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Atomic and Subatomic Decompositions in Anisotropic Function Spaces

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Abstract. This work deals with decompositions in anisotropic function spaces. Defining anisotropic atoms as smooth building blocks which are the counterpart of the atoms from the works of M. Frazier and B. Jawerth, it is shown that the study of anisotropic function spaces can be done with the help of some sequence spaces in a similar way as it is done in the isotropic case. It is also shown that the subatomic decomposition theorem for isotropic function spaces, recently proved by H. Triebel, can be extended to the anisotropic case if the mean smoothness parameter is sufficiently large.

1. Introduction

If $1 < p < \infty$ and $(s_1, \ldots, s_n)$ is an $n-$tuple of natural numbers then

$$W_p^{s,a}(\mathbb{R}^n) \;=\; \left\{ f \in S'(\mathbb{R}^n) \,:\, \|f \,|\, L_p(\mathbb{R}^n)\| + \sum_{k=1}^{n} \left\| \frac{\partial^{s_k} f}{\partial x_k^{s_k}} \,\right|\, L_p(\mathbb{R}^n) \right\| < \infty \right\}$$

is the classical anisotropic Sobolev space on $\mathbb{R}^n$. In contrast to the usual (isotropic) Sobolev space ($s_1 = \cdots = s_n$) the smoothness properties of an element from $W_p^{s,a}(\mathbb{R}^n)$ depend on the chosen direction in $\mathbb{R}^n$. The number s defined by

$$\frac{1}{s} \;=\; \frac{1}{n}\left(\frac{1}{s_1} + \cdots + \frac{1}{s_n}\right)$$

is usually called the "mean smoothness" and $a = (a_1, \ldots, a_n)$, where $a_1 = s/s_1$, ..., $a_n = s/s_n$, characterizes the anisotropy.

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Anisotropic Bessel potential spaces, or fractional Sobolev spaces, defined by

$$H_p^{s,a}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \left\| \left(\sum_{k=1}^{n} \left(1+\xi_k^2\right)^{s/(2a_k)} \widehat{f} \right)^{\vee} \middle| L_p(\mathbb{R}^n) \right\| < \infty \right\}$$

where $1 < p < \infty$, $s \in \mathbb{R}$ and $a = (a_1, \ldots, a_n)$ is a given anisotropy, generalize in a natural way the above spaces (as usual, $S'(\mathbb{R}^n)$ is the space of tempered distributions and $\widehat{f}$, $\check{f}$ are respectively the Fourier and the inverse Fourier transform of f).

Similar to the isotropic case, the study of anisotropic Bessel potential spaces $H_p^{s,a}(\mathbb{R}^n)$ for a fixed anisotropy $a = (a_1, \ldots, a_n)$, is a part of the more general theory of the spaces of $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ type. Spaces of that type (or on domains in $\mathbb{R}^n$) have been studied in great detail by S. M. Nikol'skij, [Nik77], and by O. V. Besov, V. P. Il'in and S. M. Nikol'skij, [BIN75], and it is well known that this theory has a more or less full counterpart to the basic facts (definitions, elementary properties, embeddings for different metrics, interpolation) of isotropic spaces $B_{pq}^{s}(\mathbb{R}^n)$ and $F_{pq}^{s}(\mathbb{R}^n)$ as it was presented in the works of H. Triebel, [Tri83] and [Tri92].

The anisotropic function spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ are defined in terms of Fourier analytical quasi – norms: any function $f \in S'(\mathbb{R}^n)$ is decomposed in a sum of entire analytic functions $\left(\varphi_j \widehat{f}\,\right)^{\vee}$ and this decomposition is used to introduce the spaces.

Hence, as in the isotropic case, entire analytic functions may be considered as building blocks for the spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ in the following sense: let a problem be given, for example mapping properties for PDE's or ψDE's between spaces of the above type or traces on hyperplanes, etc. First one asks what happens when the problem is applied to entire analytic functions; then the rest reduces to a discussion of convergence.

In the theory of isotropic function spaces there is a well known other type of decomposition in simple building blocks, the so – called atoms. They have a history of some twenty years and in [Tri92, 1.9], cf. also [AdH96], a historical report was given on this topic; we do not repeat it here. We only want to mention that the (smooth) atoms in isotropic $B_{pq}^{s}(\mathbb{R}^n)$ and $F_{pq}^{s}(\mathbb{R}^n)$ spaces as they were defined by M. Frazier and B. Jawerth in [FrJ85], [FrJ90] (cf. also [FJW91]), proved to be a powerful tool in the theory of function spaces. We also wish to emphasize that there exist many other types of atomic decompositions in isotropic spaces but we will not discuss this point here.

More information about this subject is given in [FrJ90], [Tri92] and [AdH96] where one can find many modifications and applications as well as comprehensive references extending the subject.

Several authors were concerned in the last years with the problem of obtaining useful decompositions of anisotropic function spaces in simple building blocks: a construction of unconditional bases in $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ spaces using Meyer wavelets was done by M. Z. Berkolaiko and I. Ya. Novikov in [BeN93] (and used then in [BeN95]); in [Din95a, Theorem 1] P. Dintelmann obtained a decomposition for anisotropic function spaces which is the counterpart of the characterization of isotropic function spaces with the help of the φ – transform of M. Frazier and B. Jawerth (see [FrJ90] and the survey [FJW91]) and used it in connection with the theory of

Fourier – multipliers for anisotropic function spaces (we will return to his result in Section 5). Our approach will be different, especially from the point of view of the localization of the building blocks.

The first aim of this paper is to introduce smooth anisotropic atoms and to obtain a decomposition theorem which extends the atomic decomposition theorem of M. Frazier and B. Jawerth, see [FrJ85] and [FrJ90], to the anisotropic function spaces $B^{s,a}_{pq}(\mathbb{R}^n)$ and $F^{s,a}_{pq}(\mathbb{R}^n)$. Roughly speaking, we will show that for any $g \in F^{s,a}_{pq}(\mathbb{R}^n)$ it is possible to find a decomposition (convergence in $S'(\mathbb{R}^n)$)

$$g = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}\, \rho^a_{\nu m} , \tag{1.1}$$

where $\rho^a_{\nu m}$ are the anisotropic atoms and $\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0 , \ m \in \mathbb{Z}^n\}$ belongs to an appropriate sequence space f^a_{pq}, such that

$$\big\| g \,|\, F^{s,a}_{pq}(\mathbb{R}^n) \big\| \ \sim \ \big\| \lambda \,|\, f^a_{pq} \big\| \tag{1.2}$$

(and a similar assertion for $B^{s,a}_{pq}(\mathbb{R}^n)$ spaces).

Hence the study of function spaces can be done with the help of some sequence spaces in an analogous way as it is done in the isotropic case in the above cited works of M. Frazier and B. Jawerth. The necessary explanations and details are given in Section 3.

However in (1.1) (and in Theorem 3.3) no information is given about the possibility to obtain atomic decompositions in which the atoms are constructed with the help of (anisotropic) dilatations and translations from one smooth function ρ having compact support, cf. also [BeN93, Comment 2].

For isotropic function spaces this was already done by M. Frazier and B. Jawerth, see [FrJ90, 4.2], and W. Sickel, see [Sic90]. It might be possible to extend the technique of W. Sickel, at least for large values of the smoothness parameter, using the characterization of anisotropic function spaces via oscillation from the work of A. Seeger, [See89]. But to construct such a basic (or mother) function ρ for the atoms having all required properties seems to be not very easy, at least at the first glance, see the above cited papers.

We arrive at the second aim of this paper, the subatomic (or quarkonial) decomposition theorem (Theorem 3.7) which states that given $g \in F^{s,a}_{pq}(\mathbb{R}^n)$ (with s sufficiently large) it is possible to obtain the decomposition

$$g = \sum_{\beta \in \mathbb{N}^n_0} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m}\, (\beta qu)^a_{\nu m} , \tag{1.3}$$

convergence being in $S'(\mathbb{R}^n)$, with

$$\big\| g \,|\, F^{s,a}_{pq}(\mathbb{R}^n) \big\| \ \sim \ \sup_{\beta \in \mathbb{N}^n_0} 2^{ra\beta} \big\| \lambda^\beta \,|\, f^a_{pq} \big\| , \tag{1.4}$$

where $r > 0$ is large enough, $a\beta = a_1\beta_1 + \ldots + a_n\beta_n$ if β is the multi – index $(\beta_1, \ldots, \beta_n)$, $\lambda^\beta = \big\{\lambda^\beta_{\nu m} : \nu \in \mathbb{N}_0 , m \in \mathbb{Z}^n\big\}$ and where each $(\beta qu)^a_{\nu m}$ is an extremely simple building block (in particular an anisotropic atom without moment conditions), called anisotropic quark, compactly supported and which can be obtained starting from one smooth function (and a corresponding assertion for $B^{s,a}_{pq}(\mathbb{R}^n)$).

Of course in (1.3) there are infinitely many sums over $(\nu, m) \in \mathbb{N}_0 \times \mathbb{Z}^n$ but this is well compensated by (1.4) with r large. Furthermore, it turns out from the proof that the dependence of the coefficients $\lambda^{\beta}_{\nu m}$ on g is linear.

Isotropic quarks were recently introduced by H. Triebel in [Tri97] and the subatomic (quarkonial) decomposition theorem he obtained in [Tri97, Chapter 14] proved to be a very useful ingredient for the estimation of entropy numbers of compact embeddings between function spaces on fractals. Compared with the results in [Tri97, Chapter 14] our Theorem 3.7 is in fact the extension of quarkonial decompositions to anisotropic function spaces in the case of large values of the smoothness parameter, in particular for $s > 0$ if $p \geq 1$ and $q \geq 1$.

If one wishes to extend the result to all $s \in \mathbb{R}$ then a lifting argument would be needed. But while the lift operator $(id - \Delta)^{\frac{L+1}{2}}$ between isotropic function spaces causes no problem in keeping the localization of the (isotropic) quarks (and this fact was essentially used in [Tri97, 14.4]) the situation becomes difficult in case of the anisotropic lift operator. It is well known that if $\sigma \in \mathbb{R}$ then the operator

$$I_\sigma(f) \;=\; \left(\left(\sum_{k=1}^{n}\left(1+\xi_k^2\right)^{\frac{1}{2a_k}}\right)^{\sigma}\widehat{f}\right)^{\vee}$$

maps $F^{s,a}_{pq}(\mathbb{R}^n)$ isomorphically onto $F^{s-\sigma,a}_{pq}(\mathbb{R}^n)$ and $\left\|I_\sigma(\,\cdot\,) \mid F^{s-\sigma,a}_{pq}(\mathbb{R}^n)\right\|$ is an equivalent quasi−norm on $F^{s,a}_{pq}(\mathbb{R}^n)$ (similar result for $B^{s,a}_{pq}(\mathbb{R}^n)$), see [Leo86]; it is clear that I_σ causes a lot of troubles in keeping the localization of the anisotropic quarks and this is the reason why we will restrict ourselves to large values of the smoothness parameter.

The main results of this work, the atomic and the subatomic decomposition, seem to be of interest for their own sake but they also pave the way to some extensions of the results from [TrW96b] (cf. also [Tri97, 18.2;18.6]) to anisotropic d−sets as they are defined in [Tri97, 5.2;5.3]. If Γ is an anisotropic d−set we will shed new light on the spaces $L_p(\Gamma)$ and on the results from [Tri97, 18.7;22.2] using anisotropic function spaces. But we shift this task to a later paper.

Briefly about the organizing of the manuscript. In Section 2 we set up notation and terminology and summarize some basic facts on anisotropic function spaces. In Section 3 the main results are presented with comments but without proofs. Section 4 will be concerned with the extension to anisotropic function spaces of some powerful tools (especially a theorem on local means) from the isotropic case. All these results are used in Section 5 where we prove the results announced in Section 3.

2. Definitions and basic facts

2.1. Notation

As usual, $\mathbb{R}^n$ denotes the n−dimensional real euclidean space, $\mathbb{N}$ are the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{C}$ stands for the complex numbers.

If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is a multi−index its length is $|\alpha| = \alpha_1 + \cdots + \alpha_n$, the derivatives D^α have the usual meaning and if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ then $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex−valued rapidly decreasing C^∞ functions on $\mathbb{R}^n$ equipped with the usual topology. By $S'(\mathbb{R}^n)$ we denote its topological dual, the space of all tempered distributions on $\mathbb{R}^n$. If $\varphi \in S(\mathbb{R}^n)$ then $\widehat{\varphi} = F\varphi$ and $\check{\varphi} = F^{-1}\varphi$ are respectively the Fourier and inverse Fourier transform of φ. One extends F and F^{-1} in the usual way from $S(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$.

We adopt here and in the sequel the following convention: if there is no danger of confusion we omit $\mathbb{R}^n$ in $S(\mathbb{R}^n)$ and in the other spaces below.

For a normed or quasi−normed space X we denote by $\|x \,|\, X\|$ the norm of the vector x. Recall that X is quasi−normed when the triangle inequality is weakend to $\|x + y \,|\, X\| \le c\,(\|x \,|\, X\| + \|y \,|\, X\|)$ for some $c \ge 1$ independent of x and y.

If $0 < p \le \infty$ then L_p denotes the usual Lebesgue space on $\mathbb{R}^n$ quasi−normed by $\| \cdot \,|\, L_p\|$.

All unimportant positive constants are denoted with c, occasionally with additional subscripts within the same formulas. The equivalence "$\text{term}_1 \sim \text{term}_2$" means that there exist two constants $c_1, c_2 > 0$ such that $c_1\,\text{term}_1 \le \text{term}_2 \le c_2\,\text{term}_1$.

2.2. Anisotropic distance functions

Through the whole work $n \ge 2$ and $a = (a_1, \ldots, a_n)$ will designate a given anisotropy, that is a fixed n−tuple of positive numbers with $a_1 + \cdots + a_n = n$. We will denote $a_{\min} = \min\{a_i : 1 \le i \le n\}$ and $a_{\max} = \max\{a_i : 1 \le i \le n\}$. If $a = (1, \ldots, 1)$ we speak about the "isotropic case".

