THE DISTRIBUTION OF EIGENFREQUENCIES OF ANISOTROPIC FRACTAL DRUMS

WALTER FARKAS AND HANS TRIEBEL

Abstract

Let Γ be an anisotropic fractal. The aim of the paper is to investigate the distribution of the eigenvalues of the fractal differential operator

$$(-\Delta)^{-1} \circ \mathrm{tr}^{\Gamma}$$

acting in the classical Sobolev space $\mathring{W}_2^1(\Omega)$ where Ω is a bounded C^∞ domain in the plane $\mathbb{R}^2$ with $\Gamma \subset \Omega$. Here $-\Delta$ is the Dirichlet Laplacian in Ω and tr^{Γ} is closely related to the trace operator tr_{Γ}.

1. *Introduction*

Motivated by some aspects of boundary value problems for partial differential equations, several authors have recently been concerned with the study of function spaces on and of fractals. We refer mainly to the papers by A. Jonsson and H. Wallin [**7–10**] and to the book [**16**] where complete references to this topic are given.

Let Ω be a bounded domain in $\mathbb{R}^2$ having C^∞ boundary $\partial\Omega$ and let $0 < d_A < 2$. An anisotropic d_A-set $\Gamma \subset \Omega$ having anisotropic deviation $0 \leqslant a \leqslant 1$ is, roughly speaking, a compact set which can be covered for any $j \in \mathbb{N}_0$ with $N_j \sim 2^{jd_A}$ rectangles R_{jl} $(l = 1, \ldots, N_j)$ with $\mathrm{vol}\, R_{jl} \sim 2^{-2j}$, having sides parallel to the axes and side lengths $r_1^{j,l}$, $r_2^{j,l}$ satisfying

$$2^{-j(1+a)} \leqslant r_2^{j,l} \leqslant r_1^{j,l} \leqslant 2^{-j(1-a)}$$

for any $l = 1, \ldots, N_j$. This concept was introduced in [**16**, 5.2]. If Γ is such an anisotropic d_A-set then there exists a uniquely determined Radon measure μ in $\mathbb{R}^2$ with $\mathrm{supp}\,\mu = \Gamma$ and $\mu(\Gamma \cap R_{jl}) = (\mathrm{vol}\, R_{jl})^{d_A/2}$ if $j \in \mathbb{N}_0$ and $l = 1, \ldots, N_j$ (see [**16**, 5.5]).

Let $(-\Delta)^{-1}$ be the inverse of the Dirichlet Laplacian in Ω. Let $W_2^1(\Omega)$ be the usual Sobolev space and let $\mathring{W}_2^1(\Omega) = \{f \in W_2^1(\Omega) : \mathrm{tr}_{\partial\Omega} f = 0\}$. The operator tr^{Γ},

$$(tr^{\Gamma} f)(\varphi) = \int_{\Gamma} (\mathrm{tr}_{\Gamma} f)(\gamma)\, (\varphi \mid \Gamma)(\gamma)\, d\mu(\gamma), \quad \varphi \in D(\Omega), \tag{1.1}$$

makes sense as a mapping from $\mathring{W}_2^1(\Omega)$ into $D'(\Omega)$ and it turns out that the fractal differential operator $T = (-\Delta)^{-1} \circ \mathrm{tr}^{\Gamma}$ generates a compact, non-negative, self-adjoint operator in $\mathring{W}_2^1(\Omega)$. Furthermore, as proved in [**16**, 30.7], there exist positive constants $c_1 > 0$ and $c_2 > 0$ such that for the positive eigenvalues $\lambda_k(T)$ of T

$$c_1 k^{-(d_A+2a)/d_A} \leqslant \lambda_k(T) \leqslant c_2 k^{-d_A/(d_A+2a)}, \quad k \in \mathbb{N}. \tag{1.2}$$

(For the first inequality in (1.2) it is additionally required that Γ is a so-called proper anisotropic d_A-set; see [**16**, Definition 5.11].) If the deviation $a = 0$ then we have $\lambda_k(T) \sim k^{-1}$. This means that the Weyl exponent occurs also in the case of proper anisotropic d_A-sets in the plane with deviation zero. This fact is not a surprise since those fractals are close to compact isotropic d_A-sets as described in [**16**, 3.1]. On the other hand, when $a > 0$ the two exponents in (1.2) are not equal.

Received 14 July 1997.

1991 *Mathematics Subject Classification* 46E35, 35P15, 28A90.

J. London Math. Soc. (2) 60 (1999) 224–236

The study of operators of type $(-\Delta)^{-1}\circ \mathrm{tr}^{\Gamma}$ is motivated in a natural way by the so-called fractal drums: the problem of finding the eigenfrequencies of a vibrating membrane (interpreted as a bounded domain Ω in the plane $\mathbb{R}^2$), fixed at its boundary, having the whole mass concentrated on some fractal compact set $\Gamma\subset\Omega$, can be reduced to the study of eigenvalues of operators of that type. More information about this subject is given in [**16**], especially in Sections 26.2 and 30.1–30.5, where one can find some modifications and a detailed discussion about this topic as well as further references extending the problem.

The aim of this paper is to discuss the sharpness of (1.2) and to shed some new light on these estimates. Restricting ourselves to the class of regular anisotropic fractals (anisotropic generalisations of the Cantor set in the plane) we prove that there exist two constants $C_1, C_2 > 0$ such that for all $k\in\mathbb{N}$,

$$C_1 k^{-\rho_1} \leqslant \lambda_k(T) \leqslant C_2 k^{-\rho_2} \tag{1.3}$$

for appropriate numbers ρ_1 and ρ_2 satisfying $(d_A+2a)/d_A > \rho_1 \geqslant \rho_2 > d_A/(d_A+2a)$ where $\lambda_k(T)$ are, again, the eigenvalues of the operator $T=(-\Delta)^{-1}\circ\mathrm{tr}^{\Gamma}$ acting in $\mathring{W}^1_2(\Omega)$. This means, in particular, that the estimates in (1.2) are not sharp in general. Furthermore, we will indicate a large class of regular anisotropic fractals for which $\lambda_k(T)\sim k^{-1}$, the so-called strongly regular anisotropic fractals.

