0}$ is a system from Section 9.2 associated to the covering defined by the sequence $(M_k)_{k\in\mathbb{N}_0}$ above. Consequently, the above spaces are a subclass of Besov spaces of generalised smoothness as introduced in Section 10.1.
Thus, in this way function spaces with 'positive' generalised smoothness whose elements are at least L_p-functions can be described.
Many results are known for the spaces $\mathbf{B}^{\alpha,N}_{p,q}$, for example embedding theorems - see [Ka81], [Go80], [Go84b], [Go85] or [Go92], trace theorems - see [Ka78], [Ka79], [Go79],

[Go80] and characterisations by differences and moduli of continuity - see [Go76], [Ka77b], [Ka80].
The last one leads to the following characterisation or definition, often used by M. L. Goldman.
Let $\lambda : (0,1) \to \mathbb{R}^+$ be a non-decreasing, continuous function with $\lim\limits_{t\downarrow 0} \lambda(t) = 0$, $M \in \mathbb{N}$ and $1 \leqslant p\,,\, q \leqslant \infty$. Let

$$B^{\lambda}_{p,q}(\mathbb{R}^n) = \left\{ f \in L_p \,:\, \Big(\int_0^1 \big(\frac{\omega_p^M(f,t)}{\lambda(t)}\big)^q \,\frac{d\lambda(t)}{\lambda(t)}\Big)^{1/q} < \infty \right\}$$

where

$$\omega_p^M(f,t) = \sup_{|h|<t} \|\Delta_h^M u(\cdot)\,|\,L_p\|$$

and $\Delta_h^M = \Delta_h^1 \Delta_h^{M-1}$ where $\Delta_h^1 u(x) = u(x+h) - u(x)$.
If, in addition, $t \mapsto \lambda(t)t^{-M}$ is increasing and $t \mapsto \lambda(t)t^{-\delta}$ is almost decreasing then

$$B^{\lambda}_{p,q}(\mathbb{R}^n) = B^{\alpha,N}_{p,q}(\mathbb{R}^n)$$

with $\alpha_j = 2^j$, $N_j = h_j^{-1}$, $\lambda(h_j) = 2^{-j}\lambda(1)$, compare [Go76], [KaLi87, Theorem 8.2] or, for a similar form, see [Ka80].

• In [Tr77, Chapter 2] a general covering method was also introduced and used to define and investigate general function spaces of Besov-Hardy-Sobolev type $B^{s(x)}_{p,q}$ and $F^{s(x)}_{p,q}$ on $\mathbb{R}^n$. This approach was used also in [Go80]. It contains isotropic spaces, anisotropic spaces, spaces with dominating mixed derivatives and some other. In case of the usual weight sequence $(2^j)_{j\in\mathbb{N}_0}$ all these special spaces were studied in detail in [ScTr87], [Tr83], [Tr92].
However the general approach was not developed further in its full generality.

• Other function spaces of generalised smoothness appear as a result of real interpolation with a function parameter - see [Mer86] and [CoFe86]. In these papers, a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ belongs to the class $\mathcal{B}$ if ψ is continuous, $\psi(1) = 1$ and for all $t \in (0,\infty)$ one has

$$\bar{\psi}(t) = \sup_{s>0} \frac{\psi(ts)}{\psi(s)} < \infty.$$

Let $1 < p,\, q < \infty$, let $\psi \in \mathcal{B}$ and $(\varphi_j)_{j\in\mathbb{N}_0}$ is the usual resolution of unity, associated to the symbol $|\xi|^2$ of the Laplacian and to the sequence $N_j^{|\cdot|^2,2} = 2^j$. Then

$$B^{\psi}_{p,q}(\mathbb{R}^n) = \big\{ u \in \mathcal{S}' : \big(\sum_{j=0}^{\infty} \psi(2^j)^q \|\varphi_j(D)u\,|\,L_p\|^q \big)^{1/q} < \infty \big\}.$$

One has the following interpolation result

$$\big(L_p(\mathbb{R}^n), W^k_p(\mathbb{R}^n)\big)_{\rho_1,q} = B^{\rho}_{p,q}(\mathbb{R}^n) \quad \text{where} \quad \rho(t) = (\rho_1(t^{-k}))^{-1}\ .$$

Here $\psi(t) = t^s \log{(1+t)}^b$ with $s < 0$ is now an admissible function too, related to the sequence $\alpha_j = 2^{js} j^b$ which does not fulfill $(\alpha_j^{-1})_{j\in\mathbb{N}_0} \in l_{q'}$. On the other hand, the decomposition is always fixed by the sequence $(N_j)_{j\in\mathbb{N}_0} = (2^j)_{j\in\mathbb{N}_0}$.

With $\sigma_j = \psi(2^j)$, we have

$$\bar{\psi}(1/2)\sigma_j \leqslant \sigma_{j+1} \leqslant \bar{\psi}(2)\sigma_j \quad ,$$

which means $(\sigma_j)_{j\in\mathbb{N}_0}$ is an admissible sequence and consequently these spaces are covered by Definition 10.1.1.

• Generalising fractal d-sets to (d,Ψ)-sets, D. E. Edmunds and H. Triebel considered in [EdTr98] and [EdTr99] function spaces of generalised smoothness, related to Example 9.1.7.
Let Ψ be a positive monotone function on the interval $(0,1]$ such that there are some positive constants b_0^* and b_1^* with $b_0^*\Psi(2^{-j}) \leqslant \Psi(2^{-2j}) \leqslant b_1^*\Psi(2^{-j})$ for any $j \in \mathbb{N}_0$. Then

$$B_{p,q}^{(s,\Psi)} = \left\{u \in \mathcal{S}' \,:\, \|u \,|\, B_{p,q}^{(s,\Psi)}\| = \Big(\sum_{j=0}^{\infty}(2^{js}\Psi(2^{-j}))^q \,\|\varphi_j(D)u \,|\, L_p\|^q\Big)^{1/q} < \infty\right\}$$

(modification if $q=\infty$). Here $(\varphi_j)_{j\in\mathbb{N}_0}$ is again the usual resolution of unity associated to the sequence $N_j = 2^j$.
Including the F-spaces an extensive study of these scales of spaces - embeddings, lifting properties, subatomic decompositions, local means, function spaces on fractals, entropy numbers and applications - was done by S. Moura in [Mo99] and [Mo01].
Again $\sigma_j = 2^{js}\Psi(2^{-j})$ is an admissible sequence and $N_j = 2^j$ is strongly increasing and of bounded growth. So, for all admissible parameters p, q, these spaces are covered by Definition 10.2.3, too.

• The above described function spaces were generalised and studied by M. Bricchi in his thesis, see [Bri02]. He replaced the function Ψ by a slowly varying function H (see the next subsection) and obtained atomic and subatomic decompositions. Again, these spaces are covered by our Definition 10.2.3.

• In [OpTr99] generalised smoothness of 'logarithmic' order was used to describe general embeddings of Pohozhaev-Trudinger type. The spaces under consideration in [OpTr99] are defined as

$$H^{\sigma,\alpha}(L_p)(\mathbb{R}^n) = \{u : u \in L_p \quad \text{and} \quad u = g_{\sigma,\alpha} * f \;,\; f \in L_p\}$$

with $\mathcal{F}g_{\sigma,\alpha}(\xi) = (1+|\xi|^2)^{-\sigma/2}(1+\log(1+|\xi|^2))^{-\alpha}$, $\sigma \geqslant 0$, α real.
Similarly they defined spaces $H^{\sigma,\alpha}(L_{p,r}(\log L)^{\beta})(\mathbb{R}^n)$ where L_p is replaced by some suitable Lorentz-Zygmund space.
The first case is again covered by our definition, compare Theorem 10.1.15 and Corollary 10.1.16 with $a(\xi) = \mathcal{F}g_{\sigma,\alpha}(\xi)$.

10.3.2 Examples of representatives in function spaces of generalised smoothness

It is generally accepted that one of the main reason for introducing and studying function spaces is the fact that one wants to measure smoothness of functions. Hence, it is of interest to give some specific examples of functions belonging to spaces $B_{p,q}^{\sigma,N}$ and $F_{p,q}^{\sigma,N}$ showing in particular the diversity of these spaces from the usual Besov and Triebel-Lizorkin spaces.

Those examples where discussed in all details in [Bri02, Section 2.2.2] so we will mention only the results obtained there without proofs.
To present briefly the examples we need the following

Definition 10.3.1 *Let H be a positive function defined on $(0,1]$ which is monotone. If there exists some $s \in (0,1)$ such that*

$$\lim_{r\to 0} \frac{H(sr)}{H(r)} = 1$$

then H is called slowly varying.

Trivially, positive measurable functions with positive limit at 0 (in particular) positive constants are slowly varying.
The first non-trivial example is $L_1(r) = |\log r|$. The iterated logarithms $L_2(r) = \log|\log r|$,

$$L_k(r) = \underbrace{\log \ldots \log}_{k-1} |\log r|$$

are also slowly varying.
Note that a slowly varying function may oscillate, an example being

$$H(r) = e^{|\log r|^{1/3} \cos(|\log r|)^{1/3}}.$$

In the examples below, let H a smooth slowly varying function with either $H(0+) = 0$ or $H(0+) = \infty$. Let $\sigma = (2^{js} H(2^{-j}))_{j\in\mathbb{N}_0}$ and let $N = (2^j)_{j\in\mathbb{N}_0}$.
For simplicity in the examples below we will use the notation

$$B^{s,H}_{p,q} = B^{\sigma,N}_{p,q}.$$

Example 10.3.2 It is well-known that the Dirac distribution $\delta \in B^{n/p-n}_{p,q}$ (classical space) if, and only if, $q = \infty$.
(i) If H is decreasing then $\delta \notin B^{n/p-n,H}_{p,\infty}$ as shown in [Bri02, Example 2.2.24].
(ii) If H is increasing and $(H(2^{-k}))_{k\in\mathbb{N}_0} \in l_q$ then $\delta \in B^{n/p-n,H}_{p,q}$.
Hence, if $q \neq \infty$, one gets that$B^{n/p-n}_{p,q}$ is strictly smaller than $B^{n/p-n,H}_{p,q}$ under the assumption $(H(2^{-k}))_{k\in\mathbb{N}_0} \in l_q$.

Example 10.3.3 Let $a^2 + b^2 > 0$ and $b \geqslant 0$ and define

$$f_{a,b}(x) = \varphi(x)\,|x|^a\,(-\log|x|)^{-b}$$

where φ is a smooth cut-off function with support near the origin. This example was discussed by T. Runst and W. Sickel in [RuSi96, Lemma 1, p. 44] to give examples of representatives of classical $B^s_{p,q}$ and $F^s_{p,q}$.
If $H(r) = |\log r|^{\frac{qb-2}{q}}$ and if $qb < 1$ then $f_{a,b} \in B^{n/p+a,H}_{p,q}$ but $f_{a,b} \notin B^{n/p+a}_{p,q}$.

Example 10.3.4 In this example we restrict our attention to the one-dimensional case, i.e. all functions are defined in $\mathbb{R}$. Let χ the characteristic function of $(-1/2, 1/2]$ and consider its convolution iterates

$$M_r(x) = \underbrace{(\chi * \ldots * \chi)}_{r-\text{times}}(x).$$

In general M_r is a piecewise polynomial of order r with respect to the grid $r/2+k$, $k \in \mathbb{Z}$. In [Bri02, Example 2.2.27] it was shown the following:
(i) If H is decreasing, then (Dirac's distribution) $\delta \notin B^{r-1+1/p,H}_{p,\infty}(\mathbb{R})$ and moreover $M_r \in B^{r-1+1/p}_{p,\infty}(\mathbb{R}) \setminus B^{r-1+1/p,H}_{p,\infty}(\mathbb{R})$.
(ii) If H is increasing and $(H(2^{-k})_{k\in\mathbb{N}_0})_{k\in\mathbb{N}_0} \in l_q$ for some $q \neq \infty$ then
$M_r \in B^{r-1+1/p,H}_{p,q}(\mathbb{R}) \setminus B^{r-1+1/p}_{p,q}(\mathbb{R})$.

11 Local means

11.1 Preliminaries

Assumption 11.1.1 From now on we will assume $N = (N_j)_{j\in\mathbb{N}_0}$ is a sequence of real positive numbers such that there exist two numbers $1 < \lambda_0 \leqslant \lambda_1$ with

$$\lambda_0 N_j \leqslant N_{j+1} \leqslant \lambda_1 N_j \quad , \quad \text{for any} \quad j \in \mathbb{N}_0. \tag{11.1}$$

In particular N is strongly increasing and of bounded growth.
We would like to point out that the condition $\lambda_0 > 1$ plays a key role in all the following considerations.

Remark 11.1.2 The assumption concerning λ_0 is not so restrictive. Let $(M_j)_{j\in\mathbb{N}_0}$ be strongly increasing and of bounded growth and let $(\beta_j)_{j\in\mathbb{N}_0}$ be an admissible sequence. Defining

$$N_j = M_{j\kappa_0} \quad \text{and} \quad \sigma_j = \beta_{j\kappa_0}$$

it is easy to see that the sequence $(N_j)_{j\in\mathbb{N}_0}$ satisfies (11.1) with $\lambda_0 = 2$ and

$$B^{\beta,M}_{p,q} = B^{\sigma,N}_{p,q} \quad \text{and} \quad F^{\beta,M}_{p,q} = F^{\sigma,N}_{p,q}.$$

This observation is similar to that in [Ka88, Remark 1].

Assumption 11.1.3 To avoid technical complications we will assume

$$N_1 \geqslant \lambda_1. \tag{11.2}$$

We should note that there is no loss of generality in assuming (11.2). Indeed, since $\lambda_0 > 1$, there exists an $m \in \mathbb{N}$ such that $\lambda_0^m N_0 \geqslant \lambda_1$. Let

$$m_1 = \min\{m \in \mathbb{N} \,:\, \lambda_0^m N_0 \geqslant \lambda_1\}$$

and so $N_{m_1} \geqslant \lambda_0^{m_1} N_0 \geqslant \lambda_1$. If we would not have $N_1 \geqslant \lambda_1$ then in all considerations below one has to replace N_1 with N_{m_1}.

Assumption 11.1.4 We will always denote $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ an admissible sequence, this means that there are two constants $0 < d_0 \leqslant d_1 < \infty$ such that

$$d_0\,\sigma_j \leqslant \sigma_{j+1} \leqslant d_1 \sigma_j \quad \text{for any} \quad j \in \mathbb{N}_0. \tag{11.3}$$

Under the above conditions on the sequences N and σ, the aim of this section is to give equivalent quasi-norms for the spaces $B^{\sigma,N}_{p,q}$ $(0 < p,q \leqslant \infty)$ and $F^{\sigma,N}_{p,q}$ $(0 < p < \infty$, $0 < q \leqslant \infty)$ in terms of maximal functions, local means, and atomic decompositions.

11.2 Equivalent quasi-norms

Let $N = (N_j)_{j\in\mathbb{N}_0}$ be an admissible sequence of positive numbers satisfying (11.1) with $\lambda_0 > 1$. Then there exists an natural number $l_0 = l_0(\lambda_0, \lambda_1)$ with

$$\lambda_0^{l_0} > \lambda_1. \tag{11.4}$$

We choose an $l_0 > 1$ satisfying (11.4) and fix it from now on.
Let $\mu_0, \mu \in \mathcal{S}$ be two *positive* functions on $\mathbb{R}^n$ such that

$$\mu_0(\xi) = 1 \quad \text{if} \quad |\xi| \leqslant N_1 \quad \text{and} \quad \operatorname{supp}\mu_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant \lambda_0\, N_1\} \tag{11.5}$$

and

$$\mu(\xi) = 1 \quad \text{if} \quad \frac{1}{\lambda_1} \leqslant |\xi| \leqslant \lambda_1 \quad \text{and} \quad \operatorname{supp}\mu \subset \{\xi \in \mathbb{R}^n : \frac{1}{\lambda_0^{l_0}} \leqslant |\xi| \leqslant \lambda_0^{l_0}\}. \tag{11.6}$$

For any $j \geqslant 1$ we define

$$\mu_j(\xi) = \mu(N_j^{-1}\xi), \quad \xi \in \mathbb{R}^n.$$

Remark 11.2.1 Using (11.1) it is easy to see that we have

$$\operatorname{supp}\mu_j \subset \{\xi \in \mathbb{R}^n : N_{j-l_0} \leqslant |\xi| \leqslant N_{j+l_0}\} \quad \text{for any} \quad j \geqslant 1.$$

This shows that for each fixed $j_0 \in \mathbb{N}$ the set $\operatorname{supp}\mu_{j_0}$ has a non-empty intersection with at most $4l_0 + 1$ different supports of the functions μ_j.
Moreover, a simple computation shows that for any multi-index α there is a constant c_α (depending on μ but not on j) such that

$$|D^\alpha \mu_j(\xi)| \leqslant c_\alpha \langle\xi\rangle^{-|\alpha|} \quad \text{for any} \quad \xi \in \mathbb{R}^n \quad \text{and any} \quad j \in \mathbb{N}.$$

Note that the family $(\mu_j)_{j\in\mathbb{N}_0}$ does not - in general - satisfy a condition of type (9.10) - resolution of "unity". However, we have a counterpart of (9.10) which reads as follows:

$$\mu_0(\xi) + \sum_{j=1}^{\infty} \mu_j(\xi) \geqslant 1 \quad \text{for any} \quad \xi \in \mathbb{R}^n. \tag{11.7}$$

Indeed, the sum in (11.7) is finite and each function μ_j, $j \in \mathbb{N}_0$, is positive. If now $|\xi| \leqslant N_1$ then $\mu_0(\xi) = 1$; if there is a $j_0 \geqslant 2$ such that $N_{j_0-1} \leqslant |\xi| \leqslant N_{j_0+1}$ then it follows $\frac{1}{\lambda_1} N_{j_0} \leqslant |\xi| \leqslant \lambda_1 N_{j_0}$ and this implies $\mu_{j_0}(\xi) = 1$.

Theorem 11.2.2 *Under the above assumptions on the sequences $(N_j)_{j\in\mathbb{N}_0}$, $(\sigma_j)_{j\in\mathbb{N}_0}$ and on the functions μ_0 and μ, we have:*
(i) *if $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$ then*

$$\|f \mid B^{\sigma,N}_{p,q}\|_\mu = \|\mu_0(D)f \mid L_p\| + \left(\sum_{j=1}^{\infty} \sigma_j^q \, \|\mu_j(D)f \mid L_p\|^q\right)^{1/q}$$

(with the usual modification if $q = \infty$) is an equivalent quasi-norm in $B^{\sigma,N}_{p,q}$;
(ii) *if $0 < p < \infty$ and $0 < q \leqslant \infty$ then*

$$\|f \mid F^{\sigma,N}_{p,q}\|_\mu = \|\mu_0(D)f \mid L_p\| + \left\| \left(\sum_{j=1}^{\infty} \sigma_j^q \, |\mu_j(D)f(\cdot)|^q\right)^{1/q} \Big| \, L_p \right\| \tag{11.8}$$

(with the usual modification if $q = \infty$) is an equivalent quasi-norm in $F^{\sigma,N}_{p,q}$.

Proof. We will indicate the proof in the more complicated case of F-spaces. To do this we will apply Proposition 10.2.2.
Let $(\varphi_j)_{j\in\mathbb{N}_0}$ be a smooth partition of unity satisfying (9.7) - (9.10), with $c_\varphi = 1$, and let $\|f \mid F^{\sigma,N}_{p,q}\|_\varphi$ be the quasi-norm from Definition 10.2.3.
Choose $t > \frac{n}{2} + \frac{n}{\min(p,q)}$ an integer. Since for any $j \in \mathbb{N}_0$ clearly $\mu_j(\xi) = 1$ on $\operatorname{supp}\varphi_j$ we get applying Proposition 10.2.2

$$\begin{aligned}
\|f \mid F^{\sigma,N}_{p,q}\|_\varphi &= \|\sigma_j \mathcal{F}^{-1}[\varphi_j\mu_j\,\mathcal{F}f] \mid L_p(l_q)\| \\
&= \|\mathcal{F}^{-1}\varphi_j\,\mathcal{F}\left(\sigma_j\,\mathcal{F}^{-1}[\mu_j\mathcal{F}f]\right) \mid L_p(l_q)\| \\
&\leqslant c\sup_{j\in\mathbb{N}_0}\|\varphi_j(2N_{j+l_0}\cdot) \mid H^t_2\| \cdot \|f \mid F^{\sigma,N}_{p,q}\|_\mu \\
&\leqslant c' \cdot \|f \mid F^{\sigma,N}_{p,q}\|_\mu \qquad (11.9)
\end{aligned}$$

where $\|f \mid F^{\sigma,N}_{p,q}\|_\mu$ is the quasi-norm from (11.8) and we have used the fact that for any α with $|\alpha| \leqslant t$ there exists a constant $c_\alpha > 0$ with

$$\|D^\alpha\varphi_j(N_{j+1}\cdot) \mid L_2\| \leqslant c_\alpha \quad \text{for any} \quad j \in \mathbb{N}_0$$

as a simple consequence of properties (9.8) and (9.9).
To prove the reverse inequality we note that due to the support properties of the functions μ_j we have for any $j \in \mathbb{N}_0$

$$\mu_j = \sum_{k=-2l_0}^{2l_0} \mu_j\varphi_{j+k}$$

where $\varphi_{-2l_0} = \varphi_{-2l_0+1} = ... = \varphi_{-1} = 0$. Then one has to apply again Proposition 10.2.2 and to make appropriate changes in (11.9).
Consequently, $\|f \mid F^{\sigma,N}_{p,q}\|_\varphi$ and $\|f \mid F^{\sigma,N}_{p,q}\|_\mu$ are equivalent. ∎

11.3 Maximal functions and local means

11.3.1 Some preparatory results

Before stating the main result of this subsection, see Theorem 11.3.4 below, we have to give some auxiliary results.
For any smooth function μ and for any $t > 0$ we will use the notation

$$\mu_t(x) = t^{-n}\mu(t^{-1}x). \qquad (11.10)$$

The next Lemma will play a key role in our further considerations.

Lemma 11.3.1 *Let $M \geqslant -1$ an integer and*

$$\mathcal{S}_M = \{\mu \in \mathcal{S} \,:\, D^\alpha\widehat{\mu}(0) = 0 \quad \textit{for any} \quad |\alpha| \leqslant M\}.$$

For any $L > 0$ there exists a constant $C_L > 0$ such that

$$\begin{aligned}
\sup_{z\in\mathbb{R}^n}|(\mu_t * \eta)(z)|(1+|z|)^L &\leqslant C_L \cdot t^{M+1} \cdot \max_{M+1\leqslant|\beta|\leqslant\max(M+1,L+1)}\|D^\beta\widehat{\mu} \mid L_\infty\| \cdot \\
&\quad \cdot \max_{|\gamma|\leqslant L+1}\int_{\mathbb{R}^n}(1+|\xi|)^{M+1}|D^\gamma\widehat{\eta}(\xi)|\,d\xi \qquad (11.11)
\end{aligned}$$

for any $t \in (0,1]$, for any $\mu \in \mathcal{S}_M$ and any $\eta \in \mathcal{S}$.

Proof. By elementary properties of the Fourier transform it is easy to show that for any $L > 0$ there exists a constant c_L such that for any $g \in \mathcal{S}$

$$\sup_{z\in\mathbb{R}^n} |g(z)|(1+|z|)^L \leqslant c_L \cdot \max_{|\alpha|\leqslant L+1} \|D^\alpha \widehat{g} \,|\, L_1\|. \tag{11.12}$$

Taking $t \in (0,1]$, $\mu \in \mathcal{S}_M$ and $\eta \in \mathcal{S}$ and inserting $g = \mu_t * \eta$ in (11.12) we have in particular,

$$\sup_{z\in\mathbb{R}^n} |(\mu_t * \eta)(z)| \leqslant c_L \cdot \max_{|\alpha|\leqslant L+1} \|D^\alpha \widehat{\mu_t * \eta} \,|\, L_1\|. \tag{11.13}$$

Applying Leibniz's product rule for differentiation we have

$$\begin{aligned} |D^\alpha[\widehat{\mu_t * \eta}(\xi)]| &\leqslant c_\alpha \sum_{|\delta|+|\gamma|=|\alpha|} |D^\delta[\widehat{\mu}(t\xi)]| \cdot |D^\gamma \widehat{\eta}(\xi)| \\ &= c_\alpha \sum_{|\delta|+|\gamma|=\alpha} t^{|\delta|} \cdot |(D^\delta \widehat{\mu})(t\xi)| \cdot |D^\gamma \widehat{\eta}(\xi)|. \end{aligned} \tag{11.14}$$

Fix now $\delta \leqslant \alpha$. Recall $D^\alpha \widehat{\mu}(0) = 0$ for any $|\alpha| \leqslant M$. Then for any δ with $|\delta| \leqslant M$ we have by Taylor's expansion theorem (with some positive constant c_δ)

$$|(D^\delta \widehat{\mu})(t\xi)| \leqslant c_\delta \max_{|\beta|=M+1} \|D^\beta \widehat{\mu} \,|\, L_\infty\| \cdot (t|\xi|)^{M-|\delta|+1}$$

and so

$$t^{|\delta|} \cdot |(D^\delta \widehat{\mu})(t\xi)| \leqslant c_\delta t^{M+1} \left(\max_{|\beta|=M+1} \|D^\beta \widehat{\mu} \,|\, L_\infty\| \right) \cdot (1+|\xi|)^{M+1} \quad \text{for any} \quad |\delta| \leqslant M. \tag{11.15}$$

We have now to distinguish if $M \geqslant L$ or not.
If $M \geqslant L$ then clearly the desired estimate (11.11) is a simple consequence of (11.13) using (11.14) and (11.15).
If $M < L$ then for a multi-index $\delta \leqslant \alpha$ it might happen $M+1 \leqslant |\delta| \leqslant L+1$. Then for any such a δ we have (recall $0 < t \leqslant 1$)

$$t^{|\delta|} \cdot |(D^\delta \widehat{\mu})(t\xi)| \leqslant c_\delta \cdot t^{M+1} \cdot \max_{M+1\leqslant|\beta|\leqslant L+1} \|D^\beta \widehat{\mu} \,|\, L_\infty\|. \tag{11.16}$$

Using (11.15) and (11.16) in (11.14), the inequality (11.11) follows again from (11.13). ■

Another result which we will use is the following

Lemma 11.3.2 *Let $0 < p, q \leqslant \infty$, $\rho > 0$. For any sequence $(g_j)_{j\in\mathbb{N}_0}$ of nonnegative measurable functions denote*

$$G_j(x) = \sum_{m=0}^{\infty} 2^{-|j-m|\rho} g_m(x), \quad x \in \mathbb{R}^n.$$

Then there exist some positive constants $c_1 = c(q, \varrho)$ and $c_2 = c_2(p, q, \varrho)$ such that

$$\|(G_j)_{j\in\mathbb{N}_0} \,|\, L_p(l_q)\| \leqslant c_1 \|(g_j)_{j\in\mathbb{N}_0} \,|\, L_p(l_q)\|$$

and

$$\|(G_j)_{j\in\mathbb{N}_0} \,|\, l_q(L_p)\| \leqslant c_2 \|(g_j)_{j\in\mathbb{N}_0} \,|\, l_q(L_p)\|.$$

The above lemma is well known and widely used. A proof can be found for example in [Ry99, Lemma 2]. We do not go into further details.