The action of $t \in [0, \infty)$ on $x \in \mathbb{R}^n$ is defined by the formula

$$t^a x \;=\; (t^{a_1} x_1, \ldots, t^{a_n} x_n)\;. \tag{2.1}$$

For $t > 0$ and $s \in \mathbb{R}$ we put $t^{sa}x = (t^s)^a x$. In particular we write $t^{-a}x = \left(t^{-1}\right)^a x$ and $2^{-ja}x = \left(2^{-j}\right)^a x$.

Definition 2.1. An *anisotropic distance function* is a continuous function $u : \mathbb{R}^n \to \mathbb{R}$ with the properties $u(x) > 0$ if $x \ne 0$ and $u(t^a x) = t u(x)$ for all $t > 0$ and all $x \in \mathbb{R}^n$.

It is easy to see that $u_\lambda : \mathbb{R}^n \to \mathbb{R}$ defined by

$$u_\lambda(x) \;=\; \left(\sum_{i=1}^n |x_i|^{\lambda/a_i} \right)^{1/\lambda} \tag{2.2}$$

is an anisotropic distance function for every $0 < \lambda < \infty$. If $x \in \mathbb{R}^n$ then $u_2(x)$ is usually called the anisotropic distance of x to the origin, see [ScT87, 4.2.1].

It is well known, see [Din95b, 1.2.3] and [Yam86, 1.4], that any two anisotropic distance functions u and u' are equivalent (in the sense that there exist constants $c, c' > 0$ such that $c\,u(x) \le u'(x) \le c'\,u(x)$ for all $x \in \mathbb{R}^n$) and that if u is an anisotropic distance function there exists a constant $c > 0$ such that $u(x + y) \le c\,(u(x) + u(y))$ for all $x, y \in \mathbb{R}^n$.

We are interested to use smooth anisotropic distance functions. Remark that for appropriate values of λ we can obtain arbitrary (finite) smoothness of the function u_λ from above (cf. [Din95b, 1.2.4]). A standard method concerning the construction of anisotropic distance functions in $C^\infty(\mathbb{R}^n \backslash \{0\})$ was given by E. M. Stein and S. Wainger in [StW78].

The lemma below will play an essential role in our considerations. Given the anisotropy $a = (a_1, \ldots, a_n)$ and the multi−index $\alpha = (\alpha_1, \ldots, \alpha_n)$ we use the notation $a\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n$.

Lemma 2.2. *There exists an anisotropic distance function* $|\cdot|_a \in C^\infty(\mathbb{R}^n \backslash \{0\})$ *with the following property: for any real number* s *and for any multi−index* α *there exists a constant* $c = c(s, \alpha) > 0$ *such that*

$$\left| D^\alpha \left(|x|_a^s \right) \right| \;\le\; c\, |x|_a^{s - a\alpha} \quad \textit{for all} \quad x \in \mathbb{R}^n \backslash \{0\}\,. \tag{2.3}$$

Proof. We have only to recall the construction of M. Yamazaki in [Yam86, 1.4/3,8]: for $x \neq 0$ one can define $|x|_a$ as the unique positive number t such that

$$\frac{x_1^2}{t^{2a_1}} + \cdots + \frac{x_n^2}{t^{2a_n}} \;=\; 1 \tag{2.4}$$

and then put $|0|_a = 0$ for $x = 0$. □

Given the anisotropy $a = (a_1, \ldots, a_n)$ through the whole work we will keep the notation $|\cdot|_a$ for a fixed anisotropic distance function as in the lemma above.

Let us remark that to work with an anisotropic distance function $|\cdot|_a$ satisfying (2.3) is natural since denoting $|\cdot|$ the euclidean distance in $\mathbb{R}^n$ for every real number s and for any multi−index α there exists a constant $c > 0$ such that $\left| D^\alpha \left(|x|^s \right) \right| \le c\, |x|^{s - |\alpha|}$ for all $x \in \mathbb{R}^n \backslash \{0\}$ (see [RuS96, Lemma 2.3.1/(20)]).

2.3. Anisotropic function spaces

Let φ_0 be a C^∞ function on $\mathbb{R}^n$ defined by $\varphi_0(x) = 1$ if $|x|_a \le 1$, such that $\operatorname{supp} \varphi_0 \subset \{x \in \mathbb{R}^n \,:\, |x|_a \le 2\}$ and $\varphi_j(x) = \varphi_0(2^{-ja}x) - \varphi_0\big(2^{(-j+1)a}x\big)$ if $j \in \mathbb{N}$. Then $\sum_{j=0}^\infty \varphi_j(x) = 1$ if $x \in \mathbb{R}^n$ and $\big(\varphi_j\big)_{j \in \mathbb{N}_0}$ is a smooth anisotropic dyadic resolution of unity, cf. [ScT87, 4.2].

For $f \in S'$ since $\varphi_j \widehat{f}$ is compactly supported the Paley−Wiener−Schwartz theorem provides that $\big(\varphi_j \widehat{f}\,\big)^\vee$ is an entire analytic function on $\mathbb{R}^n$.

Definition 2.3. (i) Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$; then

$$B^{s,a}_{pq} \;=\; \left\{ f \in S' \,:\, \left\| f \,|\, B^{s,a}_{pq} \right\| = \left(\sum_{j=0}^\infty 2^{jsq} \left\| \big(\varphi_j \widehat{f}\,\big)^\vee \,|\, L_p \right\|^q \right)^{1/q} < \infty \right\} \tag{2.5}$$

(with the usual modification if $q = \infty$).

(ii) Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$; then

$$(2.6)\quad F^{s,a}_{pq} = \left\{ f \in S' : \left\| f \mid F^{s,a}_{pq} \right\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \left| \left(\varphi_j \widehat{f} \right)^{\vee}(\cdot) \right|^q \right)^{1/q} \Big| L_p \right\| < \infty \right\}$$

(with the usual modification if $q = \infty$).

Of course the quasi – norms in (2.5) and (2.6) depend on the chosen system $\left(\varphi_j\right)_{j\in\mathbb{N}_0}$. But this is not the case for the spaces $B^{s,a}_{pq}$ and $F^{s,a}_{pq}$ (in the sense of equivalent quasi – norms) and that is the reason why we omit in our notation the subscript $\left(\varphi_j\right)_{j\in\mathbb{N}_0}$.

A systematic treatment of the theory of (isotropic) B^s_{pq} and F^s_{pq} spaces may be found in the books of H. Triebel [Tri83], [Tri92], for a more recent account of the theory we refer the reader also to [EdT96] and [RuS96]. A survey on the basic results for the (anisotropic) spaces $B^{s,a}_{pq}$ and $F^{s,a}_{pq}$ may be found in [ScT87, 4.2.1 – 4.2.4] and [Joh95, 2.1 – 2.2]. In this context we refer to the works of S. M. Nikol'skij [Nik77], O. V. Besov, V. P. Il'in and S. M. Nikol'skij [BIN75], B. Stöckert and H. Triebel [StT79], M. Yamazaki [Yam86], A. Seeger [See89], P. Dintelmann [Din95b, 1.2.8 – 1.2.10], etc.

An extension of (2.6) to $p = \infty$ is not reasonable; in [Tri92, 1.5.2] this point was discussed in detail.

Both $B^{s,a}_{pq}$ and $F^{s,a}_{pq}$ are quasi – Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$).

As in the isotropic case, see [Tri83, 2.3.3], the embeddings $S \hookrightarrow B^{s,a}_{pq} \hookrightarrow S'$ and $S \hookrightarrow F^{s,a}_{pq} \hookrightarrow S'$ hold true for all admissible values of p, q, s. If $s \in \mathbb{R}$ and $0 < p < \infty$, $0 < q < \infty$ then S is dense in $B^{s,a}_{pq}$ and $F^{s,a}_{pq}$, see [Yam86, 3.5] and [Din95b, 1.2.10].

We want to point out that if $1 < p < \infty$, $s \in \mathbb{R}$ then $F^{s,a}_{p2}$ is the anisotropic Bessel potential space $H^{s,a}_p$; a proof can be found in [StT79, Remark 11] (see also [Tri77, 2.5.2]) and in [Yam86, 3.11].

It will be very useful to remark that if denoting for each $k \in \{1, \ldots, n\}$, $s_k = s/a_k$ and

$$(2.7)\quad H^{s,a}_{x_k,p} = \left\{ f \in S' : \left\| f \mid H^{s,a}_{x_k,p} \right\| = \left\| \left(\left(1 + \xi_k^2\right)^{s_k/2} \widehat{f} \right)^{\vee} \Big| L_p \right\| < \infty \right\}$$

by [Nik77, 9.1] we have (in the sense of equivalent quasi – norms):

$$(2.8)\quad H^{s,a}_p = \bigcap_{k=1}^{n} H^{s,a}_{x_k,p} \quad \text{and} \quad \left\| f \mid H^{s,a}_p \right\| = \sum_{k=1}^{n} \left\| f \mid H^{s,a}_{x_k,p} \right\|$$

and if $s_k \in \mathbb{N}$ then (in the sense of equivalent quasi – norms)

$$(2.9)\quad H^{s,a}_{x_k,p} = \left\{ f \in S' : \left\| f \mid H^{s,a}_{x_k,p} \right\| = \| f \mid L_p \| + \left\| \frac{\partial^{s_k} f}{\partial x_k^{s_k}} \Big| L_p \right\| < \infty \right\}.$$

In particular, if $s_1, \ldots, s_n \in \mathbb{N}$ then (in the sense of equivalent quasi – norms) $H^{s,a}_p = W^{s,a}_p$ is the classical anisotropic Sobolev space.

3. The main results

3.1. Anisotropic atoms and the atomic decomposition theorem

Recall $a = (a_1, \ldots, a_n)$ is a given anisotropy and let $\mathbb{Z}^n$ be the lattice of all points in $\mathbb{R}^n$ with integer−valued components. If $\nu \in \mathbb{N}_0$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ we denote $Q^a_{\nu m}$ the rectangle in $\mathbb{R}^n$ centered at $2^{-\nu a}m = \big(2^{-\nu a_1}m_1, \ldots, 2^{-\nu a_n}m_n\big)$ which has sides parallel to the axes and side lengths respectively $2^{-\nu a_1}, \ldots, 2^{-\nu a_n}$. Remark that Q^a_{0m} is a cube with side length 1. If $Q^a_{\nu m}$ is such a rectangle in $\mathbb{R}^n$ and $c > 0$ then $c\,Q^a_{\nu m}$ is the rectangle in $\mathbb{R}^n$ concentric with $Q^a_{\nu m}$ and with side lengths respectively $c\,2^{-\nu a_1}, \ldots, c\,2^{-\nu a_n}$.

If E is a Lebesgue measurable subset of $\mathbb{R}^n$ then $|E|$ denotes its Lebesgue measure; recall our notation: $a\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi−index.

We are now prepared to introduce the anisotropic atoms.

Definition 3.1. (i) Let $K \in \mathbb{R}$, $c > 1$; a function $\rho : \mathbb{R}^n \to \mathbb{C}$ for which there exist all derivatives $D^\alpha\rho$ if $a\alpha \le K$ (continuous if $K \le 0$) is called an *anisotropic* 1_K *−atom* if:

$$\operatorname{supp}\rho \subset c\,Q^a_{0m} \quad \text{for some} \quad m \in \mathbb{Z}^n\,, \tag{3.1}$$

$$|D^\alpha\rho(x)| \;\le\; 1 \quad \text{if} \quad a\alpha \;\le\; K\,. \tag{3.2}$$

(ii) Let $s \in \mathbb{R}$, $0 < p \le \infty$, $K, L \in \mathbb{R}$, $c > 1$; a function $\rho : \mathbb{R}^n \to \mathbb{C}$ for which there exist all derivatives $D^\alpha\rho$ if $a\alpha \le K$ (continuous if $K \le 0$) is called an *anisotropic* $(s,p)_{K,L}$ *−atom* if:

$$\operatorname{supp}\rho \subset c\,Q^a_{\nu m} \quad \text{for some} \quad \nu \in \mathbb{N}\,,\quad m \in \mathbb{Z}^n\,, \tag{3.3}$$

$$|D^\alpha\rho(x)| \;\le\; \big|Q^a_{\nu m}\big|^{\frac{s}{n} - \frac{1}{p} - \frac{a\alpha}{n}} \quad \text{if} \quad a\alpha \;\le\; K\,, \tag{3.4}$$

$$\int_{\mathbb{R}^n} x^\beta\rho(x)\,dx \;=\; 0 \quad \text{if} \quad a\beta \;\le\; L\,. \tag{3.5}$$

If the atom ρ is located at $Q^a_{\nu m}$ $\big($that means $\operatorname{supp}\rho^a_{\nu m} \subset c\,Q^a_{\nu m}$ with $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $c > 1\big)$ then we will write it $\rho^a_{\nu m}$.

We give some technical explanations.

The value of the number $c > 1$ in (3.1) and (3.3) is unimportant. It simply makes clear that at the level ν some controlled overlapping of the supports of $\rho^a_{\nu m}$ must be allowed.

Since $\big|Q^a_{\nu m}\big| = 2^{-\nu n}$ condition (3.4) may be written as

$$|D^\alpha\rho(x)| \;\le\; 2^{-\nu\left(s - \frac{n}{p}\right)}2^{\nu a\alpha} \quad \text{if} \quad a\alpha \;\le\; K \tag{3.6}$$

and if $K \le 0$ then (3.4) is $|\rho(x)| \le 2^{-\nu\left(s - \frac{n}{p}\right)}$.

The moment conditions (3.5) can be reformulated as $D^\beta\widehat{\rho}(0) = 0$ if $a\beta \le L$, which shows that a sufficiently strong decay of $\widehat{\rho}$ at the origin is required. If $L < 0$ then (3.5) simply means that there are no moment conditions.

