Operational Risk Quantification using Extreme Value Theory and Copulas: From Theory to Practice

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Abstract

In this paper we point out several pitfalls of the standard methodologies for quantifying operational losses. Firstly, we use Extreme Value Theory to model real heavy-tailed data. We show that using the Value-at-Risk as a risk measure may lead to a mis-estimation of the capital requirements. In particular, we examine the issues of stability and coherence and relate them to the degree of heavy-tailedness of the data. Secondly, we introduce dependence between the business lines using Copula Theory. We show that standard economic thinking about diversification may be inappropriate when infinite-mean distributions are involved.

Keywords: Extreme Value Theory, Copula Theory, Value-at-Risk, Sub-additivity, Coherence.

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Introduction

On January 24 2008, Société Générale, one of the largest bank in Europe, was thrown into turmoil after it revealed that a rogue employee, Jérôme Kerviel, had executed a series of "elaborate, fictitious transactions" that cost the company more than 4.9 billion euros, the biggest loss ever recorded in the financial industry by a single trader.

Events similar to that one continue to shake the world's financial markets, and raise the awareness of banks, trading houses and regulatory agencies regarding the inherent operational risks involved in trading operations.

In an attempt to provide a regulatory framework to handle operational risk, the Basel Committee on Banking Supervision published in 2004 and updated in 2007 a new Basel Accord which opens the door to operational risk, defined by the Basel Committee of Banking Supervision (2004) as

> *"the risk of direct or indirect loss resulting from inadequate or failed internal processes, people and systems or from external events. This definition includes legal risk, but excludes strategic and reputational risk."*

This paper deals with the most elaborate approach for calculating the operational risk capital requirements, the Loss Distribution Approach (LDA), which relies on internal simulations of potential loss distributions.

Several books have been published on operational risk quantification, see McNeil et al. (2005) and Panjer (2006). They stress the relevance of the Extreme Value Theory (EVT) in the LDA, and give an accurate overview of the mathematical methods currently available for this purpose. In addition, concrete studies on operational loss data have been

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performed over the past few years. We refer in particular to Moscadelli (2004), De Fontnouvelle et al. (2004) and Dutta and Perry (2007).

In this paper we present the challenges of quantifying operational losses, and some shortfalls of the standard methodologies. Our results are based on real operational risk data, therefore we hope that this analysis will contribute to link purely theoretical properties of heavy-tailed distributions to the practical analysis of operational risk.

In a first part, we show that EVT provides us with appropriate tools to fit our data and model their heavy-tailedness. We calculate capital charges at the enterprise and business line level. We investigate the consequences of using the Value-at-Risk (VaR) as a risk measure and discuss the implications of dealing with heavy-tailed data. In particular, we examine the stability and accuracy of the capital charges as calculated following the Basel II requirements.

In a second part, we use copulas to aggregate the losses coming from the different business lines. We discuss the role of the copula and compare it to that of the degree of heavy-tailedness of the data.

Our analysis is based on data[4] collected from an individual bank database over a time-window of 4 years, between 2002 and 2006, categorized in 9 business lines and 7 event types. The overall number of observations is 7514.

[4]We would like to thank Deloitte AG for providing us with the data used in this paper. Their origin will not be revealed for confidentiality purposes.

1 Modeling Operational Risks using EVT

1.1 Why use EVT?

To start, let us perform a preliminary analysis of our data using classical statistical tools. We assume here that our losses are independent and identically distributed (iid) realizations of random variables stemming from a single distribution function F.

Figure 1 shows the time series of aggregated losses, ordered by their time of discovery.

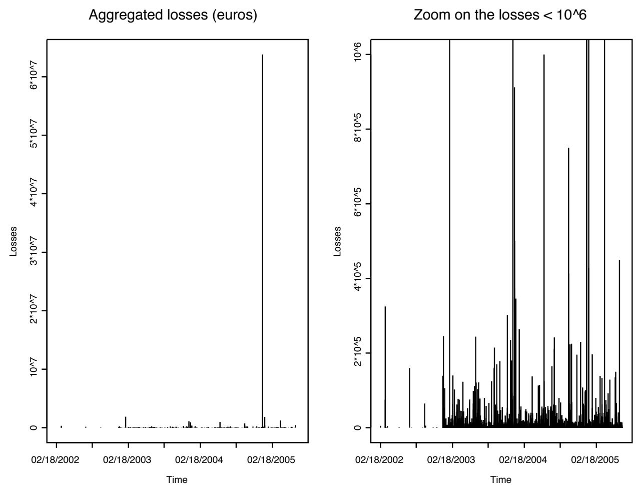

Figure 1: *Time series of aggregated data*

We can observe several stylized effects which are in line with Embrechts et al. (2003) and Chavez-Demoulin and Embrechts (2004): presence of several extreme losses, high skewness (estimated equal to 64.17) and kurtosis (estimated equal to 114), non-stationarity of the data, potential reporting bias[5]. Thus, the overall data used in this analysis appear to capture the usual trends that characterize operational risk.

[5] Frachot et al. (2004a) evaluate the impact of the reporting bias on capital charges estimates.