In Section 2 we present the basic facts concerning regular anisotropic fractals. The main result, containing the precise formulation of (1.3), is presented with comments in Section 3. The proof is in Section 5, whereas Section 4 contains some preparatory facts for the proof.

All unimportant positive constants are denoted with c, occasionally with additional subscripts within the same formulae. The equivalence '$\mathrm{term}_1\sim\mathrm{term}_2$' means that there exist two constants $c_1, c_2>0$ independent of the variables in the two terms such that $c_1\,\mathrm{term}_1 \leqslant \mathrm{term}_2 \leqslant c_2\,\mathrm{term}_1$.

2. *Preliminaries*

2.1. *Regular anisotropic fractals*

Let $\mathbb{N}$ denote the natural numbers and let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Let $Q=[0,1]\times[0,1]$ and let the log be taken with respect to the base 2, let $1<K_1<K_2$ be natural numbers, let

$$a_1=\frac{2\log K_1}{\log(K_1K_2)},\quad a_2=\frac{2\log K_2}{\log(K_1K_2)},\quad \kappa=\tfrac{1}{2}\log(K_1K_2)$$

and let $a=1-a_1=a_2-1$. Let $(A_m)_{m=1}^N$ be $N\geqslant 2$ contractions of $\mathbb{R}^2$ into itself specified by

$$A_m : x=(x_1,x_2)\longmapsto(\eta_1^m 2^{-\kappa(1-a)}x_1, \eta_2^m 2^{-\kappa(1+a)}x_2)+x^m \tag{2.1}$$

for every $m=1,\ldots,N$ where $\eta_1^m,\eta_2^m\in\{-1,+1\}$ (including possible reflections). We assume that $A_mQ\subset Q$ for all $m=1,\ldots,N$; $A_m\mathring{Q}\cap A_{m'}\mathring{Q}=\varnothing$ if $m\neq m'$ and $\sum_{m=1}^N \mathrm{vol}\,A_mQ<1$. We suppose, in addition, that the rectangles A_mQ are located in the columns as indicated in Figure 1. Let

$$AQ=(AQ)^1=\bigcup_{m=1}^N A_mQ;\quad (AQ)^0=Q;$$

$$(AQ)^\nu = A((AQ)^{\nu-1})=\bigcup_{1\leqslant m_1,\ldots,m_\nu\leqslant N} A_{m_1}\circ\ldots\circ A_{m_\nu}Q;\quad \nu\in\mathbb{N}.$$

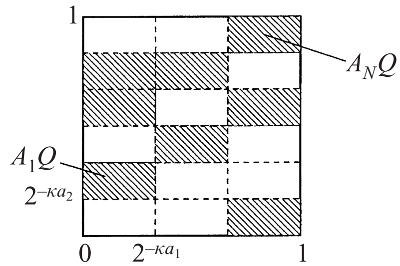

FIGURE 1.

This sequence of sets is monotonically decreasing and by [**5**, Theorem 8.3] its limit

$$\Gamma = (AQ)^\infty = \bigcap_{\nu\in\mathbb{N}} (AQ)^\nu = \lim_{\nu\to\infty} (AQ)^\nu$$

is the uniquely determined fractal generated by the contractions $(A_m)_{m=1}^N$. Fractals constructed in this way are anisotropic generalisations of the Cantor set in $\mathbb{R}^2$ and were called generalised Sierpinski carpets in [**11**] (since Sierpinski's universal curve is a special case of this construction) and regular anisotropic fractals in [**16**, 4.18].

Let n_l denote the number of rectangles $A_m Q$ in the lth column, $l = 1, \dots, K_1$. Throughout this paper we will assume that $n_l \geqslant 1$ for any $l = 1, \dots, K_1$ (in each column at least one rectangle $A_m Q$ is located).

The Hausdorff dimension (see [**6**, 2.2] for definition) of Γ is

$$\dim_H \Gamma = \frac{1}{\log K_1} \log\left(\sum_{l=1}^{K_1} n_l^{\log K_1/\log K_2}\right) \tag{2.2}$$

and the box-counting dimension (see [**6**, 3.1] for definition) of Γ is

$$\dim_B \Gamma = 1 + \frac{\log(N/K_1)}{\log K_2}. \tag{2.3}$$

Proofs of (2.2) and (2.3) are given in [**11**]; see also [**6**, Example 9.11]. Notice that in this type of example the Hausdorff dimension depends not only on the number of rectangles selected at each stage but also on their relative position. Moreover, it is clear that $\dim_H \Gamma$ and $\dim_B \Gamma$ are not, in general, equal.

Let $(A_m)_{m=1}^N$ be the $N \geqslant 2$ affine maps introduced in (2.1). The affine dimension of $\Gamma = (AQ)^\infty$ (see [**16**, 4.12]) is the uniquely determined positive number $d_A = \dim_A \Gamma$ such that

$$\sum_{m=1}^N (\operatorname{vol} A_m Q)^{d_A/2} = 1. \tag{2.4}$$

By construction we have $N = 2^{\kappa d_A}$.

DEFINITION 2.1. If $n_1 = \dots = n_{K_1} = N 2^{-\kappa(1-a)}$ (in any column there is the same number of rectangles) then we call Γ a *strongly* regular anisotropic fractal.