Let again $(N_j)_{j\in\mathbb{N}_0}$ be a sequence satisfying (11.1) with $\lambda_0 > 1$. We will also need

Lemma 11.3.3 *Let* $0 < \varrho \leqslant 1$ *and* $(b_j)_{j\in\mathbb{N}_0}$, $(a_j)_{j\in\mathbb{N}_0}$ *be two sequences taking values in* $(0,\infty]$ *respectively* $(0,\infty)$. *Assume that for some* $A_0 > 0$

$$\lim_{j\to\infty} a_j N_j^{-A_0} \quad \textit{exists in} \quad \mathbb{R} \tag{11.17}$$

and that for any $A > 0$ *there is a positive constant* C_A *such that*

$$a_j \leqslant C_A \sum_{l=j}^{\infty} (N_j N_l^{-1})^A b_l a_l^{1-\varrho} \quad , \quad j \in \mathbb{N}_0. \tag{11.18}$$

Then for any $A > 0$ *we have*

$$a_j^{\varrho} \leqslant C_A \sum_{l=j}^{\infty} (N_j N_l^{-1})^{A\varrho} b_l \quad , \quad j \in \mathbb{N}_0 \tag{11.19}$$

with the same constant C_A.

Proof. For any $j \in \mathbb{N}_0$ put $D_{j,A} = \sup_{m\geqslant j}\big((N_j N_m^{-1})^A a_m\big)$. By (11.18) we have

$$\begin{aligned} D_{j,A} &\leqslant \sup_{m\geqslant j}\Big((N_j N_m^{-1})^A \cdot\ C_A \cdot \sum_{l=m}^{\infty} (N_m N_l^{-1})^A b_l a_l^{1-\varrho}\Big) \\ &\leqslant C_A \cdot \sum_{l=j}^{\infty} (N_j N_l^{-1})^A b_l a_l^{1-\varrho} \leqslant C_A \cdot \sum_{l=j}^{\infty} (N_j N_l^{-1})^{A\varrho} b_l \,(D_{j,A})^{1-\varrho}. \end{aligned}$$

Consequently

$$a_j^{\varrho} \leqslant (D_{j,A})^{\varrho} \leqslant C_A \cdot \sum_{l=j}^{\infty} (N_j N_l^{-1})^{A\varrho} b_l \tag{11.20}$$

provided that $D_{j,A}$ is finite, which is satisfied by (11.17) at least for $A \geqslant A_0$. Thus we have proved (11.19) for $A \geqslant A_0$ and therefore also for $A < A_0$ with constant C_{A_0} since the right-hand side of (11.19) decreases as A increases.

Now let $A < A_0$ and assume that the right-hand side of (11.19) is finite (otherwise there is nothing to prove). By (11.19) with constant C_{A_0} for $m \geqslant j$

$$\begin{aligned} (N_j N_m^{-1})^A a_m &\leqslant (N_j N_m^{-1})^A \cdot C_{A_0}^{1/\varrho} \Big(\sum_{l=m}^{\infty} (N_m N_l^{-1})^{A\varrho} b_l\Big)^{1/\varrho} \\ &\leqslant C_{A_0}^{1/\varrho} \Big(\sum_{l=m}^{\infty} (N_j N_m^{-1})^{A\varrho} b_l\Big)^{1/\varrho} \end{aligned}$$

hence $D_{j,A} < \infty$, and we can use (11.20) which gives the desired estimate with constant C_A. ■

11.3.2 The theorem: equivalent quasi-norms based on maximal functions and local means

We are now prepared for the main result of this section.
Let k_0 and $k \in \mathcal{S}$, let $K \geqslant -1$ an integer such that

$$|\widehat{k_0}(\xi)| > 0 \quad \text{for} \quad |\xi| \leqslant N_1, \tag{11.21}$$

$$|\widehat{k}(\xi)| > 0 \quad \text{for} \quad \frac{1}{\lambda_1} \leqslant |\xi| \leqslant \lambda_1, \tag{11.22}$$

and

$$\int_{\mathbb{R}^n} x^\alpha k(x)\, dx = 0 \quad \text{for any} \quad |\alpha| \leqslant K. \tag{11.23}$$

Here (11.21) and (11.22) are Tauberian conditions, while (11.23) (which is in fact $D^\alpha \widehat{k}(0) = 0$ for any $|\alpha| \leqslant K$) are moment conditions on k.
If $K = -1$ then (11.23) simply means that there are no moment conditions.
For any $r > 0$, $f \in \mathcal{S}'$, and any $x \in \mathbb{R}^n$ consider J. Peetre's *maximal functions*:

$$(k_0^* f)_r(x) = \sup_{z \in \mathbb{R}^n} \frac{|(k_0 * f)(z)|}{(1+|x-z|)^r} \tag{11.24}$$

and for $j \geqslant 1$

$$(k_{N_j^{-1}}^* f)_r(x) = \sup_{z \in \mathbb{R}^n} \frac{|(k_{N_j^{-1}} * f)(z)|}{(1+N_j|x-z|)^r}. \tag{11.25}$$

We recall the notation $k_{N_j^{-1}}(x) = N_j^n\, k(N_j x)$ - see (11.10). Usually $(k_{N_j^{-1}} * f)(x)$ is called *local mean*.

Theorem 11.3.4 *Let $(N_j)_{j\in\mathbb{N}_0}$ be an admissible sequence with $\lambda_0 > 1$ and $(\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence.*
Let

$$K > -1 + \frac{\log_2 d_1}{\log_2 \lambda_0} \tag{11.26}$$

and let k_0 and k functions from $\mathcal{S}$ satisfying conditions (11.21) - (11.23) *from above.*
Let $0 < p < \infty$, respectively $0 < p \leqslant \infty$, let $0 < q \leqslant \infty$, and let $r > \frac{n}{\min(p,q)}$, respectively $r > \frac{n}{p}$.
Then there exist two constants c, $c' > 0$ such that for all $f \in \mathcal{S}'$

$$\|(k_0^* f)_r \,|\, L_p\| + \| \left(\sigma_j (k_{N_j^{-1}}^* f)_r\right)_{j\in\mathbb{N}} \,|\, L_p(l_q)\| \leqslant c\, \|f \,|\, F_{p,q}^{\sigma,N}\| \tag{11.27}$$

and

$$\|f \,|\, F_{p,q}^{\sigma,N}\| \leqslant c' \left(\|(k_0 * f \,|\, L_p\| + \| \left(\sigma_j (k_{N_j^{-1}} * f)\right)_{j\in\mathbb{N}} \,|\, L_p(l_q)\| \right), \tag{11.28}$$

respectively

$$\|(k_0^* f)_r \,|\, L_p\| + \| \left(\sigma_j (k_{N_j^{-1}}^* f)_r\right)_{j\in\mathbb{N}} \,|\, l_q(L_p)\| \leqslant c\, \|f \,|\, B_{p,q}^{\sigma,N}\| \tag{11.29}$$

and

$$\|f \,|\, B_{p,q}^{\sigma,N}\| \leqslant c' \left(\|(k_0 * f \,|\, L_p\| + \| \left(\sigma_j (k_{N_j^{-1}} * f)\right)_{j\in\mathbb{N}} \,|\, l_q(L_p)\| \right). \tag{11.30}$$

Remark 11.3.5 Note that the above inequalities are valid for all $f \in \mathcal{S}'$.
It is easy to see that for any $x \in \mathbb{R}^n$ and any $f \in \mathcal{S}'$ we have $|(k_{N_j^{-1}} * f)(x)| \leqslant (k^*_{N_j^{-1}} f)_r(x)$. This shows that the right-hand side in (11.28) is less than the left-hand side in (11.27).
Consequently the left-hand side in (11.27) and the right-hand side in (11.28) are equivalent quasi-norms in $F^{\sigma,N}_{p,q}$.
Of course a corresponding assertion is valid for the spaces $B^{\sigma,N}_{p,q}$, now based on (11.29) and (11.30).

11.3.3 Proof of Theorem 11.3.4

We will present here the proof of the inequalities (11.27) and (11.28). The inequalities (11.29) and (11.30) can be proved in a similar manner interchanging the roles of the quasi-norms in L_p and l_q.
Step 1. Take any pair of functions θ_0 and $\theta \in \mathcal{S}$ such that

$$|\widehat{\theta}_0(\xi)| > 0 \quad \text{if} \quad |\xi| \leqslant N_1,$$

and

$$|\widehat{\theta}(\xi)| \geqslant C > 0 \quad \text{if} \quad \frac{1}{\lambda_1} \leqslant |\xi| \leqslant \lambda_1 \tag{11.31}$$

and define for any $r > 0$ the functions $(\theta_0^* f)_r$ and $(\theta^*_{N_j^{-1}} f)_r$ as in (11.24) and (11.25), where $\theta_{N_j^{-1}}(x) = N_j^n \theta(N_j x)$.
We will prove in this step that there is a constant $c > 0$ such that for any $f \in \mathcal{S}'$

$$\begin{aligned} &\|(k_0^* f)_r \,|\, L_p\| + \left\| \left(\sigma_j (k^*_{N_j^{-1}} f)_r\right)_{j\in\mathbb{N}} \,|\, L_p(l_q)\right\| \\ \leqslant\;& c\left(\|(\theta_0^* f)_r \,|\, L_p\| + \left\| \left(\sigma_j (\theta^*_{N_j^{-1}} f)_r\right)_{j\in\mathbb{N}} \,|\, L_p(l_q)\right\| \right). \end{aligned} \tag{11.32}$$

Take $(\varphi_j^N)_{j\in\mathbb{N}_0}$ a fixed partition of unity associated to $(N_j)_{j\in\mathbb{N}_0}$, that means $(\varphi_j^N)_{j\in\mathbb{N}_0}$ has the properties (9.7) - (9.10) with $c_\varphi = 1$.
We define the functions $\psi_j \in C_0^\infty(\mathbb{R}^n)$, $j \in \mathbb{N}_0$, by

$$\widehat{\psi_0}(\xi) = \frac{\varphi_0^N(\xi)}{\widehat{\theta}_0(\xi)} \quad \text{and} \quad \widehat{\psi_j}(\xi) = \frac{\varphi_j^N(\xi)}{\widehat{\theta}(N_j^{-1}\xi)} \quad \text{for} \quad j \in \mathbb{N}. \tag{11.33}$$

Due to the properties of the functions θ_0 and θ the functions $\widehat{\psi_0}$ and $\widehat{\psi_j}$ are well defined and it is easy to see that for any $j \in \mathbb{N}$ we have $\operatorname{supp} \widehat{\psi_j} \subset \{\xi \in \mathbb{R}^n : N_{j-1} \leqslant |\xi| \leqslant N_{j+1}\}$.
Moreover, applying the rule of differentiation for a product of functions, using (9.9) and (11.31) it follows that for any multi-index α there is a constant $c_\alpha > 0$ such that for any $j \geqslant 1$

$$|D^\alpha \widehat{\psi_j}(\xi)| \leqslant c_\alpha \langle \xi \rangle^{-|\alpha|} \quad \text{for any} \quad \xi \in \mathbb{R}^n. \tag{11.34}$$

From (11.33) clearly

$$1 = \widehat{\theta}_0(\xi)\widehat{\psi_0}(\xi) + \sum_{j=1}^\infty \widehat{\psi_j}(\xi)\, \widehat{\theta}(N_j^{-1}\xi)$$

and so for any $f \in \mathcal{S}'$

$$f = \psi_0 * \theta_0 * f + \sum_{m=1}^{\infty} \psi_m * \theta_{N_m^{-1}} * f.$$

Consequently, we have for any $j \geqslant 1$

$$k_{N_j^{-1}} * f = k_{N_j^{-1}} * \psi_0 * \theta_0 * f + \sum_{m=1}^{\infty} k_{N_j^{-1}} * \psi_m * \theta_{N_m^{-1}} * f. \tag{11.35}$$

For a fixed $m \geqslant 1$ one has

$$\begin{aligned}
|(k_{N_j^{-1}} * \psi_m * \theta_{N_m^{-1}} * f)(y)| &\leqslant \int_{\mathbb{R}^n} |(k_{N_j^{-1}} * \psi_m)(z)| \cdot |(\theta_{N_m^{-1}} * f)(y-z)|\, dz \\
&\leqslant (\theta^*_{N_m^{-1}} f)_r(y) \cdot \int_{\mathbb{R}^n} |(k_{N_j^{-1}} * \psi_m)(z)| \cdot (1+N_m|z|)^r\, dz \\
&= (\theta^*_{N_m^{-1}} f)_r(y) \cdot I_{jm}.
\end{aligned} \tag{11.36}$$

We are going now to obtain convenient estimates from above for the integral I_{jm} in (11.36).
First, let $m \leqslant j$.
After a change of variables, inserting $k_{N_j^{-1}}(x) = N_j^n k(N_j x)$ we have

$$\begin{aligned}
I_{jm} &= \int_{\mathbb{R}^n} |(k_{N_j^{-1}} * \psi_m)(z)| \cdot (1+N_m|z|)^r\, dz \\
&= N_m^{-n} \int_{\mathbb{R}^n} |(k_{N_j^{-1}} * \psi_m)(N_m^{-1}u)| \cdot (1+|u|)^r\, du \\
&= N_m^{-n} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} N_j^n N_m^{-n} k(N_j N_m^{-1} u - N_j N_m^{-1} v) \psi_m(N_m^{-1} v)\, dv \right| \cdot (1+|u|)^r\, du \\
&= N_m^{-n} \int_{\mathbb{R}^n} | \left(k_{N_j^{-1} N_m} * \psi_m(N_m^{-1} \cdot) \right)(u)| \cdot (1+|u|)^r\, du
\end{aligned}$$

where again $k_t(x) = t^{-n} k(t^{-1}x)$. It follows that for some positive constant c (independent of j and m)

$$I_{jm} \leqslant c\, N_m^{-n} \sup_{u \in \mathbb{R}^n} \left(|\big(k_{N_j^{-1} N_m} * \psi_m(N_m^{-1} \cdot)\big)(u)| \cdot (1+|u|)^{r+n+1} \right).$$

We may apply Lemma 11.3.1 with $t = N_j^{-1} N_m \leqslant 1$, $\mu = k \in \mathcal{S}_K$ (k has K moment conditions), $\eta = \psi_m(N_m^{-1} \cdot)$; taking $L = r+n+1$ we obtain (with some positive constant c_1)

$$\begin{aligned}
I_{jm} &\leqslant c_1\, N_m^{-n}\, (N_j^{-1} N_m)^{K+1} \max_{K+1 \leqslant |\beta| \leqslant \max(K+1, r+n+2)} \|D^\beta \widehat{k} \,|\, L_\infty\| \cdot \\
&\qquad \cdot \max_{|\alpha| \leqslant r+n+2} \int_{\mathbb{R}^n} (1+|\xi|)^{K+1} |D^\alpha [\psi_m(N_m^{-1} \cdot)]^{\widehat{}}(\xi)|\, d\xi \\
&\leqslant c_2\, N_m^{-n}\, (N_j^{-1} N_m)^{K+1} \max_{|\alpha| \leqslant r+n+2} \int_{\mathbb{R}^n} (1+|\xi|)^{K+1} |D^\alpha [\widehat{\psi_m}(N_m \xi)]|\, N_m^n\, d\xi \\
&= c_2\, (N_j^{-1} N_m)^{K+1} \max_{|\alpha| \leqslant r+n+2} \int_{\mathbb{R}^n} (1+|\xi|)^{K+1} N_m^{|\alpha|}\, |(D^\alpha \widehat{\psi_m})(N_m \xi)|\, d\xi.
\end{aligned}$$

Due to the localisation of the support of $\widehat{\psi_m}$ the last integral is in fact taken over the set $\{\xi \in \mathbb{R}^n : \frac{1}{\lambda_1} \leqslant |\xi| \leqslant \lambda_1\}$. Using (11.34) we get

$$I_{jm} \leqslant c_3\,(N_j^{-1}N_m)^{K+1} \max_{|\alpha|\leqslant r+n+2} \int_{\frac{1}{\lambda_1}\leqslant|\xi|\leqslant\lambda_1} (1+|\xi|)^{K+1} N_m^{|\alpha|}\,(1+N_m|\xi|)^{-|\alpha|}\,d\xi$$

which is

$$I_{jm} \leqslant c\,(N_j^{-1}N_m)^{K+1} \tag{11.37}$$

with some positive constant $c > 0$ independent of j and m.

Let now $m > j$.

Then, again making use of changing of variables, and inserting $k_{N_j^{-1}}(x) = N_j^n k(N_j x)$ we have

$$\begin{aligned}
I_{jm} &= \int_{\mathbb{R}^n} |(k_{N_j^{-1}} * \psi_m)(z)|(1+N_m|z|)^r\,dz \\
&\leqslant (N_j^{-1}N_m)^r \int_{\mathbb{R}^n} |(k_{N_j^{-1}} * \psi_m)(z)|(1+N_j|z|)^r\,dz \\
&= (N_j^{-1}N_m)^r\,N_j^{-n} \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} k_{N_j^{-1}}(N_j^{-1}u - v)\psi_m(v)dv\right| (1+|u|)^r\,du \\
&= (N_j^{-1}N_m)^r\,N_j^{-n} \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \psi_m(N_j^{-1}y)\,k(u-y)\,dy\right| (1+|u|)^r\,du \\
&= (N_j^{-1}N_m)^r\,N_j^{-n} \int_{\mathbb{R}^n} |(\psi_m(N_j^{-1}\cdot) * k)(u)|\,(1+|u|)^r\,du.
\end{aligned}$$

Consequently there exists a constant $c > 0$ independent of j and m such that for any $L \geqslant r+n+1$

$$I_{jm} \leqslant c\,(N_j N_m^{-1})^{-r}\,N_j^{-n} \cdot \sup_{u\in\mathbb{R}^n} \left(|(\psi_m(N_j^{-1}\cdot) * k)(u)|\,(1+|u|)^L\right). \tag{11.38}$$

To use again Lemma 11.3.1 we define for any $m \geqslant 1$ the function

$$\psi^{(m)}(u) = \psi_m(N_m^{-1}u) \quad , \quad u \in \mathbb{R}^n.$$

Then $\widehat{\psi^{(m)}}(\xi) = N_m^n\,\widehat{\psi_m}(N_m\xi)$ and $\operatorname{supp}\widehat{\psi^{(m)}} \subset \{\xi : \frac{1}{\lambda_1} \leqslant |\xi| \leqslant \lambda_1\}$ which implies in particular

$$D^\alpha\widehat{\psi^{(m)}}(0) = 0 \quad \text{for any multi-index} \quad \alpha. \tag{11.39}$$

Moreover, by (11.34) it follows that for any multi-index α, there is a constant c_α such that

$$|D^\alpha\widehat{\psi^{(m)}}(\xi)| \leqslant c_\alpha\,N_m^n \quad \text{for any} \quad m \geqslant 1 \quad \text{and for any} \quad \xi \in \mathbb{R}^n. \tag{11.40}$$

Writing, as usual, $\psi_t^{(m)}(x) = t^{-n}\psi^{(m)}(t^{-1}x)$ we have

$$(\psi_m(N_j^{-1}\cdot) * k)(u) = (\psi^{(m)}(N_m N_j^{-1}\cdot) * k)(u) = (N_j N_m^{-1})^n\,(\psi^{(m)}_{N_j N_m^{-1}} * k)(u).$$

So (11.38) becomes

$$I_{jm} \leqslant c\,(N_j N_m^{-1})^{-r}\,N_j^{-n}(N_j N_m^{-1})^n \cdot \sup_{u\in\mathbb{R}^n} \left(|(\psi^{(m)}_{N_j N_m^{-1}} * k)(u)|(1+|u|)^L\right). \tag{11.41}$$

Since $m > j$ we can use Lemma 11.3.1 with $t = N_j N_m^{-1}$, $\mu = \psi^{(m)} \in \mathcal{S}_M$ (where M can be chosen arbitrary large due to (11.39)) and $\eta = k$ and obtain

$$\sup_{u\in\mathbb{R}^n} \left(|(\psi^{(m)}_{N_jN_m^{-1}} * k)(u)|(1+|u|)^L \right) \leqslant c_L \, (N_jN_m^{-1})^{M+1} \cdot$$
$$\max_{M+1\leqslant|\beta|\leqslant\max(M+1,L+1)} \|D^\beta \widehat{\psi^{(m)}} \mid L_\infty\| \cdot \max_{|\alpha|\leqslant L+1} \int_{\mathbb{R}^n} (1+|\xi|)^{M+1} |D^\alpha \widehat{k}(\xi)| \, d\xi$$

and using (11.40) we have with a positive constant $c' > 0$

$$\sup_{u\in\mathbb{R}^n} \left(|(\psi^{(m)}_{N_jN_m^{-1}} * k)(u)|(1+|u|)^L \right) \leqslant c' \, (N_jN_m^{-1})^{M+1} \, N_m^n. \tag{11.42}$$

Inserting the last inequality in (11.41) we finally obtain

$$I_{jm} \leqslant c \, (N_jN_m^{-1})^{-r} \, N_j^{-n} (N_jN_m^{-1})^n (N_jN_m^{-1})^{M+1} \, N_m^n = c \, (N_jN_m^{-1})^{-r+M+1}. \tag{11.43}$$

Recall that by (11.39) we may choose M as large as we want. We chose M an integer of the form

$$M = -1 + 2r + s \quad \text{with a real } s \text{ satisfying} \quad s \log_2 \lambda_0 + \log_2 d_0 > 0 \tag{11.44}$$

(note that such an s exists due to the fact that $\lambda_0 > 1$) and (11.43) can be written

$$I_{jm} \leqslant c \, (N_jN_m^{-1})^{s+r}. \tag{11.45}$$

Further, note that for all $x, y \in \mathbb{R}^n$

$$\begin{aligned} (\theta^*_{N_m^{-1}} f)_r(y) &\leqslant (\theta^*_{N_m^{-1}} f)_r(x)(1+N_m|x-y|)^r \\ &\leqslant (\theta^*_{N_m^{-1}} f)_r(x) \cdot \max(1, (N_j^{-1}N_m)^r) \cdot (1+N_j|x-y|)^r. \end{aligned}$$

Inserting the last inequality in (11.36), then dividing by $(1+N_j|x-y|)^r$ and using the estimates (11.37) and (11.45) for I_{jm} we have

$$\begin{aligned} &\sup_{y\in\mathbb{R}^n} \frac{|(k_{N_j^{-1}} * \psi_m * \theta_{N_m^{-1}} * f)(y)|}{(1+N_j|x-y|)^r} \\ \leqslant\ & c\, (\theta^*_{N_m^{-1}} f)_r(x) \cdot \max(1, (N_j^{-1}N_m)^r) \cdot \begin{cases} (N_j^{-1}N_m)^{K+1} & \text{if } m \leqslant j \\ (N_jN_m^{-1})^{s+r} & \text{if } m > j \end{cases} \\ =\ & c' \, (\theta^*_{N_m^{-1}} f)_r(x) \cdot \begin{cases} (N_j^{-1}N_m)^{K+1} & \text{if } m \leqslant j \\ (N_jN_m^{-1})^{s} & \text{if } m > j \end{cases} . \end{aligned} \tag{11.46}$$

Note that in the above computations we did not use moment conditions for the function $\widehat{\psi_1}$. So, replacing ψ_1 and θ_1 with ψ_0 and θ_0 we get the similar estimate

$$\sup_{y\in\mathbb{R}^n} \frac{|(k_{N_j^{-1}} * \psi_0 * \theta_0 * f)(y)|}{(1+N_j|x-y|)^r} \leqslant c \, (\theta^*_0 f)_r(x) \, (N_j^{-1}N_0)^{K+1}. \tag{11.47}$$

Using now (11.46) and (11.47) in (11.35), after multiplying with σ_j we have

$$\begin{aligned} \sigma_j (k^*_{N_j^{-1}} f)_r(x) &\leqslant c \, (\theta^*_0 f)_r(x) \sigma_j N_j^{-(K+1)} \\ &\quad + c' \sum_{m=1}^{\infty} (\theta^*_{N_m^{-1}} f)_r(x) \cdot \begin{cases} \sigma_j (N_j^{-1}N_m)^{K+1} & \text{if } m \leqslant j \\ \sigma_j (N_jN_m^{-1})^{s} & \text{if } m > j \end{cases} \end{aligned} \tag{11.48}$$

with some positive constants c, c' independent of j and m.
Let $m < j$. Then, after using (11.1) and (11.3) we have

$$\begin{aligned}\sigma_j (N_j^{-1} N_m)^{K+1} &\leqslant d_1^{j-m} \sigma_m \cdot \lambda_0^{-(j-m)(K+1)} \\ &= \sigma_m \cdot 2^{-(j-m)[-\log_2 d_1 + (K+1)\log_2 \lambda_0]}.\end{aligned}$$

Let now $m \geqslant j$. Again using (11.1) and (11.3) we have

$$\begin{aligned}\sigma_j (N_j N_m^{-1})^{s} &\leqslant d_0^{-(m-j)} \sigma_m \cdot \lambda_0^{-(m-j)s} \\ &= \sigma_m \cdot 2^{-(m-j)(\log_2 d_0 + s \log_2 \lambda_0)}\end{aligned}$$

Moreover,

$$\sigma_j \, N_j^{-(K+1)} \leqslant d_1^j \, \sigma_0 \cdot \lambda_0^{-j(K+1)} N_0^{-(K+1)} = \sigma_0 \, N_0^{-(K+1)} \, 2^{-j[-\log_2 d_1 + (K+1)\log_2 \lambda_0]}$$

Note that due to (11.44) and to our assumption on K we have

$$\varrho = \min\{-\log_2 d_1 + (K+1)\log_2 \lambda_0 \, , \, s \log_2 \lambda_0 + \log_2 d_0\} > 0.$$

Inserting the last two estimates in (11.48) we get for all $f \in \mathcal{S}'$, all $x \in \mathbb{R}^n$ and all $j \in \mathbb{N}$

$$\sigma_j (k^*_{N_j^{-1}} f)_r(x) \leqslant c \, \sigma_0 \, (\theta^*_0 f)_r(x) \, 2^{-j\varrho} + c' \sum_{m=1}^{\infty} \sigma_m \, (\theta^*_{N_m^{-1}} f)_r(x) \cdot 2^{-|j-m|\varrho}.$$

Again for $j = 1$ we did not use moment conditions to obtain this estimate so we can replace $k_{N_1^{-1}}$ with k_0 and get

$$(k^*_0 f)_r(x) \leqslant c \, (\theta^*_0 f)_r(x) + c' \sum_{m=1}^{\infty} \sigma_m \, (\theta^*_{N_m^{-1}} f)_r(x) \cdot 2^{-m\varrho}.$$

The estimate (11.32) is now as a simple consequence of the elementary Lemma 11.3.2. Consequently we have finished the proof of the inequality (11.32).