The reason for the normalizing factor in (3.2) and (3.4) is that there exists a constant $c > 0$ such that for all these atoms we have $\|\rho \,|\, B^{s,a}_{pq}\| \le c$, $\|\rho \,|\, F^{s,a}_{pq}\| \le c$. Hence, as in the isotropic case, atoms are normalized building blocks satisfying some moment conditions.

Our construction of anisotropic atoms which generalize isotropic atoms as they are in the works of M. Frazier and B. Jawerth, is slightly related to the concept of anisotropic building blocks (compactly supported and satisfying some norming and some moment conditions) used by P. Soardi in [Soa83] to define anisotropic Hardy spaces and to study the relations of these spaces to anisotropic Lipschitz and Campanato – Morrey spaces.

We introduce now the sequence spaces b_{pq} and f^a_{pq}.

If $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $Q^a_{\nu m}$ is a rectangle as above let $\chi_{\nu m}$ be the characteristic function of $Q^a_{\nu m}$; if $0 < p \le \infty$ let $\chi^{(p)}_{\nu m} = 2^{\nu n/p}\chi_{\nu m}$ (obvious modification if $p = \infty$) be the L_p – normalized characteristic function of $Q^a_{\nu m}$.

Definition 3.2. Let $0 < p \le \infty$, $0 < q \le \infty$. Then:

(i) b_{pq} is the collection of all sequences $\lambda = \{\lambda_{\nu m} \in \mathbb{C} \,:\, \nu \in \mathbb{N}_0,\, m \in \mathbb{Z}^n\}$ such that

$$\|\lambda \,|\, b_{pq}\| \;=\; \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q} \tag{3.7}$$

(with the usual modification if $p = \infty$ and/or $q = \infty$) is finite;

(ii) f^a_{pq} is the collection of all sequences $\lambda = \{\lambda_{\nu m} \in \mathbb{C} \,:\, \nu \in \mathbb{N}_0,\, m \in \mathbb{Z}^n\}$ such that

$$\|\lambda \,|\, f^a_{pq}\| \;=\; \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \left|\lambda_{\nu m}\chi^{(p)}_{\nu m}(\,\cdot\,)\right|^q \right)^{1/q} \,\Bigg|\, L_p \right\| \tag{3.8}$$

(with the usual modification if $p = \infty$ and/or $q = \infty$) is finite.

One can easily see that b_{pq} and f^a_{pq} are quasi – Banach spaces and using $\|\chi^{(p)}_{\nu m} \,|\, L_p\| = 1$ it is clear that comparing $\|\lambda \,|\, b_{pq}\|$ and $\|\lambda \,|\, f^a_{pq}\|$ the roles of the quasi – norms in L_p and l_q are interchanged.

Let $d_+ = \max(d, 0)$. For $0 < p \le \infty$ and $0 < q \le \infty$ we introduce the abbreviations

$$\sigma_p \;=\; n\left(\frac{1}{p} - 1\right)_+ \quad \text{and} \quad \sigma_{pq} \;=\; n\left(\frac{1}{\min(p,q)} - 1\right)_+ . \tag{3.9}$$

Theorem 3.3. *Let $0 < p < \infty$ (respectively $0 < p \le \infty$), $0 < q \le \infty$, $s \in \mathbb{R}$ and let K, $L \in \mathbb{R}$ be such that*

$$K \;\ge\; a_{\max} + s \quad \textit{if} \quad s \;\ge\; 0\,, \tag{3.10}$$

$$L \;\ge\; \sigma_{pq} - s \quad (\textit{respectively} \quad L \ge \sigma_p - s). \tag{3.11}$$

Then $g \in S'$ belongs to $F^{s,a}_{pq}$ (respectively $B^{s,a}_{pq}$) if, and only if, it can be represented as

$$g \;=\; \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}\rho^a_{\nu m}\,, \tag{3.12}$$

convergence being in S', *where* $\rho^a_{\nu m}$ *are anisotropic* 1_K *– atoms* ($\nu = 0$) *or anisotropic* $(s,p)_{K,L}$ *– atoms* ($\nu \in \mathbb{N}$) *and* $\lambda \in f^a_{pq}$ (*respectively* $\lambda \in b_{pq}$) *where* $\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0\,,\, m \in \mathbb{Z}^n\}$.

Furthermore, $\inf \big\| \lambda \mid f^a_{pq} \big\|$ $\big($*respectively* $\inf \big\| \lambda \mid b_{pq} \big\|\big)$, *where the infimum is taken over all admissible representations* (3.12), *is an equivalent quasi – norm in* $F^{s,a}_{pq}$ $\big($*respectively* $B^{s,a}_{pq}\big)$.

The convergence in S' can be obtained as a by – product of the proof using the same method as in [Tri97, 13.9] so we will not stress this point. We refer to the above theorem as to the atomic decomposition theorem in anisotropic function spaces.

Remark 3.4. Let $d > 0$ be given, let $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ be fixed and let us denote by $R^a_{\nu m}$ a rectangle with sides parallel to the axes, centered at $x^{\nu m}$ where

$$\big| x^{\nu m}_i - 2^{-\nu a_i} m_i \big| \;\le\; d\, 2^{-\nu a_i} \quad \text{for all} \quad i \in \{1, \ldots, n\}\,, \tag{3.13}$$

and with side lengths respectively $2^{-\nu a_1}, \ldots, 2^{-\nu a_n}$.

Then let $c > 0$ be chosen in dependence of d such that for every choice of $\nu \in \mathbb{N}_0$ and all choices of $x^{\nu m}$ in (3.13) we have

$$\bigcup_{m \in \mathbb{Z}^n} c\, R^a_{\nu m} \;=\; \mathbb{R}^n\,. \tag{3.14}$$

It will be clear from the proof that we may replace in Definition 3.1 the rectangle $Q^a_{\nu m}$ by $R^a_{\nu m}$, the number c being that from (3.14). A similar remark in the isotropic case was very useful in the work of H. Triebel and H. Winkelvoss, [TrW96a], cf. also [EdT96, 2.2.3].

We shift the proof of the theorem to Section 5 but let us make here some remarks. The first part of the proof, in which the atoms are constructed and where it is shown that the decomposition (3.12) holds, is essentially based on an anisotropic version of a resolution of unity of Calderon type, cf. [FJW91, 5.12]; this construction is the anisotropic counterpart of what was done in [FJW91, Theorem 5.11].

To prove the second part we will use a theorem on local means in anisotropic function spaces, the technique of maximal functions and an inequality of Fefferman – Stein type. All needed results are presented in the next section.

3.2. Anisotropic quarks and the subatomic decomposition theorem

In this subsection we will asume that $|\cdot|_a \in C^\infty(\mathbb{R}^n \backslash \{0\})$ is an anisotropic distance function according to (2.3) satisfying in addition

$$\{x \in \mathbb{R}^n \;:\; |x|_a \le 2\} \subset [-\pi, \pi]^n\,. \tag{3.15}$$

The above restriction, which is of technical nature, was introduced by P. Dintelmann in [Din95a] and seems to be a natural one compared with the isotropic case.

For $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ let $Q^a_{\nu m}$ be the rectangles introduced in Definition 3.1. Let in particular Q^a_{00} be the cube with side length 1 centered at the origin and let $2^a Q^a_{00}$ be the rectangle concentric with Q^a_{00} and with side lengths respectively $2^{a_1}, \ldots, 2^{a_n}$.

Definition 3.5. Let $\psi \in S$ be such that

$$\operatorname{supp}\psi \subset 2^a Q^a_{00} \quad \text{and} \quad \sum_{k\in\mathbb{Z}^n} \psi(x-k) = 1 \quad \text{if} \quad x \in \mathbb{R}^n \tag{3.16}$$

and let for any $\beta \in \mathbb{N}^n_0$, $\psi^\beta(x) = x^\beta \psi(x)$. If $0 < p \le \infty$ and $s \in \mathbb{R}$ then

$$(\beta qu)^a_{\nu m}(x) \;=\; 2^{-\nu\left(s-\frac{n}{p}\right)} \psi^\beta(2^{\nu a}x - m) \tag{3.17}$$

is called an *anisotropic* $(s,p)-\beta$ *– quark related to* $Q^a_{\nu m}$.

Remark 3.6. It is easy to see that up to normalizing constants the anisotropic $(s,p)-\beta-$quarks are anisotropic $(s,p)_{K,L}-$atoms for any given $K \in \mathbb{R}$ and any given $L < 0$. Moreover, the normalizing constants by which the anisotropic $(s,p)-\beta-$quark must be divided to become an anisotropic $(s,p)_{K,L}-$atom can be estimated from above by $c\,2^{\kappa a\beta}$ where $c > 0$ and $\kappa > 0$ are independent of β (recall the notation $a\beta = a_1\beta_1 + \cdots + a_n\beta_n$ where $\beta = (\beta_1, \ldots, \beta_n)$ is a multi–index).

We will use below the sequence spaces b_{pq} and f^a_{pq} with respect to the sequences $\lambda^\beta = \{\lambda^\beta_{\nu m} \in \mathbb{C} \,:\, \nu \in \mathbb{N}_0\,,\, m \in \mathbb{Z}^n\}$ where now $\beta \in \mathbb{N}^n_0$ is a multi–index and we will keep the notation $(\beta qu)^a_{\nu m}$ for an anisotropic $(s,p)-\beta-$quark related to the rectangle $Q^a_{\nu m}$. The numbers σ_p and σ_{pq} have the same meaning as in (3.9).

Theorem 3.7. *Let* $0 < p < \infty$ *(respectively* $0 < p \le \infty$*),* $0 < q \le \infty$ *and* $s > \sigma_{pq}$ *(respectively* $s > \sigma_p$*). There exists a number* $\kappa > 0$ *with the following property: let* $r > \kappa$*; then* $g \in S'$ *belongs to* $F^{s,a}_{pq}$ $\big($*respectively* $B^{s,a}_{pq}\big)$ *if, and only if, it can be represented as*

$$g \;=\; \sum_{\beta\in\mathbb{N}^n_0} \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda^\beta_{\nu m}\, (\beta qu)^a_{\nu m} \;, \tag{3.18}$$

convergence being in S' *(first* m*, then* ν*, then* β*), and*

$$\sup_{\beta\in\mathbb{N}^n_0} 2^{r a\beta} \left\|\lambda^\beta \,|\, f^a_{pq}\right\| \;<\; \infty \quad \left(\textit{respectively} \quad \sup_{\beta\in\mathbb{N}^n_0} 2^{r a\beta} \left\|\lambda^\beta \,|\, b_{pq}\right\| < \infty\right). \tag{3.19}$$

Furthermore, the infimum in (3.19) *over all admissible representations* (3.18) *is an equivalent quasi – norm in* $F^{s,a}_{pq}$ $\big($*respectively* $B^{s,a}_{pq}\big)$*.*

The technique developed in Section 5 to prove the above theorem is that of H. Triebel from [Tri97, 14.15]. However the proof given there covered only isotropic $B-$spaces; the considerations in Section 5 show that the method can be extended to $F-$spaces.

To show that $g \in F^{s,a}_{pq}$ $\big($respectively $g \in B^{s,a}_{pq}\big)$ can be decomposed as in (3.18) with (3.19) we need not the assumption $s > \sigma_{pq}$ $\big($respectively $s > \sigma_p\big)$. This restriction is needed only to prove the converse assertion.

4. Results on anisotropic function spaces

4.1. Prerequisites

If $z \in \mathbb{R}^n$ and $t > 0$ then the set $\Omega^a(z,t) = \{y \in \mathbb{R}^n : |y - z|_a \le t\}$ is the (closed) anisotropic ball centered at z and (anisotropic) radius t. The Lebesgue measure of such an anisotropic ball is $|\Omega^a(z,t)| = c\, t^n$ with c independent of t.

If f is a complex−valued locally integrable function on $\mathbb{R}^n$ then

$$M^a f(x) \;=\; \sup \frac{1}{|\Omega^a|} \int_{\Omega^a} |f(y)|\, dy \tag{4.1}$$

is the anisotropic Hardy−Littlewood maximal function, where the supremum is taken over all anisotropic balls Ω^a containing x.

Let $1 < p < \infty$, $1 < q \le \infty$. There exists a constant $c > 0$ such that

$$\big\| \big(M^a f_j\big)_{j \in \mathbb{N}_0} \,|\, L_p\big(l_q\big)\big\| \;\le\; c\, \big\| f \,|\, L_p\big(l_q\big)\big\| \tag{4.2}$$

for all sequences $f = \big(f_j\big)_{j \in \mathbb{N}_0}$ of complex−valued locally Lebesgue integrable functions on $\mathbb{R}^n$.

Comments and further informations to the isotropic version of (4.2) which essentially goes back to C. Fefferman and E. M. Stein, see [FeS71], may be found in [Tri92, 2.2.2] and [Tri83, 1.2.3].

B. Stöckert and H. Triebel remarked in [StT79, p. 257] that the maximal function in (4.1) is equivalent to a maximal function where in (4.1) we can take rectangles with sides parallel to the axes (containing x) and that the iterative application of the isotropic one dimensional case ($n = 1$) leads to (4.2). Different proofs of (4.2) can be found in the works of M. Yamazaki [Yam86, 2.2], A. Seeger [See89] and P. Dintelmann [Din95b, A.1.3 – A.1.4].

For $0 < p \le \infty$ let again $\sigma_p = n\big(\frac{1}{p} - 1\big)_+$.

To prove the theorem below one has only to adapt the method from [Tri83, 1.5.2].