REMARK 2.2. If Γ is strongly regular then

$$\dim_H \Gamma = \dim_B \Gamma = \frac{d_A + 2a}{1+a}$$

as a simple consequence of (2.2), (2.3) and (2.4). Furthermore, Γ is an isotropic d-set (see [**16**, 3.1] for definition) where $d = (d_A + 2a)/(1+a)$. It will be clear from what follows that the typical number which also appears in the case of arbitrary regular anisotropic fractals is

$$d = \frac{d_A + 2a}{1+a}.$$

THEOREM 2.3 [**16**, 4.15]. *Let Γ be the regular anisotropic fractal introduced above having the affine dimension d_A according to* (2.4). *Then there exists a Radon measure μ in $\mathbb{R}^2$ uniquely determined with* $\operatorname{supp}\mu = \Gamma$ *and*

$$\mu(\Gamma \cap A_{m_1} \circ \ldots \circ A_{m_j} Q) = (\operatorname{vol} A_{m_1} \circ \ldots \circ A_{m_j} Q)^{d_A/2} \tag{2.5}$$

for all $j \in \mathbb{N}$ and all $m_1, \ldots, m_j \in \{1, \ldots, N\}$.

DEFINITION 2.4. Let $n_{\max} = \max\{n_l : 1 \leqslant l \leqslant K_1\}$ and $n_{\min} = \min\{n_l : 1 \leqslant l \leqslant K_1\}$. Then there exist two numbers $\lambda^+ \geqslant 0$ and $\lambda^- \leqslant 0$ such that

$$n_{\max} = N 2^{-\kappa(1-a)} 2^{\kappa(1-a)\lambda^+} \quad \text{and} \quad n_{\min} = N 2^{-\kappa(1-a)} 2^{\kappa(1-a)\lambda^-}. \tag{2.6}$$

We call λ^+ the *upper mass concentration factor* of Γ and λ^- the *lower mass concentration factor* of Γ since these numbers give information about the distribution of the rectangles in Figure 1 and about the structure of Γ.

REMARK 2.5. Clearly $\lambda^+ < 1$ since we assumed that $n_l \geqslant 1$ for every $l \in \{1, \ldots, K_1\}$. Note also that if $\lambda^+ = \lambda^- = 0$ then Γ is strongly regular according to Definition 2.1.

It is clear that for any $j \in \mathbb{N}$ there are $N^j = 2^{j\kappa d_A}$ rectangles of type $R_j = A_{m_1} \circ \ldots \circ A_{m_j} Q$, having side lengths $2^{-j\kappa(1-a)}$, $2^{-j\kappa(1+a)}$ belonging to $(AQ)^j$. Let R_j be such a rectangle. We subdivide R_j in rectangles E_{jl} having side lengths $2^{-(j+m)\kappa(1-a)}$ and $2^{-j\kappa(1+a)}$ such that

$$2^{-j\kappa(1+a)} \sim 2^{-(j+m)\kappa(1-a)}.$$

The rectangles E_{jl} are almost squares; it is immaterial for what follows to assume that E_{jl} are squares, which means that

$$2^{-j\kappa(1+a)} = 2^{-(j+m)\kappa(1-a)}. \tag{2.7}$$

The lemma below gives information about the mass concentration in E_{jl} ($l = 1, \ldots, 2^{j\kappa 2a}$).

LEMMA 2.6. *There exist constants $c_1, c_2 > 0$ such that the measure of any square E_{jl} can be estimated by*

$$c_1 2^{-j\kappa(d_A + 2a(1-\lambda^-))} \leqslant \mu(\Gamma \cap E_{jl}) \leqslant c_2 2^{-j\kappa(d_A + 2a(1-\lambda^+))}. \tag{2.8}$$

Proof. For $j \in \mathbb{N}$ let $\nu = j + m$ in (2.7). Then the rectangle R_j contains $N^{\nu - j}$ rectangles R_ν with side lengths $2^{-\nu\kappa(1-a)}$, $2^{-\nu\kappa(1+a)}$ belonging to $(AQ)^\nu$ which are obtained from R_j after $\nu - j$ steps of iteration.

Hence any square E_{jl} contains at most $n_{\max}^{\nu-j}$ of the rectangles R_ν; see Figure 2. Clearly $\mu(\Gamma \cap E_{jl}) \leqslant c n_{\max}^{\nu-j} \mu(\Gamma \cap R_\nu)$. By (2.4), (2.5), (2.6) and $\nu(1-a) = j(1+a)$ the estimate from above in (2.8) follows from

$$\mu(\Gamma \cap E_{jl}) \leqslant c N^{\nu - j} 2^{-(\nu - j)\kappa(1-a)(1-\lambda^+)} 2^{-\nu\kappa d_A} = c 2^{-j\kappa d_A} 2^{-j\kappa 2a(1-\lambda^+)}.$$

The estimate from below can be obtained in the same way. □

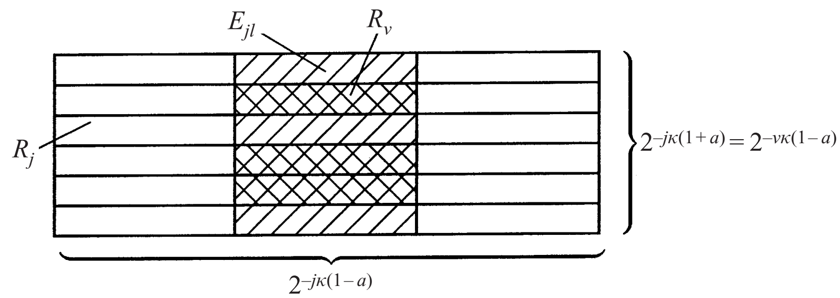

FIGURE 2.

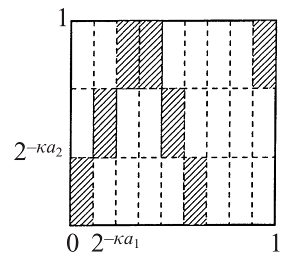

FIGURE 3.

As a simple consequence of $\nu(1-a) = j(1+a)$ we obtain the following corollary.