Step 2. Take again $(\varphi_j^N)_{j\in\mathbb{N}_0}$ a fixed partition of unity associated to $(N_j)_{j\in\mathbb{N}_0}$, that means $(\varphi_j^N)_{j\in\mathbb{N}_0}$ has the properties (9.7) - (9.10) with $c_\varphi = 1$.
For a fixed $j \in \mathbb{N}_0$ let

$$\Phi_j(\xi) = \sum_{m=0}^{j} \varphi_m^N(\xi).$$

Using the properties of the system $(\varphi_j^N)_{j\in\mathbb{N}_0}$ we have $\Phi_j(\xi) = 1$ if $|\xi| < N_{j-1}$, $\Phi_j(\xi) = 0$ if $|\xi| > N_{j+1}$ and for any multi-index α there exists a constant c_α (independent of j) such that

$$|D^\alpha \Phi_j(\xi)| \leqslant c_\alpha \, \langle \xi \rangle^{-|\alpha|}.$$

Let us consider now the function Ψ_j defined by

$$\widehat{\Psi_j}(\xi) = \frac{\Phi_j(\xi)}{\widehat{k_0}(N_j^{-1}\xi)} \quad , \; j \in \mathbb{N}_0 \, . \tag{11.49}$$

Note that for $|\xi| \leqslant N_{j+1} \leqslant \lambda_1 N_j$ it follows $N_j^{-1}|\xi| \leqslant \lambda_1 \leqslant N_1$ and due to the assumption (11.21) on k_0 this shows that Ψ_j is well defined.
Clearly one has $\operatorname{supp} \widehat{\Psi_j} \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant N_{j+1}\}$ since $\Phi_j(\xi) = 0$ if $|\xi| > N_{j+1}$.
Moreover, applying Leibniz's rule for differentiation of a product we get that for any multi-index γ there exists a constant $c_\gamma > 0$ independent of j such that

$$|D^\gamma[\widehat{\Psi_j}(N_j\xi)]| \leqslant \sum_{\delta\leqslant\gamma} c_{\gamma\delta}\; D^\delta[\Phi_j(N_j\xi)]\; D^{\gamma-\delta}\left[\frac{1}{\widehat{k_0}(\xi)}\right] \leqslant c_\gamma. \tag{11.50}$$

From (11.49) we get that for any $f \in \mathcal{S}'$ we have

$$\Psi_j * N_j^n\, k_0(N_j\,\cdot) * f = \check{\Phi}_j * f.$$

We will use later on the notation $(k_0)_{N_j^{-1}}(x) = N_j^n k_0(N_j x)$.
On the other hand for a fixed j we define for any $m \geqslant j+1$ the functions ψ_m by

$$\widehat{\psi_m}(\xi) = \frac{\varphi_m^N(\xi)}{\widehat{k}(N_m^{-1}\xi)}$$

in analogy to the first step, see (11.33), now with k instead of θ. Consequently we have for any $f \in \mathcal{S}'$ (and for a fixed j)

$$f = \Psi_j * (k_0)_{N_j^{-1}} * f + \sum_{m=j+1}^{\infty} \psi_m * k_{N_m^{-1}} * f$$

and this implies

$$k_{N_j^{-1}} * f = (\Psi_j * (k_0)_{N_j^{-1}}) * (k_{N_j^{-1}} * f) + \sum_{m=j+1}^{\infty} (k_{N_j^{-1}} * \psi_m) * (k_{N_m^{-1}} * f). \tag{11.51}$$

Now $(\Psi_j * (k_0)_{N_j^{-1}})(z) = (k_0 * \Psi_j(N_j^{-1}\cdot))(N_j z)$ and we may apply Lemma 11.3.1 with $t=1$, $M=-1$, $\mu = k_0$ and $\eta = \Psi_j(N_j^{-1}\cdot)$. So for any $L > 0$

$$\begin{aligned}
&\sup_{z\in\mathbb{R}^n}\Big(|(\Psi_j * (k_0)_{N_j^{-1}})(z)| \cdot (1+N_j|z|)^L\Big)\\
=\;& \sup_{z\in\mathbb{R}^n}\big(|(k_0 * \Psi_j(N_j^{-1}\cdot))(N_j z)| \cdot (1+N_j|z|)^L\big)\\
\leqslant\;& c_L \cdot 1^0 \cdot \max_{0\leqslant|\beta|\leqslant\max(0,L+1)} \|D^\beta \widehat{k_0}\,|\,L_\infty\| \cdot \max_{|\gamma|\leqslant L+1} \int_{\mathbb{R}^n} |D^\gamma[\Psi_j(N_j^{-1}\cdot)]\,\widehat{}\,(\xi)\,|\,d\xi\\
\leqslant\;& c' \max_{|\gamma|\leqslant L+1} N_j^n \int_{|\xi|\leqslant\lambda_1} |D^\gamma[\widehat{\Psi_j}(N_j\xi)]|\,d\xi\\
\leqslant\;& c''\, N_j^n
\end{aligned}$$

where $c'' > 0$ is independent of j and in the last two inequalities we have used the properties of the function $\widehat{\Psi_j}$, in particular the localisation of its support and (11.50).
Consequently, for any $L > 0$ there is a positive constant $C_L > 0$ such that

$$|(\Psi_j * (k_0)_{N_j^{-1}})(z)| \leqslant C_L\, \frac{N_j^n}{(1+N_j|z|)^L} \quad , \quad z \in \mathbb{R}^n. \tag{11.52}$$

Writing for any $m \geqslant j+1$

$$(k_{N_j^{-1}} * \psi_m)(u) = (N_j N_m^{-1})^n \, (\psi^{(m)}_{N_j N_m^{-1}} * k)(N_j u)$$

where $\psi^{(m)} = \psi_m(N_m^{-1}u)$ we get as in Step 1, compare (11.38) and (11.42), that for any M and L there is a constant $c >$ independent of j and m

$$|(k_{N_j^{-1}} * \psi_m)(z)| \leqslant c \, \frac{N_j^n \, (N_j N_m^{-1})^M}{(1+N_j|z|)^L} \quad , \quad z \in \mathbb{R}^n. \tag{11.53}$$

Inserting the estimates (11.52) and (11.53) with $r = L$ in (11.51) we get for all $f \in \mathcal{S}'$, $y \in \mathbb{R}^n$ and $j \in \mathbb{N}$

$$\begin{aligned} |(k_{N_j^{-1}} * f)(y)| \;&\leqslant\; c \int_{\mathbb{R}^n} \frac{N_j^n}{(1+N_j|y-z|)^r} \, |(k_{N_j^{-1}} * f)(z)| \, dz \\ &\quad + c' \sum_{m=j+1}^{\infty} \int_{\mathbb{R}^n} \frac{N_j^n \, (N_j N_m^{-1})^M}{(1+N_j|y-z|)^r} \, |(k_{N_m^{-1}} * f)(z)| \, dz \\ &\leqslant\; C \sum_{m=j}^{\infty} N_j^n \, (N_j N_m^{-1})^M \int_{\mathbb{R}^n} \frac{|(k_{N_m^{-1}} * f)(z)|}{(1+N_j|y-z|)^r} \, dz. \end{aligned} \tag{11.54}$$

Fix now any $\varrho \in (0,1]$. We divide both sides of (11.54) by $(1+N_j|x-y|)^r$, then: in the left-hand side we take the supremum over $y \in \mathbb{R}^n$ and in the right hand-side we use the inequalities

$$(1+N_j|x-y|)(1+N_j|y-z|) \geqslant 1+N_j|x-z|, \tag{11.55}$$

$$|(k_{N_m^{-1}} * f)(z)| \leqslant |(k_{N_m^{-1}} * f)(z)|^{\varrho} \cdot [(k^*_{N_m^{-1}} f)_r(x)]^{1-\varrho} \cdot (1+N_m|x-z|)^{r(1-\varrho)},$$

and

$$\frac{(1+N_m|x-z|)^{r(1-\varrho)}}{(1+N_j|x-z|)^r} \leqslant \frac{(N_j^{-1}N_m)^r}{(1+N_m|x-z|)^{r\varrho}}$$

and get for all $f \in \mathcal{S}'$, all $x \in \mathbb{R}^n$ and all $j \in \mathbb{N}$

$$(k^*_{N_j^{-1}} f)_r(x) \;\leqslant\; c \sum_{m=j}^{\infty} (N_j N_m^{-1})^A \int_{\mathbb{R}^n} \frac{N_m^n \cdot |(k_{N_m^{-1}} * f)(z)|^{\varrho}}{(1+N_m|x-z|)^{r\varrho}} \, dz \cdot [(k^*_{N_m^{-1}} f)_r(x)]^{1-\varrho} \tag{11.56}$$

where $A = M - r + n$ can be still taken arbitrary large.
Quite analogously one proves for all $f \in \mathcal{S}'$ the estimate

$$\begin{aligned} (k_0^* f)_r(x) \;&\leqslant\; c \Big(\int_{\mathbb{R}^n} \frac{|(k_0 * f)(z)|^{\varrho}}{(1+|x-z|)^{r\varrho}} \, dz \cdot [(k_0^* f)_r(x)]^{1-\varrho} \\ &\quad + \sum_{m=1}^{\infty} N_m^{-A} \int_{\mathbb{R}^n} \frac{N_m^n \cdot |(k_{N_m^{-1}} * f)(z)|^{\varrho}}{(1+N_m|x-z|)^{r\varrho}} \cdot [(k_0^* f)_r(x)]^{1-\varrho} \Big) \, dz. \end{aligned} \tag{11.57}$$

At this moment we need Lemma 11.3.3. We fix $x \in \mathbb{R}^n$ and apply Lemma 11.3.3 with

$$a_m = (k^*_{N_m^{-1}} f)_r(x), \quad m \in \mathbb{N}, \quad a_0 = (k_0^* f)_r(x),$$

$$b_m = \int_{\mathbb{R}^n} \frac{N_m^n \, |(k_{N_m^{-1}} * f)(z)|^\varrho}{(1 + N_m|x-z|)^{r\varrho}} \, dz, \quad b_0 = \int_{\mathbb{R}^n} \frac{|(k_0 * f)(z)|}{(1 + |x-z|)^{r\varrho}} \, dz.$$

The assumption (11.17) is satisfied with A_0 equal to the order of the distribution $f \in \mathcal{S}'$. The estimates (11.56) and (11.57) take the form (11.18). Consequently (11.19) is true and this means that for every $A > 0$ there is a constant $c_A > 0$ such that

$$(k^*_{N_j^{-1}} f)_r(x)^\varrho \leqslant c_A \sum_{m=j}^{\infty} (N_j N_m^{-1})^{A\varrho} \cdot \int_{\mathbb{R}^n} \frac{N_m^n \, |(k_{N_m^{-1}} * f)(z)|^\varrho}{(1 + N_m|x-z|)^{r\varrho}} \, dz \tag{11.58}$$

together with the corresponding estimate for $(k_0^* f)_r(x)$. Note that c_A in (11.58) is independent of $f \in \mathcal{S}'$, $x \in \mathbb{R}^n$, $j \in \mathbb{N}$ and $\varrho \in (0, 1]$ because of Lemma 11.3.3.
Further note that (11.58) is also true for $\varrho > 1$ with a simpler proof. It suffices to take $r + n$ instead of r, apply Hölder's inequality in m and in z and finally the inequality (11.55). We omit the details.
It is possible to choose ϱ so that

$$\frac{n}{r} < \varrho < \min(p, q) \quad \text{(respectively} \quad \frac{n}{r} < \varrho < p \quad \text{for Besov spaces).}$$

We make such a choice and fix ϱ for the rest of the proof.
Now the function $z \mapsto \frac{1}{(1+|z|)^{r\varrho}}$ is in L_1 and we may use the majorant property for the Hardy - Littlewood maximal operator $\mathcal{M}$, see E. M. Stein and G. Weiss [StWe71, Chapter 2,(3.9)],

$$\left(|g|^\varrho * \frac{1}{(1 + |\cdot|)^{r\varrho}} \right)(x) \leqslant \mathcal{M}(|g|^\varrho)(x) \cdot \left\| \frac{1}{(1 + |\cdot|)^{r\varrho}} \,|\, L_1 \right\| .$$

It follows from (11.58) that

$$(k^*_{N_j^{-1}} f)_r(x)^\varrho \leqslant c \sum_{m=j}^{\infty} (N_j N_m^{-1})^{A\varrho} \cdot \mathcal{M}(|k_{N_m^{-1}} * f|^\varrho)(x) \tag{11.59}$$

together with the corresponding estimate for $(k_0^* f)_r(x)$.
We use again that for $m \geqslant j$

$$N_j N_m^{-1} \leqslant \lambda_0^{-(m-j)} = 2^{-(m-j)\log_2 \lambda_0}$$

and

$$\sigma_j \leqslant d_0^{-(m-j)} \sigma_m = 2^{-(m-j)\log_2 d_0}$$

and so (11.59) becomes (with some positive constant c)

$$\sigma_j^\varrho \, (k^*_{N_j^{-1}} f)_r(x)^\varrho \leqslant c \sum_{m=j}^{\infty} 2^{-(m-j)(A\varrho \log_2 \lambda_0 + \varrho \log_2 d_0)} \sigma_m^\varrho \cdot \mathcal{M}(|k_{N_m^{-1}} * f|^\varrho)(x). \tag{11.60}$$

We can choose $A > 0$ large enough such that

$$\varepsilon = A\varrho \log_2 \lambda_0 + \varrho \log_2 d_0 > 0.$$

We apply now Lemma 11.3.2 with

$$g_j(x) = \sigma_j^{\varrho}\, \mathcal{M}(|k_{N_j^{-1}} * f|^{\varrho})(x), \quad j \in \mathbb{N}, \quad g_0 = \mathcal{M}(|k_0 * f|^{\varrho})$$

in $L_{p/\varrho}(l_{q/\varrho})$ and get from (11.60)

$$\begin{aligned} &\|(k_0^*)_r \,|\, L_p\| + \| \left(\sigma_j\, (k^*_{N_j^{-1}} f)_r\right)_{j\in\mathbb{N}} \,|\, L_p(l_q)\| \\ \leqslant\ & c \left(\|\mathcal{M}_{\varrho}(k_0 * f) \,|\, L_p\| + \| \left(\sigma_j\, \mathcal{M}_{\varrho}(k_{N_j^{-1}} * f)\right)_{j\in\mathbb{N}} \,|\, L_p(l_q)\| \right) \end{aligned}$$

where we used the notation $\mathcal{M}_{\varrho}(g) = \mathcal{M}(|g|^{\varrho})^{1/\varrho}$.
By the maximal inequality of C. Fefferman and E. M. Stein, see [FeSt71], we know that $\mathcal{M}_{\varrho}$ is a bounded operator

$$\mathcal{M}_{\varrho} : L_p(l_q) \to L_p(l_q), \quad \varrho < p < \infty, \quad \varrho < q \leqslant \infty \tag{11.61}$$

(respectively $\mathcal{M}_{\varrho} : l_q(L_p) \to l_q(L_p), \quad \varrho < p \leqslant \infty, \quad 0 < q \leqslant \infty$). Our choice of ϱ enables us to apply (11.61) and we obtain (with some positive constant C)

$$\begin{aligned} &\|(k_0^* f)_r \,|\, L_p\| + \| \left(\sigma_j\, (k^*_{N_j^{-1}} f)_r\right)_{j\in\mathbb{N}} \,|\, L_p(l_q)\| \\ \leqslant\ & C \left(\|k_0 * f \,|\, L_p\| + \| \left(\sigma_j\, (k_{N_j^{-1}} * f)\right)_{j\in\mathbb{N}} \,|\, L_p(l_q)\| \right) \quad \text{for any} \quad f \in \mathcal{S}' \end{aligned} \tag{11.62}$$

A corresponding inequality is obtained for the spaces $l_q(L_p)$.

Step 3. Let μ_0 and $\mu \in \mathcal{S}$ be two positive functions on $\mathbb{R}^n$ satisfying (11.5) and (11.6). Let

$$\widehat{\theta_0} = \mu_0 \quad \text{and} \quad \widehat{\theta} = \mu.$$

We have successively

$$\begin{aligned} &\|(k_0^* f)_r \,|\, L_p\| + \| \left(\sigma_j (k^*_{N_j^{-1}} f)_r\right)_{j\in\mathbb{N}} \,|\, L_p(l_q)\| \\ \leqslant\ & c \left(\|(\theta_0^* f)_r \,|\, L_p\| + \| \left(\sigma_j (\theta^*_{N_j^{-1}} f)_r\right)_{j\in\mathbb{N}} \,|\, L_p(l_q)\| \right) \\ \leqslant\ & c_1 \left(\|\theta_0 * f \,|\, L_p\| + \| \left(\sigma_j\, (\theta_{N_j^{-1}} * f)\right)_{j\in\mathbb{N}} \,|\, L_p(l_q)\| \right) \\ \leqslant\ & c_2\, \|f \,|\, F^{\sigma,N}_{p,q}\| \end{aligned}$$

where the first inequality is (11.32), see Step 1, the second inequality is (11.62) (with θ_0 and θ instead of k_0 and k), see Step 2, and finally the last inequality is nothing else than (11.8), see Theorem 11.2.2, since $\theta_0 * f = (\mu_0 \widehat{f})^{\vee}$ and $\theta_{N_j^{-1}} * f = (\mu_j \widehat{f})^{\vee}$. Consequently we have proved (11.27).
Moreover,

$$\begin{aligned} \|f \,|\, F^{\sigma,N}_{p,q}\| \ \leqslant\ & c \left(\|(\theta_0^* f)_r \,|\, L_p\| + \| \left(\sigma_j (\theta^*_{N_j^{-1}} f)_r\right)_{j\in\mathbb{N}} \,|\, L_p(l_q)\| \right) \\ \leqslant\ & c_1 \left(\|(k_0^* f)_r \,|\, L_p\| + \| \left(\sigma_j (k^*_{N_j^{-1}} f)_r\right)_{j\in\mathbb{N}} \,|\, L_p(l_q)\| \right) \\ \leqslant\ & c_2 \left(\|(k_0 * f) \,|\, L_p\| + \| \left(\sigma_j (k_{N_j^{-1}} * f)\right)_{j\in\mathbb{N}} \,|\, L_p(l_q)\| \right) \quad , \end{aligned}$$

where the first inequality is an obvious consequence of (11.8), see Theorem 11.2.2, the second inequality is (11.32), see Step 1, with the roles of k_0 and k respectively θ_0 and θ interchanged, and finally the last inequality is (11.62), see Step 2.
Consequently we have proved (11.28), too. ■

11.4 Comments

We would like to point out that in the above proof we used at several places the fact that the sequence N is of bounded growth.
The above proof has as a starting point the technique used by H.-Q. Bui, M. Paluszyński and M. Taibleson, see [BPT96] and [BPT97], and the simplified version of their papers given by V. Rychkov in [Ry99]. However, due to the general structure of the sequences $(N_j)_{j\in\mathbb{N}_0}$ satisfying (11.1) there are some significant differences compared with their proofs.
First, the key Lemma 11.3.1 is related to Lemma 2.1 in [BPT96] and to Lemma 1 in [Ry99] but we needed to indicate the dependence of μ and η of the factor that multiplies t^{M+1}.
Secondly, the argument of Step 1 and the idea of proving (11.32) goes essentially back to J. Peetre, see [Pe75]. Compared with the classical situation ($N_j = 2^j$ for any $j \in \mathbb{N}_0$) and with the proof in [Ry99], to estimate the integral I_{jm} in (11.36) for $m > j$ we had to take into account that the functions ψ_m are not generated from a single function ψ. This caused complications which were solved applying Lemma 11.3.1 in the form which was stated.
As a third observation we point out, see Step 2, that in order to prove (11.62) we had to introduce the function Φ_j to obtain the equality (11.51). This allowed us to avoid the dilation argument from [Ry99] which could not work in the case of general sequences $(N_j)_{j\in\mathbb{N}_0}$.
Finally, note that the above technique to prove the estimate (11.58) was used in the classical case ($N_j = 2^j$ for any $j \in \mathbb{N}_0$) by J.-O. Strömberg and A. Torchinsky in [StTo89, Chapter 5, Theorem 2(a)].

Remark 11.4.1 Theorem 11.3.4 paves the way to the proof of the atomic decomposition theorem, see next section, but it is of independent interest since it covers the classical results of H.-Q. Bui, M. Paluszyński and M. Taibleson, see [BPT96] and [BPT97], the theorem on local means from [Tr92, Theorem 2.4.6], and the theorem on local means from [Mo99] and [Mo01]. This is discussed in the next Example.

Example 11.4.2 (The classical case) As we have already mentioned several times in this work if $N_j = 2^j$, and $\sigma_j = 2^{js}$, $s \in \mathbb{R}$, then the spaces $B^{\sigma,N}_{p,q}$ and $F^{\sigma,N}_{p,q}$ are the classical spaces $B^s_{p,q}$ and $F^s_{p,q}$.
Condition (11.1) is fulfilled with $\lambda_0 = \lambda_1 = 2$. Moreover, condition (11.3) is fulfilled with $d_0 = d_1 = 2^s$.
The restriction (11.26) in the theorem on local means is then $K > -1 + s$. Note that if $s < 0$ there are no moment conditions needed. Theorem 11.3.4 coincides with the result of H.-Q. Bui, M. Paluszyński and M. Taibleson as it was already mentioned.

Example 11.4.3 (The spaces $B^{(s,\Psi)}_{p,q}$ and $F^{(s,\Psi)}_{p,q}$) In Example 9.1.7 we mentioned that if $s \in \mathbb{R}$ is fixed and

$$\sigma_j = 2^{js}\Psi(2^{-j}) \quad , \quad j \in \mathbb{N}_0 \quad ,$$

where Ψ is a positive monotone function on $(0,1]$ such that there are positive constants b_0^* and b_1^* with

$$b_0^*\,\Psi(2^{-j}) \leqslant \Psi(2^{-2j}) \leqslant b_1^*\,\Psi(2^{-j}) \quad \text{for all} \quad j \in \mathbb{N}_0$$

then σ is an admissible sequence with $d_0 = b_0\,2^s$ and $d_1 = b_1\,2^s$ in (9.2), where $b_0 = \min(b_0^*, 1, \Psi(2^{-1})\Psi(1)^{-1})$ and $b_1 = \max(b_1^*, 1, \Psi(2^{-1})\Psi(1)^{-1})$.
If, in addition $N_j = 2^j$ then the spaces $B^{\sigma,N}_{p,q}$ and $F^{\sigma,N}_{p,q}$ are the spaces $B^{(s,\Psi)}_{p,q}$ and $F^{(s,\Psi)}_{p,q}$ considered in [Mo99] and [Mo01].
Condition (11.1) is fulfilled with $\lambda_0 = \lambda_1 = 2$ and condition (11.3) is fulfilled with $d_0 = b_0\,2^s$ and $d_1 = b_1\,2^s$.
The restriction (11.26) in the theorem on local means is then $K > -1 + s + \log_2 b_1$.
Our condition concerning the constants K slightly different (because of the additional log-terms) from that in the atomic decomposition of S. Moura, see [Mo01, Theorem 1.18], which was proved directly for the spaces $B^{(s,\Psi)}_{p,q}$ and $F^{(s,\Psi)}_{p,q}$.

Example 11.4.4 (ψ-Bessel potential spaces) Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an admissible continuous negative definite function, compare Definition 6.1.1, and assume that there exists $r_0 \in (0,1]$ such that $\xi \mapsto \psi(\xi)\langle\xi\rangle^{-2r_0}$ is increasing in $|\xi|$. If $(N_j)_{j\in\mathbb{N}_0}$ is the sequence associated to ψ and to $r = 2$, compare (6.5), then for any $j \in \mathbb{N}_0$ one has

$$2^{\frac{1}{w}}\,N_j \leqslant N_{j+1} \leqslant 2^{\frac{1}{r_0}}\,N_j \quad ,$$

compare Lemma 6.2.2 and Lemma 7.2.3.
Based on Corollary 7.1.4 we have for $s \in \mathbb{R}$ and $1 < p < \infty$ the equality $H^{\psi,s}_p = F^{\sigma^s,N}_{p,2}$ where $\sigma^s = (2^{js})_{j\in\mathbb{N}_0}$.
Condition (11.1) is fulfilled with $\lambda_0 = 2^{\frac{1}{w}}$ and $\lambda_1 = 2^{\frac{1}{r_0}}$ and condition (11.3) is fulfilled with $d_0 = d_1 = 2^s$.
The restriction (11.26) in the theorem on local means is then $K > -1 + sw$. Note that if $sw < 0$, i.e. if $s < 0$ (recall $w > 0$), there are no moment conditions needed.
Note also that if $\psi(\xi) = f(1 + |\xi|^2)$ (for an appropriate Bernstein function f) then $K > -1 + s$ since then $w = 1$.

12 Atomic decompositions

12.1 Preliminaries: N-atoms and sequence spaces

In this section we will consider again $N = (N_j)_{j\in\mathbb{N}_0}$ an admissible sequence with bounded growth which satisfies (11.1) with $\lambda_0 > 1$.
Let $\mathbb{Z}^n$ be the lattice of all points in $\mathbb{R}^n$ with integer-valued components.
If $\nu \in \mathbb{N}_0$ and $m = (m_1, ..., m_n) \in \mathbb{Z}^n$ we denote $Q_{\nu m}$ the cube in $\mathbb{R}^n$ centred at $N_\nu^{-1}m = (N_\nu^{-1}m_1, ..., N_\nu^{-1}m_n)$ which has sides parallel to the axes and side length N_ν^{-1}.
If $Q_{\nu m}$ is such a cube in $\mathbb{R}^n$ and $c > 0$ then $cQ_{\nu m}$ is the cube in $\mathbb{R}^n$ concentric with $Q_{\nu m}$ and with side length cN_ν^{-1}.

We are now prepared to introduce the N-atoms (associated to the sequence N).