Theorem 4.1. *Let Ω be a compact subset of $\mathbb{R}^n$, $0 < p \le \infty$. Let $r \ge 0$ and let w be a weight function for which there exists a constant $c > 0$ such that*

$$0 \;< w(x) \;\le\; c\, w(y)\big(1 + |x - y|_a\big)^r \quad \text{for all} \quad x\,,\, y \in \mathbb{R}^n\,. \tag{4.3}$$

If $s > r + \frac{n}{2} + \sigma_p$ then there exists a constant $c > 0$ such that

$$\big\| w\big(m\widehat{f}\big)^\vee \,|\, L_p\big\| \;\le\; c\, \big\| m \,|\, H_2^{s,a}\big\| \cdot \big\| wf \,|\, L_p\big\| \tag{4.4}$$

for all $m \in H_2^{s,a}$ and all $f \in S'$, $\operatorname{supp} \widehat{f} \subset \Omega$ with $wf \in L_p$.

If $0 < p \le \infty$, $0 < q \le \infty$ and $\Omega = \big(\Omega_j\big)_{j \in \mathbb{N}_0}$ is a collection of compact subsets in $\mathbb{R}^n$ we define the space $L_p^\Omega\big(l_q\big)$ as the collection of all systems $f = \big(f_j\big)_{j \in \mathbb{N}_0} \subset S'$ such that $\operatorname{supp} \widehat{f_j} \subset \Omega_j$ if $j \in \mathbb{N}_0$ which satisfy $\big\| f \,|\, L_p\big(l_q\big)\big\| < \infty$. By [Tri83, 1.6.1] $L_p^\Omega\big(l_q\big)$ is a quasi−Banach space with quasi−norm $\big\| f \,|\, L_p\big(l_q\big)\big\|$ if $f = \big(f_j\big)_{j \in \mathbb{N}_0}$.

Theorem 4.2. *Let $0 < p < \infty$, $0 < q \leq \infty$. For every $j \in \mathbb{N}_0$ let $R_j > 0$ be a given number, let $\Omega_j = \{\xi \in \mathbb{R}^n : |\xi|_a \leq R_j\}$ and let $\Omega = (\Omega_j)_{j \in \mathbb{N}_0}$.*

(i) *If $0 < t < \min(p, q)$ then there exists a constant $c > 0$ such that*

$$\left\| \left(\sup_{z \in \mathbb{R}^n} \frac{|f_j(\,\cdot - z)|}{1 + \left|R_j^a z\right|_a^{n/t}} \right)_{j \in \mathbb{N}_0} \middle| L_p(l_q) \right\| \leq c \left\| f \mid L_p(l_q) \right\| \tag{4.5}$$

for all $f = (f_j)_{j \in \mathbb{N}_0} \in L_p^\Omega(l_q)$.

(ii) *If $s > \frac{n}{2} + \frac{n}{\min(p,q)}$ then there exists a constant $c > 0$ such that*

$$\left\| \left((m_j \widehat{f}_j)^\vee \right)_{j \in \mathbb{N}_0} \middle| L_p(l_q) \right\| \leq c \sup_{j \in \mathbb{N}_0} \left\| m_j(R_j^a\, \cdot\,) \mid H_2^{s,a} \right\| \cdot \left\| f \mid L_p(l_q) \right\| \tag{4.6}$$

for all $(m_j)_{j \in \mathbb{N}_0}$ such that $m_j(R_j^a \cdot) \in H_2^{s,a}$ if $j \in \mathbb{N}_0$ and all $f = (f_j)_{j \in \mathbb{N}_0} \in L_p^\Omega(l_q)$.

The above result is the anisotropic counterpart of [Tri83, 1.6.2; 1.6.3] and the proof can be done in the same manner as there making standard anisotropic changes, see also [Tri92, 2.2.4], [BeN93, Proposition 1] and [Din95b, A.1.4].

4.2. Equivalent quasi−norms

If $(\psi_j)_{j \in \mathbb{N}_0} \subset S$ and $r > 0$ we define the maximal functions

$$(\psi_j^* f)_r(x) = \sup_{z \in \mathbb{R}^n} \frac{\left|(\psi_j \widehat{f})^\vee (x - z)\right|}{1 + \left|2^{ja} z\right|_a^r} \tag{4.7}$$

for $f \in S'$ where $x \in \mathbb{R}^n$ and $j \in \mathbb{N}_0$. The above maximal function essentially goes back to J. Peetre, see [P75] and [P76]; see also [Tri83, 2.3.6/Remark 2] and [Yam86].

The result below is the anisotropic version of [Tri92, 2.3.2] and it can be obtained as a simple consequence of (4.5), cf. also [Din95b, A.1.5].

Theorem 4.3. *Let $(\varphi_j)_{j \in \mathbb{N}_0}$ be a smooth anisotropic dyadic resolution of unity.*

(i) *Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $r > \frac{n}{p}$. Then*

$$B_{pq}^{s,a} = \left\{ f \in S' : \left\| f \mid B_{pq}^{s,a} \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| (\varphi_j^* f)_r \mid L_p \right\|^q \right)^{1/q} < \infty \right\} \tag{4.8}$$

(modification if $q = \infty$) in the sense of equivalent quasinorms.

(ii) *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $r > \frac{n}{\min(p,q)}$. Then*

$$F_{pq}^{s,a} = \left\{ f \in S' : \left\| f \mid F_{pq}^{s,a} \right\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \left| (\varphi_j^*)_r(\,\cdot\,) \right|^q \right)^{1/q} \middle| L_p \right\| < \infty \right\} \tag{4.9}$$

(modification if $q = \infty$) in the sense of equivalent quasi−norms.

4.3. General characterizations

We present now a rather general, but highly technical, characterization of the space $F^{s,a}_{pq}$ (respectively of $B^{s,a}_{pq}$) and this characterization is the anisotropic counterpart of [Tri92, 2.4.1] (respectively [Tri92, 2.5.1]).

In the theorem below φ needs not be an element of S and it is not immediately clear what is meant by $\big(\varphi(2^{-ja}\,\cdot\,)\widehat{f}\big)^\vee$. It is defined via limiting procedures as in Step 3 of the proof in [Tri92, 2.4.1].

The numbers σ_p and σ_{pq} have the same meaning as in (3.9).

Given the anisotropy $a = (a_1, \dots, a_n)$ recall our notations $2^{ka}x = \big(2^{ka_1}x_1, \dots, 2^{ka_n}x_n\big)$ and $2^{-ja}x = \big(2^{-ja_1}x_1, \dots, 2^{-ja_n}x_n\big)$.

Let $h \in S$ and $G \in S$ with

$$\operatorname{supp} h \subset \{x \in \mathbb{R}^n : |x|_a \le 2\}\,, \quad h(x) = 1 \quad \text{if} \quad |x|_a \le 1\,, \tag{4.10}$$

$$\operatorname{supp} G \subset \left\{x \in \mathbb{R}^n : \frac{1}{4} \le |x|_a \le 4\right\}\,, \quad G(x) = 1 \quad \text{if} \quad \frac{1}{2} \le |x|_a \le 2\,. \tag{4.11}$$

Theorem 4.4. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. Let s_0, s_1 be two real numbers with*

$$s_0 + \sigma_{pq} < s < s_1 \quad \text{and} \quad s_1 > \sigma_p\,. \tag{4.12}$$

Let φ_0 and φ be two complex-valued C^∞ functions on $\mathbb{R}^n$ and $\mathbb{R}^n\backslash\{0\}$, respectively, which satisfy the following Tauberian conditions:

$$|\varphi_0(x)| > 0 \quad \text{if} \quad |x|_a \le 2\,, \tag{4.13}$$

$$|\varphi(x)| > 0 \quad \text{if} \quad \frac{1}{2} \le |x|_a \le 2\,. \tag{4.14}$$

Let $r > \frac{n}{\min(p,q)}$ and assume:

$$\int_{\mathbb{R}^n} \left| \left(\frac{\varphi(\,\cdot\,)h(\,\cdot\,)}{|\cdot|_a^{s_1}} \right)^\vee (y) \right| (1 + |y|_a)^r\, dy < \infty\,, \tag{4.15}$$

$$\sup_{k \in \mathbb{N}} 2^{-ks_0} \int_{\mathbb{R}^n} \left| \big(\varphi(2^{ka}\,\cdot\,)G(\,\cdot\,)\big)^\vee (y) \right| (1 + |y|_a)^r\, dy < \infty\,, \tag{4.16}$$

$$\sup_{k \in \mathbb{N}} 2^{-ks_0} \int_{\mathbb{R}^n} \left| \big(\varphi_0(2^{ka}\,\cdot\,)G(\,\cdot\,)\big)^\vee (y) \right| (1 + |y|_a)^r\, dy < \infty\,. \tag{4.17}$$

Let $\varphi_j(x) = \varphi\big(2^{-ja}x\big)$ if $x \in \mathbb{R}^n\backslash\{0\}$ and $j \in \mathbb{N}$.

Then

$$\left\| \left(\sum_{j=0}^\infty 2^{jsq} \left| \big(\varphi_j \widehat{f}\big)^\vee (\,\cdot\,) \right|^q \right)^{1/q} \Bigg| L_p \right\| \tag{4.18}$$

(usual modification if $q = \infty$) is an equivalent quasi-norm in $F^{s,a}_{pq}$.

Proof. The proof is following the lines of the proof of [Tri92, 2.4.1]. We indicate briefly the necessary modifications.

Starting with a smooth anisotropic dyadic resolution of unity we have to make usual anisotropic changes and to use the maximal function from (4.7) to obtain counterparts of (14 – 19) in [Tri92, 2.4.1].

Since $r > \frac{n}{\min(p,q)}$ we may use the maximal inequality (4.5); then we use the vector valued multiplier theorem (4.6) and obtain counterparts of (21 – 23) in [Tri92, 2.4.1]. The term with $j = 0$ is critical but it may be incorporated by the lemma below.

Using again the maximal function (4.7) and the multiplier theorem (4.4) with weight $w(y) = (1 + |y|_a)^r$ we obtain anisotropic counterparts of (24 – 29) in [Tri92, 2.4.1].

Counterparts of (30 – 33) in [Tri92, 2.4.1] can be obtained in the same manner using in addition the embedding for different metrics $F^{s,a}_{pq} \hookrightarrow F^{\sigma,a}_{1q}$ which holds for $0 < p < 1$, $\sigma = s - \sigma_p$, see [Joh95, (2.14)].

To obtain counterparts of (36 – 45) in [Tri92, 2.4.1] we have to use the anisotropic version of the Plancherel – Polya – Nikol'skij inequality from [Yam86, 2.13] and the Fefferman – Stein inequality (4.2).

Finally, to complete the proof, we have to replace Remark 1 from the end of the proof of [Tri92, 2.4.1] by the lemma below. □

Lemma 4.5. *Let $0 < p < \infty$ and $\sigma \in S$, $\sigma(x) = 1$ if $|x|_a \le 1$, $\operatorname{supp} \sigma \subset \{x \in \mathbb{R}^n : |x|_a \le 2\}$. If $s_1 > \sigma_p$ then there exists a constant $c > 0$ such that*

$$\big\| \big(|\cdot|_a^{s_1} \sigma(\cdot) \widehat{g}(\cdot) \big)^{\vee} \,|\, L_p \big\| \;\le\; c \,\big\| g \,|\, L_p \big\| \tag{4.19}$$

for all $g \in S' \cap L_p$ with $\operatorname{supp} \widehat{g} \subset \{\xi \in \mathbb{R}^n : |\xi|_a \le 1\}$.

Proof. By the unweighted version ($r = 0$, $w(x) \equiv 1$) of the multiplier theorem (4.4) the estimate is valid if $\psi(x) = |x|_a^{s_1} \sigma(x) \in H_2^{v,a}$ where $v > \frac{n}{2} + \sigma_p$. Since $s_1 > \sigma_p$ we may choose $\frac{n}{2} + s_1 > v > \frac{n}{2} + \sigma_p$.

Let now χ be a C^∞ function on $\mathbb{R}^n$ such that $\chi(x) = 0$ if $|x|_a \le 1$ and $\chi(x) = 1$ if $|x|_a \ge 2$ and let $\psi_j(x) = |x|_a^{s_1} \sigma(x) \chi\big(2^{ja} x\big)$ for every $j \in \mathbb{N}$.

The sequence $\big(\psi_j\big)_{j \in \mathbb{N}}$ is fundamental in $H_2^{v,a}$; by (2.8) it is sufficient to prove that it is fundamental in each $H^{v,a}_{x_k,2}$ where $1 \le k \le n$.

If $v_k = \frac{v}{a_k} \in \mathbb{N}$ this can be done by straightforward calculations using (2.9) and the estimate $\big| D^\alpha \big(|x|_a^{s_1} \big) \big| \le c\, |x|_a^{s_1 - a\alpha}$ for any $|x|_a > 0$.

If $v_k = \frac{v}{a_k} \notin \mathbb{N}$ it is a matter of interpolation: we write $s_k = \theta m_k$ with $\theta \in (0,1)$, $m_k \in \mathbb{N}$ and use the interpolation result $(L_2, L_2(w_k))_{\theta,2} = L_2(w)$ with $w_k(\xi) = \big(1 + \xi_k^2\big)^{m_k/2}$ and $w = w_k^\theta$ which is a simple consequence of [Tri78, 1.18.4] where $L_2(w_k)$ and $L_2(w)$ are the weighted L_2 spaces with weights respectively w_k and w. □

Remark that an isotropic version of the assertion $\psi(x) = |x|_a^{s_1} \sigma(x) \in H_2^{v,a}$ where $v > \frac{n}{2} + \sigma_p$ is presented in a more general context in [RuS96, Lemma 2.3.1/1].

It is clear that following the lines of the above proof we can obtain an anisotropic counterpart of [Tri92, 2.5.1] for $B^{s,a}_{pq}$ spaces.

As in [Tri92] conditions (4.15 – 4.17) can be reformulated.