COROLLARY 2.7. *There exist constants $c_1, c_2 > 0$ such that for any square E_{jl} of side length $2^{-\nu\kappa(1-a)}$ we have*

$$c_1 2^{-\nu\kappa(1-a)\,d(\lambda^-)} \leqslant \mu(\Gamma \cap E_{jl}) \leqslant c_2 2^{-\nu\kappa(1-a)\,d(\lambda^+)} \tag{2.9}$$

where

$$d(\lambda^-) = \frac{d_A + 2a(1-\lambda^-)}{1+a}, \quad d(\lambda^+) = \frac{d_A + 2a(1-\lambda^+)}{1+a}.$$

REMARK 2.8. In the situation in Figure 1, by changing the roles of columns and rows we may suppose that in each column precisely one rectangle $A_m Q$ is located; then $1 < a_1 < 2$ (which is equivalent to $K_1 > K_2$). Let

$$\frac{K_1}{K_2} = 2k+1, \quad k \in \mathbb{N}$$

be an odd natural number and let the rectangles $A_m Q$ be arranged as depicted in Figure 3 where we choose in the counterpart of (2.1) always $\eta_2^m = 1$ and we choose $\eta_1^m = 1$ in the first K_2 columns, $\eta_1^m = -1$ in the second K_2 columns (additional reflection), then again $\eta_1^m = 1$ in the third K_2 columns and so on. Under these assumptions the resulting anisotropic fractal Γ is the graph of a continuous function; for a proof see [**16**, 4.21]. It is clear that Γ may be interpreted as a generalisation of Hironaka's curve (briefly presented in [**11**]). The Hausdorff dimension of Γ is $\dim_H \Gamma = 2 - a_2/a_1$; see [**16**, 4.22]. It is not difficult to see that Γ is a strongly regular anisotropic fractal with affine dimension $d_A = a_1 = 1-a$.

2.2. *The spaces* $L_p(\Gamma)$

Some preliminaries. We recall here the definition of Besov spaces on $\mathbb{R}^2$. Let φ_0 be a C^∞ function on $\mathbb{R}^2$, $\varphi_0(x) = 1$ if $|x| \leqslant 1$; $\operatorname{supp}\varphi_0 \subset \{x \in \mathbb{R}^2 : |x| \leqslant 2\}$ and let $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$ if $j \in \mathbb{N}$. Then $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ if $x \in \mathbb{R}^2$ and $(\varphi_j)_{j\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity.

Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $s \in \mathbb{R}$; the Besov space $B^s_{pq}(\mathbb{R}^2)$ consists of all tempered distributions $f \in S'(\mathbb{R}^2)$ for which the quasi-norm

$$\|f \,|\, B^s_{pq}(\mathbb{R}^2)\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^\vee \,|\, L_p(\mathbb{R}^2)\|^q\right)^{1/q}$$

(with the usual modification if $q = \infty$) is finite. Here $\hat{g} = Fg$ and $\check{g} = F^{-1}g$ are respectively the Fourier and inverse Fourier transform on $S'(\mathbb{R}^2)$. These are quasi-Banach spaces (Banach spaces if $p \geqslant 1$ and $q \geqslant 1$) which are independent of the choice of $(\varphi_j)_{j\in\mathbb{N}_0}$. The space $H^s_2(\mathbb{R}^2) = B^s_{22}(\mathbb{R}^2)$ is the fractional Sobolev space. Function spaces of $F^s_{pq}(\mathbb{R}^2)$ type, $0 < p < \infty$, are defined by changing the roles of the spaces $L_p(\mathbb{R}^2)$ and l_q in the definition above but we do not stress this point here.

A systematic treatment of the theory of $B^s_{pq}(\mathbb{R}^2)$ and $F^s_{pq}(\mathbb{R}^2)$ spaces may be found in **[14]** and **[15]**; for a more recent account we refer also to **[4]** and **[12]**. These two scales of function spaces include many well-known classical spaces such as Sobolev spaces, Hölder–Zygmund spaces and inhomogeneous Hardy spaces.

The structure theorem. If Γ is a closed set with Lebesgue measure $|\Gamma| = 0$ and if $s \in \mathbb{R}$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ then we define

$$B^{s,\Gamma}_{pq}(\mathbb{R}^2) = \{f \in B^s_{pq}(\mathbb{R}^2) : f(\varphi) = 0 \text{ if } \varphi \in S(\mathbb{R}^2),\ \varphi \,|\, \Gamma = 0\}$$

where $\varphi \,|\, \Gamma$ is the restriction of φ to Γ. We have $\operatorname{supp} f \subset \Gamma$ if $f \in B^{s,\Gamma}_{pq}(\mathbb{R}^2)$ in any case. Note also that if $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ and $s > 2((1/p) - 1)_+$ (if $b \in \mathbb{R}$ then $b_+ = \max(b, 0)$) then $B^s_{pq}(\mathbb{R}^2) \hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^2)$ [**15**, Remark 2.3.2/3] and hence $B^{s,\Gamma}_{pq}(\mathbb{R}^2) = \{0\}$ is trivial. In other words, only values $s \leqslant 2((1/p) - 1)_+$ (in particular $s \leqslant 0$ if $1 \leqslant p \leqslant \infty$) are of interest.

Let Γ be the regular anisotropic fractal constructed above. The L_p-spaces on Γ, $0 < p \leqslant \infty$, are introduced in the usual way with respect to the underlying Radon measure μ on Γ according to Theorem 2.3. If $1 \leqslant p \leqslant \infty$ then any $f_\Gamma \in L_p(\Gamma)$ can be interpreted as a tempered distribution $f \in S'(\mathbb{R}^2)$ given by

$$f(\varphi) = \int_\Gamma f_\Gamma(\gamma)\,(\varphi \,|\, \Gamma)(\gamma)\, d\mu(\gamma), \quad \varphi \in S(\mathbb{R}^2). \tag{2.10}$$