Definition 12.1.1 **(i)** *Let $M \in \mathbb{R}$, $c^* > 1$. A function $\rho : \mathbb{R}^n \to \mathbb{C}$ for which there exist all derivatives $D^\alpha \rho$ if $|\alpha| \leqslant M$ (continuous if $M \leqslant 0$) is called an 1_M-N-atom if:*

$$\operatorname{supp} \rho \subset c^* Q_{0m} \quad \textit{for some} \quad m \in \mathbb{Z}^n, \tag{12.1}$$

$$|D^\alpha \rho(x)| \leqslant 1 \quad \textit{if} \quad |\alpha| \leqslant M. \tag{12.2}$$

(ii) *Let $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ be an admissible sequence, let $0 < p \leqslant \infty$, $M, L \in \mathbb{R}$, $c^* > 1$. A function $\rho : \mathbb{R}^n \to \mathbb{C}$ for which there exist all derivatives $D^\alpha \rho$ if $|\alpha| \leqslant M$ (continuous if $M \leqslant 0$) is called an $(\sigma, p)_{M,L}$-N-atom if:*

$$\operatorname{supp} \rho \subset c^* Q_{\nu m} \quad \textit{for some} \quad \nu \in \mathbb{N}, m \in \mathbb{Z}^n, \tag{12.3}$$

$$|D^\alpha \rho(x)| \leqslant \sigma_\nu^{-1} N_\nu^{\frac{n}{p} + |\alpha|} \quad \textit{if} \quad |\alpha| \leqslant M, \tag{12.4}$$

$$\int_{\mathbb{R}^n} x^\gamma \rho(x) dx = 0 \quad \textit{if} \quad |\gamma| \leqslant L. \tag{12.5}$$

If the atom ρ is located at $Q_{\nu m}$ (that means $\operatorname{supp} \rho \subset c^* Q_{\nu m}$ with $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $c > 1$) then we will denote it by $\rho_{\nu m}$.

This concept generalises the smooth (isotropic) atoms from the works of M. Frazier and B. Jawerth, [FrJa85] and [FrJa90], which correspond to $N_\nu = 2^\nu$ and $\sigma_\nu = 2^{\nu s}$ with real s.
We give some technical explanations.
The value of the number $c^* > 1$ in (12.1) and (12.3) is unimportant. It simply makes clear that at the level ν some controlled overlapping of the supports of $\rho_{\nu m}$ must be allowed.
The moment conditions (12.5) can be reformulated as $D^\gamma \widehat{\rho}(0) = 0$ if $|\gamma| \leqslant L$, which shows that a sufficiently strong decay of $\widehat{\rho}$ at the origin is required. If $L < 0$ then (12.5) simply means that there are no moment conditions.
The reason for the normalising factor in (12.2) and (12.4) is that there exists a constant $c > 0$ such that for all these atoms we have $\|\rho \,|\, B^{\sigma,N}_{p,q}\| \leqslant c$ and $\|\rho \,|\, F^{\sigma,N}_{p,q}\| \leqslant c$. Hence, as in the classical case, atoms are normalised building blocks satisfying some moment conditions.

Before we will state the atomic decomposition theorem we have to introduce the sequence spaces $b_{p,q}$ and $f^N_{p,q}$.
If $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $Q_{\nu m}$ is a cube as above let $\chi_{\nu m}$ be the characteristic function of $Q_{\nu m}$; if $0 < p \leqslant \infty$ let

$$\chi^{(p)}_{\nu m} = N_\nu^{n/p} \chi_{\nu m}$$

(obvious modification if $p = \infty$) be the L_p-normalised characteristic function of $Q_{\nu m}$.

Definition 12.1.2 *Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$. Then:*
(i) *$b_{p,q}$ is the collection of all sequences $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ such that*

$$\|\lambda \,|\, b_{p,q}\| = \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q}$$

(with the usual modification if $p = \infty$ and/or $q = \infty$) is finite;

(ii) $f^N_{p,q}$ *is the collection of all sequences* $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ *such that*

$$\|\lambda \,|\, f^N_{p,q}\| = \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m} \chi^{(p)}_{\nu m}(\cdot)|^q \right)^{1/q} \,|\, L_p \right\|$$

(with the usual modification if $p = \infty$ *and/or* $q = \infty$*) is finite.*

One can easily see that $b_{p,q}$ and $f^N_{p,q}$ are quasi-Banach spaces and using $\|\chi^{(p)}_{\nu m} \,|\, L_p\| = 1$ it is clear that comparing $\|\lambda \,|\, b_{p,q}\|$ and $\|\lambda \,|\, f^N_{p,q}\|$ the roles of the quasi-norms in L_p and l_q are interchanged.

12.2 The atomic decomposition theorem

We are able now to state the main result of this chapter and of this section.

Theorem 12.2.1 *Let* $N = (N_j)_{j \in \mathbb{N}_0}$ *be an admissible sequence with* $\lambda_0 > 1$ *in* (11.1) *and let* $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ *be an admissible sequence.*
Let $0 < p < \infty$*, respectively* $0 < p \leqslant \infty$*,* $0 < q \leqslant \infty$*, and let* M*,* $L \in \mathbb{R}$ *such that*

$$M > \frac{\log_2 d_1}{\log_2 \lambda_0} \tag{12.6}$$

and

$$L > -1 + n \left(\frac{\log_2 \lambda_1}{\log_2 \lambda_0} \frac{1}{\min(1,p,q)} - 1 \right) - \frac{\log_2 d_0}{\log_2 \lambda_0} \quad , \tag{12.7}$$

respectively

$$L > -1 + n \left(\frac{\log_2 \lambda_1}{\log_2 \lambda_0} \frac{1}{\min(1,p)} - 1 \right) - \frac{\log_2 d_0}{\log_2 \lambda_0}. \tag{12.8}$$

Then $g \in \mathcal{S}'$ *belongs to* $F^{\sigma,N}_{p,q}$*, respectively to* $B^{\sigma,N}_{p,q}$*, if and only if, it can be represented as*

$$g = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \rho_{\nu m} \quad , \tag{12.9}$$

convergence being in $\mathcal{S}'$*, where* $\rho_{\nu m}$ *are* 1_M*-*N*-atoms (*$\nu = 0$*) or* $(\sigma, p)_{M,L}$*-*N*-atoms (*$\nu \in \mathbb{N}$*) and* $\lambda \in f^N_{p,q}$*, respectively* $\lambda \in b_{p,q}$*, where* $\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$*.*
Furthermore, $\inf \|\lambda \,|\, f^N_{p,q}\|$*, respectively* $\inf \|\lambda \,|\, b_{p,q}\|$*, where the infimum is taken over all admissible representations* (12.9)*, is an equivalent quasi-norm in* $F^{\sigma,N}_{p,q}$*, respectively* $B^{\sigma,N}_{p,q}$*.*

The convergence in $\mathcal{S}'$ can be obtained as a by-product of the proof using the same method as in [Tr97, Theorem 13.7], compare also the discussion in [Tr01], so we will not stress this point. We refer to the above theorem as to the atomic decomposition theorem in function spaces of generalised smoothness.

Before giving the proof let us make here some remarks. The first part of the proof, that one in which the atoms are constructed and where it is shown that the decomposition (12.9) holds, is essentially based on a version of a resolution of unity of Calderon type, cf. [FJW91, Lemma 5.12].
To prove the second part we will use the theorem on local means, see Theorem 11.3.4, the technique of maximal functions and an inequality of Fefferman - Stein type.

12.3 Proof of the atomic decomposition theorem

12.3.1 An auxiliary result: a partition of unity of Calderon type

We will need the following

Lemma 12.3.1 *Let $N = (N_j)_{j\in\mathbb{N}_0}$ be an admissible sequence with $\lambda_0 > 1$ in* (11.1) *and let $(\Omega_j^N)_{j\in\mathbb{N}_0}$ be the associated covering of $\mathbb{R}^n$ with $J = \kappa_0 = 1$, see* (9.5) *and* (9.6). *Let also $(\varphi_j^N)_{j\in\mathbb{N}_0} \in \Phi^N$ be fixed with $c_\varphi = 1$ and $L \geqslant 0$ be also fixed.*
Then there exist functions θ_0, $\theta \in \mathcal{S}$ with:

$$\operatorname{supp}\theta_0\ ,\ \operatorname{supp}\theta \subset \{x\in\mathbb{R}^n\ :\ |x|\leqslant 1\}, \tag{12.10}$$

$$|\widehat{\theta_0}(\xi)| \geqslant c_0 > 0 \quad \textit{if} \quad |\xi| \leqslant N_1, \tag{12.11}$$

$$|\widehat{\theta}(\xi)| \geqslant c > 0 \quad \textit{if} \quad \frac{1}{\lambda_1} \leqslant |\xi| \leqslant \lambda_1, \tag{12.12}$$

$$\int_{\mathbb{R}^n} x^\gamma \theta(x)dx = 0 \quad \textit{if} \quad |\gamma| \leqslant L, \tag{12.13}$$

and

$$\widehat{\theta_0}(\xi)\widehat{\psi_0}(\xi) + \sum_{j=1}^{\infty} \widehat{\theta}(N_j^{-1}\xi)\,\widehat{\psi_j}(\xi) = 1 \quad \textit{for all} \quad \xi\in\mathbb{R}^n \tag{12.14}$$

where the functions $\psi_j \in \mathcal{S}$ are defined by

$$\widehat{\psi_0}(\xi) = \frac{\varphi_0^N(\xi)}{\widehat{\theta_0}(\xi)} \quad \textit{and} \quad \widehat{\psi_j}(\xi) = \frac{\varphi_j^N(\xi)}{\widehat{\theta}(N_j^{-1}\xi)} \quad \textit{for} \quad j\in\mathbb{N}. \tag{12.15}$$

Let us mention that the difference to the classical result is due to the fact that the functions ψ_j are in general not obtained simply by dilation from a fixed function ψ.

Proof. Recall $(\varphi_j^N)_{j\in\mathbb{N}_0} \in \Phi^N$ is fixed.
Let, as in [FrJa85, Theorem 2.6], $\Theta \in \mathcal{S}$ be a real-valued radial function satisfying

$$\operatorname{supp}\Theta \subset \{x\in\mathbb{R}^n\ :\ |x|\leqslant 1\} \quad \text{and} \quad \widehat{\Theta}(0) = 1.$$

Then for some $1 > \varepsilon > 0$ we have $\widehat{\Theta}(\xi) \geqslant 1/2$ for all ξ satisfying $|\xi| < \varepsilon\lambda_1$. Then

$$\theta(x) = \varepsilon^{-n}\,(-\Delta)^L\,\Theta\left(\frac{x}{\varepsilon}\right)$$

satisfies requirements (12.10) - (12.14).
Since $\widehat{\theta}(N_j^{-1}\xi) \geqslant c > 0$ for $\lambda_1^{-1}\,N_j \leqslant |\xi| \leqslant \lambda_1\,N_j$, using $\lambda_1^{-1}N_j \leqslant N_{j-1}$ and $N_{j+1} \leqslant \lambda_1\,N_j$ one has $\widehat{\theta}(N_j^{-1}\xi) \geqslant c > 0$ for any $\xi \in \operatorname{supp}\varphi_j^N \subset \{\xi\in\mathbb{R}^n\ :\ N_{j-1} \leqslant |\xi| \leqslant N_{j+1}\}$.
Consequently, the functions ψ_j are well defined for $j \geqslant 1$ and

$$\sum_{j=1}^{\infty} \widehat{\theta}(N_j^{-1}\xi)\,\widehat{\psi_j}(\xi) = 1 \quad \text{for any} \quad \xi\in\mathbb{R}^n\backslash\operatorname{supp}\varphi_0^N.$$

Similarly one has to find θ_0 with $\widehat{\theta_0}(\xi) \geqslant c > 0$ for any $\xi \in \operatorname{supp}\varphi_0^N$. Taking a function $\Theta \in \mathcal{S}$ such that $\widehat{\Theta}(\xi) \geqslant 1/2$ for $|\xi| \leqslant \delta N_1$ the function $\theta_0 = \delta^{-n}\Theta(x/\delta)$ satisfies the above mentioned requirement. Now one has to define the corresponding function ψ_0 and the proof is complete. ∎

Remark 12.3.2 From the proof of the above Lemma it is clear that for a given system $(\varphi_j^N)_{j\in\mathbb{N}_0} \in \Phi^N$ and fixed functions θ_0, $\theta \in \mathcal{S}$ the associated system $(\psi_j)_{j\in\mathbb{N}_0}$ from (12.15) satisfies

$$\widehat{\psi_j}(\xi) \geqslant 0 \quad \text{and} \quad \operatorname{supp} \widehat{\psi_j} \subset \{\xi \in \mathbb{R}^n \,:\, N_{j-1} \leqslant |\xi| \leqslant N_{j+1}\} \quad \text{for any} \quad j \geqslant 1.$$

An easy application of Leibniz's rule shows that for any $\gamma \in \mathbb{N}_0^n$ there is a constant $c_\gamma > 0$ (independent of j) such that

$$|D^\gamma \widehat{\psi_j}(\xi)| \leqslant c_\gamma \, \langle \xi \rangle^{-|\gamma|} \quad \text{for any} \quad \xi \in \mathbb{R}^n.$$

Consequently, each function $\widehat{\psi_j}$ is a Fourier multiplier in L_p, as a simple application of the scalar version of Proposition 0.2.1.

12.3.2 Proof of Theorem 12.2.1

Part I. Let $g \in F^{\sigma,N}_{p,q}$; we use the method of M. Frazier, B. Jawerth and G. Weiss from [FJW91, Theorem 5.11] to construct atoms and to decompose g as in (12.9).
Let θ_0, θ, ψ_0 and ψ_ν ($\nu \geqslant 1$) functions in $\mathcal{S}$ satisfying (12.10)-(12.14).
Using $\widehat{\theta}(N_\nu^{-1}\xi) = N_\nu^n [\theta(N_\nu \cdot)]\widehat{\ }(\xi)$ we have

$$g = \theta_0 * \psi_0 * g + \sum_{\nu=1}^{\infty} \theta_{N_\nu^{-1}} * \psi_\nu * g$$

and using the definition of the cubes $Q_{\nu m}$ we obtain the following equality in $\mathcal{S}'$:

$$g(x) = \sum_{m\in\mathbb{Z}^n} \int_{Q_{0m}} \theta_0(x-y)(\psi_0 * g)(y)dy + \sum_{\nu=1}^{\infty} \sum_{m\in\mathbb{Z}^n} N_\nu^n \int_{Q_{\nu m}} \theta(N_\nu(x-y))(\psi_\nu * g)(y)dy.$$

We define for every $\nu \in \mathbb{N}$ and all $m \in \mathbb{Z}^n$

$$\lambda_{\nu m} = C_\theta \, \sigma_\nu \, N_\nu^{-\frac{n}{p}} \sup_{y\in Q_{\nu m}} |(\psi_\nu * g)(y)| \tag{12.16}$$

where $C_\theta = \max\big\{ \sup_{|x|\leqslant 1} |D^\alpha \theta(x)| \,:\, |\alpha| \leqslant K \big\}$. Define also

$$\rho_{\nu m}(x) = \frac{1}{\lambda_{\nu m}} \, N_\nu^n \int_{Q_{\nu m}} \theta(N_\nu(x-y))(\psi_\nu * g)(y)\, dy \tag{12.17}$$

if $\lambda_{\nu m} \neq 0$ and $\rho_{\nu m} = 0$ otherwise.
Similarly we define for every $m \in \mathbb{Z}^n$ the numbers λ_{0m} and the functions ρ_{0m} taking in (12.16) and (12.17) $\nu = 0$ and replacing ψ_ν and θ by ψ_0 and θ_0, respectively.

It is obvious that (12.9) is satisfied and it follows by straightforward calculations, using the properties of the functions θ_0, θ, ψ_0 and ψ_ν, that ρ_{0m} are 1_M-N-atoms and that $\rho_{\nu m}$ are $(\sigma, p)_{M,L}$-N-atoms for $\nu \in \mathbb{N}$.
Finally, we will show that there exists a constant $c > 0$ such that $\|\lambda \,|\, f^N_{pq}\| \leqslant c \, \|g \,|\, F^{\sigma,N}_{p,q}\|$.

We have for a fixed $\nu \in \mathbb{N}$:

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi^{(p)}_{\nu m}(x) &= C_\theta\, \sigma_\nu\, N_\nu^{-\frac{n}{p}} \sum_{m\in\mathbb{Z}^n} \sup_{y\in Q_{\nu m}} |(\psi_\nu * g)(y)| \cdot N_\nu^{\frac{n}{p}} \chi_{\nu m}(x)\\ &\leqslant c'\, \sigma_\nu \Big(\sup_{|z|\leqslant c\, N_\nu^{-1}} \frac{|(\psi_\nu * g)(x-z)|}{(1+N_\nu|z|)^r}\,(1+N_\nu|z|)^r \Big)\\ &\leqslant c''\, \sigma_\nu\, (\psi_\nu^* g)_r(x)\end{aligned}$$

since $|x-y| \leqslant c\, N_\nu^{-1}$ for $x, y \in Q_{\nu m}$ and $\sum_{m\in\mathbb{Z}^n} \chi_{\nu m}(x) = 1$. Here $r > \frac{n}{\min(p,q)}$ and $(\psi_\nu^* g)_r$ is the maximal function of J. Peetre, compare (11.25). It follows

$$\sum_{\nu=1}^{\infty} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}\, \chi^{(p)}_{\nu m}(\cdot)|^q \leqslant c \sum_{\nu=1}^{\infty} \sigma_\nu^q\, (\psi_\nu^* g)_r(\cdot)^q \tag{12.18}$$

(with the usual modification if $q=\infty$) where c is a positive constant.

Now we have to use (12.18) and its counterpart for $\nu = 0$ (which can be obtained by a similar calculation) and get

$$\|\lambda \,|\, f^N_{p,q}\| \leqslant c \left\| \Big(\sum_{\nu=0}^{\infty} \sigma_\nu^q\, (\psi_\nu^* g)_r(\cdot)^q \Big)^{1/q} \,|\, L_p \right\| \leqslant c'\, \|g \,|\, F^{\sigma,N}_{p,q}\| \tag{12.19}$$

(with the usual modification if $q=\infty$) and this completes the proof of the first part of the theorem if we would be able to justify the last inequality in (12.19).
But the last inequality in (12.19) is nothing else than a simple application of Proposition 10.2.1 taking in that theorem $f = (f_\nu)_{\nu\in\mathbb{N}_0}$ where for any $\nu \in \mathbb{N}_0$ the function f_ν is $\sigma_\nu(\psi_\nu * g)$ and the domain Ω_ν is $\{\xi \in \mathbb{R}^n : |\xi| \leqslant N_{\nu+1}\}$ and recalling the definition of the maximal functions from (11.24) and (11.25).

<u>*Part* II.</u> Reciprocally, assume now g can be represented by (12.9), with M and L satisfying (12.6) and (12.7), respectively. We will show that $g \in F^{\sigma,N}_{p,q}$ and that $\|g \,|\, F^{\sigma,N}_{p,q}\| \leqslant c\, \|\lambda \,|f^N_{pq}\|$ for some constant $c > 0$.
Let k_0 and k be two functions in $\mathcal{S}$ such that $\operatorname{supp} k_0, \operatorname{supp} k \subset \{x \in \mathbb{R}^n : |x| \leqslant 1\}$ and $|\widehat{k_0}(\xi)| > 0$ for $|\xi| \leqslant N_1$, $|\widehat{k}(\xi)| > 0$ for $\frac{1}{\lambda_1} \leqslant |\xi| \leqslant \lambda_1$ and

$$\int_{\mathbb{R}^n} x^\alpha k(x)\, dx = 0 \quad \text{for any} \quad |\alpha| \leqslant K. \tag{12.20}$$

Our intention is to apply Theorem 11.3.4. Let K enough large such that $K \geqslant M-1$. Temporarily let $\nu, j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$ be fixed; we start finding convenient estimates for $\sigma_j\, (k_{N_j^{-1}} * \rho_{\nu m})(x)$.
Step II.1 Let $j \geqslant \nu$ and let again $k_{N_j^{-1}}(x) = N_j^n\, k(N_j x)$. Then

$$\sigma_j\, (k_{N_j^{-1}} * \rho_{\nu m})(x) = \sigma_j \int_{|y|\leqslant 1} k(y)\, \rho_{\nu m}(x - N_j^{-1} y) dy.$$

Due to (12.3) the above integral is non zero only if $|x - N_j^{-1}y - N_\nu^{-1}m| \leqslant c\, N_\nu^{-1}$ which implies that x is located in some $cQ_{\nu m}$ since

$$|x - N_\nu^{-1}m| \leqslant |x - N_j^{-1}y - N_\nu^{-1}m| + |N_j^{-1}y| \leqslant c\, N_\nu^{-1} + N_j^{-1} \leqslant c'\, N_\nu^{-1}.$$

According to (12.4) the derivatives $D^\alpha \rho_{\nu m}$ exist if $|\alpha| \leqslant M$ so we can use Taylor's expansion theorem of order M for the function $w \mapsto \rho_{\nu m}(w)$ on the set $B(x, N_j^{-1})$ (the ball centred at x and of radius N_j^{-1}).
We put then $w = x - N_j^{-1}y$ and noting that if $z \in B(x, N_j^{-1})$ then $z \in cQ_{\nu m}$ we get the expansion

$$\rho_{\nu m}(x - N_j^{-1}y) = \sum_{|\alpha| \leqslant M-1} c_\alpha (x - N_j^{-1}y - z)^\alpha D^\alpha \rho_{\nu m}(z) + R_M(x,y) \tag{12.21}$$

where

$$\begin{aligned} |R_M(x,y)| &\leqslant c \max_{|\alpha|=M} \left(N_j^{-|\alpha|} \sup_{z \in c' Q_{\nu m}} |D^\alpha \rho_{\nu m}(z)| \right) \\ &\leqslant c' N_j^{-M} \sigma_\nu^{-1} N_\nu^{\frac{n}{p}+M} \widetilde{\chi}_{\nu m}(x) \\ &= c' \sigma_\nu^{-1} (N_j^{-1} N_\nu)^M \widetilde{\chi}^{(p)}_{\nu m}(x) \end{aligned} \tag{12.22}$$

for some $c, c' > 0$ where $\widetilde{\chi}^{(p)}_{\nu m}$ is the p-normalised characteristic function of some cube $cQ_{\nu m}$.
Recall K is enough large such that $K \geqslant M - 1$; using the moment conditions for the function k we obtain $\int_{\mathbb{R}^n} (x - N_j^{-1}y - z)^\alpha \, k(y)dy = 0$ for all α such that $|\alpha| \leqslant M - 1$. Hence (12.21) and (12.22) yield

$$\left| \sigma_j \, (k_{N_j^{-1}} * \rho_{\nu m})(x) \right| \leqslant c \, \sigma_j \, \sigma_\nu^{-1} \, (N_j^{-1} N_\nu)^M \, \widetilde{\chi}^{(p)}_{\nu m}(x). \tag{12.23}$$

Using now (11.3) and (11.1) we have for $j \geqslant \nu$

$$\sigma_j \, \sigma_\nu^{-1} \leqslant d_1^{j-\nu} = 2^{-(j-\nu)(-\log_2 d_1)} \quad \text{and} \quad N_j^{-1} N_\nu \leqslant \lambda_0^{-(j-\nu)} = 2^{-(j-\nu)\log_2 \lambda_0}.$$

Inserting the last estimates in (12.23) we get

$$\left| \sigma_j \, (k_{N_j^{-1}} * \rho_{\nu m})(x) \right| \leqslant c \, 2^{-(j-\nu)(-\log_2 d_1 + M \log_2 \lambda_0)} \, \widetilde{\chi}^{(p)}_{\nu m}(x) = 2^{-(j-\nu)\delta} \, \widetilde{\chi}^{(p)}_{\nu m}(x) \tag{12.24}$$

for $\delta = -\log_2 d_1 + M \log_2 \lambda_0$. Clearly $\delta > 0$ since M satisfies the estimate (12.6).

Step II.2 Let now $j < \nu$. We chose K in (12.20) enough large such that, in addition, $K \geqslant L$. Then

$$\sigma_j \, (k_{N_j^{-1}} * \rho_{\nu m})(x) = \sigma_j \, N_j^n \int_{\mathbb{R}^n} k(N_j y) \, \rho_{\nu m}(x - y) dy \tag{12.25}$$

and due to the support localisation of k the above integration can be restricted to the set $\{y \in \mathbb{R}^n \, : \, |y| \leqslant N_j^{-1}\}$.
We remark also that by our assumption on j and ν and to the support localisation for $\rho_{\nu m}$ one has

$$|x - N_\nu^{-1} m| \leqslant |x - y - N_\nu^{-1} m| + |y| \leqslant c \, N_\nu^{-1} + N_j^{-1} \leqslant c' \, N_j^{-1}$$

and this implies that if the above integral is non-zero then x is located in some cB_{jm} where $B_{jm} = \{z \in \mathbb{R}^n \, : \, |z - N_\nu^{-1} m| \leqslant N_j^{-1}\}$.