Corollary 4.6. *Let p, q, s, s_0, s_1 and r be the same numbers as in the theorem. Let φ_0, φ be two complex – valued C^∞ functions on $\mathbb{R}^n$ and $\mathbb{R}^n \backslash \{0\}$, respectively, which* *satisfy the Tauberian conditions* (4.13), (4.14) *and let* $v > r + \frac{n}{2}$. *Assume:*

$$\left\| \frac{\varphi(\,\cdot\,)h(\,\cdot\,)}{|\cdot|_a^{s_1}} \,\Big|\, H_2^{v,a} \right\| \;<\; \infty\,, \tag{4.20}$$

$$\sup_{k\in\mathbb{N}} 2^{-ks_0} \big\| \varphi\big(2^{ka}\,\cdot\,\big) G(\,\cdot\,) \,|\, H_2^{v,a} \big\| \;<\; \infty\,, \tag{4.21}$$

$$\sup_{k\in\mathbb{N}} 2^{-ks_0} \big\| \varphi_0\big(2^{ka}\,\cdot\,\big) G(\,\cdot\,) \,|\, H_2^{v,a} \big\| \;<\; \infty\,, \tag{4.22}$$

where h, G *have the same meaning as in* (4.10), (4.11).

Let $\varphi_j(x) = \varphi\big(2^{-ja}x\big)$ *if* $x \in \mathbb{R}^n\backslash\{0\}$ *and* $j \in \mathbb{N}$. *Then* (4.18) *is an equivalent quasi-norm in* $F_{pq}^{s,a}$.

Proof. The result is a simple consequence of the inequality

$$\left\| (1 + |\cdot|_a)^r \widehat{\psi} \,|\, L_1 \right\| \le c\; \|\psi \,|\, H_2^{v,a}\|$$

for all $\psi \in H_2^{v,a}$ where $v > r + \frac{n}{2}$ which can be proved as in [ScT87, 1.7.5]. □

The counterpart of the above corollary for $B_{pq}^{s,a}$ spaces reads as follows:

Corollary 4.7. *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ *and let* s_0, s_1 *be two real numbers with* $s_0 + \sigma_p < s < s_1$ *and* $s_1 > \sigma_p$. *Let* φ_0, φ *be two complex-valued* C^∞ *functions on* $\mathbb{R}^n$ *and* $\mathbb{R}^n\backslash\{0\}$, *respectively, which satisfy the Tauberian conditions* (4.13), (4.14); *let* $v > \sigma_p + \frac{n}{2}$ *and assume* (4.20), (4.21) *and* (4.22) *are satisfied. Let* $\varphi_j(x) = \varphi\big(2^{-ja}x\big)$ *if* $x \in \mathbb{R}^n\backslash\{0\}$ *and* $j \in \mathbb{N}$.

Then $\left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \big(\varphi_j \widehat{f}\,\big)^\vee \,|\, L_p \right\|^q\right)^{1/q}$ (*usual modification if* $q = \infty$) *is an equivalent quasi-norm in* $B_{pq}^{s,a}$.

4.4. Local means

For the anisotropy $a = (a_1, \ldots, a_n)$ we will use the notation

$$x + t^a y \;=\; \big(x_1 + t^{a_1} y_1,\, \ldots,\, x_n + t^{a_n} y_n\big)\,.$$

If $B^a = \{y \in \mathbb{R}^n \,:\, |y|_a \le 1\}$ is the anisotropic unit ball in $\mathbb{R}^n$, k is a C^∞ function on $\mathbb{R}^n$, $\operatorname{supp} k \subset B^a$ then we introduce the local means (cf. [Tri92, 2.4.6/1])

$$k(t,f)(x) \;=\; \int_{\mathbb{R}^n} k(y)\, f\big(x + t^a y\big)\, dy \;=\; t^{-n} \int_{\mathbb{R}^n} k(t^{-a}(z-x))\, f(z)\, dz \tag{4.23}$$

which make sense for any $f \in S'$ (appropriately interpreted).

Lemma 4.8. *Let* $s_1 > 0$ *be a given number and assume* $k \in S$ *such that there exists a constant* $c > 0$ *with* $\big|\widehat{k}(\xi)\big| \le c\,|\xi|_a^{s_1}$ *for* ξ *near zero. Then* $\int_{\mathbb{R}^n} x^\alpha k(x)\, dx = 0$ *for all* $\alpha \in \mathbb{N}_0^n$ *with* $a\alpha < s_1$.

Proof. We have to prove that $D^\alpha \widehat{k}(0) = 0$ for all $\alpha \in \mathbb{N}_0^n$ with $a\,\alpha < s_1$. If $s_1 < a_{\min}$ the assertion is clear so let us assume that $s_1^* = [s_1/a_{\min}] \geq 1$.

By Taylor's expansion theorem we get for ξ near zero

$$\widehat{k}(\xi) \;=\; \sum_{|\alpha|<s_1^*} \frac{D^\alpha \widehat{k}(0)}{\alpha!}\,\xi^\alpha + \sum_{|\alpha|=s_1^*} R_\alpha(\xi) \tag{4.24}$$

where $|R_\alpha(\xi)| \leq c_\alpha\,|\xi^\alpha| \leq c'_\alpha\,|\xi|_a^{a\alpha}$ for ξ small and for some constants c_α, $c'_\alpha > 0$.

By standard limit arguments we arrive at $D^\alpha \widehat{k}(0) = 0$ for all α with $|\alpha| < s_1^*$ and this leads immediately to the conclusion. □

Theorem 4.9. *Let k_0 and k be two C^∞ functions on $\mathbb{R}^n$ such that*

$$\operatorname{supp} k_0 \subset B^a\,, \qquad \left|\widehat{k_0}(\xi)\right| \;>\; 0 \quad \textit{if} \quad |\xi|_a \;\leq\; 2\,, \tag{4.25}$$

$$\operatorname{supp} k \subset B^a\,, \qquad \left|\widehat{k}(\xi)\right| \;>\; 0 \quad \textit{if} \quad \frac{1}{2} \;\leq\; |\xi|_a \;\leq\; 2\,. \tag{4.26}$$

Let $s_1 > 0$ and assume that there exists a constant $c > 0$ such that

$$\left|\widehat{k}(\xi)\right| \;\leq\; c\,|\xi|_a^{s_1} \quad \textit{for} \quad \xi \quad \textit{near zero}\,. \tag{4.27}$$

(i) *Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. If $s_1 > \max(s, \sigma_p) + \sigma_p$ then*

$$\big\|k_0(1,f)\,|\,L_p\big\| + \left(\sum_{j=1}^{\infty} 2^{jsq}\big\|k\big(2^{-j},f\big)\,|\,L_p\big\|^q\right)^{1/q} \tag{4.28}$$

(usual modification if $q = \infty$) is an equivalent quasi-norm in $B_{pq}^{s,a}$.

(ii) *Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. If $s_1 > \max(s, \sigma_p) + \frac{n}{\min(p,q)}$ then*

$$\big\|k_0(1,f)\,|\,L_p\big\| + \left\|\left(\sum_{j=1}^{\infty} 2^{jsq}\,\big|k\big(2^{-j},f\big)(\,\cdot\,)\big|^q\right)^{1/q} \,\Big|\, L_p\right\| \tag{4.29}$$

(usual modification if $q = \infty$) is an equivalent quasi-norm in $F_{pq}^{s,a}$.

Proof. We sketch the proof for F-spaces. By our assumption on s_1 we find numbers s_2, v and r such that $s_2 > \max(s, \sigma_p)$, $r > \frac{n}{\min(p,q)}$ and $v > \frac{n}{2} + r$ such that

$$s_1 \;>\; s_2 + v - \frac{n}{2}\,. \tag{4.30}$$

The functions $\check{k_0}$ and $\check{k}$ fulfill the Tauberian conditions (4.13), (4.14) and we will identify them respectively with φ_0 and φ in Corollary 4.6, now with s_2 in place of s_1.

To prove (4.20) we use (2.8). Fix $i \in \{1, \ldots, n\}$.

If $v_i = \frac{v}{a_i} \in \mathbb{N}$ then we recall (2.9); clearly $\widehat{k}(\,\cdot\,)h(\,\cdot\,)|\cdot|_a^{-s_2} \in L_2$ where h is the function from (4.10). Let $l_i \in \{0, \ldots, v_i\}$; then by Lemma 2.2 there exists a constant $c > 0$ such that for ξ near zero

$$\left|D_i^{v_i - l_i}\left(|\xi|_a^{-s_2}\right)\right| \;\leq\; c\,|\xi|_a^{-s_2 - a_i(v_i - l_i)}\,. \tag{4.31}$$

Using the above lemma we find also a constant $c > 0$ such that for ξ near zero

$$\left| D_i^{l_i} \left(\widehat{k}(\xi) h(\xi) \right) \right| \;\le\; c\, |\xi|_a^{s_1 - a_i l_i} \,. \tag{4.32}$$

By (4.31) and (4.32) using (4.30) we get $D_i^{v_i}\left(\widehat{k}(\,\cdot\,)h(\,\cdot\,)|\cdot|_a^{-s_2}\right) \in L_2$ which is just what we want.

If $v_i = \frac{v}{a_i} \notin \mathbb{N}$ we have to use the interpolation result which was mentioned at the end of the proof of Lemma 4.5 and this completes the proof of (4.20).

To prove (4.21) let $s_0 \in \mathbb{R}$ with $s_0 + \sigma_{pq} < s$ and let G be the function from (4.11). If $m \in \mathbb{N}$ then $\psi_m(\,\cdot\,) = \check{k}(2^{ma}\cdot\,)\, G(\,\cdot\,)$ has a compact support which is at most $\Omega = \left\{ x \in \mathbb{R}^n \,:\, \frac{1}{4} \le |x|_a \le 4 \right\}$.

If $j \in \mathbb{N}$ such that $j > 1 + \left[\frac{v}{a_i}\right]$ for all $i \in \{1, \ldots, n\}$ then there exists a constant $c > 0$ such that

$$\left\| \psi_m \,|\, H_2^{v,a} \right\| \;\le\; c \sum_{a\alpha \le j} \| D^\alpha \psi_m \,|\, L_\infty \| \,. \tag{4.33}$$

To prove (4.33) we have only to recall (2.8) and to distinguish between $v_i = \frac{v}{a_i} \in \mathbb{N}$ and $v_i = \frac{v}{a_i} \notin \mathbb{N}$.

Let $l \in \mathbb{N}$ be such that $j - l \le s_0$ and such that $\left| D^\alpha \check{k}(x) \right| \le c(1 + |x|_a)^{-l}$ for all $x \in \mathbb{R}^n$ and any $\alpha \in \mathbb{N}_0^n$ with $a\,\alpha \le j$.

Then there exist constants $c,\, c' > 0$ independent of $m \in \mathbb{N}$ such that for any $x \in \Omega$ we have $|D^\alpha \psi_m(x)| \le c\, 2^{ma\alpha}(1 + |2^{ma}x|_a)^{-l} \le c'\, 2^{m(j-l)} \le c'\, 2^{ms_0}$ if $\alpha \in \mathbb{N}_0^n$ with $a\,\alpha \le j$.

It follows that there exists a constant $c > 0$ independent of $m \in \mathbb{N}$ such that

$$\sum_{a\alpha \le j} \| D^\alpha \psi_m \,|\, L_\infty \| \;\le\; c\, 2^{ms_0} \,. \tag{4.34}$$

The condition (4.21) is now a simple consequence of (4.33) and (4.34). A similar argument can be used to check (4.22).

For B−spaces one has to use Corollary 4.7 and to make obvious changes above. □

Remark 4.10. The isotropic counterpart of the above result can be found in [Tri92, 2.4.6, 2.5.3]; instead of (4.27) it is used the representation $k = \Delta^N k^0$ (with $2N = s_1$ sufficiently large and $k^0 \in S$) which is in fact $\widehat{k}(\xi) = |\xi|^{2N}\widehat{k^0}(\xi)$. But this assumption was taken only for simplicity and it can be replaced by (4.27) with the euclidean distance $|\cdot|$ instead of $|\cdot|_a$.

Remark 4.11. Examples of functions k as in the theorem can be constructed as in [FrJ85, p. 783].

The advantage of (4.23) compared with $\left(\varphi_j \widehat{f}\,\right)^\vee$ from Definition 2.3 is its strictly local nature: in order to calculate $k(t,f)(x)$ in a given point $x \in \mathbb{R}^n$ one needs only a knowledge of $f(z)$ in an anisotropic ball $\Omega^a(x,t)$. This observation will be of great service for us in the proof of the atomic decomposition theorem.

5. Proofs

5.1. Proof of the atomic decomposition theorem

We begin with some preparations. Our theorem is based on the following lemma which provides the existence of an anisotropic resolution of unity of Calderon type and which is a generalization of the result from [FJW91, 5.12].

Lemma 5.1. *Let θ_0, $\theta \in S$ be functions with:*

$$\left|\widehat{\theta_0}(\xi)\right| \; > \; 0 \quad \textit{if} \quad |\xi|_a \; \le \; 2\,, \tag{5.1}$$

$$|\widehat{\theta}(\xi)| \; > \; 0 \quad \textit{if} \quad \frac{1}{2} \; \le \; |\xi|_a \; \le \; 2\,. \tag{5.2}$$

Then there exist functions φ_0, $\varphi \in S$ and a positive number $\delta \le 2$ such that

$$\operatorname{supp}\varphi_0 \subset \{\xi \in \mathbb{R}^n \,:\, |\xi|_a \le 2\} \quad \textit{and} \quad |\varphi_0(\xi)| > 0 \;\; \textit{if} \;\; |\xi|_a \; \le \; \delta\,, \tag{5.3}$$

$$\operatorname{supp}\varphi \subset \left\{\xi \in \mathbb{R}^n \,:\, \frac{1}{2} \le |\xi|_a \le 2\right\} \quad \textit{and} \quad |\varphi(\xi)| > 0 \;\; \textit{if} \;\; \frac{3}{5} \; \le \; |\xi|_a \; \le \; \frac{5}{3}\,, \tag{5.4}$$

and

$$\widehat{\theta_0}(\xi)\varphi_0(\xi) + \sum_{\nu=1}^{\infty} \widehat{\theta}(2^{-\nu a}\xi)\varphi(2^{-\nu a}\xi) \; = \; 1 \quad \textit{for all} \quad \xi \in \mathbb{R}^n\,. \tag{5.5}$$

The proof is classical and it can be done using the technique of M. Frazier, B. Jawerth, G. Weiss from [FJW91, Lemma 6.9] adapted to our purpose so we do not go into details.