THEOREM 2.9. *Let Γ be a regular anisotropic fractal with upper mass concentration factor λ^+ and affine dimension d_A. If $1 \leqslant p \leqslant \infty$ and $(1/p) + (1/p') = 1$ then (in the sense of* (2.10))

$$L_p(\Gamma) \hookrightarrow B^{-(2-d(\lambda^+))/p'}_{p\infty}(\mathbb{R}^2), \quad d(\lambda^+) = \frac{d_A + 2a(1-\lambda^+)}{1+a}. \tag{2.11}$$

Moreover, if Γ is strongly regular and if $1 < p \leqslant \infty$ then (in the sense of (2.10))

$$L_p(\Gamma) = B^{-(2-d)/p',\Gamma}_{p\infty}(\mathbb{R}^2), \quad d = \frac{d_A + 2a}{1+a}. \tag{2.12}$$

Proof. We consider a square $Q(x,t)$ centred at $x \in \mathbb{R}^2$ and with side length $2^{-\nu\kappa(1-a)}$; by the right-hand side of (2.9) we have $\mu(\Gamma \cap Q(x,t)) \leqslant ct^{d(\lambda^+)}$.

To prove (2.11) one has now to follow the lines of the proof of [**16**, Theorem 18.15/Step 1] making the above modification.

If Γ is strongly regular then $d(\lambda^+) = d$ and Γ becomes an isotropic d-set; then (2.12) is in fact [**17**, Theorem 2/(8)]. □

Traces. Assume that Γ is a regular anisotropic fractal; if $\varphi \in S(\mathbb{R}^2)$ then $\mathrm{tr}_\Gamma \varphi = \varphi | \Gamma$ makes sense pointwise. If $0 < p,\ q < \infty$ and $s \in \mathbb{R}$ then $\mathrm{tr}_\Gamma B^s_{pq}(\mathbb{R}^2) \hookrightarrow L_p(\Gamma)$ must be understood as follows. There exists a positive number $c > 0$ such that for any $\varphi \in S(\mathbb{R}^2)$

$$\|\mathrm{tr}_\Gamma \varphi \,|\, L_p(\Gamma)\| \leqslant c \|\varphi \,|\, B^s_{pq}(\mathbb{R}^2)\|.$$

Since $S(\mathbb{R}^2)$ is dense in $B^s_{pq}(\mathbb{R}^2)$ this inequality can be extended by completion to any $f \in B^s_{pq}(\mathbb{R}^2)$ and the resulting function is denoted $\mathrm{tr}_\Gamma f$.

In addition, the equality $\mathrm{tr}_\Gamma B^s_{pq}(\mathbb{R}^2) = L_p(\Gamma)$ means that any $f_\Gamma \in L_p(\Gamma)$ is the trace of a suitable $g \in B^s_{pq}(\mathbb{R}^2)$ on Γ and

$$\|f_\Gamma \,|\, L_p(\Gamma)\| \sim \inf\{\|g \,|\, B^s_{pq}(\mathbb{R}^2)\| : \mathrm{tr}_\Gamma g = f_\Gamma\}.$$

THEOREM 2.10. *Let Γ be a regular anisotropic fractal with upper mass concentration factor λ^+ and affine dimension d_A. If $1 \leqslant p \leqslant \infty$ then*

$$\mathrm{tr}_\Gamma B^{(2-d(\lambda^+))/p}_{p1}(\mathbb{R}^2) \hookrightarrow L_p(\Gamma), \quad d(\lambda^+) = \frac{d_A + 2a(1-\lambda^+)}{1+a}. \tag{2.13}$$

Moreover, if Γ is strongly regular and if $1 \leqslant p < \infty$ then

$$\mathrm{tr}_\Gamma B^{(2-d)/p}_{p1}(\mathbb{R}^2) = L_p(\Gamma), \quad d = \frac{d_A + 2a}{1+a}. \tag{2.14}$$

Proof. If $p = \infty$ we have $B^0_{\infty 1}(\mathbb{R}^2) \hookrightarrow C(\mathbb{R}^2)$ and (2.13) is obvious (here $C(\mathbb{R}^2)$ is the space of all uniformly continuous bounded functions on $\mathbb{R}^2$). To prove (2.13) for $p < \infty$ one has to repeat the arguments from [**16**, Theorem 18.15/Step 2] with $d(\lambda^+)$ instead of $d_A/(1+a)$. In addition, (2.14) is [**17**, Theorem 2/(9)]. □

Theorems 2.9 and 2.10 pave the way to our main result (which is presented in the next section) but we hope they are also of independent interest. They are the anisotropic counterparts of [**17**, Theorems 2 and 3] (see also [**16**, 18.2, 18.6]), and complement the results from [**16**, 18.15, 18.17].

3. *The main result*

As usual, Ω stands for a bounded domain in $\mathbb{R}^2$ with C^∞ boundary and $D'(\Omega)$ denotes the space of all complex-valued distributions on Ω. Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $s \in \mathbb{R}$; the space $B^s_{pq}(\Omega)$ is defined as the restriction of $B^s_{pq}(\mathbb{R}^2)$ to Ω, that is,

$$B^s_{pq}(\Omega) = \{f \in D'(\Omega) : \text{there exists a } g \in B^s_{pq}(\mathbb{R}^2) \text{ with } g|\Omega = f\},$$
$$\|f \,|\, B^s_{pq}(\Omega)\| = \inf \|g \,|\, B^s_{pq}(\mathbb{R}^2)\|$$

where the infimum is taken over all $g \in B^s_{pq}(\mathbb{R}^2)$ such that its restriction to Ω, denoted by $g|\Omega$, coincides in $D'(\Omega)$ with f. In particular $B^1_{22}(\Omega) = W^1_2(\Omega)$.

In the sequel $\Gamma \subset \Omega$ will be a regular anisotropic fractal and we shall not distinguish between f_Γ as an element of some $L_p(\Gamma)$ and as the distribution belonging to some $B^{-s}_{p\infty}(\Omega)$ according to (2.11).