Since k is a smooth function on $\mathbb{R}^n$ we may use Taylor's expansion theorem of order L for the function $w \mapsto k(w)$ on the set $B(z_x, N_j N_\nu^{-1}) = \{w \in \mathbb{R}^n : |w - z_x| \leqslant N_j N_\nu^{-1}\}$, where $z_x = z(j, \nu, m, x) = N_j(N_\nu^{-1} m - x)$.
After that we let $w = N_j y$ and get

$$k(N_j y) = \sum_{|\alpha| \leqslant L} c_\alpha \, (N_j y - z_x)^\alpha D^\alpha k(z_x) + R_L(y, x) \tag{12.26}$$

where

$$|R_L(y, x)| \leqslant c \, (N_j N_\nu^{-1})^{L+1}$$

for some positive constant c since k is smooth and has compact support.
By the moment conditions (12.5) we have $\int_{\mathbb{R}^n} (N_j y - z_x)^\alpha \rho_{\nu m}(x - y) dy = 0$ if $|\alpha| \leqslant L$ since we have chosen $K \geqslant L$; using (12.26) we may replace (12.25) by:

$$\begin{aligned} |\sigma_j \, (k_{N_j^{-1}} * \rho_{\nu m})(x)| &\leqslant \sigma_j \, N_j^n \int_{|y| \leqslant N_j^{-1}} |R_L(y, x)| \, |\rho_{\nu m}(x - y)| \, dy \\ &\leqslant c \, \sigma_j \, N_j^n \, (N_j N_\nu^{-1})^{(L+1)} \int_{|y| \leqslant N_j^{-1}} |\rho_{\nu m}(x - y)| \, dy. \end{aligned}$$

Using (12.4) to estimate $\rho_{\nu m}$ we get

$$|\sigma_j \, (k_{N_j^{-1}} * \rho_{\nu m})(x)| \leqslant c \, \sigma_j \, N_j^n \, (N_j N_\nu^{-1})^{(L+1)} \sigma_\nu^{-1} \, N_\nu^{\frac{n}{p}} \int_{|y| \leqslant N_j^{-1}} \widetilde{\chi}_{\nu m}(x - y) \, dy \tag{12.27}$$

where $\widetilde{\chi}_{\nu m}$ is the characteristic function of some cube $cQ_{\nu m}$.
Let now χ^{jm} be the characteristic function of the ball cB_{jm} where x is located; by a straightforward computation we have:

$$\int_{|y| \leqslant N_j^{-1}} \widetilde{\chi}_{\nu m}(x - y) \, dy \leqslant c \, N_\nu^{-n} \, \chi^{jm}(x). \tag{12.28}$$

Due to condition (12.7) on L we may choose an $\omega < \min(1, p, q)$ such that

$$\omega > \frac{n \log_2 \lambda_1}{\log_2 d_0 + (L + 1 + n) \log_2 \lambda_0}. \tag{12.29}$$

Denoting $\mathcal{M}\chi_{\nu m}$ the Hardy - Littlewood maximal function of $\chi_{\nu m}$ we get

$$\chi^{jm}(\cdot) \leqslant c \, (N_j^{-1} N_\nu)^{\frac{n}{\omega}} \, (\mathcal{M}\chi_{\nu m}(\cdot))^{1/\omega}. \tag{12.30}$$

Finally, using (12.28) and (12.30), the estimate (12.27) becomes:

$$|\sigma_j \, (k_{N_j^{-1}} * \rho_{\nu m})(x)| \leqslant c \, \sigma_j \, N_j^n \, (N_j N_\nu^{-1})^{(L+1)} \sigma_\nu^{-1} \, N_\nu^{-n} \, (N_j^{-1} N_\nu)^{\frac{n}{\omega}} \, \left(\mathcal{M}\widetilde{\chi}_{\nu m}^{(p)}(x)\right)^{1/\omega} \tag{12.31}$$

where again $\widetilde{\chi}_{\nu m}^{(p)}$ is the p-normalised characteristic function of some cube $cQ_{\nu m}$.
Using now (11.3) and (11.1) we have for $j < \nu$

$$\sigma_j \, \sigma_\nu^{-1} \leqslant d_0^{-(\nu - j)} = 2^{-(\nu - j)(\log_2 d_0)},$$

$$N_j N_\nu^{-1} \leqslant \lambda_0^{-(\nu-j)} = 2^{-(\nu-j)\log_2\lambda_0} \quad \text{and} \quad N_j^{-1} N_\nu \leqslant \lambda_1^{\nu-j} = 2^{-(\nu-j)(-\log_2\lambda_1)}$$

so that (12.31) becomes

$$\begin{aligned} & |\sigma_j\,(k_{N_j^{-1}} * \rho_{\nu m})(x)| \\ \leqslant\ & c\,2^{-(\nu-j)\log_2 d_0}\,2^{-(\nu-j)(L+1+n)\log_2\lambda_0}\,2^{-(\nu-j)\frac{n}{\omega}(-\log_2\lambda_1)}\,\big(\mathcal{M}\widetilde{\chi}^{(p)}_{\nu m}(x)\big)^{1/\omega} \\ =\ & c\,2^{-(\nu-j)\varepsilon}\,\big(\mathcal{M}\widetilde{\chi}^{(p)}_{\nu m}(x)\big)^{1/\omega} \end{aligned} \tag{12.32}$$

where

$$\varepsilon = \log_2 d_0 + (L+1+n)\log_2\lambda_0 - \frac{n}{\omega}\log_2\lambda_1 > 0$$

due to our choice of ω, see (12.29).
Remark that the terms with $j=0$ and/or $\nu=0$ can also be covered by the technique in steps II.1-2.

Step II.3 Using (12.24) and (12.32) we get for $0 < q \leqslant 1$:

$$\Big|\sigma_j(k_{N_j^{-1}} * \sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\rho_{\nu m})(x)\Big|^q \leqslant c\sum_{\nu\leqslant j}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^q\,2^{-\delta(j-\nu)q}\,\widetilde{\chi}^{(p)q}_{\nu m}(x) +$$

$$+c'\sum_{\nu>j}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^q\,2^{-\varepsilon(\nu-j)q}\,\big(\mathcal{M}\widetilde{\chi}^{(p)}_{\nu m}(x)\big)^{q/\omega}$$

with δ, $\varepsilon > 0$, with the usual modification if $1 < q \leqslant \infty$.
We sum over j, take the $\frac{1}{q}$-th power and then the L_p- quasi-norm and obtain that

$$\Big\|\big(\sum_{j=1}^{\infty}\sigma_j^q\,|(k_{N_j^{-1}} * \sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\rho_{\nu m})(\cdot)|^q\big)^{1/q}\,|\,L_p\Big\|$$

can be estimated from above by

$$\begin{aligned} & c\,\Big\|\big(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^q\,\widetilde{\chi}^{(p)}_{\nu m}(\cdot)^q\big)^{1/q}\,|\,L_p\Big\| \\ +\ & c'\,\Big\|\big(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^q\,\big(\mathcal{M}\widetilde{\chi}^{(p)}_{\nu m}(\cdot)\big)^{q/\omega}\big)^{1/q}\,|\,L_p\Big\| \end{aligned} \tag{12.33}$$

with the usual modification if $q=\infty$.
The first term of (12.33) is just what we want since $\widetilde{\chi}^{(p)}_{\nu m}$ can be replaced by $\chi^{(p)}_{\nu m}$.
With $h_{\nu m} = \lambda_{\nu m}\,\chi^{(p)}_{\nu m}$ the second term of (12.33) can be written as:

$$c''\,\Big\|\big(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\mathcal{M}h^{\omega}_{\nu m}(\cdot)^{q/\omega}\big)^{\omega/q}\,|\,L_{p/\omega}\Big\|^{\frac{1}{\omega}}$$

(usual modification if $q=\infty$). Recall $1 < \frac{p}{\omega} < \infty$ and $1 < \frac{q}{\omega} \leqslant \infty$ so that we can apply the Fefferman - Stein inequality and obtain again what we want.
The term with $j=0$ can be incorporated by the same technique. ■

12.4 Comments

Remark 12.4.1 For spaces $B^{\omega}_{p,q}$ of positive smoothness, defined in the spirit of M. L. Goldman, see Section 10.3, an atomic decomposition in the sense of M. Frazier and B. Jawerth was described by Yu. V. Netrusov in [Net89]. There are no moment conditions in his characterisation - in contrast to the case $0 < p \leqslant 1$ in the above theorem. The reason is, that Yu. V. Netrusov defined the spaces $B^{\omega}_{p,q}$ in a slightly different way which insures a-priori the embedding $L_p \hookrightarrow B^{\omega}_{p,q}$ for all admissible parameters $0 < p \leqslant \infty$.

Remark 12.4.2 Let $d > 0$ be given, let $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ fixed and let us denote $R_{\nu m}$ a cube with sides parallel to the axes, centred at $x^{\nu m}$ where

$$|x^{\nu m} - N_{\nu}^{-1} m| \leqslant d\, N_{\nu}^{-1}, \tag{12.34}$$

and with side length N_{ν}^{-1}.
Then let $c > 0$ be chosen in dependence of d such that for every choice of $\nu \in \mathbb{N}_0$ and all choices of $x^{\nu m}$ in (12.34) we have

$$\bigcup_{m\in\mathbb{Z}^n} cR_{\nu m} = \mathbb{R}^n. \tag{12.35}$$

It is clear from the previous proof that we may replace in Definition 12.1.1 the cube $Q_{\nu m}$ by $R_{\nu m}$, with the number c being from (12.35).
A similar remark in the classical case ($N_{\nu} = 2^{\nu}$ and $\sigma_{\nu} = 2^{\nu s}$, $s \in \mathbb{R}$) turned out to be very useful in the work of H. Triebel and H. Winkelvoß, [TrWi96].

Example 12.4.3 (The classical case) Let again $N_j = 2^j$, and $\sigma_j = 2^{js}$, $s \in \mathbb{R}$ and consider the classical spaces $B^s_{p,q}$ and $F^s_{p,q}$.
As we have already mentioned, condition (11.1) is fulfilled with $\lambda_0 = \lambda_1 = 2$ and condition (11.3) is fulfilled with $d_0 = d_1 = 2^s$.
The restrictions (12.6), (12.7) respectively (12.8), and their counterparts, in the atomic decomposition theorem are $M > s$ and

$$L > -1 + n\left(\frac{1}{\min(1,p,q)} - 1\right) - s$$

respectively

$$L > -1 + n\left(\frac{1}{\min(1,p)} - 1\right) - s$$

which are essentially the restrictions from the works of M. Frazier and B. Jawerth, cf. also the formulation in [Tr97].

Example 12.4.4 (The spaces $B^{(s,\Psi)}_{p,q}$ and $F^{(s,\Psi)}_{p,q}$) Let $s \in \mathbb{R}$ and Ψ be a positive monotone function on $(0,1]$ such that there are positive constants b_0^* and b_1^* with

$$b_0^*\,\Psi(2^{-j}) \leqslant \Psi(2^{-2j}) \leqslant b_1^*\,\Psi(2^{-j}) \quad \text{for all} \quad j \in \mathbb{N}_0$$

Let again

$$\sigma_j = 2^{js}\Psi(2^{-j}) \quad , \quad j \in \mathbb{N}_0 \quad ;$$

then we have already mentioned that σ is an admissible sequence with $d_0 = b_0\, 2^s$ and $d_1 = b_1\, 2^s$ in (9.2), where $b_0 = \min(b_0, 1, \Psi(2^{-1})\Psi(1)^{-1})$ and $b_1 = \max(b_1, 1, \Psi(2^{-1})\Psi(1)^{-1})$.
If, in addition, $N_j = 2^j$ then the spaces $B^{\sigma,N}_{p,q}$ and $F^{\sigma,N}_{p,q}$ are the spaces $B^{(s,\Psi)}_{p,q}$ and $F^{(s,\Psi)}_{p,q}$ considered in [Mo99] and [Mo01].
Again condition (11.1) is fulfilled with $\lambda_0 = \lambda_1 = 2$ and condition (11.3) is fulfilled with $d_0 = b_0\, 2^s$ and $d_1 = b_1\, 2^s$.
The restrictions (12.6), (12.7) respectively (12.8), in the atomic decomposition theorem are $M > s + \log_2 b_1$ and

$$L > -1 + n\left(\frac{1}{\min(1,p,q)} - 1\right) - s - \log_2 b_0$$

respectively

$$L > -1 + n\left(\frac{1}{\min(1,p)} - 1\right) - s - \log_2 b_0.$$

Here we have to mention again that our conditions concerning the constants M and L are slightly different (because of the additional log-terms) from those in the atomic decomposition of S. Moura, see [Mo01, Theorem 1.18], which was proved directly for the spaces $B^{(s,\Psi)}_{p,q}$ and $F^{(s,\Psi)}_{p,q}$ themselves.

Example 12.4.5 (ψ-Bessel potential spaces) Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an admissible continuous negative definite function, compare Definition 6.1.1, and assume that there exists $r_0 \in (0,1]$ such that $\xi \mapsto \psi(\xi)\langle\xi\rangle^{-2r_0}$ is increasing in $|\xi|$. If $(N_j)_{j\in\mathbb{N}_0}$ is the sequence associated to ψ and to $r = 2$, compare (6.5), then for any $j \in \mathbb{N}_0$ one has

$$2^{\frac{1}{w}} N_j \leqslant N_{j+1} \leqslant 2^{\frac{1}{r_0}} N_j \quad ,$$

compare Lemma 6.2.2 and Lemma 7.2.3.
Based on Corollary 7.1.4 we have for $s \in \mathbb{R}$ and $1 < p < \infty$ the equality $H^{\psi,s}_p = F^{\sigma^s,N}_{p,2}$ where $\sigma^s = (2^{js})_{j\in\mathbb{N}_0}$.
Condition (11.1) is fulfilled with $\lambda_0 = 2^{\frac{1}{w}}$ and $\lambda_1 = 2^{\frac{1}{r_0}}$ and condition (11.3) is fulfilled with $d_0 = d_1 = 2^s$
The restrictions (12.6) and (12.7) in the atomic decomposition theorem are $M > sw$ and

$$L > -1 + n\left(\frac{w}{r_0} - 1\right) - sw.$$

Note that if $s > n\left(\frac{1}{r_0} - \frac{1}{w}\right)$ then there are no moment conditions needed in the atomic decomposition theorem.
Finally, let us mention that if $\psi(\xi) = f(1+|\xi|^2)$ (for an appropriate Bernstein function f) then the restrictions are $M > s$ and

$$L > -1 + n\left(\frac{1}{r_0} - 1\right) - s.$$

since then $w = 1$.

Chapter IV. Pseudo-differential operators related to an admissible continuous negative definite function

13 Introduction to Chapter IV

The notion of pseudo-differential operators has grown in the seventies in order to obtain sharp *a-priori* estimates for the solutions of partial differential equations. It turns out that pseudo-differential operators are one of the most powerful tools in attacking various problems such as the existence and uniqueness of boundary value problems, regularity of solutions etc.
We will treat below pseudo-differential operators from a slightly different point of view, namely, we are interested in mapping properties and in pseudo-differential operators related to a continuous negative definite function in order to discuss whether such pseudo-differential operators extend to a generator of a Feller semigroup or to an L_p-sub-Markovian semigroup.
In particular, as a by-product, we will show that pseudo-differential operators with exotic symbols operate on function spaces of generalised smoothness. This will be obtained using the atomic decomposition theorem presented in the previous chapter.

We start this chapter with an introductory section on pseudo-differential operators, see **Section 14.** We set up terminology and recall the fundamental definition of Hörmander's class of symbols $S^{\mu}_{\rho,\delta}$. We briefly recall some fundamental mapping properties for associate pseudo-differential operators acting on function spaces of Bessel potential type or, more general, in spaces of $B^s_{p,q}$ and $F^s_{p,q}$ type.

In **Section 15** we are interested in the so-called exotic symbols (and in the associated operators).
It was observed by C.-H. Ching in [Ch72] that exotic pseudo-differential operators do not necessarily map L_2 into L_2. We recall his example in Example 14.1.4.
In connection with Bony's application of exotic pseudo-differential operators to non-linear problems exotic symbols attracted more and more attention.
Our main result of this section is Theorem 15.1.1 in which we show that if $a \in S^0_{1,1}$ is an exotic symbol and if s satisfies some reasonable assumptions then $a(\cdot, D)$ maps (function space of generalised smoothness) $F^{\sigma^s,N}_{p,q}$ linear and bounded into itself.
In Subsection 15.3 we point out that this covers some important previous results of T. Runst, R. H. Torres and Y. Meyer.

We conclude the last chapter with **Section 16** in which we start with an admissible continuous negative ψ and introduce a class of symbols (related to ψ) which is a refinement of the classical symbols classes $S^{\mu}_{\rho,\delta}$. Finally, in Subsection 16.4 we discuss conditions under which the associated pseudo-differential operators generate sub-Markovian semi-groups. The results are rather sketchy presented but we think these considerations are a starting point for further investigation.

14 Basic facts on pseudo-differential operators

The aim of this first section is to recall some basic definitions and results from the theory of pseudo-differential operators. We refer to the books of L. Hörmander, see [Hö85], of M. E. Taylor, see [Ta81], and of H. Kumano-go, see [Ku74].

14.1 Hörmander's class of symbols

We will start this Section recalling the definition of the class of symbols $S^{\mu}_{\rho,\delta}$, class which was introduced by L. Hörmander, see [Hö67].

Definition 14.1.1 *Let $\mu \in \mathbb{R}$ and $0 \leqslant \delta \leqslant \rho \leqslant 1$. Then $S^{\mu}_{\rho,\delta}$ (Hörmander's class of symbols) is the collection of all complex-valued C^{∞} functions $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that for any multi-index β and for any multi-index α there exists $c_{\beta\alpha} > 0$ with*

$$|D^{\beta}_x D^{\alpha}_{\xi} a(x,\xi)| \leqslant c_{\beta\alpha} \langle\xi\rangle^{\mu} \cdot \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|} \quad \textit{for any} \quad x \in \mathbb{R}^n,\, \xi \in \mathbb{R}^n. \tag{14.1}$$

Two further generalisations of (14.1) should be mentioned. Sometimes (14.1) is required only for some derivatives, say with $|\beta| \leqslant B$ and $|\alpha| \leqslant A$ for given B and $A \geqslant 0$. The following notation will be used in the next Section.

Notation 14.1.2 For numbers $B, A \in [0,\infty]$ (note $B = \infty$ and/or $A = \infty$ is allowed) define $S^{\mu}_{\rho,\delta}(B,A)$ to be the collection of all $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that for any multi-indices β and α with $\beta \leqslant B$ and $|\alpha| \leqslant A$ there exists a constant $c_{\beta\alpha} > 0$ such that (14.1) is satisfied.
It is clear that $S^{\mu}_{\rho,\delta}(\infty,\infty) = S^{\mu}_{\rho,\delta}$.

Secondly, instead of $x \in \mathbb{R}^n$ in (14.1) it is required that (14.1) holds only for x in a compact set and $c_{\beta\alpha}$ may depend on this compact set. However, we will not follow this aspect.

It is easy to see that

$$S^{\mu}_{1,0} \subset S^{\mu}_{\rho,\delta} \subset S^{\mu'}_{\rho',\delta'} \quad (\mu \leqslant \mu', \rho \geqslant \rho', \delta \leqslant \delta')$$

and that for any ρ and δ one has $\bigcap\limits_{\mu\in\mathbb{R}} S^{\mu}_{\rho,\delta} = \bigcap\limits_{\mu\in\mathbb{R}} S^{\mu}_{1,0}$.
Hence we may write

$$S^{-\infty} = \bigcap_{\mu\in\mathbb{R}} S^{\mu}_{\rho,\delta}.$$

The next example can be found in [Ku74, Chapter 2].

Example 14.1.3 **(i)** For $\mu \in \mathbb{N}_0$ and $a_{\alpha} : \mathbb{R}^n \to \mathbb{R}$, $a_{\alpha} \in C^{\infty}_b$ the function

$$\sum_{|\alpha|\leqslant\mu} a_{\alpha}(x)\,\xi^{\alpha}$$

belongs to the class $S^{\mu}_{1,0}$.
(ii) For any $\mu \in \mathbb{R}$ the mapping $\xi \mapsto (1+|\xi|^2)^{\mu/2}$ is a symbol in the class $S^{\mu}_{1,0}$.
(iii) Let $g(\xi) = i\xi_1 + \sum\limits_{j=1}^{n} \xi_j^2$ and let $\varphi \in C^{\infty}_0(\mathbb{R}^n)$ such that $\varphi(x) = 0$ if $|x| \leqslant 1$ and $\varphi(x) = 0$ if $|x| \geqslant 2$. Then the function $\varphi\,\frac{1}{g}$ belongs to the symbol class $S^{-1}_{1/2,0}$.

The next example goes back to C.-H. Ching, see [Ch72].

Example 14.1.4 Let $(\eta_k)_{k\in\mathbb{N}}$ be a sequence of elements from $\mathbb{R}^n$ such that $|\eta_k| = 3\cdot 5^k$ for any $k \in \mathbb{N}$ and let χ be a function in $C_0^\infty(\mathbb{R}^n)$ such that $\chi(\xi) = 1$ if $2 \leqslant |\xi| \leqslant 4$ and such that $\operatorname{supp}\chi \subset \{\xi \in \mathbb{R}^n : 1 \leqslant |\xi| \leqslant 5\}$. Let $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$,

$$a(x,\xi) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}\, e^{-i\eta_k\cdot x}\, \chi(5^{-k}\xi);$$

then a belongs to the symbol class $S^0_{1,1}$.
Symbols from $S^0_{1,1}$ are called *exotic*. In the last years they have attracted much attention and we will also treat them in the next section.

If $\mu \in \mathbb{R}$, $0 \leqslant \delta \leqslant \rho \leqslant 1$ and if $a \in S^\mu_{\rho,\delta}$ then

$$a(x,D)u = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, a(x,\xi)\, \widehat{u}(\xi)\, d\xi \quad , \quad x \in \mathbb{R}^n$$

is the corresponding pseudo-differential operator acting on $\mathcal{S}$.
It turns out that pseudo-differential associated to symbols from $S^\mu_{1,1}$ have mapping properties which are different from those generated by symbols from $S^\mu_{\rho,\delta}$ with $\delta < 1$.

14.2 Some mapping properties

In discussing mapping properties for pseudo-differential operators, acting on Sobolev spaces or Bessel potential spaces, or, more general on spaces of $B^s_{p,q}$ and $F^s_{p,q}$ type we will restrict ourselves to the symbol class $S^\mu_{1,\delta}$ with $0 \leqslant \delta \leqslant 1$.
It is a known fact that the assumption $\rho = 1$ is natural in this context, see for example M. E. Taylor [Ta81, Chapter XI] and cf. [Tr92, Section 6.2]

14.2.1 Pseudo-differential operators with symbols in $S^\mu_{1,\delta}$ $(\delta < 1)$

It can be easily seen that if $a \in S^\mu_{1,\delta}$ then $a(x,D)$ maps $\mathcal{S}$ into $\mathcal{S}$. If $\delta < 1$ then one can apply a duality argument to show that $a(x,D)$ maps also $\mathcal{S}'$ into $\mathcal{S}'$.
For $a \in S^\mu_{1,\delta}$ one can define the seminorms $|a|^{(\mu)}_{(l,k)}$ for $l,k \in \mathbb{N}_0$ by

$$|a|^{(\mu)}_{(l,k)} = \max_{|\beta|\leqslant l, |\alpha|\leqslant k} \sup_{x,\xi\in\mathbb{R}^n} \left\{ |D^\beta_x D^\alpha_\xi a(x,\xi)| \cdot \langle\xi\rangle^{-\mu} \langle\xi\rangle^{|\alpha|-\delta\,|\beta|} \right\}. \tag{14.2}$$

Theorem 14.2.1 *Let $0 \leqslant \delta < 1$ and $a \in S^0_{1,\delta}$. Then for all p with $1 < p < \infty$ there exist integers l and k and a constant $c > 0$, all independent of a, such that*

$$\|a(\cdot,D)u\,|\,L_p\| \leqslant c\,|a|^{(0)}_{(l,k)} \cdot \|u\,|\,L_p\| \quad \textit{for any} \quad u \in L_p.$$

One should note that $\delta < 1$.
This result was proved by R. A. Illner in 1975, see [Il75]. Later on, several authors, among we mention G. Bourdaud, see [Bou82] and M. Nagase, see [Na83] considered non-regular symbols and got weaker assumptions on a.

However, the result is sharp with respect to the parameter p; as shown in [UnBo65a], there exist functions $m \in S^0_{1,0}$ which are not Fourier multipliers in L_1 and L_∞. Consequently, for the corresponding pseudo-differential operators Theorem 14.2.1 does not hold.
Using Theorem 14.2.1, one gets

Corollary 14.2.2 *Let $0 \leqslant \delta < 1$ and $a \in S^0_{1,\delta}$. Let $1 < p < \infty$, let $\mu \in \mathbb{R}$ and $s \in \mathbb{R}$. Then there exist integers l and k and a constant $c > 0$ independent of a such that*

$$\left\| a(\cdot, D)u \mid H^s_p \right\| \leqslant c\, |a|^{(\mu)}_{(l,k)} \cdot \left\| u \mid H^{s+\mu}_p \right\| \quad \textit{for any} \quad u \in H^{s+\mu}_p.$$

Moreover one has the following:

Theorem 14.2.3 *Let $\mu \in \mathbb{R}$, $0 \leqslant \delta < 1$, and let $a \in S^\mu_{1,\delta}$. Let $s \in \mathbb{R}$, and $0 < q \leqslant \infty$. If $0 < p < \infty$ then $a(\cdot, D)$ is a linear bounded map from $F^{s+\mu}_{p,q}$ into $F^s_{p,q}$.*

Note that the above result (and its obvious counterpart for B spaces, $0 < p \leqslant \infty$) has many forerunners. A complete proof is given for example in [Tr92, Theorem 6.2.2].

As we mentioned in the introduction, pseudo-differential operators have been studied preferably in an L_2 setting, see the books mentioned at the beginning of the section. But there has also been done a lot to study several types of pseudo-differential operators in L_p with $1 < p < \infty$. We mention here, without going into further details, the works of R. Beals, see [Be79a] and [Be79b], of H. Kumano-go and M. Nagase [KuNa78], of M. Nagase, see [Na77]-[Na86], of A. Nagel and E. M. Stein, see [NaSt78] and [NaSt79], of M. Yamazaki, see [Yam85] and [Yam86b].
Further contributions to the L_p-theory of pseudo-differential operators are due to to G. Bourdaud, see [Bou88b], to A. Miyachi, see [Mi87a]-[Mi88], to L. Päivärinta and E. Somersalo, see [PäSo], to T. Muramuta and M. Nagase, see [MuNa79], to W. Rouhuai and L. Chengzhang, see [RoCh84], to M. Sugimoto, see [Su88a].
Some of the above mentioned papers deal with the problem to weaken the smoothnes assumptions for the symbol $a \in S^\mu_{\rho,\delta}$ in particular with respect to the x variable.

Mapping properties for pseudo-differential operators in $B^s_{p,q}$ and $F^s_{p,q}$ spaces have been studied by G. Bourdaud, see [Bou82], G. Gibbons, see [Gi78], H.-Q. Bui, see [Bu83], M. Yamazaki, see [Yam83]-[Yam86a], J. Marschall, see [Mar88]-[Mar91], K. Yabuta, see [Yab88a], [Yab88b], M. Sugimoto, see [Su88b], M. Frazier, Y.-S. Han, B. Jawerth and G. Weiss, see [FHJW87] and by J. Alvarez and J. Hounie, see [AlHo90].
An extension of boundary value problems for pseudo-differential operators from L_p spaces to $B^s_{p,q}$ and $F^s_{p,q}$ spaces has been given by J. Franke, see [Fr85a], [Fr85b] and also G. Grubb [Gr90], [Gr89].

14.2.2 Pseudo-differential operators with symbols in $S^\mu_{1,1}$

There is a striking difference between mapping properties for non-exotic pseudo-differential operators, i.e. operators with symbols in $S^\mu_{1,\delta}$ with $\delta < 1$ on the one hand and mapping properties for exotic pseudo-differential operators.
It was observed by C.-H. Ching in [Ch72] that exotic pseudo-differential operators do not necessarily map L_2 into L_2. More precisely, he showed that the pseudo-differential operator associated to the symbol from Example 14.1.4 does not map L_2 into L_2.

But at the beginning of the eighties, Y. Meyer proved that pseudo-differential operators with exotic symbols map $H^s_p = F^s_{p,2}$ ($s > 0$ and $1 < p < \infty$) into itself.