Remark 5.2. A construction as in [FrJ85, p. 783] can be used to prove that if $L \ge 0$ is a given number there exists a function $\theta \in S$ satisfying (5.2) such that, in addition,

$$\operatorname{supp}\theta \subset \{x \in \mathbb{R}^n \,:\, |x|_a \le 1\}\,, \tag{5.6}$$

$$\int_{\mathbb{R}^n} x^\beta \theta(x)\, dx \; = \; 0 \quad \text{if} \quad a\beta \; \le \; L \tag{5.7}$$

(cf. also Remark 4.11).

An instrument which will be of considerable use is an anisotropic version of Taylor's expansion theorem. Recall the binomial notation $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ if $x \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi−index.

Theorem 5.3. *Let $A \ge 0$ be a given number. Let U be an open convex subset of $\mathbb{R}^n$ and assume that $f : U \to \mathbb{C}$ is a function such that $D^\alpha f$ exists for all $\alpha \in \mathbb{N}_0^n$ with $a\alpha \le A + a_{\max}$.*

Let $y \in U$ and $t > 0$ be such that $\Omega^a(y,t) = \{z \in \mathbb{R}^n \,:\, |z - y|_a \le t\} \subset U$.

Then there exists a constant $c > 0$ such that for all $x \in \Omega^a(y,t)$ it follows

$$f(x) \;=\; \sum_{a\alpha \le A} \frac{1}{\alpha!} D^\alpha f(y)\,(x-y)^\alpha + R_A(x) \tag{5.8}$$

with

$$|R_A(x)| \le c \sum_{a\alpha > A}^{A+a_{\max}} t^{a\alpha} \sup_{z \in \Omega^a(y,t)} |D^\alpha f(z)| \tag{5.9}$$

where the notation $\sum_{a\alpha>A}^{A+a_{\max}}$ means that the sum is taken over all $\alpha \in \mathbb{N}_0^n$ such that $A < a\alpha \le A + a_{\max}$.

We will refer to this result as to the anisotropic Taylor expansion theorem of (anisotropic) order A on the set $\Omega^a(y,t)$.

The main idea in proving the above theorem is an iterative application of the one dimensional classical Taylor expansion theorem. In lack of a convincing reference we sketch a proof in the case $n = 2$.

Applying Taylor's classical expansion theorem with Lagrange remainder there exists $\xi_1 = \xi_1(x_1)$ between x_1 and y_1 such that

$$f(x_1,x_2) \;=\; \sum_{\alpha_1=0}^{[A/a_1]} \frac{1}{\alpha_1!} D_1^{\alpha_1} f(y_1,x_2)\,(x_1-y_1)^{\alpha_1} + R_{1+[A/a_1]}(x)$$

where

$$R_{1+[A/a_1]}(x) \;=\; c_1\, D_1^{1+[A/a_1]} f(\xi_1,x_2)\,(x_1-y_1)^{1+[A/a_1]}\,.$$

Applying now Taylor's classical expansion theorem with Lagrange remainder to each $D_1^{\alpha_1} f(y_1,\,\cdot\,)$ there exists $\xi_2^{\alpha_1} = \xi_2^{\alpha_1}(x_2)$ between x_2 and y_2 such that

$$\begin{aligned} D_1^{\alpha_1} f(y_1,x_2) \;=\; & \sum_{\alpha_2=0}^{[(A-a_1\alpha_1)/a_2]} \frac{1}{\alpha_2!} D_1^{\alpha_1} D_2^{\alpha_2} f(y_1,y_2)\,(x_2-y_2)^{\alpha_2} \\ & + R_{\alpha_1,1+[(A-a_1\alpha_1)/a_2]}(x) \end{aligned}$$

where the remainder $R_{\alpha_1,1+[(A-a_1\alpha_1)/a_2]}(x)$ is

$$c_2(\alpha_1)\, D_1^{\alpha_1} D_2^{1+[(A-a_1\alpha_1)/a_2]} f(y_1,\xi_2^{\alpha_1})\,(x_2-y_2)^{1+[(A-a_1\alpha_1)/a_2]}\,.$$

If $n = 2$ the expansion (5.8) with (5.9) is now a simple consequence of the last four relations and of

$$R_A(x) \;=\; R_{1+[A/a_1]}(x) + \sum_{\alpha_1=0}^{[A/a_1]} \frac{1}{\alpha_1!}\,(x_1-y_1)^{\alpha_1} R_{\alpha_1,1+[(A-a_1\alpha_1)/a_2]}(x)\,.$$

The general case can be treated in a similar manner.

P r o o f (of the atomic decomposition theorem). We present here the proof for $F_{pq}^{s,a}$ spaces; the proof for $B_{pq}^{s,a}$ spaces is simpler and is obtained essentially by interchanging the roles of the L_p and l_q quasi−norms in the proof below.

Part I. Let $g \in F^{s,a}_{pq}$; we use the method of M. Frazier, B. Jawerth and G. Weiss from [FJW91, Theorem 5.11] to construct atoms and to decompose g as in (3.12).

Let θ_0, θ, φ_0 and φ be functions in S satisfying (5.1 – 5.7). Then $\big(\varphi_0\widehat{g}\big)^\vee$ and $\big(\varphi(2^{-\nu a}\cdot\,)\widehat{g}\big)^\vee$ are entire analytic functions; using $\widehat{\theta}(2^{-\nu a}\xi) = 2^{\nu n}F(\theta(2^{\nu a}\cdot))(\xi)$ we obtain the following equality in S':

$$
\begin{aligned}
g(x) \;&=\; \sum_{m\in\mathbb{Z}^n}\int_{Q^a_{0m}} \theta_0(x-y)(\varphi_0\widehat{g}\,)^\vee(y)\,dy \\
&\quad+\sum_{\nu=1}^{\infty}\sum_{m\in\mathbb{Z}^n} 2^{\nu n}\int_{Q^a_{\nu m}} \theta(2^{\nu a}(x-y))(\varphi(2^{-\nu a}\cdot\,)\widehat{g})^\vee(y)\,dy\,.
\end{aligned}
\tag{5.10}
$$

We define for every $\nu \in \mathbb{N}$ and all $m \in \mathbb{Z}^n$

$$
\lambda_{\nu m} \;=\; C\,2^{\nu(s-\frac{n}{p})}\sup_{y\in Q^a_{\nu m}}\big|(\varphi(2^{-\nu a}\cdot\,)\widehat{g}\,)^\vee(y)\big|
\tag{5.11}
$$

where

$$
C \;=\; \max_{a\alpha\le K}\;\sup_{|x|_a\le 1}|D^\alpha\theta(x)|
$$

and

$$
\rho^a_{\nu m}(x) \;=\; \frac{1}{\lambda_{\nu m}}\,2^{\nu n}\int_{Q^a_{\nu m}} \theta(2^{\nu a}(x-y))(\varphi(2^{-\nu a}\cdot\,)\widehat{g}\,)^\vee(y)\,dy\,.
\tag{5.12}
$$

Similarly we define for every $m \in \mathbb{Z}^n$ the numbers λ_{0m} and the functions ρ^a_{0m} taking in (5.11) and (5.12) $\nu = 0$ and replacing φ and θ by φ_0 and θ_0, respectively.

It is obvious that (3.12) is satisfied and it follows by straightforward calculations, using the properties of the functions θ_0, θ, φ_0 and φ, that ρ^a_{0m} are anisotropic 1_K-atoms and that $\rho^a_{\nu m}$ are anisotropic $(s,p)_{K,L}-$atoms for $\nu \in \mathbb{N}$.

Finally, we will show that there exists a constant $c > 0$ such that $\big\|\lambda \,|\, f^a_{pq}\big\| \le c\,\big\|g\,|\,F^{s,a}_{pq}\big\|$. We have for a fixed $\nu \in \mathbb{N}$ that

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(p)}_{\nu m}(x) &= C\,2^{\nu(s-\frac{n}{p})}\sum_{m\in\mathbb{Z}^n}\sup_{y\in Q^a_{\nu m}}\big|(\varphi(2^{-\nu a}\cdot\,)\widehat{g}\,)^\vee(y)\big|\cdot 2^{\nu\frac{n}{p}}\chi_{\nu m}(x) \\
&\le c'\,2^{\nu s}\left(\sup_{|z|_a\le c\,2^{-\nu}}\frac{\big|(\varphi(2^{-\nu a}\cdot\,)\widehat{g}\,)^\vee(x-z)\big|}{(1+|2^{\nu a}z|_a)^r}\,(1+|2^{\nu a}z|_a)^r\right) \\
&\le c''\,2^{\nu s}\big(\varphi^*_\nu g\big)_r(x)
\end{aligned}
\tag{5.13}
$$

since $|x-y|_a \le c\,2^{-\nu}$ for $x,\, y \in Q^a_{\nu m}$ and $\sum_{m\in\mathbb{Z}^n}\chi_{\nu m}(x) = 1$. Here $\varphi_\nu = \varphi(2^{-\nu a}\cdot\,)$, $r > \frac{n}{\min(p,q)}$ and $\big(\varphi^*_\nu g\big)_r$ is the maximal function from (4.7). It follows

$$
\sum_{\nu=1}^{\infty}\sum_{m\in\mathbb{Z}^n}\big|\lambda_{\nu m}\,\chi^{(p)}_{\nu m}(\,\cdot\,)\big|^q \;\le\; c\sum_{\nu=1}^{\infty}2^{\nu sq}\big(\varphi^*_\nu g\big)_r(\,\cdot\,)^q
\tag{5.14}
$$

(with the usual modification if $q = \infty$) where c is a positive constant.

Now we have to use (5.14), its counterpart for $\nu = 0$ (which can be obtained by a similar calculation), Theorem 4.3 and get

$$\left\|\lambda \mid f^a_{pq}\right\| \le c \left\| \left(\sum_{\nu=0}^{\infty} 2^{\nu s q} (\varphi^*_\nu g)_r(\,\cdot\,)^q \right)^{1/q} \middle| L_p \right\| \;\le\; c' \left\|g \mid F^{s,a}_{pq}\right\| \tag{5.15}$$

(with the usual modification if $q = \infty$) and this completes the proof of the first part of the theorem.

Part II. Reciprocally, assume now that g can be represented by (3.12), with K and L satisfying respectively (3.10) and (3.11). We will show that $g \in F^{s,a}_{pq}$ and that $\left\|g \mid F^{s,a}_{pq}\right\| \le c \left\|\lambda \,|\, f^a_{pq}\right\|$ for some constant $c > 0$.

Let k_0 and k be two C^∞ functions on $\mathbb{R}^n$ as in Theorem 4.9 and let $s_1 > 0$ be in that theorem enough large such that we also have $s_1 > K$.

Temporarily let $\nu\,, j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$ be fixed; we start finding convenient estimates for $2^{js}\, k\big(2^{-j}, \rho^a_{\nu m}\big)(x)$.

Step II.1. Let $j \ge \nu$ and let

$$2^{js}\, k\big(2^{-j}, \rho^a_{\nu m}\big)(x) \;=\; 2^{js} \int_{|y|_a \le 1} k(y)\, \rho^a_{\nu m}\big(x + 2^{-ja} y\big)\, dy\,. \tag{5.16}$$

Let us remark that in this case x is located in some $c\, Q^a_{\nu m}$ (else, the integral above would be certainly zero by (3.3)).

Suppose first $s \ge 0$ and let $A = K - a_{\max} \ge s \ge 0$. The derivatives $D^\alpha \rho^a_{\nu m}$ exist if $a\,\alpha \le K$ so we can use the anisotropic Taylor expansion theorem of order A for the function $w \mapsto \rho(w)$ on the set $\Omega^a\big(x, 2^{-j}\big)$. We then put $w = x + 2^{-ja} y$ and get the expansion

$$\rho^a_{\nu m}\big(x + 2^{-ja} y\big) \;=\; \sum_{a\alpha \le A} c_\alpha \big(x + 2^{-ja} y - z\big)^\alpha D^\alpha \rho^a_{\nu m}(z) + R_K(x, y) \tag{5.17}$$

where

$$|R_K(x, y)| \;\le\; c \sum_{a\alpha > A}^{K} 2^{-ja\alpha} \sup_{z \in c' Q^a_{\nu m}} |D^\alpha \rho^a_{\nu m}(z)| \tag{5.18}$$

for some c, $c' > 0$. We may choose $A' > A$ such that for all α with $A < a\alpha \le K$ we have $a\alpha > A'$. Using this remark and the estimate (3.6) the inequality (5.18) becomes

$$\begin{aligned} |R_K(x, y)| \;&\le\; c \sum_{a\alpha > A'}^{K} 2^{-ja\alpha} 2^{-\nu(s - \frac{n}{p})} 2^{\nu a\alpha}\, \widetilde{\chi}_{\nu m}(x) \\ &\le\; c' 2^{-\nu s} 2^{(\nu - j)A'}\, \widetilde{\chi}^{(p)}_{\nu m}(x) \end{aligned} \tag{5.19}$$

where $\widetilde{\chi}^{(p)}_{\nu m}$ is the $p-$normalized characteristic function of some rectangle $c\, Q^a_{\nu m}$.