To avoid any misunderstanding we emphasise that the trace operator has two different meanings which we distinguish by tr_Γ and tr^Γ if extra clarity is desirable. If, for example, $1 < p < \infty$, then

$$\mathrm{tr}_\Gamma : B^{(2-d(\lambda^+))/p}_{p1}(\Omega) \longrightarrow L_p(\Gamma) \tag{3.1}$$

by (2.13) and

$$\mathrm{tr}^\Gamma : B^{(2-d(\lambda^+))/p}_{p1}(\Omega) \longrightarrow B^{-(2-d(\lambda^+))/p'}_{p\infty}(\Omega) \tag{3.2}$$

if one also applies (2.11). The latter can be rephrased by asking for an optimal extension of tr^Γ considered as a mapping from $D(\Omega)$ into $D'(\Omega)$ given by (1.1).

Recall that $(-\Delta)^{-1}$ stands for the inverse of the Dirichlet Laplacian in Ω.

THEOREM 3.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with C^∞ boundary. Let $\Gamma \subset \Omega$ be a regular anisotropic fractal having respectively upper and lower mass concentration factors λ^+ and λ^- according to* (2.6) *and having affine dimension d_A according to* (2.4). *Let*

$$d(\lambda^-) = \frac{d_A + 2a(1-\lambda^-)}{1+a}, \quad d(\lambda^+) = \frac{d_A + 2a(1-\lambda^+)}{1+a}$$

and

$$d = \frac{d_A + 2a}{1+a}. \tag{3.3}$$

Let tr^Γ *be the trace operator in the interpretation* (3.2) *and* (1.1), *whereas* tr_Γ *stands for the trace operator according to* (3.1).

(i) *The operator $T = (-\Delta)^{-1} \circ \mathrm{tr}^\Gamma$ is compact, non-negative, self-adjoint in $\mathring{W}^1_2(\Omega)$, has null space $N(T) = \{f \in \mathring{W}^1_2(\Omega) : \mathrm{tr}_\Gamma f = 0\}$ and is generated by the quadratic form in $\mathring{W}^1_2(\Omega)$*

$$\int_\Gamma f(\gamma)\,\overline{g(\gamma)}\,d\mu(\gamma) = (Tf, g)_{W^1_2(\Omega)} \tag{3.4}$$

where $f \in \mathring{W}^1_2(\Omega)$, $g \in \mathring{W}^1_2(\Omega)$ and μ is the Radon measure according to Theorem 2.3.

(ii) *There exist constants $c_1, c_2 > 0$ such that the positive eigenvalues $\lambda_k(T)$ of T, repeated according to multiplicity and ordered by their magnitude, can be estimated by*

$$c_1 k^{-d(\lambda^-)/d} \leqslant \lambda_k(T) \leqslant c_2 k^{-d(\lambda^+)/d}, \quad k \in \mathbb{N}. \tag{3.5}$$

Furthermore, if Γ is strongly regular then there are constants $c_1, c_2 > 0$ such that $c_1 k^{-1} \leqslant \lambda_k(T) \leqslant c_2 k^{-1}$.

The proof of the theorem is in Section 5 but here we make some comments.

REMARK 3.2. According to Definition 2.4 and Remark 2.5 we have $0 \leqslant \lambda^+ < 1$ and this implies that $d(\lambda^+)/d > d_A/(d_A + 2a)$. Hence the estimate from above in (3.5) is an improvement of the estimate from above in (1.2).

REMARK 3.3. If $-1 < \lambda^- \leqslant 0$ then $d(\lambda^-)/d < (d_A + 2a)/d_A$ and so the estimate from below obtained in (3.5) is better than the estimate from below in (1.2).

REMARK 3.4. By [**16**, 30.2] (isotropic fractal drum) it is not a surprise that if $\lambda^+ = \lambda^- = 0$ then the two exponents in (3.5) are both -1 since in this case the regular anisotropic fractal Γ becomes an isotropic d-set, where d is the number from (3.3).

4. *Prerequisites for the proof of Theorem* 3.1

4.1. *Basic facts about entropy numbers and approximation numbers*

Let A and B be two quasi-Banach spaces and let $T: A \longrightarrow B$ be linear. Just as for the Banach space case, T will be called bounded or continuous if

$$\|T\| := \sup\{\|Tx \,|\, B\| : x \in A, \|x \,|\, A\| \leqslant 1\} < \infty.$$

The family of all such T will be denoted by $L(A, B)$ or $L(A)$ if $A = B$. Otherwise terminology which is standard in the context of Banach spaces will be taken over without special comments to the quasi-Banach situation. If B is a quasi-Banach space then $U_B = \{b \in B: \|b \,|\, B\| \leqslant 1\}$ stands for the unit ball in B.

DEFINITION 4.1. Let A, B be two quasi-Banach spaces and let $T \in L(A, B)$. Then for all $k \in \mathbb{N}$, the kth entropy number $e_k(T)$ of T is defined by

$$e_k(T) = \inf\left\{\varepsilon > 0 : T(U_A) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon U_B) \text{ for some } b_1, \ldots, b_{2^{k-1}} \in B\right\}.$$

This formulation, which simply generalises to quasi-Banach spaces what has previously been done for Banach spaces, coincides with the definition given in [**4**, 1.3.1] where the reader can find further comments and historical references.

LEMMA 4.2. *Let A, B, C be quasi-Banach spaces, let $T \in L(A, B)$ and let $V \in L(B, C)$.*

(i) $\|T\| \geqslant e_1(T) \geqslant e_2(T) \geqslant \ldots \geqslant 0$*;* $e_1(T) = \|T\|$ *if B is a Banach space.*

(ii) *For all $k, l \in \mathbb{N}$,*

$$e_{k+l-1}(V \circ T) \leqslant e_k(V)\, e_l(T). \tag{4.1}$$

A proof can be found in [**4**, Lemma 1.3.1/1]. In the case of quasi-Banach spaces it may happen that $\|T\| > e_1(T)$.