Afterwards exotic pseudo-differential operators attracted more and more attention, in particular in connection with Bony's application of exotic pseudo-differential operators to non-linear problems. Corresponding investigations covering general spaces of $B^s_{p,q}$ and $F^s_{p,q}$ type were given by T. Runst in [Ru85], and R. H. Torres in [To90]; see also G. Bourdaud [Bou88a], L. Hörmander [Hö85] - [Hö88] for further other aspects.
In the next chapter we will obtain a mapping property for exotic pseudo-differential operators acting on function spaces of generalised smoothness.
This will allow us to cover most of the known mapping properties for exotic pseudo-differential operators, see in particular Subsection 15.3.1.

15 Exotic pseudo-differential operators on function spaces of generalised smoothness

15.1 A mapping theorem

We start fixing the assumptions and the terminology.
In the following we will assume that $(N_j)_{j\in\mathbb{N}_0}$ is a sequence of non-negative numbers such that there exist $1 < \lambda_0 \leqslant \lambda_1$ with $\lambda_0 N_j \leqslant N_{j+1} \leqslant \lambda_1 N_j$ for any $j \in \mathbb{N}_0$. For simplicity we will assume $N_0 = 1$.
Then Q_{00} denotes the cube centred at the origin having side-length 1.
Recall that if $\nu \in \mathbb{N}$ and $m = (m_1, ..., m_n) \in \mathbb{Z}^n$ we denote $Q_{\nu m}$ the cube in $\mathbb{R}^n$ centred at $N_\nu^{-1} m = (N_\nu^{-1} m_1, ..., N_\nu^{-1} m_n)$ which has sides parallel to the axes and side length N_ν^{-1}.
Considering $\sigma^s = (2^{js})_{j\in\mathbb{N}_0}$ and $1 < p < \infty$, $M, L \in \mathbb{R}$, $c^* > 1$, we will call a function $\rho : \mathbb{R}^n \to \mathbb{C}$ for which there exist all derivatives $D^\alpha \rho$ if $|\alpha| \leqslant M$ (continuous if $M \leqslant 0$) and such that (12.3), and (12.5) are satisfied, and if

$$|D^\alpha \rho(x)| \leqslant 2^{-\nu s} N_\nu^{\frac{n}{p}+|\alpha|} \quad \text{if} \quad |\alpha| \leqslant M,$$

an $(s,p)_{M,L}$-atom instead of $(\sigma^s, p)_{M,L}$-atom, compare Definition 12.1.1.

The moment conditions (12.5) can be reformulated as $D^\gamma \widehat{\rho}(0) = 0$ if $|\gamma| \leqslant L$. Recall that if $L < 0$ then (12.5) simply means that there are no moment conditions.

Note that if $\sigma_j = 2^{js}$ for any $j \in \mathbb{N}_0$ then conditions (12.6) and (12.7) from the atomic decomposition theorem for the space $F^{\sigma^s,N}_{p,q}$ can be reformulated as

$$M > \frac{s}{\log_2 \lambda_0}$$

and

$$L > -1 + n\left(\frac{\log_2 \lambda_1}{\log_2 \lambda_0}\frac{1}{\min(1,p,q)} - 1\right) - \frac{s}{\log_2 \lambda_0}. \tag{15.1}$$

Now we are able to formulate the main result of this Section. Recall the notation $\sigma^s = (2^{js})_{j\in\mathbb{N}_0}$.

Theorem 15.1.1 *Let $a \in S^0_{1,1}$ an exotic symbol.*
Let $(N_j)_{j\in\mathbb{N}_0}$ be a sequence with $N_0 = 1$ and such that for some $1 < \lambda_0 \leqslant \lambda_1$ one has $\lambda_0 N_j \leqslant N_{j+1} \leqslant \lambda_1 N_j$, $j \in \mathbb{N}$. Let $0 < q \leqslant \infty$.
(i) *If $0 < p < \infty$ and if $s \in \mathbb{R}$ is such that*

$$\frac{s}{\log_2 \lambda_0} > n\left(\frac{\log_2 \lambda_1}{\log_2 \lambda_0} \cdot \frac{1}{\min(1,p,q)} - 1\right) \tag{15.2}$$

then the pseudo-differential operator $a(\cdot, D)$ maps $F^{\sigma^s,N}_{p,q}$ linear and bounded into itself in the sense that there exists a constant $c > 0$ such that

$$\left\| a(\cdot, D)u \,|\, F^{\sigma^s,N}_{p,q} \right\| \leqslant \left\| u \,|\, F^{\sigma^s,N}_{p,q} \right\| \quad \textit{for any} \quad u \in F^{\sigma^s,N}_{p,q}.$$

(ii) *If $0 < p \leqslant \infty$ and if $s \in \mathbb{R}$ is such that*

$$\frac{s}{\log_2 \lambda_0} > n\left(\frac{\log_2 \lambda_1}{\log_2 \lambda_0} \cdot \frac{1}{\min(1,p)} - 1\right) \tag{15.3}$$

then the pseudo-differential operator $a(\cdot, D)$ maps $B^{\sigma^s,N}_{p,q}$ linear and bounded into itself.

Remark 15.1.2 We will shift the proof of Theorem 15.1.1 to the next section since we think it is interesting for its own sake but let us make here some comments.
Of course we will concentrate ourselves on the more complicated case of F-spaces.
We will apply the atomic decomposition theorem proved in the previous chapter. Due to restriction (15.2) we will not need moment conditions in the atomic decomposition theorem applied to the space $F^{\sigma^s,N}_{p,q}$ compare Theorem 12.2.1 and (15.1).
Choosing $M > s/\log \lambda_0$ large enough and L satisfying (15.1), L large enough and let any $u \in F^{\sigma^s,N}_{pq}$ decomposed as (compare (12.9))

$$u = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\, \rho_{\nu m}$$

where $\rho_{\nu m}$ are 1_M-atoms for $\nu = 0$ or $(s,p)_{M,L}$-atoms for $\nu \in \mathbb{N}$ and $\lambda = (\lambda_{\nu m})_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n} \in f^N_{p,q}$.
The *main idea of the proof* is to show that we can obtain a decomposition

$$a(x,D)u = \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \theta_{jk}\, d_{jk} \tag{15.4}$$

where d_{jk} are 1_K-atoms for $j = 0$ or $(s,p)_{K,-1}$-atoms for $j \in \mathbb{N}$ (no moment conditions) where $K > s/\log_2 \lambda_0$ and $\theta = (\theta_{jk})_{j\in\mathbb{N}_0,k\in\mathbb{Z}^n} \in f^N_{p,q}$, with $\left\|\theta \,|\, f^N_{p,q}\right\| \leqslant c \left\|\lambda \,|\, f^N_{p,q}\right\|$ (recall the definition of the sequence space $f^N_{p,q}$ from Definition 12.1.2).

Remark 15.1.3 It should be expected that the above theorem holds in a more general context, that one in which the sequence $(2^{js})_{j\in\mathbb{N}_0}$ is replaced by a more general admissible sequence $(\sigma_j)_{j\in\mathbb{N}_0}$ and in which restrictions (15.2) and (15.3) are modified appropriately.
Since we are mainly interested in applying the above theorem to ψ-Bessel potential spaces (where ψ is an admissible continuous negative definite function) we will restrict ourselves to the form stated above. The more general assertion will be shifted to a later work.

After proving the theorem, we will discuss some particular cases of the above mapping result in Subsection 15.3.

15.2 Proof of Theorem 15.1.1

Let us recall, compare Notation 14.1.2, that if $B, A \in [0,\infty]$ (note $B=\infty$ and/or $A=\infty$ is allowed) then $S^0_{1,1}(B,A)$ is the collection of all $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that for any multi-indices β and α with $\beta \leqslant B$ and $|\alpha| \leqslant A$ there exists a constant $c_{\beta\alpha} > 0$ such that

$$|D^\beta_x D^\alpha_\xi a(x,\xi)| \leqslant c_{\beta\alpha} \langle\xi\rangle^{-|\alpha|+|\beta|} \quad \text{for any} \quad x \in \mathbb{R}^n,\, \xi \in \mathbb{R}^n. \tag{15.5}$$

It is clear that $S^0_{1,1}(\infty,\infty) = S^0_{1,1}$ is the class of exotic symbols.
It is clear that if $a \in S^0_{1,1}$ then $a \in S^0_{1,1}(B,A)$ for any $B,\ A \geqslant 0$.

15.2.1 Mapping of 1_M-atoms

We will first show that if $B,\ A \geqslant 0$ and if $a \in S^0_{1,1}(B,A)$ then the pseudo-differential operator $a(x,D)$ maps an 1_M-atom located at Q_{0m} into a so-called molecule, if M is appropriately chosen, more precisely we have Lemma 15.2.1 and Corollary 15.2.2 from below.

Lemma 15.2.1 *Let $B, A \geqslant 0$ and let $a \in S^0_{1,1}(B,A)$. Let $M \geqslant B+n+1$.
Then for any multi-index γ with $|\gamma| \leqslant B$ there exists a $C>0$ (which may depend on B, A, n, and γ) such that*

$$|D^\gamma \left(a(x,D)\,\rho(x)\right)| \leqslant C\,\langle x\rangle^{-A} \quad \textit{for any} \quad x \in \mathbb{R}^n, \tag{15.6}$$

and for any 1_M-atom ρ located at Q_{00}.

<u>Proof.</u> Let B, A and a as stated above and let $M \geqslant B+n+1$.
We will do the proof in several steps.
Step 1. For any multi-index η and for any multi-index δ with $|\delta| \leqslant M$ there exists a constant $C>0$ (which may depend on η and on δ)

$$|\xi^\delta\, D^\eta \widehat{\rho}(\xi)| \leqslant C \quad \text{for any} \quad \xi \in \mathbb{R}^n, \tag{15.7}$$

and for any 1_M-atom ρ located at Q_{00}.
To prove (15.7), we start noting that for any $\eta \in \mathbb{N}^n_0$, and for any 1_M-atom located at Q_{00}, the function $y \mapsto y^\eta\,\rho(y)$ is (up to constants) again an 1_M-atom located at Q_{00}. Consequently, the integral $\int_{\mathbb{R}^n} e^{-iy\cdot\xi}\, y^\eta\, \rho(y)\,dy$ is uniformly convergent and hence

$$D^\eta \widehat{\rho}(\xi) = c_n \int_{\mathbb{R}^n} e^{-iy\cdot\xi}\, y^\eta\, \rho(y)\,dy$$

for some constant $c_n > 0$ depending only on n.
Then for any $\delta \in \mathbb{N}^n_0$ with $|\delta| \leqslant M$ we have integrating by parts (note that the integral is only on $\operatorname{supp}\rho$)

$$\begin{aligned} \xi^\delta\, D^\eta \widehat{\rho}(\xi) &= c_n \int_{\mathbb{R}^n} \xi^\delta\, e^{-iy\cdot\xi}\, y^\eta\, \rho(y)\,dy = c' \int_{\mathbb{R}^n} D^\delta_y(e^{-iy\cdot\xi})\, y^\eta\, \rho(y)\,dy \\ &= c'' \int_{\mathbb{R}^n} e^{-iy\cdot\xi} \cdot D^\delta_y(y^\eta\, \rho(y))\,dy \end{aligned}$$

and consequently $|\xi^\delta\, D^\eta \widehat{\rho}(\xi)| \leqslant C$ for any $\xi \in \mathbb{R}^n$ where the constant $C > 0$ is independent of ρ and this proves (15.7).

Step 2. It is well known, or easy to show, that given $M \geqslant 0$ there are two constants $c_1, c_2 > 0$ such that

$$c_1\,(1+|\xi|^2)^{M/2} \leqslant \sum_{|\delta|\leqslant M} |\xi^\delta| \leqslant c_2\,(1+|\xi|^2)^{M/2} \quad \text{for any} \quad \xi \in \mathbb{R}^n,\ \xi \neq 0. \tag{15.8}$$

For the first inequality we refer to [Ja02, Section 2.3] where it was mentioned that a lower bound for the constant c_1 is $(n+1)^{-M/2}$.
The second inequality, which is true for all $\xi \in \mathbb{R}^n$, can be immediately obtained multiplying the elementary inequalities

$$|\xi_k^{\delta_k}| = (|\xi_k|^2)^{\frac{\delta_k}{2}} \leqslant (1+|\xi_1|^2+...+|\xi_n|^2)^{\frac{\delta_k}{2}} \quad \text{if} \quad \xi = (\xi_1,...,\xi_n) \in \mathbb{R}^n.$$

Using the estimate (15.7) for any δ with $|\delta| \leqslant M$, and summing over δ we get from (15.8) that for any multi-index η there is some constant $c > 0$ with

$$|D^\eta \widehat{\rho}(\xi)| \leqslant c\,(1+|\xi|^2)^{-M/2} \quad \text{for any} \quad \xi \in \mathbb{R}^n, \tag{15.9}$$

and for any 1_M-atom ρ located at Q_{00}.
Note that the last inequality implies the fact that if $a \in S^0_{1,1}(B,A)$ and if ρ is an 1_M-atom located at Q_{00}, then the integral

$$a(x,D)\rho(x) = c_n \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, a(x,\xi)\,\widehat{\rho}(\xi)\,d\xi$$

is convergent since $M \geqslant B+n+1 > n$.

Step 3. For any multi-index γ with $|\gamma| \leqslant B$ there exists a constant $C > 0$ (which may depend on γ) such that

$$|D^\gamma\,(a(x,D)\,\rho(x))\,| \leqslant C \quad \text{for any} \quad x \in \mathbb{R}^n, \tag{15.10}$$

and for any 1_M-atom ρ located at Q_{00}.
Indeed, applying Leibniz's rule for differentiation of a product, and using the inequality (15.5) for the symbol a, we get successively

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} D_x^\gamma\left(e^{ix\cdot\xi} a(x,\xi)\right)\cdot \widehat{\rho}(\xi)\,d\xi\right| &\leqslant \sum_{\delta+\eta=\gamma} c_{\delta\eta} \int_{\mathbb{R}^n} |D_x^\delta e^{ix\cdot\xi}|\cdot|D_x^\eta a(x,\xi)|\cdot|\widehat{\rho}(\xi)|\,d\xi \\
&\leqslant \sum_{\delta+\eta=\gamma} c'_{\delta\eta} \int_{\mathbb{R}^n} |\xi^\delta|\cdot\langle\xi\rangle^{|\eta|}\cdot|\widehat{\rho}(\xi)|\,d\xi \\
&\leqslant c'' \int_{\mathbb{R}^n} \langle\xi\rangle^{|\gamma|}\cdot\langle\xi\rangle^{-M}\,d\xi \leqslant C
\end{aligned}$$

since $M - |\gamma| \geqslant M - B > n$. From the last inequality we immediately get (15.10).

Step 4. For any multi-index α, with $|\alpha| \leqslant A$, there exists a constant $C > 0$ (which may depend on α) such that

$$|x^\alpha\,(a(x,D)\,\rho(x))\,| \leqslant C \quad \text{for any} \quad x \in \mathbb{R}^n, \tag{15.11}$$

and for any 1_M-atom ρ located at Q_{00}.
Since $\xi \mapsto a(x,\xi)\widehat{\rho}(\xi)$ can be controlled using estimates (15.9) and (15.5) we obtain by partial integration

$$\begin{aligned}\left|\int_{\mathbb{R}^n} D^\alpha_\xi(e^{ix\cdot\xi})\cdot(a(x,\xi)\widehat{\rho}(\xi))\,d\xi\right| &\leqslant c\int_{\mathbb{R}^n}\left|e^{ix\cdot\xi}\right|\sum_{\delta+\eta=\alpha} c_{\delta\eta}\,|D^\delta_\xi a(x,\xi)|\cdot|D^\eta_\xi\widehat{\rho}(\xi)|\,d\xi\\ &\leqslant c'\sum_{\delta+\eta=\alpha}\int_{\mathbb{R}^n}\langle\xi\rangle^{-|\delta|}\,\langle\xi\rangle^{-M}\,d\xi\leqslant C\end{aligned}$$

so the above integral converges uniformly and we obtain (15.11).

Step 5. For any multi-index γ, with $|\gamma|\leqslant B$, and for any multi-index α, with $|\alpha|\leqslant A$, there exists a constant $C>0$ (which may depend on γ and on α) such that

$$|x^\alpha\ (D^\gamma a(x,D)\,\rho(x))\,|\leqslant C\quad\text{for any}\quad x\in\mathbb{R}^n, \tag{15.12}$$

and for any 1_M-atom ρ located at Q_{00}.
For $|\gamma|\leqslant B$ we have from Step 3

$$\begin{aligned}D^\gamma_x\,(a(x,D)\,\rho(x)) &= c\int_{\mathbb{R}^n} D^\gamma_x\left(e^{ix\cdot\xi}\,a(x,\xi)\right)\cdot\widehat{\rho}(\xi)\,d\xi\\ &= \sum_{\delta+\eta=\gamma} c_{\delta\eta}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\,\xi^\delta\cdot D^\eta_x a(x,\xi)\cdot\widehat{\rho}(\xi)\,d\xi,\end{aligned}$$

for some positive constants $c_{\delta\eta}$. Using (15.9) and Step 4, we get after partial integration

$$\begin{aligned}|x^\alpha\,D^\gamma_x\,(a(x,D)\,\rho(x))\,| &= |\sum_{\delta+\eta=\gamma} c_{\delta\eta}\int_{\mathbb{R}^n} D^\alpha_\xi(e^{ix\cdot\xi})\cdot\xi^\delta\cdot D^\eta_x a(x,\xi)\cdot\widehat{\rho}(\xi)\,d\xi|\\ &\leqslant \sum_{\delta+\eta=\gamma} c_{\delta\eta}\int_{\mathbb{R}^n}|e^{ix\cdot\xi}|\cdot|D^\alpha_\xi[D^\eta_x a(x,\xi)\cdot\xi^\delta\widehat{\rho}(\xi)]|\,d\xi\\ &\leqslant \sum_{\delta+\eta=\gamma} c_{\delta\eta}\sum_{\alpha_1+\alpha_2=\alpha} c_{\alpha_1\alpha_2}\int_{\mathbb{R}^n}|D^{\alpha_1}_\xi D^\eta_x a(x,\xi)|\cdot|D^{\alpha_2}_\xi(\xi^\delta\widehat{\rho}(\xi))|\,d\xi.\end{aligned}$$

Using Step 2 we obtain after applying Leibniz's product rule

$$|D^{\alpha_2}_\xi(\xi^\delta\cdot\widehat{\rho}(\xi))|\leqslant\sum_{\alpha_3+\alpha_4=\alpha_2} c_{\alpha_3\alpha_4}\,|\xi^{\delta-\alpha_3}|\cdot|D^{\alpha_4}\widehat{\rho}(\xi)|\leqslant c\,\langle\xi\rangle^{|\delta|}\cdot\langle\xi\rangle^{-M}$$

so that

$$|D^{\alpha_1}_\xi D^\eta_x a(x,\xi)|\cdot|D^{\alpha_2}_\xi(\xi^\delta\widehat{\rho}(\xi))|\leqslant c\,\langle\xi\rangle^{-|\alpha_1|+|\eta|}\,\langle\xi\rangle^{|\delta|}\cdot\langle\xi\rangle^{-M}\leqslant c\,\langle\xi\rangle^{|\gamma|}\cdot\langle\xi\rangle^{-M}$$

which immediately implies (15.12) since $|\gamma|-M\leqslant B-M<-n$.

Step 6. Finally, to get (15.6), it is enough to apply (15.12) for any multi-index α with $|\alpha|\leqslant A$, to sum then the obtained inequalities and to use the equivalence (15.8) from Step 2. ■

Corollary 15.2.2 *Let $B, A \geqslant 0$ and let $a \in S^0_{1,1}(B,A)$. Let $M \geqslant B+n+1$. Then for any multi-index γ with $|\gamma| \leqslant B$ there exists a constant $C > 0$ (which may depend on B, A, n, and γ) such that*

$$|D^\gamma\left(a(x,D)\rho_{0m}(x)\right)| \leqslant C\,\langle x-m\rangle^{-A} \quad \text{for any} \quad x \in \mathbb{R}^n,\, m \in \mathbb{Z}^n, \tag{15.13}$$

and for any 1_M-atom ρ_{0m} located at Q_{0m}.

Proof. Let us first remark that an easy translation argument shows that if ρ_{0m} is an 1_M-atom located at Q_{0m} then the function ρ defined by $x \mapsto \rho(x) = \rho_{0m}(x+m)$, is an 1_M-atom located at Q_{00} and $\widehat{\rho_{0m}}(\xi) = e^{-im\cdot\xi}\,\widehat{\rho}(\xi)$.
If we consider the symbol a_m, defined by $a_m(x,\xi) = a(x+m,\xi)$, we immediately get from (15.5) that for appropriate multi-indices β and α there exists a constant $c > 0$, independent of m, such that

$$|D^\beta_x D^\alpha_\xi a_m(x,\xi)| \leqslant c\,\langle\xi\rangle^{-|\alpha|+|\beta|}$$

for any $x, \xi \in \mathbb{R}^n$ so that a_m is again a symbol in $S^0_{1,1}(B,A)$. Consequently, from

$$\begin{aligned} a(x,D)\rho_{0m}(x) &= c_n \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\cdot\widehat{\rho_{0m}}(\xi)\,d\xi \\ &= c_n \int_{\mathbb{R}^n} e^{i(x-m)\cdot\xi} a(x,\xi)\cdot\widehat{\rho}(\xi)\,d\xi \\ &= c_n \int_{\mathbb{R}^n} e^{i(x-m)\cdot\xi} a_m(x-m,\xi)\cdot\widehat{\rho}(\xi)\,d\xi \\ &= a_m(x-m,D)\rho(x-m), \end{aligned}$$

using the previous lemma for the symbol $a_m \in S^0_{1,1}(B,A)$ we get the estimate (15.13). ∎

Remark 15.2.3 Note that the constant C in (15.6) as well as the constant C in (15.13) do not depend on the 1_M-atom ρ.

15.2.2 Mapping of atoms located at $Q_{\nu m}$

Now we will show that if $B, A \geqslant 0$ and if $a \in S^0_{1,1}(B,A)$ then the pseudo-differential operator $a(x,D)$ maps an $(s,p)_{M,L}$-atom located at $Q_{\nu m}$ into a molecule, for an appropriate M, more precisely we have Lemma 15.2.4 and Corollary 15.2.5 from below.
The next result is an analogue to Corollary 15.2.2 for atoms located at $Q_{\nu 0}$.

Lemma 15.2.4 *Let $B, A \geqslant 0$ and let $a \in S^0_{1,1}(B,A)$.
There exist numbers $M \geqslant B+n+1$ and $L \geqslant 0$ with the following property: for any multi-index γ with $|\gamma| \leqslant B$ there exists a constant $C > 0$ (which may depend on B, A, n, and γ) such that*

$$|D^\gamma\left(a(x,D)\rho_{\nu 0}(x)\right)| \leqslant C\,2^{-\nu s}\,N_\nu^{\frac{n}{p}+|\gamma|}\langle N_\nu x\rangle^{-A} \quad \text{for any} \quad x \in \mathbb{R}^n, \tag{15.14}$$

and for any $(s,p)_{M,L}$-atom $\rho_{\nu 0}$ located at $Q_{\nu 0}$, $\nu \in \mathbb{N}$.

Proof. We will do the proof in two steps.
Step 1. There exist numbers $\varepsilon_1 \geqslant 0$ and $\varepsilon_2 \geqslant 0$ such that for any multi-indices β and α, with $|\beta| \leqslant B$ and $|\alpha| \leqslant A$, there exists a constant $c > 0$ (which may depend on β and α) such that for any $\nu \in \mathbb{N}_0$, and any $x, \xi \in \mathbb{R}^n, \xi \neq 0$

$$|D_x^\beta D_\xi^\alpha a(x, N_\nu \xi)| \leqslant c\, N_\nu^{-|\alpha|+|\beta|} \frac{\langle \xi \rangle^{\varepsilon_1}}{|\xi|^{\varepsilon_2}}. \tag{15.15}$$

To prove the estimate (15.15) note that replacing ξ with $N_\nu \xi$ in (15.5) we have only to control $(N_\nu^{-1} + |\xi|)^{-|\alpha|+|\beta|}$.
If $-|\alpha| + |\beta|$ is positive we have $0 \leqslant -|\alpha| + |\beta| \leqslant B$ and then $(N_\nu^{-1} + |\xi|)^{-|\alpha|+|\beta|} \leqslant c\, \langle \xi \rangle^{-|\alpha|+|\beta|}$, with some constant $c = c(\beta, \alpha)$ independent of ν.
If $-|\alpha| + |\beta|$ is negative we have $0 \geqslant -|\alpha| + |\beta| \geqslant -A$ and then $(N_\nu^{-1} + |\xi|)^{-|\alpha|+|\beta|} \leqslant |\xi|^{-|\alpha|+|\beta|}$. So, if $|\xi|$ is small we have $(N_\nu^{-1} + |\xi|)^{-|\alpha|+|\beta|} \leqslant |\xi|^{-A}$ and if $|\xi|$ is large we have $(N_\nu^{-1} + |\xi|)^{-|\alpha|+|\beta|} \leqslant C$ for an appropriate constant C.
Now it is easy to find numbers ε_1 and ε_2 depending only on B, A, and n such that (15.15) is satisfied. In fact (up to constants) one has $\varepsilon_1 \sim B + A$ and $\varepsilon_2 \sim A$.