Recall $s_1 > K$ so we may use Lemma 4.8 and obtain $\int_{\mathbb{R}^n} (x + 2^{-ja} y - z)^\alpha\, k(y)\, dy = 0$ for all α such that $a\alpha \le A < s_1$. Hence (5.17) and (5.19) yield

$$\left|2^{js}\, k\big(2^{-j}, \rho^a_{\nu m}\big)(x)\right| \;\le\; c\, 2^{-(j-\nu)(A'-s)}\, \widetilde{\chi}^{(p)}_{\nu m}(x) \quad \text{where} \quad A' \;>\; s\,. \tag{5.20}$$

If now $s < 0$ we have to use only the estimate $|\rho^a_{\nu m}(x)| \le 2^{-\nu(s-\frac{n}{p})}\ \widetilde{\chi}_{\nu m}(x)$ from (3.6) and get

$$\big|2^{js}\, k\big(2^{-j}, \rho^a_{\nu m}\big)(x)\big| \ \le\ c\, 2^{-(j-\nu)(-s)}\, \widetilde{\chi}^{(p)}_{\nu m}(x)\,. \tag{5.21}$$

So, by (5.20) and (5.21), it is clear that we arrive in any case at

$$\big|2^{js}\, k\big(2^{-j}, \rho^a_{\nu m}\big)(x)\big| \ \le\ c\, 2^{-(j-\nu)\delta}\, \widetilde{\chi}^{(p)}_{\nu m}(x) \quad \text{for some} \quad \delta\ >\ 0\,. \tag{5.22}$$

Step II.2. Let now $j < \nu$; by a change of variables we have

$$2^{js}\, k\big(2^{-j}, \rho^a_{\nu m}\big)(x) \ =\ 2^{js}\, 2^{jn} \int_{\mathbb{R}^n} k\big(2^{ja} y\big)\, \rho^a_{\nu m}(x+y)\, dy\,. \tag{5.23}$$

Clearly the integration above can be restricted to the set $\big\{y \in \mathbb{R}^n : |y|_a \le 2^{-j}\big\}$; we remark also that by our assumption on j and ν, x is located in some $c\,\Omega_{jm}$ where $\Omega_{jm} = \big\{z \in \mathbb{R}^n : |z - 2^{-\nu a} m|_a \le 2^{-j}\big\}$ is the anisotropic ball centered at $2^{-\nu a} m$ and radius 2^{-j} (else the integral above would be certainly zero). This can be easily proved if we recall the definition of the rectangle $Q^a_{\nu m}$, the assumption (3.3) and use the generalized triangle inequality for $|\cdot|_a$.

Since k is a smooth function on $\mathbb{R}^n$ we may use the anisotropic Taylor expansion theorem of order L for the function $w \mapsto k(w)$ on the set $\Omega^a(z_x, 2^{j-\nu})$, where $z_x = z(j,\nu,m,x) = 2^{ja}(2^{-\nu a} m - x)$. After that we let $w = 2^{ja} y$ and get

$$k\big(2^{ja} y\big) \ =\ \sum_{a\alpha \le L} c_\alpha \big(2^{ja} y - z_x\big)^{\alpha} D^\alpha k(z_x) + R_L(y,x) \tag{5.24}$$

where

$$|R_L(y,x)| \ \le\ c \sum_{a\alpha > L}^{L + a_{\max}} 2^{(j-\nu) a\alpha} \tag{5.25}$$

for some positive constant c.

By the moment conditions (3.5) we have $\int_{\mathbb{R}^n} (2^{ja} y - z_x)^\alpha \rho^a_{\nu m}(x+y)\, dy = 0$ if $a\,\alpha \le L$; using (5.24) we may replace (5.23) by

$$2^{js}\, k\big(2^{-j}, \rho^a_{\nu m}\big)(x) \ =\ 2^{js}\, 2^{jn} \int_{\mathbb{R}^n} R_L(y,x)\, \rho^a_{\nu m}(x+y)\, dy \tag{5.26}$$

where the integration can be restricted to the set $\big\{y \in \mathbb{R}^n : |y|_a \le 2^{-j}\big\}$. As in the first step we may choose $L' > L$ such that for all α with $L < a\,\alpha \le L + a_{\max}$ we have $a\,\alpha > L'$. Hence by (5.25) and (3.6) we get

$$\begin{aligned} \big|2^{js}\, k\big(2^{-j}, \rho^a_{\nu m}\big)(x)\big| &\le c\, 2^{js}\, 2^{jn} \sum_{a\alpha > L'}^{L + a_{\max}} 2^{(j-\nu)a\alpha} \int_{|y|_a \le 2^{-j}} |\rho^a_{\nu m}(x+y)|\, dy \\ &\le c'\, 2^{js}\, 2^{jn}\, 2^{(j-\nu)L'}\, 2^{-\nu(s-\frac{n}{p})} \int_{|y|_a \le 2^{-j}} \widetilde{\chi}_{\nu m}(x+y)\, dy \\ &= c'\, 2^{(j-\nu)(L'+s)}\, 2^{jn}\, 2^{\nu\frac{n}{p}} \int_{|y|_a \le 2^{-j}} \widetilde{\chi}_{\nu m}(x+y)\, dy \end{aligned} \tag{5.27}$$

where $\widetilde{\chi}_{\nu m}$ is the characteristic function of some rectangle $c\,Q^a_{\nu m}$.

Let now χ^{jm} be the characteristic function of the anisotropic ball $c\,\Omega_{jm}$ where x is located; by straightforward computation we have

$$\int_{|y|_a \le 2^{-j}} \widetilde{\chi}_{\nu m}(x+y)\,dy \;\le\; 2^{-\nu n}\,\chi^{jm}(x)\,. \tag{5.28}$$

Recall $L' > L \ge \sigma_{pq} - s$; we may choose an $\omega < \min(1,p,q)$ such that $L' + s > n\left(\frac{1}{\omega} - 1\right) > \sigma_{pq}$. Denoting, as usual, $M^a \chi^\omega_{\nu m}$ the anisotropic Hardy–Littlewood maximal function of $\chi^\omega_{\nu m}$ we get

$$\chi^{jm}(\,\cdot\,) \le c\,2^{(\nu-j)\frac{n}{\omega}}\big(M^a\chi^\omega_{\nu m}(\,\cdot\,)\big)^{1/\omega}. \tag{5.29}$$

Finally, using (5.28) and (5.29), the estimate (5.27) becomes

$$\big|2^{js}\,k\big(2^{-j},\rho^a_{\nu m}\big)(x)\big| \;\le\; c\,2^{(j-\nu)(L'+s-n(\frac{1}{\omega}-1))}\big(M^a\chi^{(p)\omega}_{\nu m}(x)\big)^{1/\omega} \tag{5.30}$$

which is in fact

$$\big|2^{js}\,k\big(2^{-j},\rho^a_{\nu m}\big)(x)\big| \le c\,2^{-(\nu-j)\varepsilon}\big(M^a\chi^{(p)\omega}_{\nu m}(x)\big)^{1/\omega} \quad \text{for some} \quad \varepsilon > 0\,. \tag{5.31}$$

Remark that the terms with $j = 0$ and/or $\nu = 0$ can also be covered by the technique in steps II.1 – 2.

Step II.3. Using (5.22) and (5.31) we get for $0 < q \le 1$

$$\begin{aligned} &\left|2^{js}k\left(2^{-j},\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\rho^a_{\nu m}\right)(x)\right|^q \\ &\le\; c\sum_{\nu\le j}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^q\,2^{-\delta(j-\nu)q}\;\widetilde{\chi}^{(p)q}_{\nu m}(x) \\ &\quad + c'\sum_{\nu>j}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^q\,2^{-\varepsilon(\nu-j)q}\big(M^a\chi^{(p)\omega}_{\nu m}(x)\big)^{q/\omega} \end{aligned} \tag{5.32}$$

for some δ, $\varepsilon > 0$, with the usual modification if $1 < q \le \infty$.

We sum over j, take the $\frac{1}{q}$–th power and then the L_p–quasi–norm and obtain that

$$\left\|\left(\sum_{j=1}^{\infty}2^{jsq}\left|k\left(2^{-j},\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\rho^a_{\nu m}\right)(\,\cdot\,)\right|^q\right)^{1/q}\,\Big|\,L_p\right\| \tag{5.33}$$

can be estimated from above by

$$\begin{aligned} &c\left\|\left(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^q\;\widetilde{\chi}^{(p)}_{\nu m}(\,\cdot\,)^q\right)^{1/q}\,\Big|\,L_p\right\| \\ &+ c'\left\|\left(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^q\,\big(M^a\chi^{(p)\omega}_{\nu m}(\,\cdot\,)\big)^{q/\omega}\right)^{1/q}\,\Big|\,L_p\right\| \end{aligned} \tag{5.34}$$

with the usual modification if $q = \infty$.

The first term of (5.34) is just what we want since $\widetilde{\chi}^{(p)}_{\nu m}$ can be replaced by $\chi^{(p)}_{\nu m}$. With $h_{\nu m} = \lambda_{\nu m}\, \chi^{(p)}_{\nu m}$ the second term of (5.34) can be written as

$$c'' \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} M^a h^{\omega}_{\nu m}(\,\cdot\,)^{q/\omega} \right)^{\omega/q} \,\Big|\, L_{p/\omega} \right\|^{\frac{1}{\omega}} \tag{5.35}$$

(usual modification if $q = \infty$). Recall $1 < \frac{p}{\omega} < \infty$ and $1 < \frac{q}{\omega} \le \infty$ so that we can apply the Fefferman – Stein inequality (4.2) and obtain again what we want.

The term with $j = 0$ can be incorporated by the same technique. □

5.2. Proof of the subatomic decomposition theorem

For the given anisotropy $a = (a_1, \dots, a_n)$ let $|\cdot|_a \in C^\infty(\mathbb{R}^n \backslash \{0\})$ be an anisotropic distance function according to (3.15). We begin with a preparation.

Theorem 5.4. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, let $\big(\varphi_j\big)_{j \in \mathbb{N}_0}$ be a smooth anisotropic dyadic resolution of unity and let $\rho \in S$ be such that $\rho(x) = 1$ if $|x|_a \le 2$ and $\operatorname{supp} \rho \subset [-\pi, \pi]^n$. The operators $U_\varphi : F^{s,a}_{pq} \to f^a_{pq}$ and $T_\rho : f^a_{pq} \to F^{s,a}_{pq}$ defined by*

$$U_\varphi(g) \;=\; \Big\{ (2\pi)^{-n/2}\, 2^{\nu\left(s - \frac{n}{p}\right)} (\varphi_\nu \widehat{g}\,)^\vee (2^{-\nu a} m) \;:\; \nu \in \mathbb{N}_0\,,\, m \in \mathbb{Z}^n \Big\} \tag{5.36}$$

if $g \in F^{s,a}_{pq}$ and

$$T_\rho(\lambda) \;=\; \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}\, 2^{-\nu\left(s - \frac{n}{p}\right)} \check{\rho}(2^{\nu a}\;\cdot\; - m) \tag{5.37}$$

if $\lambda \in f^a_{pq}$, $\lambda = \{\lambda_{\nu m} \,:\, \nu \in \mathbb{N}_0\,, m \in \mathbb{Z}^n\}$ are bounded.

Furthermore, $(T_\rho \circ U_\varphi)(g) = g$ for any $g \in F^{s,a}_{pq}$ and $\big\| U_\varphi(\,\cdot\,) \,|\, f^a_{pq} \big\|$ is an equivalent quasi – norm on $F^{s,a}_{pq}$.

The same result holds for $B^{s,a}_{pq}$ spaces with $0 < p \le \infty$ and with b_{pq} in place of f^a_{pq}.

Remark 5.5. This theorem is due to P. Dintelmann [Din95a, Theorem 1] and is the anisotropic counterpart of the characterization of isotropic function spaces by the φ – transform of M. Frazier and B. Jawerth, see [FrJ90] and [FJW91]; he considered in the cited paper more general distance functions but for our purpose the above form of his result will be sufficient. We have to remark that he made the proof using the density of S in $B^{s,a}_{pq}$ and $F^{s,a}_{pq}$ (see 2.3) and, in consequence, restricted to $0 < p < \infty$, $0 < q < \infty$. There is no problem to obtain his result for all admissible values of parameters; this can be done, for example, as in [Tri97, 14.15].

P r o o f (of the subatomic decomposition theorem). As usually we present here the proof for $F^{s,a}_{pq}$ spaces; of course the same can be done for $B^{s,a}_{pq}$.

Step 1. Assume that $g \in S'$ is given by (3.18) with (3.19). We show that $g \in F^{s,a}_{pq}$ and that there exists a constant $c > 0$ (independent of g) such that

$$\big\| g \,|\, F^{s,a}_{pq} \big\| \;\le\; c \sup_{\beta \in \mathbb{N}^n_0} 2^{r a \beta} \big\| \lambda^\beta \,|\, f^a_{pq} \big\| \,. \tag{5.38}$$

Since for any given $K \in \mathbb{R}$ and any given $L < 0$ the anisotropic $(s,p) - \beta-$quarks are anisotropic $(s,p)_{K,L}-$atoms multiplied by normalizing constants which can be estimated from above by $c\, 2^{\kappa a \beta}$ where $c > 0$ and $\kappa > 0$ are independent of β, it follows from the atomic decomposition theorem that for any fixed $\beta \in \mathbb{N}^n_0$ it follows that

$$g^\beta \;=\; \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m}\, (\beta q u)^a_{\nu m} \tag{5.39}$$

converges in S', $g^\beta \in F^{s,a}_{pq}$ and $\big\| g^\beta \,|\, F^{s,a}_{pq} \big\| \le c\, 2^{\kappa a \beta} \big\| \lambda^\beta \,|\, f^a_{pq} \big\|$, where $c > 0$ and $\kappa > 0$ are independent of β (and of course of g). So, for $r > \kappa$

$$\big\| g^\beta \mid F^{s,a}_{pq} \big\| \;\le\; c\, 2^{-(r-\kappa) a \beta} \sup_{\beta \in \mathbb{N}^n_0} 2^{r a \beta} \big\| \lambda^\beta \mid f^a_{pq} \big\| \tag{5.40}$$

where $c > 0$ is independent of β. Applying now the $t-$triangle inequality to $\Big\| \sum_{\beta \in \mathbb{N}^n_0} g^\beta \;\Big|\; F^{s,a}_{pq} \Big\|$, where $t = \min(1,p,q)$, we obtain that $g = \sum_{\beta \in \mathbb{N}^n_0} g^\beta$ converges in $F^{s,a}_{pq}$ and that for some $c > 0$ we have (5.38) and this completes the first part of the proof.