REMARK 4.3. Since the numbers $e_k(T)$ decrease as k increases and are non-negative, $\lim_{k\to\infty} e_k(T)$ exists and plainly equals

$$\inf\{\varepsilon > 0 : T(U_A) \text{ can be covered by finitely many } B\text{-balls of radius } \varepsilon\}.$$

Recall that $T \in L(A, B)$ is compact if and only if for every $\varepsilon > 0$ there is a finite ε-net in B covering $T(U_A)$. Hence $T \in L(A, B)$ is compact if and only if $\lim_{k\to\infty} e_k(T) = 0$.

Let A be a complex quasi-Banach space and $T \in L(A)$ be a compact map. We know from [**4**, Theorem 1.2] that the spectrum of T, apart from the point 0, consists solely of eigenvalues of finite algebraic multiplicity; let $\{\lambda_k(T) : k \in \mathbb{N}\}$ be the sequence of all non-zero eigenvalues of T, repeated according to algebraic multiplicity and ordered so that

$$|\lambda_1(T)| \geqslant |\lambda_2(T)| \geqslant \ldots \geqslant 0. \tag{4.2}$$

If T has only $m(<\infty)$ distinct eigenvalues and M is the sum of their algebraic multiplicities, we put $\lambda_k(T)=0$ for $k>M$.

Perhaps the most useful connection for our purposes between the eigenvalues of the operator T and its entropy numbers is the following theorem.

THEOREM 4.4. *Let T and $\{\lambda_k(T): k\in\mathbb{N}\}$ be as above. Then*

$$|\lambda_k(T)| \leqslant \sqrt{2}\, e_k(T). \tag{4.3}$$

A proof of this result, originally proved by Carl in [**1**] (see also [**2**]) is given in [**4**, Theorem 1.3.4].

DEFINITION 4.5. Let A, B be two quasi-Banach spaces and let $T\in L(A,B)$. Then given any $k\in\mathbb{N}$, the kth approximation number $a_k(T)$ of T is defined by

$$a_k(T) = \inf\{\|T-L\| : L\in L(A,B), \operatorname{rank} L < k\}$$

where $\operatorname{rank} L$ is the dimension of the range of L.

These numbers have various properties similar to those of the entropy numbers: we have $\|T\| = a_1(T) \geqslant a_2(T) \geqslant \ldots \geqslant 0$ and a counterpart of the multiplication property (4.1) for entropy numbers; see [**4**, Lemma 1.3.1/2]. On the other hand there are radical differences between entropy numbers and approximation numbers (see [**4**, Remark 1.3.2/6] and [**3**, II.2.3]), but we do not go into further details.

The approximation numbers have important connections with eigenvalues, the picture being clearest in a Hilbert space setting.

THEOREM 4.6. *Let H be a Hilbert space and let $T\in L(H)$ be a compact, non-negative and self-adjoint operator. Then the approximation numbers $a_k(T)$ of T coincide with its eigenvalues* (*ordered as in* (4.2)).

A proof can be found in [**3**, II.5.10]; see also [**4**, p. 21].

4.2. *Compactness of embeddings into $L_p(\Gamma)$*

We will use the following improvement of [**16**, 22.2].

THEOREM 4.7. *Let Γ be a regular anisotropic fractal having upper mass concentration factor λ^+, affine dimension d_A and let $d=(d_A+2a)/(1+a)$. Let $0<p_1\leqslant p_2\leqslant\infty$, $0<q\leqslant\infty$ and*

$$s(p_1,p_2) = \frac{2}{p_1} - \frac{d(\lambda^+)}{p_2}. \tag{4.4}$$

If $s>0$ then the trace operator

$$\operatorname{tr}_\Gamma : B^{s(p_1,p_2)+s}_{p_1 q}(\mathbb{R}^2) \longrightarrow L_{p_2}(\Gamma)$$

is compact and there exists a constant $c>0$ such that for any $k\in\mathbb{N}$,

$$e_k(\operatorname{tr}_\Gamma : B^{s(p_1,p_2)+s}_{p_1 q}(\mathbb{R}^2) \longrightarrow L_{p_2}(\Gamma)) \leqslant c k^{-(s/d)-(1/p_1)+(1/p_2)}.$$

Proof. One has only to repeat the arguments from [**16**, 22.2] using (4.4) instead of [**16**, (22.1)]. □

5. *Proof of Theorem* 3.1

Step 1: Part (i) of the theorem is covered by [**16**, Theorem 30.7]. In particular from [**16**, (30.25)] it follows that there exists a constant $c > 0$ such that

$$\|\mathrm{tr}_\Gamma f \,|\, L_2(\Gamma)\| \leqslant c\|f \,|\, W_2^1(\Omega)\| \tag{5.1}$$

for any $f \in \mathring{W}_2^1(\Omega)$ and from (3.4) we have

$$\|\mathrm{tr}_\Gamma f \,|\, L_2(\Gamma)\| = \|\sqrt{T} f \,|\, W_2^1(\Omega)\| \tag{5.2}$$

where $\sqrt{T} = T^{1/2}$.

Step 2: To prove the estimate from above in (3.5) we factorise the operator T by $T = \mathrm{id}_2 \circ (-\Delta)^{-1} \circ \mathrm{id}_1 \circ \mathrm{tr}_\Gamma$ with

$$\begin{aligned} \mathrm{tr}_\Gamma &: \mathring{W}_2^1(\Omega) \longrightarrow L_2(\Gamma) \\ \mathrm{id}_1 &: L_2(\Gamma) \longrightarrow B_{2\infty}^{-(2-d(\lambda^+))/2}(\Omega) \\ (-\Delta)^{-1} &: B_{2\infty}^{-(2-d(\lambda^+))/2}(\Omega) \longrightarrow B_{2\infty}^{2-(2-d(\lambda^+))/2}(\Omega) \\ \mathrm{id}_2 &: B_{2\infty}^{2-(2-d(\lambda^+))/2}(\Omega) \longrightarrow \mathring{W}_2^1(\Omega). \end{aligned} \tag{5.3}$$

The boundedness of tr_Γ in the first line of (5.3) is (5.1) and the embedding id_1 is (2.11).