Step 2. If now $\rho_{\nu 0}$ is an $(s, p)_{M,L}$-atom located at $Q_{\nu 0}$ then the function ρ defined by

$$\rho(x) = 2^{\nu s}\, N_\nu^{-\frac{n}{p}}\, \rho_{\nu 0}(N_\nu^{-1} x)$$

is an 1_M-atom located at Q_{00} having moment conditions up to order L; this can be easily checked by a direct computation. Moreover, arguing as in Step 1 in Lemma 15.2.1, we get for any η with $|\eta| \leqslant L$, there exists a constant $c_\eta > 0$ such that

$$|D^\eta \widehat{\rho}(\xi)| \leqslant c_\eta\, |\xi|^L\, \langle \xi \rangle^{-M}. \tag{15.16}$$

Note that the numbers M and L are at our disposal and the constant c_η in (15.16) is independent of ρ.
So we may write

$$\rho_{\nu 0}(x) = 2^{-\nu s}\, N_\nu^{\frac{n}{p}}\, \rho(N_\nu x)$$

and this implies $\widehat{\rho_{\nu 0}}(\xi) = 2^{-\nu s}\, N_\nu^{\frac{n}{p}}\, N_\nu^{-n}\, \widehat{\rho}(N_\nu^{-1} \xi)$. Obviously

$$\begin{aligned} a(x, D) \rho_{\nu 0}(x) &= c_n \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, a(x, \xi) \cdot \widehat{\rho_{\nu 0}}(\xi)\, d\xi \\ &= c_n\, 2^{-\nu s}\, N_\nu^{\frac{n}{p}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, a(x, \xi) \cdot N_\nu^{-n} \widehat{\rho}(N_\nu^{-1} \xi)\, d\xi \\ &= c_n\, 2^{-\nu s}\, N_\nu^{\frac{n}{p}} \int_{\mathbb{R}^n} e^{iN_\nu x\cdot\xi}\, a(x, N_\nu \xi) \cdot \widehat{\rho}(\xi)\, d\xi. \end{aligned} \tag{15.17}$$

Take $L = \varepsilon_2$ and $M \geqslant B + n + 1 + \varepsilon_1$. Then applying the same arguments as in Step 3 of the proof of Lemma 15.2.1 and using (15.15) we get that for any multi-index γ with $|\gamma| \leqslant B$

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} D_x^\gamma \left(e^{iN_\nu x\cdot\xi} a(x, N_\nu \xi) \right) \cdot \widehat{\rho}(\xi)\, d\xi \right| \\ &\leqslant \sum_{\delta+\eta=\gamma} c_{\delta\eta} \int_{\mathbb{R}^n} |D_x^\delta \left(e^{iN_\nu x\cdot\xi} \right)| \cdot |D_x^\eta [a(x, N_\nu \xi)]| \cdot |\widehat{\rho}(\xi)|\, d\xi \end{aligned}$$

$$\leqslant \sum_{\delta+\eta=\gamma} c'_{\delta\eta} \int_{\mathbb{R}^n} N_\nu^{|\delta|}|\xi^\delta| \cdot N_\nu^{|\eta|} \frac{\langle\xi\rangle^{\varepsilon_1}}{|\xi|^{\varepsilon_2}} \cdot |\xi|^L \langle\xi\rangle^{-M} d\xi$$
$$\leqslant c'' N_\nu^{|\gamma|} \int_{\mathbb{R}^n} \langle\xi\rangle^{|\gamma|+\varepsilon_1} \cdot \langle\xi\rangle^{-M} d\xi \leqslant C N_\nu^{|\gamma|}$$

all the constants being independent of ν.

If we consider the symbol $a_\nu(x,\xi) = a(N_\nu^{-1}x, N_\nu\xi)$ we immediately get using (15.15) that for any β and α with $|\beta| \leqslant B$ and $|\alpha| \leqslant A$ there is a constant $c > 0$ independent of ν such that

$$|D_x^\beta D_\xi^\alpha a_\nu(x,\xi)| \leqslant c_1 N_\nu^{-|\beta|+|\alpha|} |D_x^\alpha D_\xi^\beta a(N_\nu^{-1}x, N_\nu\xi)| \leqslant c \frac{\langle\xi\rangle^{\varepsilon_1}}{|\xi|^{\varepsilon_2}}, \tag{15.18}$$

for any $x, \xi \in \mathbb{R}^n$, $\xi \neq 0$. Moreover, (15.17) becomes

$$\begin{aligned} a(x,D)\rho_{\nu 0}(x) &= c_n 2^{-\nu s} N_\nu^{\frac{n}{p}} \int_{\mathbb{R}^n} e^{iN_\nu x\cdot\xi} a(N_\nu^{-1}(N_\nu x), N_\nu\xi) \cdot \widehat{\rho}(\xi) d\xi \\ &= 2^{-\nu s} N_\nu^{\frac{n}{p}} a_\nu(N_\nu x, D) \rho(N_\nu x) \end{aligned}$$

We argue as in Step 5 of Lemma 15.2.1 for the symbol a_ν using (15.18).
Using the same technique as there we get for any multi-index γ and for any multi-index α with $|\gamma| \leqslant B$ and $|\alpha| \leqslant A$ that there exists a constant $C > 0$ such that

$$|D^\gamma (a_\nu(N_\nu x, D)\rho(N_\nu x))| \leqslant C N_\nu^{|\gamma|} \langle N_\nu x\rangle^{-A} \quad \text{for any} \quad x \in \mathbb{R}^n, \tag{15.19}$$

the constant being independent of ν and of the 1_M-atom. Combining (15.17) with (15.19) we finally get the desired estimate (15.14). ■

Corollary 15.2.5 *Let B, $A \geqslant 0$ and $a \in S^0_{1,1}(B,A)$.*
There exist numbers $M \geqslant B+n+1$ and $L \geqslant 0$ with the following property: for any multi-index γ with $|\gamma| \leqslant B$ there exists a constant $C > 0$ (which may depend on B, A, n and γ) such that

$$|D^\gamma (a(x,D)\rho_{\nu m}(x))| \leqslant C 2^{-\nu s} N_\nu^{\frac{n}{p}+|\gamma|} \langle N_\nu x - m\rangle^{-A} \tag{15.20}$$

for any $x \in \mathbb{R}^n$, $m \in \mathbb{Z}^n$, and for any $(s,p)_{M,L}$-atom $\rho_{\nu m}$ located at $Q_{\nu m}$ for any $\nu \in \mathbb{N}$, $m \in \mathbb{Z}^n$.

Proof. A direct computation shows that if $\rho_{\nu m}$ is an $(s,p)_{M,L}$-atom located at $Q_{\nu m}$ then the function ρ defined by $x \mapsto \rho_\nu(x) = \rho_{\nu m}(x+N_\nu^{-1}m)$ is an $(s,p)_{M,L}$-atom located at $Q_{\nu 0}$ having moment conditions up to order L. Applying the previous lemma to ρ_ν we get the desired estimate. ■

Remark 15.2.6 Note that the constant C in (15.14) as well as the constant C in (15.20) are independent of the $(s,p)_{M,L}$-atom ρ.

15.2.3 An inequality in sequence spaces

Let $(N_j)_{j\in\mathbb{N}}$ be a sequence with $N_0 = 1$ and such that for some numbers $1 \leqslant \lambda_0 < \lambda_1$ one has $\lambda_0\, N_j \leqslant N_{j+1} \leqslant \lambda_1\, N_j$ for any $j \in \mathbb{N}_0$.
If $0 < p < \infty$ and $0 < q \leqslant \infty$ recall the definition of the sequence space $f^N_{p,q}$ from Definition 12.1.2.

Lemma 15.2.7 *Let $0 < p < \infty$, let $0 < q \leqslant \infty$ and let $A > n/\min(1, p, q)$. Let $(N_j)_{j\in\mathbb{N}}$ be a sequence with $N_0 = 1$ and such that for some numbers $1 \leqslant \lambda_0 < \lambda_1$ one has $\lambda_0\, N_j \leqslant N_{j+1} \leqslant \lambda_1\, N_j$ for any $j \in \mathbb{N}_0$.*
Let $(\lambda_{\nu m})_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} \in f^N_{pq}$ and for any $\nu \in \mathbb{N}_0$ and for any $k \in \mathbb{Z}^n$ let

$$\theta_{\nu k} = \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} \langle k - m\rangle^{-A}. \tag{15.21}$$

Then there exists a constant $c > 0$ such that

$$\left\|(\theta_{\nu k})_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n} \mid f^N_{pq}\right\| \leqslant c \left\|(\lambda_{\nu m})_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n} \mid f^N_{pq}\right\| \tag{15.22}$$

Proof. For fixed $\nu \in \mathbb{N}_0$ and $k \in \mathbb{Z}^n$, clearly

$$|\theta_{\nu k}| \leqslant \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| \langle k - m\rangle^{-A}.$$

For a fixed $k \in \mathbb{Z}^n$ we decompose $\mathbb{Z}^n$ as the union of the sets E_j,

$$\mathbb{Z}^n = \bigcup_{j=0}^{\infty} E_j,$$

where $E_j = \{m \in \mathbb{Z}^n : N_j - 1 \leqslant |m - k| < N_{j+1} - 1\}$ if $j \in \mathbb{N}_0$.
Note that if $m \in E_j$ then

$$(1 + |m - k|)^{-A} \leqslant N_j^{-A}.$$

Moreover, if $k \in \mathbb{Z}^n$ is fixed, if $m \in E_j$, if $x \in Q_{\nu k}$ and $y \in Q_{\nu m}$ there exists a constant $C > 0$ such that

$$\begin{aligned}
|y - x| &\leqslant |x - N_\nu^{-1}k| + |N_\nu^{-1}k - N_\nu^{-1}m| + |N_\nu^{-1}m - y| \\
&\leqslant c\, N_\nu^{-1} + N_\nu^{-1}|k - m| + c\, N_\nu^{-1} \\
&\leqslant C\, N_\nu^{-1} N_j
\end{aligned}$$

where C is independent of ν, k, m.
There exists an ω such that $n/A < \omega < \min(1, p, q)$. We choose such an ω and fix it.
For a fixed ν the rectangles $Q_{\nu m}$ have the volume N_ν^{-n} and are disjoint.
Let $x \in Q_{\nu k}$; using the embedding $l_\omega \hookrightarrow l_1$ we get

$$\begin{aligned}
\sum_{m\in E_j} |\lambda_{\nu m}| &\leqslant \left(\sum_{m\in E_j} |\lambda_{\nu m}|^\omega\right)^{1/\omega} \leqslant \left(N_\nu^n \int_{Q_{\nu m}} \left(\sum_{m\in E_j} |\lambda_{\nu m}|\, \chi_{\nu m}(y)\right)^\omega dy\right)^{1/\omega} \\
&\leqslant \left(N_\nu^n \int_{|y-x|\leqslant C\, N_\nu^{-1} N_j} \left(\sum_{m\in E_j} |\lambda_{\nu m}|\, \chi_{\nu m}(y)\right)^\omega dy\right)^{1/\omega} \\
&\leqslant c \left(N_j^n\, \mathcal{M}\left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\, \chi_{\nu m}\right)^\omega (x)\right)^{1/\omega}
\end{aligned} \tag{15.23}$$

where $\mathcal{M}$ is the Hardy - Littlewood maximal function and the constants are independent of ν, k, m.
Let $\chi_{\nu k}$ be the characteristic function of the rectangle $Q_{\nu k}$ and $\chi_{\nu k}^{(p)} = N_\nu^{\frac{n}{p}} \chi_{\nu k}$.
From (15.23) using $\lambda_0^j N_0 = \lambda_0^j \leqslant N_j$ with $\lambda_0 > 1$ and using $A > \frac{n}{\omega}$, we have

$$\begin{aligned}
\left|\theta_{\nu,k}\, \chi_{\nu k}^{(p)}(x)\right| &\leqslant c_1 \sum_{m\in\mathbb{Z}^n} \langle k-m\rangle^{-A} \,|\lambda_{\nu m}|\, N_\nu^{\frac{n}{p}}\, \chi_{\nu k}(x) \\
&= c_1 \sum_{j=0}^{\infty} \sum_{m\in E_j} \langle k-m\rangle^{-A} \,|\lambda_{\nu m}|\, N_\nu^{\frac{n}{p}} \chi_{\nu k}(x) \\
&\leqslant c_2 \sum_{j=0}^{\infty} N_j^{-A} \sum_{m\in E_j} |\lambda_{\nu m}|\, N_\nu^{\frac{n}{p}}\, \chi_{\nu k}(x) \\
&\leqslant c_3 \sum_{j=0}^{\infty} N_j^{-A}\, N_j^{n/\omega} \left(\mathcal{M}\left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\, \chi_{\nu m}^{(p)}\right)^{\omega}(x)\right)^{1/\omega} \chi_{\nu k}(x) \\
&\leqslant c \left(\mathcal{M}\left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\, \chi_{\nu m}^{(p)}\right)^{\omega}(x)\right)^{1/\omega} \chi_{\nu,k}(x)
\end{aligned} \tag{15.24}$$

where the constants above do not depend on ν and k.
In (15.24) we take the power q, sum over $k \in \mathbb{Z}^n$ and then over $\nu \in \mathbb{N}_0$ and get

$$\sum_{\nu=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \left|\theta_{\nu k}\, \chi_{\nu k}^{(p)}(x)\right|^q \leqslant c \sum_{\nu=0}^{\infty} (\mathcal{M} h_\nu^\omega(x))^{q/\omega} \tag{15.25}$$

where $h_\nu = \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\, \chi_{\nu m}^{(p)}$ (with the usual modification if $q = \infty$).
Taking the power $1/q$ and the L_p-norm we obtain that $\|(\theta_{\nu k})_{\nu\in\mathbb{N}_0, k\in\mathbb{Z}^n} \mid f_{p,q}^N\|$ can be estimated from above by

$$c \left\|((\mathcal{M} h_\nu^\omega(\cdot))^{1/\omega})_{\nu\in\mathbb{N}_0} \mid L_p(l_q)\right\| = c \left\|(\mathcal{M} h_\nu^\omega)_{\nu\in\mathbb{N}_0} \mid L_{p/\omega}(l_{q/\omega})\right\|^{1/\omega} \tag{15.26}$$

(with the usual modification if $q = \infty$).
To obtain (15.22) we have now only to apply the Fefferman - Stein inequality (0.8) to the right-hand side of (15.26); this can be done since $1 < p/\omega < \infty$ and $1 < q/\omega \leqslant \infty$ and so the proof is finished. ■

15.2.4 The final step

Now we are able to prove Theorem 15.1.1.
Part **(i)** We start proving the assertion for F spaces.
Let $A > n/\min(1, p, q)$ so that we may apply Lemma 15.2.7 and let $B > 0$ arbitrary.
We choose M and L large enough such that we may apply Lemma 15.2.1, Corollary 15.2.2, Lemma 15.2.4 and Corollary 15.2.5.

Step 1. Recall we assumed $N_0 = 1$. An easy computation shows that there exists a constant $c > 0$ such that for any $k, m \in \mathbb{Z}^n$ and for any $x \in \mathbb{R}^n$ the so-called Peetre inequality holds

$$\frac{1+|k|^2}{1+|x-m|^2} \leqslant c\,(1+|x-(m-k)|^2).$$

In particular, for given $k, m \in \mathbb{Z}^n$, if $x \in Q_{0,m-k}$ then for a constant $C > 0$, independent of m and k, we have

$$\frac{\langle k\rangle^A}{\langle x-m\rangle^A} \leqslant C. \tag{15.27}$$

Let $(\varphi_l)_{l\in\mathbb{Z}^n}$ be a smooth partition of unity, such that for some fixed constant $c_* > 1$ we have $\operatorname{supp}\varphi_l \subset c_*\,Q_{0l}$ for any $l \in \mathbb{Z}^n$.
For a fixed $m \in \mathbb{Z}^n$ and a fixed 1_M-atom ρ_{0m}, the mapping

$$x \mapsto d_{0,m-k}(x) = \langle k\rangle^A\,(a(x,D)\rho_{0m}(x)) \cdot \varphi_{0,m-k}(x)$$

is (up to constants) an 1_K-atom located at $Q_{0,m-k}$, where $K = M-n-1$.
This is clear since $\operatorname{supp} d_{0,m-k} \subset c_*\,Q_{0,m-k}$ and for any γ with $|\gamma| \leqslant K$ we can control the derivative $D^\gamma d_{0,m-k}$ according to (15.13) and to inequality (15.27).
Then it follows (convergence in $\mathcal{S}'$)

$$\begin{aligned} a(\cdot,D)\left(\sum_{m\in\mathbb{Z}^n}\lambda_{0m}\rho_{0m}\right) &= \sum_{m\in\mathbb{Z}^n}\lambda_{0m}\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{-A}\,\langle k\rangle^A(a(\cdot,D)\rho_{0m})\cdot\varphi_{0,m-k} \\ &= \sum_{m\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}\lambda_{0m}\,\langle k\rangle^{-A}\,d_{0,m-k} \\ &= \sum_{m\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n}\lambda_{0m}\langle l-m\rangle^{-A}\,d_{0l} = \sum_{l\in\mathbb{Z}^n}\theta_{0l}\,d_{0l} \end{aligned} \tag{15.28}$$

where θ_{0l} is given by

$$\theta_{0l} = \sum_{m\in\mathbb{Z}^n}\lambda_{0l}\,\langle l-m\rangle^{-A}.$$

Step 2. As in Step 1, using Lemma 15.2.4 and Corollary 15.2.5 we obtain for any fixed $\nu \in \mathbb{N}$

$$a(\cdot,D)\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\rho_{\nu m}\right) = \sum_{m\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n}\lambda_{\nu m}\langle l-m\rangle^{-A}\,d_{\nu l} = \sum_{l\in\mathbb{Z}^n}\theta_{\nu l}\,d_{\nu l} \tag{15.29}$$

where $\theta_{\nu l}$ is given by

$$\theta_{\nu l} = \sum_{m\in\mathbb{Z}^n}\lambda_{\nu l}\,\langle l-m\rangle^{-A}.$$

From (15.28) and (15.29) we obtain the decomposition

$$a(\cdot,D)u = \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\theta_{jk}\,d_{jk}$$

where d_{jk} are 1_K-atoms for $j = 0$ or $(s,p)_{K,-1}$-atoms for $j \in \mathbb{N}$ (no moment conditions) and $\theta = (\theta_{jk})_{j\in\mathbb{N}_0,k\in\mathbb{Z}^n}$ is defined as in (15.21), with $\left\|\theta \mid f^N_{p,q}\right\| \leqslant c\left\|\lambda \mid f^N_{p,q}\right\|$

This is exactly (15.4) announced before.
To get the conclusion we have now only to apply Lemma 15.2.7, and to apply the atomic decomposition theorem for the space $F^{\sigma^s,N}_{p,q}$.
Note that to apply the atomic decomposition theorem and to get the assertion (i) of the theorem, we need $K = M - n - 1 > s/\log_2 \lambda_0$ so that M has to be chosen large enough.

(ii) To get the assertion for B spaces we have to use the same technique as above and to replace Lemma 15.2.7 by an obvious counterpart with b_{pq} instead of $f^N_{p,q}$ in (15.22). ■

15.3 Comments and examples

Remark 15.3.1 (The classical case) Let $N_j = 2^j$, and $\sigma_j = 2^{js}$, $s \in \mathbb{R}$ and consider the classical spaces $B^s_{p,q}$ and $F^s_{p,q}$.
Then clearly $\lambda_0 = \lambda_1 = 2$ and restriction (15.2) in Theorem 15.1.1 becomes

$$s > n\left(\frac{1}{\min(1,p,q)} - 1\right).$$

Consequently, if s is as above and $a \in S^0_{1,1}$ then the associated pseudo-differential operator $a(\cdot,D)$ maps $F^s_{p,q}$ linear and bounded into $F^s_{p,q}$.
A similar assertion is true for the Besov spaces $B^s_{p,q}$ if $s > n\left(\frac{1}{\min(1,p)} - 1\right)$.
These results are known, and in their full generality were proved by T. Runst in [Ru85] using paramultiplication and R. H. Torres in [To90] using so-called molecular decompositions and the φ-transform of M. Frazier and B. Jawerth. For a historical view on partial results compare for example the literature mentioned in [Tr92, Section 6.3.2] and in the previous section.
As a corollary, if $1 < p < \infty$ and if $s > 0$ then the pseudo-differential operator $a(\cdot,D)$ with exotic symbol a maps $H^s_p = F^s_{p,2}$ linear and bounded into itself and this is the famous result of Y. Meyer, see [Mey80].

Remark 15.3.2 An extension of the mapping result stated in Theorem 15.1.1, to the whole range of the parameter s without additional assumptions on the symbol such that we would have counterparts of Theorem 14.2.3 cannot be expected. We have already mentioned that the pseudo-differential operator associated to the symbol from Example 14.1.4 fails to be continuous on $L_2 = F^0_{2,2}$.
However one can still obtain some boundedness results under some additional assumptions, we refer to the papers of L. Hörmander [Hö88] and [Hö89]. In fact he proved that if $s \leqslant 0$ and the Fourier transform $\widehat{a}(\eta,\xi)$ of $a(x,\xi)$ with respect to the variable x is small, in some sense, on the "twisted diagonal" $\{(\eta,\xi) : \eta = -\xi\}$ then $a(\cdot,D)$ maps H^s_2 into itself (see the above mentioned papers for the precise meaning). It turns out that a similar kind of behaviour of the symbol can be obtained by imposing certain conditions on the formal transpose of the operator $a(\cdot,D)$.

Remark 15.3.3 As it was already mentioned, in the L_p-theory of function spaces we do not have such strong tools as in the Hilbert space case. The idea of proving

mapping properties of pseudo-differential operators using some "nice decompositions" in function spaces is quite a natural one and is not new.
In [To90] R. H. Torres used the φ-transform of M. Frazier and B. Jawerth to discuss mapping properties of pseudo-differential operators with exotic symbols, he obtained the results stated in Remark 15.3.1.
H. Triebel used a different kind of atomic decomposition in proving mapping properties on spaces of $B^s_{p,q}$ and $F^s_{p,q}$ type, see [Tr92, Theorem 6.3.2]. However the atoms he used there are slightly different from the "classical" and from those one used in our proof above.

Example 15.3.4 (ψ-Bessel potential spaces) In the next section we are interested to apply this mapping theorem to exotic pseudo-differential operators acting on ψ-Bessel potential spaces.
Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an admissible continuous negative definite function, compare Definition 6.1.1, and assume that there exists $r_0 \in (0,1]$ such that $\xi \mapsto \psi(\xi)\langle\xi\rangle^{-2r_0}$ is increasing in $|\xi|$. If $(N_j)_{j\in\mathbb{N}_0}$ is the sequence associated to ψ and to $r = 2$, compare (6.5), then for any $j \in \mathbb{N}_0$ one has

$$2^{\frac{1}{w}} N_j \leqslant N_{j+1} \leqslant 2^{\frac{1}{r_0}} N_j \quad ,$$

compare Lemma 6.2.2 and Lemma 7.2.3.
Based on Corollary 7.1.4 we have for $s \in \mathbb{R}$ and $1 < p < \infty$ the equality $H^{\psi,s}_p = F^{\sigma^s,N}_{p,2}$ where $\sigma^s = (2^{js})_{j\in\mathbb{N}_0}$.
Condition (15.2) is now

$$s > n\left(\frac{1}{r_0} - \frac{1}{w}\right).$$

Consequently, if s is as above we obtained that the pseudo-differential operator $a(\cdot, D)$ with exotic symbol a maps the space $H^{\psi,s}_p$ into itself.
In particular if $\psi(\xi) = f(1 + |\xi|^2)$ (for an appropriate Bernstein function f) then the restriction on s is

$$s > n\left(\frac{1}{r_0} - 1\right).$$

since then $w = 1$.
We will state this example as a separate theorem in the next section.

16 Symbols and pseudo-differential operators related to an admissible continuous negative definite function

16.1 Assumptions and preliminaries

Through the whole section we will consider $\psi : \mathbb{R}^n \to \mathbb{R}$ an admissible continuous negative definite function (recall Definition 6.1.1) which satisfies the following assumption.

Assumption 16.1.1 There exists an $r_0 \in (0,1]$ such that the function $\xi \mapsto \psi(\xi)\langle\xi\rangle^{-2r_0}$ is increasing in $|\xi|$.

Recall that in particular the function $\xi \mapsto \langle\xi\rangle \log(1+\langle\xi\rangle)$ as well as the function $\xi \mapsto \langle\xi\rangle(1-\exp(-4\langle\xi\rangle))$ are admissible and satisfy the above assumption, compare Corollary 7.2.9.

Remark 16.1.2 Recall that due to the fact that ψ is admissible there exists a number $0 < w \leqslant 1$ such that for some $c_2 > 0$ we have $\psi(\xi) \leqslant c_2\,(1+|\xi|^2)^w$ for any $\xi \in \mathbb{R}^n$, compare Remark 6.1.2. In particular, if $\psi(\xi) = f(1+|\xi|^2)$ where f is an appropriate Bernstein function, then the number $w = 1$.

Remark 16.1.3 Moreover, if ψ is an admissible continuous negative definite function which satisfies Assumption 16.1.1 then for some $c_1 > 0$ we have $c_1\,(1+|\xi|^2)^{r_0} \leqslant \psi(\xi)$ for any $\xi \in \mathbb{R}^n$.

Remark 16.1.4 Note also that if ψ satisfies Assumption 16.1.1 we have the embedding result stated in Corollary 7.2.5: if $1 < p < \infty$, and if $s > \frac{1}{r_0}\frac{n}{p}$ then $H_p^{\psi,s} \hookrightarrow C_\infty$.

16.2 The class $S_{1,\delta}^{\psi,\mu}$

Starting with an admissible continuous negative definite function ψ which satisfies Assumption 16.1.1 we will first introduce in analogy to the class $S^{\mu}_{1,\delta}$, compare Definition 14.1.1, a class of symbols related to ψ.

Definition 16.2.1 *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an admissible continuous negative definite function which satisfies Assumption* 16.1.1.
Let $\mu \in \mathbb{R}$ and $0 \leqslant \delta \leqslant 1$. Then let $S_{1,\delta}^{\psi,\mu}$ be the collection of all complex valued C^∞ functions $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that for any multi-indices β and α there exists a constant $c_{\beta\alpha} > 0$ such that

$$|D_x^\beta D_\xi^\alpha a(x,\xi)| \leqslant c_{\beta\alpha}\,(1+\psi(\xi))^{\frac{\mu}{2}} \cdot \langle\xi\rangle^{-|\alpha|+\delta\,|\beta|} \quad \textit{for any} \quad x \in \mathbb{R}^n,\, \xi \in \mathbb{R}^n. \tag{16.1}$$

In particular, if $a \in S_{1,\delta}^{\psi,\mu}$ then for some constant $C_0 > 0$

$$|a(x,\xi)| \leqslant C_0\,(1+\psi(\xi))^{\frac{\mu}{2}} \quad \text{for any} \quad x \in \mathbb{R}^n,\, \xi \in \mathbb{R}^n. \tag{16.2}$$

It is clear that

$$S_{1,0}^{\psi,\mu} \subset S_{1,\delta}^{\psi,\mu} \subset S_{1,\delta'}^{\psi,\mu'} \quad \text{if} \quad \mu \leqslant \mu', \delta \leqslant \delta'.$$

In introducing the class $S_{1,\delta}^{\psi,\mu}$ we were of course motivated by the properties of the admissible function ψ and by the following example.

Example 16.2.2 **(i)** Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an admissible function and let $a(x,\xi) = \psi(\xi)$. As a direct consequence of Definition 6.1.1.(iii), in particular of inequality (6.1), we get $a \in S_{1,0}^{\psi,2}$.
(ii) Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an admissible function and $\mu \in \mathbb{R}$. Defining $a(x,\xi) = (1+\psi(\xi))^{\mu/2}$, as a direct consequence of Lemma 7.1.2, we get $a \in S_{1,0}^{\psi,\mu}$.
(iii) Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an admissible function and let $g : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $\sup_{x\in\mathbb{R}^n} |D^\gamma g(x)| \leqslant c_\gamma$ for all $\gamma \in \mathbb{N}_0^n$ (the function g may be zero on a domain in $\mathbb{R}^n$). Then the symbol $(x,\xi) \mapsto g(x)^k\,(1+\psi(\xi))^{\mu/2}$ belongs again to $S_{1,0}^{\psi,\mu}$.