Remark that in this step the restriction $s > \sigma_{pq}$ was essentially for the using of the atomic decomposition theorem with no moment conditions required for the atoms.

Step 2. If now $g \in F^{s,a}_{pq}$ we will show that we can decompose it as in (3.18) with (3.19). Let $\rho \in S$ be such that $\rho(x) = 1$ if $|x|_a \le 2$ and supp $\rho \subset [-\pi,\pi]^n$, and let ψ be the function from (3.16).

We may assume $r \in \mathbb{N}$.

By the above theorem there exists a sequence $\lambda \in f^a_{pq}$, $\lambda = \{\lambda_{\nu m} \,:\, \nu \in \mathbb{N}_0\,, m \in \mathbb{Z}^n\}$ such that

$$g(x) \;=\; \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}\, 2^{-\nu\left(s - \frac{n}{p}\right)} \sum_{k \in \mathbb{Z}^n} \check{\rho}(2^{\nu a} x - m)\, \psi\big(2^{(\nu + r) a} x - k\big) \tag{5.41}$$

if $x \in \mathbb{R}^n$ where

$$\big\| \lambda \,|\, f^a_{pq} \big\| \;\sim\; \big\| g \,|\, F^{s,a}_{pq} \big\| \,. \tag{5.42}$$

The entire analytic function $\check{\rho} \in S$ can be extended from $\mathbb{R}^n$ to $\mathbb{C}^n$.

Using $c_1\,(1 + |\xi|)^{1/a_{\max}} \le 1 + |\xi|_a \le c_2\,(1 + |\xi|)^{1/a_{\min}}$ for some constants c_1, $c_2 > 0$ which depend only on the anisotropy a (see [Leo86]), by the Paley−Wiener−Schwartz theorem we have for any $\varepsilon > 0$ and an appropriate $c_\varepsilon > 0$

$$|\check{\rho}(x + iy)| \le c_\varepsilon \exp(c|y|)(1 + |x|_a)^{-\varepsilon} \tag{5.43}$$

(see [Tri83, 1.2.1]) where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$.

Iterative application of Cauchy's representation formula in the complex plane yields

$$(5.44)\quad \check{\rho}(z) \;=\; (2\pi i)^{-n} \int_{|\zeta_1 - z_1|=1} \cdots \int_{|\zeta_n - z_n|=1} \frac{\check{\rho}(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1)\ldots(\zeta_n - z_n)}\, d\zeta_1 \,\ldots\, d\zeta_n$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. By (5.43) we obtain from (5.44) in particular

$$(5.45)\qquad \left|D^\beta \check{\rho}(x)\right| \;\le\; c'_\varepsilon\, \beta!\, (1 + |x|_a)^{-\varepsilon} \quad \text{for} \quad x \in \mathbb{R}^n$$

where c'_ε does not depend on $x \in \mathbb{R}^n$ and on the multi−index β.

For every fixed $k \in \mathbb{Z}^n$ we expand $\check{\rho}(2^{\nu a} \,\cdot\, - m)$ at the point $2^{-(\nu+r)a}k$ and obtain

$$(5.46)\qquad \check{\rho}(2^{\nu a}x - m) \;=\; \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{\nu a\beta}\, D^\beta \check{\rho}(2^{-ra}k - m)}{\beta!}\, (x - 2^{-(\nu+r)a}k)^\beta$$

and so $\check{\rho}(2^{\nu a}x - m)\, \psi\big(2^{(\nu+r)a}x - k\big)$ can be replaced in (5.41) by

$$(5.47)\qquad \sum_{\beta \in \mathbb{N}_0^n} \frac{D^\beta \check{\rho}(2^{-ra}k - m)}{\beta!}\, 2^{-ra\beta}\, \psi^\beta\big(2^{(\nu+r)a}x - k\big)\,.$$

We get

$$\begin{aligned} g(x) \;&=\; \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}\, 2^{-\nu\left(s - \frac{n}{p}\right)} \sum_{k \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{N}_0^n} \frac{D^\beta \check{\rho}(2^{-ra}k - m)}{\beta!}\, 2^{-ra\beta}\, \psi^\beta\Big(2^{(\nu+r)a}x - k\Big) \\ (5.48)\qquad &=\; \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} 2^{-\nu\left(s - \frac{n}{p}\right)} \psi^\beta\Big(2^{(\nu+r)a}x - k\Big) \left(\sum_{m \in \mathbb{Z}^n} \frac{D^\beta \check{\rho}(2^{-ra}k - m)}{\beta!}\, 2^{-ra\beta} \lambda_{\nu m} \right) \\ &=\; \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \lambda^\beta_{\nu+r,k}\, 2^{r\left(s - \frac{n}{p}\right)} (\beta qu)^a_{\nu+r,k}(x) \end{aligned}$$

where $(\beta qu)^a_{\nu k}(x) \;=\; 2^{-\nu\left(s - \frac{n}{p}\right)} \psi^\beta(2^{\nu a}x - k)$ are the anisotropic $(s,p) - \beta-$quarks $\big($related to the rectangles $Q^a_{\nu k}\big)$ and

$$(5.49)\qquad \lambda^\beta_{\nu+r,k} \;=\; 2^{-ra\beta} \sum_{m \in \mathbb{Z}^n} \frac{D^\beta \check{\rho}(2^{-ra}k - m)}{\beta!}\, \lambda_{\nu m} \;=\; 2^{-ra\beta}\, \theta^\beta_{\nu+r,k}\,.$$

We may replace in (5.48) $\nu + r$ by ν and obtain (3.18) (recall we assumed $r \in \mathbb{N}$).

Let $\theta^\beta \;=\; \big\{\theta^\beta_{\nu k} \;:\; \nu \in \mathbb{N}_0\,,\, k \in \mathbb{Z}^n\big\}$; if we would prove now that there exists a constant $c > 0$ independent of β such that

$$(5.50)\qquad \left\|\theta^\beta \,|\, f^a_{pq}\right\| \;\le\; c\, \left\|\lambda \,|\, f^a_{pq}\right\|$$

we would obtain from (5.49) and (5.42)

$$(5.51)\qquad 2^{ra\beta}\, \left\|\lambda^\beta \,|\, f^a_{pq}\right\| \;\le\; c \left\|g \,|\, F^{s,a}_{pq}\right\|$$

for some $c > 0$ independent of β and this is the counterpart of (3.19).

So it remains to prove (5.50) where the numbers $\theta^\beta_{\nu k}$ are defined from (5.49) by

$$\theta^\beta_{\nu+r,k} \;=\; \sum_{m\in\mathbb{Z}^n} \frac{D^\beta\check{\rho}(2^{-ra}k-m)}{\beta!}\,\lambda_{\nu m} \quad \text{if} \quad \nu\in\mathbb{N}_0 \quad \text{and} \quad k\in\mathbb{Z}^n\,. \tag{5.52}$$

Let now $\nu\in\mathbb{N}_0$ and $k\in\mathbb{Z}^n$ be fixed. By (5.45) there exists a constant $c>0$ independent of β with

$$\begin{aligned} \left|\theta^\beta_{\nu+r,k}\right| \;&\le\; c\sum_{m\in\mathbb{Z}^n}(1+|2^{-ra}k-m|_a)^{-\varepsilon}|\lambda_{\nu m}| \\ &=\; c\sum_{m\in\mathbb{Z}^n}(1+|2^{\nu a}(2^{-(\nu+r)a}k)-m|_a)^{-\varepsilon}\,|\lambda_{\nu m}|\,. \end{aligned} \tag{5.53}$$

We denote $x_k = 2^{-(\nu+r)a}k$ and let $m_k\in\mathbb{Z}^n$ be such that $x_k\in Q^a_{\nu,m_k}$; then clearly $\left|2^{\nu a}x_k-m_k\right|_a\le d$ for some $d>0$, where d depends only on the anisotropy and is independent of ν, k and m_k.

We decompose $\mathbb{Z}^n$ in the sets $E_j=\left\{m\in\mathbb{Z}^n\;:\;2^j-1\le|m-m_k|_a<2^{j+1}-1\right\}$ where $j\in\mathbb{N}_0$. If j is fixed, for $m\in E_j$ we have on the one hand

$$\begin{aligned} 2^j \;&\le\; 1+|m-m_k|_a \\ &\le\; c\left(1+|2^{\nu a}x_k-m_k|_a+|2^{\nu a}x_k-m|_a\right) \\ &\le\; c'\left(1+|2^{\nu a}x_k-m|_a\right) \end{aligned} \tag{5.54}$$

where $c'>0$ is independent of ν, k, m and so

$$\left(1+|2^{\nu a}x_k-m|_a\right)^{-\varepsilon} \;\le\; c\,2^{-j\varepsilon}\,. \tag{5.55}$$

On the other hand, if $x\in Q^a_{\nu+r,k}$ and $y\in Q^a_{\nu m}$ using

$$|y-x|_a \;\le\; c\left(|y-2^{-\nu a}m|_a+|2^{-\nu a}m-2^{-\nu a}m_k|_a+|2^{-\nu a}m_k-x_k|_a+|x_k-x|_a\right)$$

we get

$$|y-x|_a \;\le\; c'\,2^{-\nu}\left(1+|m-m_k|_a\right) \;\le\; C\,2^{j-\nu} \tag{5.56}$$

where $C>0$ is independent of ν, k, m but may depend on r.

Choose now $0<\omega<\min(1,p,q)$; for a fixed ν the rectangles $Q^a_{\nu m}$ have the volume $2^{-\nu n}$ and are disjoint so that using the embedding $l_\omega\hookrightarrow l_1$ and (5.56) we obtain

$$\begin{aligned} \sum_{m\in E_j}|\lambda_{\nu m}| \;&\le\; \left(\sum_{m\in E_j}|\lambda_{\nu m}|^\omega\right)^{1/\omega} \\ &=\; \left(2^{\nu n}\int_{|y-x|_a\le C\,2^{j-\nu}}\left(\sum_{m\in E_j}|\lambda_{\nu m}|\,\chi_{\nu m}(y)\right)^\omega\,dy\right)^{1/\omega} \\ &\le\; c\left(2^{jn}\,M^a\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\,\chi_{\nu m}\right)^\omega(x)\right)^{1/\omega} \end{aligned} \tag{5.57}$$

for $x \in Q^a_{\nu+r,k}$ and where M^a is the anisotropic Hardy – Littlewood maximal function and all the constants are independent of ν, k, m.

Let $\chi_{\nu+r,k}$ be the characteristic function of the rectangle $Q^a_{\nu+r,k}$ and $\chi^{(p)}_{\nu+r,k} = 2^{(\nu+r)\frac{n}{p}}\chi_{\nu+r,k}$. Using (5.55) and (5.57) in (5.53) and assuming that $\varepsilon > n/\omega$ is sufficiently large we have

$$
\begin{aligned}
\left|\theta^\beta_{\nu+r,k}\,\chi^{(p)}_{\nu+r,k}(x)\right| &\leq c_1 \sum_{m\in\mathbb{Z}^n}\big(1+|2^{\nu a}x_k - m|_a\big)^{-\varepsilon}\,|\lambda_{\nu m}|\,2^{(\nu+r)\frac{n}{p}}\chi_{\nu+r,k}(x)\\
&\leq c_2\sum_{j=0}^\infty 2^{-j\varepsilon}\sum_{m\in E_j}|\lambda_{\nu m}|\,2^{\nu\frac{n}{p}}\chi_{\nu+r,k}(x)\\
&\leq c_3\sum_{j=0}^\infty 2^{-j\left(\varepsilon-\frac{n}{\omega}\right)}\left(M^a\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\,\chi^{(p)}_{\nu m}\right)^\omega(x)\right)^{1/\omega}\chi_{\nu+r,k}(x)\\
&\leq c\left(M^a\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\,\chi^{(p)}_{\nu m}\right)^\omega(x)\right)^{1/\omega}\chi_{\nu+r,k}(x)
\end{aligned}
\tag{5.58}
$$

where the constants above do not depend on ν and k but may depend on r.

In (5.58) we take the q – th power, sum over $k \in \mathbb{Z}^n$ and then over $\nu \in \mathbb{N}_0$ and get

$$
\sum_{\nu=0}^\infty\sum_{k\in\mathbb{Z}^n}\left|\theta^\beta_{\nu k}\,\chi^{(p)}_{\nu k}(x)\right|^q \;\leq\; c\sum_{\nu=0}^\infty\big(M^a h^\omega_\nu(x)\big)^{q/\omega}
\tag{5.59}
$$

(with the usual modification if $q=\infty$) where $h_\nu = \sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\,\chi^{(p)}_{\nu m}$.

Taking the $\frac{1}{q}$ – th power and the L_p – quasi – norm we obtain that $\left\|\theta^\beta \mid f^a_{pq}\right\|$ can be estimated from above by

$$
c\left\|\left(\big(M^a h^\omega_\nu(\,\cdot\,)\big)^{1/\omega}\right)_{\nu\in\mathbb{N}_0}\;\Big|\,L_p\big(l_q\big)\right\| \;=\; c\left\|\big(M^a h^\omega_\nu(\,\cdot\,)\big)_{\nu\in\mathbb{N}_0}\;\big|\,L_{p/\omega}\big(l_{q/\omega}\big)\right\|^{1/\omega}
\tag{5.60}
$$

(with the usual modification if $q=\infty$). To obtain (5.50) we have now only to apply the Fefferman – Stein inequality (4.2) to the right – hand side of (5.60); this can be done since $1<\frac{p}{\omega}<\infty$ and $1<\frac{q}{\omega}\leq\infty$ and so the proof is finished. □

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