Recall that $(-\Delta)$ maps any space $B_{pq,0}^s(\Omega) = \{g \in B_{pq}^s(\Omega) : \mathrm{tr}_{\partial\Omega} g = 0\}$ onto $B_{pq}^{s-2}(\Omega)$ provided $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ and $s > 1/p$. This is a consequence of [**13**, 5.7.1, Remark 1] complemented by [**14**, 4.3.3–4.3.4]. By this mapping property the boundedness of $(-\Delta)^{-1}$ as indicated in the third line in (5.3) is justified. Finally, the embedding id_2 is a consequence of the elementary embedding from [**14**, Proposition 2.3.2/2].

Let $f \in \mathring{W}_2^1(\Omega)$ be an eigenfunction of T. Then it follows that f belongs also to $B_{2\infty}^{2-(2-d(\lambda^+))/2}(\Omega)$ and so it is also an eigenfunction of the operator T restricted to this space. Obviously the converse is also true. Hence the root systems of T considered in $W_2^1(\Omega)$ (or $\mathring{W}_2^1(\Omega)$ which is the same in our context) and in $B_{2\infty}^{2-(2-d(\lambda^+))/2}(\Omega)$ coincide. Then the eigenvalues of T considered in these spaces also coincide, including their multiplicities. Using the multiplication property (4.1) for entropy numbers and (5.3) there exists a constant $c > 0$ such that for all $k \in \mathbb{N}$,

$$e_k(T : B_{2\infty}^{2-(2-d(\lambda^+))/2}(\Omega) \longrightarrow B_{2\infty}^{2-(2-d(\lambda^+))/2}(\Omega)) \leqslant c e_k(\mathrm{tr}_\Gamma : B_{2\infty}^{2-(2-d(\lambda^+))/2}(\Omega) \longrightarrow L_2(\Gamma)). \tag{5.4}$$

From Theorem 4.7 (with $p_1 = p_2 = 2$) if $s > 0$ then there exists a constant $c > 0$ such that for all $k \in \mathbb{N}$,

$$e_k(\mathrm{tr}_\Gamma : B_{2\infty}^{s+(2-d(\lambda^+))/2}(\mathbb{R}^2) \longrightarrow L_2(\Gamma)) \leqslant c k^{-s/d}. \tag{5.5}$$

Inserting $s = d(\lambda^+)$ in (5.5) and using Carl's inequality (4.3), from (5.4) we obtain the estimate from above in (3.5).

Step 3: We prove that there exists a constant $c > 0$ such that the approximation numbers $a_k(\sqrt{T})$ of $\sqrt{T}$ can be estimated by

$$a_k(\sqrt{T}) \geqslant c k^{-d(\lambda^-)/2d}, \quad k \in \mathbb{N}. \tag{5.6}$$

The estimate from below in (3.5) is then a simple application of Theorem 4.6.

We rely on Lemma 2.6 and Figure 2 assuming again (2.7) without restriction of generality. By (2.4) we have $N = 2^{\kappa d_A}$ and hence by (3.3) for any $j \in \mathbb{N}$ there are

$$2^{2j\kappa a} N^j = 2^{2j\kappa a + j\kappa d_A} = 2^{j\kappa(1+a)\,d}$$

squares E_{jl} (of side length $2^{-j\kappa(1+a)}$). We put $m = m(j) = j\kappa(1+a)$ and denote the corresponding squares E_l^m. Of course $m = m(j)$ need not to be a natural number but this is immaterial for what follows. In other words, the disjoint squares E_l^m ($l = 1, \ldots, 2^{md}$) of side length 2^{-m} cover Γ and originate from the squares E_{jl} in Lemma 2.6.

Let $x^{m,l}$ be the centre of the square E_l^m. Let φ be an appropriately chosen non-trivial C^∞ function on $\mathbb{R}^2$ supported near the origin. If

$$\varphi_{ml}(x) = \varphi(2^m(x - x^{m,l}))$$

then $\operatorname{supp} \varphi_{ml} \subset E_l^m$. Furthermore, as a simple consequence of (2.9), there exists a constant $c > 0$ such that

$$\left\| \sum_{l=1}^{2^{md}} c_{ml}\, \varphi_{ml} \mid L_2(\Gamma) \right\| \geqslant c 2^{-md(\lambda^-)/2} \left(\sum_{l=1}^{2^{md}} |c_{ml}|^2 \right)^{1/2} \tag{5.7}$$

for any complex numbers c_{ml} and for any m and $l = 1, \ldots, 2^{md}$.

By (5.2) and (5.7) there exists a constant $c > 0$ such that

$$\| \sqrt{T} g_m \mid W_2^1(\Omega) \| \geqslant c 2^{-md(\lambda^-)/2} \left(\sum_{l=1}^{2^{md}} |c_{ml}|^2 \right)^{1/2} \tag{5.8}$$

for any function g_m of type

$$g_m = \sum_{l=1}^{2^{md}} c_{ml}\, \varphi_{ml},$$

the constant $c > 0$ in (5.8) being independent of m, l and the complex numbers c_{ml}. Then there exists a number $c > 0$ which is independent of m such that

$$a_{2^{md}}(\sqrt{T}) \geqslant c 2^{-md(\lambda^-)/2}. \tag{5.9}$$

Using elementary properties of approximation numbers it is easy to see that (5.9) implies (5.6) and this completes the proof of the estimate from below in (3.5).

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Bucharest University
Faculty of Mathematics
14. Academiei Str.
RO-70109 Bucharest
Romania

Friedrich Schiller Universität Jena
Mathematisches Institut
Ernst-Abbe-Platz 1–4
D-07740 Jena
Germany

Current address:
Institut für Theoretische Informatik und Mathematik
Universität der Bundeswehr
Werner-Heisenberg-Weg 39
D-85 577 Neubiberg
Germany