It is clear that if $\psi(\xi) = |\xi|^2$ then the class $S^{\psi,\mu}_{1,\delta}$ coincides with the Hörmander class $S^{\mu}_{1,\delta}$.
On the other hand, based on Remark 16.1.2 and on Remark 16.1.3, it is easy to see that our classes $S^{\psi,\mu}_{1,\delta}$ are refinements of the Hörmander classes $S^{\mu}_{1,\delta}$, more precisely we have the obvious

Corollary 16.2.3 *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an admissible continuous negative definite function which satisfies Assumption* 16.1.1 *and let* $0 \leqslant \delta \leqslant 1$.
(i) *If $\mu > 0$ then $S^{\mu r_0}_{1,\delta} \subset S^{\psi,\mu}_{1,\delta} \subset S^{\mu w}_{1,\delta}$.*
(ii) *If $\mu < 0$ then $S^{\mu r_0}_{1,\delta} \supset S^{\psi,\mu}_{1,\delta} \supset S^{\mu w}_{1,\delta}$.*
(iii) *If $\mu = 0$ then $S^{\psi,0}_{1,\delta} = S^{0}_{1,\delta}$. In particular, the class of exotic symbols $S^0_{1,1}$ coincides with $S^{\psi,0}_{1,1}$.*

Remark 16.2.4 Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an admissible continuous negative definite function which satisfies Assumption 16.1.1, let $\mu \in \mathbb{R}$, and let $0 \leqslant \delta \leqslant 1$. If $a \in S^{\psi,\mu}_{1,\delta}$ then the function $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$,

$$b(x,\xi) = (1+\psi(\xi))^{-\frac{\mu}{2}}\, a(x,\xi) \quad , \quad x,\xi \in \mathbb{R}^n,$$

belongs to the class $S^0_{1,\delta}$.
This can be easily shown using Leibniz's product rule and using Lemma 7.1.2. Indeed, for any two multi-indices β and α we have

$$\begin{aligned}
|D^\beta_x D^\alpha_\xi b(x,\xi)| &\leqslant \sum_{\alpha_1+\alpha_2=\alpha} c_{\alpha_1\alpha_2}\, |D^{\alpha_1}_\xi \left((1+\psi(\xi))^{-\frac{\mu}{2}}\right)| \cdot |D^\beta_x D^{\alpha_2}_\xi a(x,\xi)| \\
&\leqslant \sum_{\alpha_1+\alpha_2=\alpha} c'_{\beta\alpha_1\alpha_2}\, (1+\psi(\xi))^{-\frac{\mu}{2}}\, \langle\xi\rangle^{-|\alpha_1|} \cdot (1+\psi(\xi))^{\frac{\mu}{2}}\, \langle\xi\rangle^{-|\alpha_2|+\delta|\beta|} \\
&= C_{\beta\alpha}\, \langle\xi\rangle^{-|\alpha|+\delta|\beta|}.
\end{aligned}$$

For a symbol $a \in S^{\psi,\mu}_{1,\delta}$, let $a(x,D)$ the associated pseudo-differential operator,

$$a(x,D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\, \widehat{u}(\xi)\, d\xi$$

defined for $u \in \mathcal{S}$.
Consequently, any pseudo-differential operator $a(x,D)$ with symbol $a \in S^{\psi,\mu}_{1,\delta}$ maps $\mathcal{S}$ into $\mathcal{S}$ and if $\delta < 1$ also $\mathcal{S}'$ into $\mathcal{S}'$.

Remark 16.2.5 One of our aims is to provide a good L_p-theory for pseudo-differential operators generating L_p-sub-Markovian and Feller semigroups. We think the symbol class $S^{\psi,\mu}_{1,\delta}$ introduced above as a refinement of the classical Hörmander classes fits good in fulfilling this aim.

Several authors considered refinements of the Hörmander class $S^{\mu}_{1,\delta}$.
We would like to mention here the work of R. Beals [Be75] who considered more general symbols than those in the class $S^{\psi,\mu}_{1,\delta}$. More precisely, in [Be75] the author considered symbols satisfying the inequality

$$|D^\beta_x D^\alpha_\xi a(x,\xi)| \leqslant c_{\beta\alpha}\, \varphi(x,\xi)^{|\beta|} \Phi(x,\xi)^{|\alpha|} \quad \text{for any} \quad x \in \mathbb{R}^n,\, \xi \in \mathbb{R}^n$$

instead of (16.1), where φ and Φ are appropriate functions satisfying some reasonable assumptions. In [Be75] R. Beals gave a comprehensive study of the associated pseudo-differential operators including L_2-boundedness assertions, a-priori estimates and hypoellipticity etc. however only in an L_2 setting.
Our symbols from Definition 16.2.1 may be interpreted in some sense as some special cases of the symbols of R. Beals but we will treat them in an L_p-context and having in mind the problem mentioned above.

16.3 Mapping properties

16.3.1 Symbols from $S_{1,1}^{\psi,\mu}$

We start stating separately Theorem 15.1.1 in the case of of ψ-Bessel potential spaces, compare Example 15.3.4.

Theorem 16.3.1 *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an admissible continuous negative which satisfies Assumption 16.1.1.*
Let $1 < p < \infty$ and let $a \in S_{1,1}^0$ be an exotic symbol. If

$$s > n\left(\frac{1}{r_0} - \frac{1}{w}\right)$$

then the pseudo-differential operator $a(\cdot, D)$ maps the space $H_p^{\psi,s}$ linear and bounded into itself.

Corollary 16.3.2 *Let $f : (0,\infty) \to (0,\infty)$ be a Bernstein function with $\lim_{t\to\infty} f(t) = \infty$ and assume that there exists $r_0 \in (0,1]$ such that $t \mapsto f(t)\, t^{-r_0}$ is increasing.*
Let $1 < p < \infty$ and let $a \in S_{1,1}^0$ be an exotic symbol. If

$$s > n\left(\frac{1}{r_0} - 1\right)$$

then the pseudo-differential operator $a(\cdot, D)$ maps the space $H_p^{f(1+|\cdot|^2),s}$ linear and bounded into itself.

As a consequence of Theorem 16.3.1 and of Remark 16.2.4 we get the following mapping result.

Theorem 16.3.3 *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an admissible continuous negative which satisfies Assumption 16.1.1. Let $\mu \in \mathbb{R}$ and let $a \in S_{1,1}^{\psi,\mu}$.*
If $1 < p < \infty$ and if

$$s > n\left(\frac{1}{r_0} - \frac{1}{w}\right)$$

then the pseudo-differential operator $a(\cdot, D)$ maps the space $H_p^{\psi,s+\mu}$ linear and bounded into $H_p^{\psi,s}$ in the sense that there exists a constant $C > 0$ such that

$$\left\| a(\cdot, D)u \,|\, H_p^{\psi,s} \right\| \leqslant C \left\| u \,|\, H_p^{\psi,s+\mu} \right\| \quad \textit{for any} \quad u \in H_p^{\psi,s+\mu}.$$

Proof. It is clear that at least for $u \in \mathcal{S}$ we may write

$$\begin{aligned} a(x,D)u(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\,\widehat{u}(\xi) \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} (1+\psi(\xi))^{\frac{-\mu}{2}} a(x,\xi)\,(1+\psi(\xi))^{\frac{\mu}{2}} \widehat{u}(\xi)\,d\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, b(x,\xi)\,\widehat{v}(\xi)\,d\xi \\ &= b(x,D)v(x) \end{aligned}$$

where $b(x,\xi) = (1+\psi(\xi))^{\frac{-\mu}{2}} a(x,\xi)$ and $(1+\psi(\xi))^{\frac{\mu}{2}}\widehat{u}(\xi) = \widehat{v}$.
Moreover, $v \in H_p^{\psi,s}$ since

$$\begin{aligned} \left\| v \,|\, H_p^{\psi,s} \right\| &= \left\| \mathcal{F}^{-1}[(1+\psi(\cdot))^{\frac{s}{2}}\,\widehat{v}] \,|\, L_p \right\| \\ &= \left\| \mathcal{F}^{-1}[(1+\psi(\cdot))^{\frac{s}{2}} (1+\psi(\cdot))^{\frac{\mu}{2}}\,\widehat{u}] \,|\, L_p \right\| = \left\| u \,|\, H_p^{\psi,s+\mu} \right\|. \end{aligned}$$

Based on Remark 16.2.4 we have $b \in S_{1,1}^0$ and consequently we may apply Theorem 16.3.1 to get

$$\left\| a(\cdot,D)u \,|\, H_p^{\psi,s} \right\| = \left\| b(\cdot,D)v \,|\, H_p^{\psi,s} \right\| \leqslant c \left\| v \,|\, H_p^{\psi,s} \right\| = c \left\| u \,|\, H_p^{\psi,s+\mu} \right\|$$

and this completes the proof. ■

Of course we have an obvious corollary of the above theorem for the spaces $H_p^{f(1+|\cdot|^2),s}$ if $\psi = f(1+|\xi|^2)$ where f is an appropriate Bernstein function.

16.3.2 Symbols from $S_{1,\delta}^{\psi,\mu}$ ($\delta < 1$)

We start with a simple counterpart of Remark 16.2.4.

Remark 16.3.4 Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an admissible continuous negative definite function, let $\mu \in \mathbb{R}$, and let $0 \leqslant \delta \leqslant 1$. If $a \in S_{1,\delta}^{\psi,\mu}$ and $s \in \mathbb{R}$ then the function $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$,

$$b(x,\xi) = (1+\psi(\xi))^{\frac{s}{2}}\, a(x,\xi) \quad , \quad x,\xi \in \mathbb{R}^n,$$

belongs to the class $S_{1,\delta}^{\psi,\mu+s}$.
This can be easily shown as in was done in Remark 16.2.4, using Leibniz's product rule and using Lemma 7.1.2.

Theorem 16.3.5 *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an admissible continuous negative definite function, let $\mu \in \mathbb{R}$, let $0 \leqslant \delta < 1$ and let $a \in S_{1,\delta}^{\psi,\mu}$.*
If $1 < p < \infty$ and if $s \in \mathbb{R}$ then the pseudo-differential operator $a(\cdot,D)$ maps the space $H_p^{\psi,s+\mu}$ linear and bounded into $H_p^{\psi,s}$ in the sense that there exists a constant $c > 0$

$$\left\| a(\cdot,D)u \,|\, H_p^{\psi,s} \right\| \leqslant c \left\| u \,|\, H_p^{\psi,s+\mu} \right\| \quad \textit{for any} \quad u \in H_p^{\psi,s+\mu}.$$

Remark 16.3.6 One can do the proof of the above result in two ways.
The first one is given below and it is based on a direct application of the mapping result for exotic symbols.

Proof. Of course the pseudo-differential operator

$$I_\tau = (1+\psi(D))^{\frac{\tau}{2}}$$

(which has its symbol in $S^{\psi,\tau}_{1,0}$) is a lift operator for the scale of spaces $H^{\psi,s}_p$ so there is no loss of generality if we make the proof for $\mu = 0$.
So let $a \in S^0_{1,\delta}$. We have however to prove the mapping result for all real s.
Let $s \in \mathbb{R}$ arbitrary and let $\tau \in \mathbb{R}$ such that

$$s + \tau > n\left(\frac{1}{r_0} - \frac{1}{w}\right). \tag{16.3}$$

We decompose the pseudo-differential operator $a(\cdot, D)$ as follows

$$a(\cdot, D)u = I_\tau \circ (I_{-\tau} \circ a(\cdot, D) \circ I_\tau) \circ I_{-\tau} u.$$

Since $I_{-\tau}$ is linear and bounded from $H^{\psi,s}_p$ to $H^{\psi,s+\tau}_p$ and I_τ is linear and bounded from $H^{\psi,s+\tau}_p$ to $H^{\psi,s}_p$ it remains to discuss the mapping properties of $I_{-\tau} \circ a(\cdot, D) \circ I_\tau$.
Using the composition rule for pseudo-differential operators as stated for example in the book [Ku74, Chapter 2] (recall $\delta < 1$) and using the assumption on s and τ it is clear that $I_{-\tau} \circ a(\cdot, D) \circ I_\tau$ is linear and bounded from $H^{\psi,s+\tau}_p$ into itself. ■

In order to get more information on the constant in the above mapping theorem we want to indicate a second proof of the above result based on the mapping theorem of R. A. Illner stated in Theorem 14.2.1 it.
Of course, due to the assumption in R. A. Illner's result this works only for for pseudo-differential operators with symbols in $S^{\psi,\mu}_{1,\delta}$ with $\delta < 1$.

In analogy to the seminorms associated to symbols from the classical class $S^{\mu}_{1,\delta}$, see (14.2), we associate to a symbol $a \in S^{\psi,\mu}_{1,\delta}$ seminorms $|a|^{(\psi,\mu)}_{(l,k)}$ for $l, k \in \mathbb{N}_0$ by

$$|a|^{(\psi,\mu)}_{(l,k)} = \max_{|\beta|\leqslant l, |\alpha|\leqslant k} \sup_{x,\xi\in\mathbb{R}^n} \left\{ |D^\beta_x D^\alpha_\xi a(x,\xi)| \cdot (1+\psi(\xi))^{-\frac{\mu}{2}} \langle\xi\rangle^{|\alpha|-\delta|\beta|} \right\}. \tag{16.4}$$

Consequently, combining Theorem 14.2.1 with Remark 16.3.4 we get the following formulation for Theorem 16.3.5.

Theorem 16.3.7 *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an admissible continuous negative definite function, let $\mu \in \mathbb{R}$, let $0 \leqslant \delta < 1$ and let $a \in S^{\psi,\mu}_{1,\delta}$.*
If $1 < p < \infty$ and if $s \in \mathbb{R}$ then the pseudo-differential operator $a(\cdot, D)$ maps the space $H^{\psi,s+\mu}_p$ linear and bounded into $H^{\psi,s}_p$.
More precisely, there exist integers l and k and a constant $C > 0$, all independent of a, such that

$$\left\| a(\cdot, D)u \,|\, H^{\psi,s}_p \right\| \leqslant C\, |a|^{(\psi,\mu)}_{(l,k)} \left\| u \,|\, H^{\psi,s+\mu}_p \right\| \quad \text{for any} \quad u \in H^{\psi,s+\mu}_p. \tag{16.5}$$

16.4 Towards Feller semigroups and sub-Markovian semigroups generated by pseudo-differential operators with symbols in $S_{1,\delta}^{\psi,\mu}$

Now we would like to indicate how one can use the results in the previous sections in order to have some of the first steps for a reasonable L_p-theory for generators of Feller semigroups and L_p-sub-Markovian semigroups.

Let $(x,\xi) \mapsto a(x,\xi)$ be a negative definite symbol and consider on $C_0^\infty(\mathbb{R}^n;\mathbb{R})$ the operator $-a(x,D)$.
Since $C_0^\infty(\mathbb{R}^n;\mathbb{R})$ is dense in $C_\infty(\mathbb{R}^n;\mathbb{R})$ and $-a(x,D)$ satisfies the positive maximum principle, an application of Hille-Yosida-Ray theorem, see Theorem 2.1.21, "only" requires to solve for some $\lambda > 0$ the equation

$$a_\lambda(x,D)u = a(x,D)u + \lambda u = f \tag{16.6}$$

for a dense set (in $C_\infty(\mathbb{R}^n,\mathbb{R})$) of right-hand sides f in the space $C_\infty(\mathbb{R}^n,\mathbb{R})$.
This problem seems to be very difficult and we propose a different way.
The strategy should be to consider $a(x,D)$ on a larger domain where (16.6) is easier to handle. For this it is useful to know that the positive maximum principle holds on a larger domain.
The following theorem was proved in [Ho98b] and one can find it also in [Ja02, Theorem 2.6.1].

Theorem 16.4.1 *Let $D(A) \subset C_\infty(\mathbb{R}^n;\mathbb{R})$ and let $A : D(A) \to C_\infty(\mathbb{R}^n;\mathbb{R})$ be a linear operator. Assume, in addition that $C_0^\infty(\mathbb{R}^n;\mathbb{R}) \subset D(A)$ is an operator core of A in the sense that for every $u \in D(A)$ there exists a sequence $(\varphi_j)_{j\in\mathbb{N}_0} \subset C_0^\infty(\mathbb{R}^n;\mathbb{R})$ such that*

$$\lim_{j\to\infty} \|\varphi_j - u \,|\, C_\infty\| = \lim_{j\to\infty} \|A\varphi_j - Au \,|\, C_\infty\| = 0.$$

If $A|_{C_0^\infty}$ satisfies the positive maximum principle on $C_0^\infty(\mathbb{R}^n;\mathbb{R})$ then it satisfies the positive maximum principle also on $D(A)$.

Let us return to equation (16.6). Note that for solving we do not need the positive maximum principle so that we may consider complex-valued functions.
Let us start with an admissible continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ which satisfies Assumption 16.1.1. Then it follows from the embedding result stated in Corollary 7.2.5 that if

$$s > \frac{1}{r_0}\frac{n}{p} \quad \text{then} \quad H_p^{\psi,s} \hookrightarrow C_\infty \tag{16.7}$$

so that for some constant $c_1 > 0$ we have for any $u \in H_p^{\psi,s}$ the inequality $\|u \,|\, C_\infty\| \leqslant c_1 \left\|u \,|\, H_p^{\psi,s}\right\|$.
For a fixed s with (16.7) we consider now a negative definite symbol a from some $S_{1,\delta}^{\psi,\mu}$, $0 \leqslant \delta \leqslant 1$.
Of course we will assume, additionally, that $s > n\left(\frac{1}{r_0} - \frac{1}{w}\right)$ if $\delta = 1$.
Consequently using the mapping property stated in Theorem 16.3.5 we have the linear bounded operator

$$a_\lambda(\cdot,D) : H_p^{\psi,\mu+s} \to H_p^{\psi,s}$$

in the sense that for some constant $c_2 > 0$ such that for any $u \in H_p^{\psi,\mu+s}$, we have $\|a_\lambda(\cdot, D)u \mid H_p^{\psi,s}\| \leqslant c_2 \|u \mid H_p^{\psi,s}\|$.
From Remark 7.2.10 we know that for admissible continuous negative definite functions ψ satisfying Assumption 16.1.1, the space C_0^∞ is dense in $H_p^{\psi,s}$. Consequently C_0^∞ is an operator core for $a_\lambda(\cdot, D)$.
Since $H_p^{\psi,\mu+s} \subset H_p^{\psi,s} \subset C_\infty$ if $\mu \geqslant 0$, the operator $-a_\lambda(\cdot, D)$ is densely defined (in C_∞) and satisfies on $H_p^{\psi,\mu+s}$ also the positive maximum principle by Theorem 16.4.1.
So far we have reduced the problem of extending $a(\cdot, D)$ to a generator of a Feller semigroup to an application of the Hille-Yosida-Ray theorem, see Theorem 2.1.21, and this was reduced to the solving of the equation $a(x, D)u + \lambda u = f$ in the space $H_p^{\psi,\mu+s}$ for all $f \in H_p^{\psi,s}$.
Consequently, we obtain the following result.

Corollary 16.4.2 *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an admissible continuous negative definite function which satisfies Assumption* 16.1.1 *and let $a \in S_{1,\delta}^{\psi,\mu}$.*
If s is sufficiently large and if there exists a $\lambda > 0$ such that for any $f \in H_p^{\psi,s}$ the equation $a(x, D)u + \lambda u = f$ has a solution in $H_p^{\psi,\mu+s}$ then $a(\cdot, D)$ extends to a generator of a Feller semigroup.

N. Jacob in [Ja92] and W. Hoh in [Ho98a] solved the equation $a(x, D)u + \lambda u = f$ in a Hilbert space setting. Namely they got solutions in $H_2^{\psi,s}$ using essentially the existence of variational solutions and the existence of a Garding type inequality for the spaces.
In our L_p-setting, new ideas and techniques are required. Note also that we can treat, under some conditions, also exotic pseudo-differential operators.
One idea is to restrict ourselves to a subclass of $S_{1,\delta}^{\psi,\mu}$ in order to have a parametrix for the associated pseudo-differential operators.

When discussing the problem of extending $a(\cdot, D)$ to a generator of an L_p-sub-Markovian semigroup in order to apply the Hille-Yosida theorem, Theorem 2.1.20, we need instead of the positive maximum principle, the dissipativity in L_p of $a_\lambda(\cdot, D)$.
Again it seems necessary to restrict ourselves to a subclass of $S_{1,\delta}^{\psi,\mu}$ in order to have a parametrix for the associated pseudo-differential operators.

We propose the following subclass.

Definition 16.4.3 *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ an admissible continuous negative definite function which satisfies Assumption* 16.1.1, *let $\mu, \mu' \in \mathbb{R}$ with $\mu' \leqslant \mu$, let $\delta \leqslant 1$.*
Then $S^\psi(\mu, \mu', \delta)$ is the collection of all complex-valued C^∞ functions $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that $a \in S_{1,\delta}^{\psi,\mu}$ and such that for any multi-index β and for any multi-index α there exists $c_{\beta\alpha} > 0$ with

$$|D_x^\beta D_\xi^\alpha a(x, \xi)| \leqslant c_{\beta\alpha} |a(x, \xi)| \langle\xi\rangle^{-|\alpha|+\delta|\beta|} \quad \text{for any} \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n \tag{16.8}$$

and for which there exists a constant $c_0 > 0$ such that $c_0 (1 + \psi(\xi))^{\frac{\mu'}{2}} \leqslant |a(x, \xi)|$.

Example 16.4.4 Clearly if $\psi : \mathbb{R}^n \to \mathbb{R}$ is an admissible continuous negative definite function satisfying assumption 16.1.1 then $(x, \xi) \mapsto a(x, \xi) = (1 + \psi(\xi))^{\frac{\mu}{2}}$ belongs to the class $S^\psi(\mu, \mu, 0)$.

Remark 16.4.5 Note that from (16.2) we know $|a(x,\xi)| \leqslant C_0\,(1+\psi(\xi))^{\frac{\mu'}{2}}$ and so the class $S^\psi(\mu,\mu',\delta)$ is of course related to the class of hypoelliptic symbols of slowly varying strength treated by H.-G. Leopold in [Leo89a], [Leo89b] and [Le91].
He denoted his class $S(\mu,\mu',\delta)$ and this is a subclass of $S^\mu_{1,\delta}$ consisting of symbols a which satisfy $c_{\mu'}\,\langle\xi\rangle^{\mu'} \leqslant |a(x,\xi)| \leqslant c_\mu\,\langle\xi\rangle^\mu$ and which satisfy

$$|D_x^\beta\, D_\xi^\alpha a(x,\xi)| \leqslant c_{\beta\alpha}\,|a(x,\xi)|\,\langle\xi\rangle^{-|\alpha|+\delta|\beta|} \quad \text{if} \quad x,\xi \in \mathbb{R}^n,$$

and associated Besov spaces of variable order of differentiation to such symbols a and discussed applications to degenerate elliptic differential equations.
In fact due to our assumption on ψ we have $S^\psi(\mu,\mu',\delta) \subset S(\mu w,\mu' r_0,\delta)$.

Example 16.4.6 Let $\sigma(x) = s + \chi(x)$ be a real-valued function where s is a constant and $\chi \in \mathcal{S}(\mathbb{R}^n)$, let $0 < \mu' = \inf\limits_{x\in\mathbb{R}^n} \sigma(x)$ and $\mu = \sup\limits_{x\in\mathbb{R}^n} \sigma(x)$.
(i) Then symbols of type $a(x,\xi) = \langle\xi\rangle^{\sigma(x)}$, considered many years ago by B. Beauzamy in [Beu72] and by A. Unterberger and J. Bokobza in [UnBo65a], belong to the class $S(\mu,\mu',\delta)$ for any $0 < \delta < 1$, compare [Leo89b]. It is clear that modifying this classical example and consider symbols of type $a(x,\xi) = (1+\psi(\xi))^{\sigma(x)}$ we obtain additional examples for our class.
(ii) If t is a real number then symbols of type $a(x,\xi) = \langle\xi\rangle^{\sigma(x)}\,(1+\log\langle\xi\rangle^2)^{t/2}$ where investigated by A. Unterberger and J. Bokobza in [UnBo65b]. They belong to the class $S(M,M',\delta)$ where $0<\delta<1$ if $0<M'<\mu'$ and $M>\mu$.
Appropriate modification will give us examples in our classes $S^\psi(\mu,\mu',\delta)$.

Example 16.4.7 Let $g:\mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $\sup\limits_{x\in\mathbb{R}^n} |D^\gamma g(x)| \leqslant c_\gamma$ for all $\gamma \in \mathbb{N}_0^n$ (the function g may be zero on a domain in $\mathbb{R}^n$). Let $0<\mu' \leqslant \mu$ and let k be a natural number with $\mu-\mu' < 2k$.
Then the symbol $(x,\xi) \mapsto \langle\xi\rangle^{\mu'} + g(x)^{2k}\,\langle\xi\rangle^\mu$ belongs to $S(\mu,\mu',\delta)$ where $\delta = (\mu-\mu')/(2k)$, compare [Leo89b]. If μ and μ' are even numbers, then a is the symbol of a degenerate partial differential equation.
It is clear how this example can be modified to get examples in our classes $S^\psi(\mu,\mu',\delta)$

Using the inclusion $S^\psi(\mu,\mu',\delta) \subset S(\mu w,\mu' r_0,\delta)$ we get the existence of a parametrix for the pseudo-differential operators with symbols in $S^\psi(\mu,\mu',\delta)$ based on the classical result of L. Hörmander stated for example in [Ku74, Chapter 2].
This should be the main tool in discussing the L_p dissipativity as well as solutions in $H_p^{\psi,\mu+s}$ for the equation $a(x,D)u + \lambda u = f$ as discussed above.
One idea is to use the existence of the parametrix, the precise estimate on the constant in inequality (16.5) and the theorem stated in [Ku74, Chapter 2], in which the symbol of the composition of two pseudo-differential operators is expressed in dependence of the initial symbols with help of the seminorms from (14.2).

We consider that these aspects are an interesting subject for further investigation.

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**Ehrenwörtliche Versicherung
über die selbständige Anfertigung der Habilitationsschrift**

Hiermit erkläre ich, daß ich die vorgelegte Arbeit selbständig und nur unter Verwendung der angegebenen Hilfsmittel und Literatur angefertigt habe.

München, den 30. Dezember 2002 Erich Walter Farkas