We fit classical distributions to the data using the criteria of maximum likelihood, starting from light-tailed distributions (e.g. Weibull), to medium-tailed distributions (e.g. Gamma, exponential and lognormal). We adapt the Kolmogorov-Smirnov (K-S) test[6] and the Anderson-Darling (A-D) test[7] to measure the distance between the empirical and theoretical distribution functions only in the tail area, after the 93% quantile. Table 1 reports the goodness-of-fit values as well as the critical values, computed using Monte-Carlo simulations.

	K-S	A-D[a]	c.v$_{5\%}$ K-S	c.v$_{5\%}$ A-D
Exponential	0.43	$+\infty$	0.007	0.20
Weibull	0.07	$+\infty$	0.007	0.23
Gamma	0.07	$+\infty$	0.007	0.21
Lognormal	0.015	3.78	0.007	0.23

[a]By construction, A-D test yields infinite values when the theoretical distribution has a finite endpoint which is below that of the empirical distribution.

Table 1: *Classical distributional fittings*

All p-values are equal to zero. The lognormal distribution is the only model for which the A-D test value is finite. However, it clearly does not reproduce the heavy-tailedness of the data. The failure of conventional distributions for our purpose is due to rare and extreme events, which are not captured by usual statistical tools.

1.2 Analysis of the aggregated losses using EVT

The key attraction of Extreme-Value Theory (EVT) is that it focuses on the analysis of the tail area of the distribution, providing appropriate methods for modeling extreme losses and their impact in insurance, finance and quantitative risk management. Thus, it has

[6]See Eadie et al. (1971)
[7]See Anderson and Darling (1952)

developed very quickly over the last decade. Specific conditions on the data are required to use EVT, which explains why it is still open to criticisms, discussed in Embrechts et al. (2003) and Embrechts et al. (2004). McNeil et al. (2005) detail in chapter 10 of their book the statistical methods used in EVT, applied to finance and insurance. The original mathematical theory is available in Embrechts et al. (2002a).

EVT provides an accurate model for exceedances over a high threshold by the means of the Generalized Pareto Distribution (GPD), defined as follows:

$$GPD_{\xi,\beta}(x) = \begin{cases} 1 - (1 + \frac{\xi x}{\beta})^{\frac{-1}{\xi}}, & \xi \neq 0 \\ \\ 1 - exp(-\frac{x}{\beta}), & \xi = 0 \end{cases}$$

where $\beta > 0$, $x \geq 0$ when $\xi \geq 0$ and $0 \leq x \leq \frac{-\beta}{\xi}$ when $\xi < 0$. The parameters ξ and β are referred to, respectively, as the shape and scale parameters. We refer to McNeil et al. (2005) for explanations on the role and properties of each parameter.

The main theorem of that approach has been developed by Balkema and de Haan (1974) and Pickands (1975). It states that the following statement holds under assumptions which are satisfied for most of the classical distributions - we refer to McNeil et al. (2005) p.277 for the details -.

Theorem 1.1. (Balkema-De Haan 1974 and Pickands 1975) *Under certain conditions, the excess distribution $F_u(x)$ converges toward the GPD as the threshold is raised to the right endpoint of the distribution:*

$$\lim_{u \to x_F} \sup_{0 \leq x < x_F - u} |F_u(x) - GPD_{\xi,\beta(u)}(x)| = 0,$$

where the excess distribution is defined as in p. 276 of McNeil et al. (2005), by

$$F_u(x) = P(X - u \leq x | X > u) = \frac{F(x+u) - F(u)}{1 - F(u)}$$

for $0 \leq x < x_F - u$, and $x_F \leq \infty$ is the right endpoint of F.

In light of this theorem, we fit separately the body and the tail of the distribution. We use conventional inference to model the left part of the distribution and the GPD to model the tail.

1.2.1 Tail estimation

We use a mean-excess plot, as defined in p.279 of McNeil et al. (2005) to assess the validity of modeling the tail by a power-law distribution. Figure 2 represents the mean excess function of the aggregated losses. The graph presents a clear upwards linearity, hence the data has a reasonable chance to be well modeled by a GPD distribution.

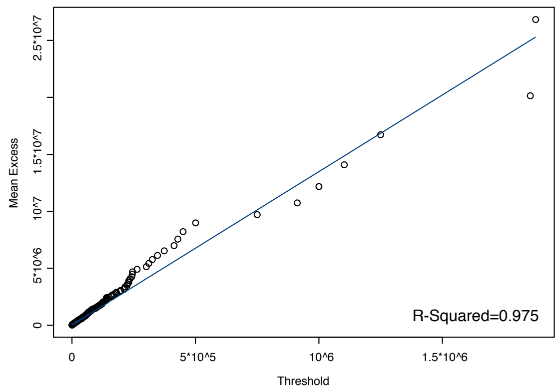

Figure 2: *Mean excess plot of the aggregated data*

Additionally, we use the Hill method to get a rough estimate of the shape parameter. The Hill estimator $\hat{\xi}_k = \frac{1}{\hat{\alpha_k}}$, as defined in Hill (1975) and detailed in McNeil et al. (2005) and Koedijk et al. (1990), has become a benchmark in the literature due to its easy implementation and asymptotic unbiasedness. It has been shown to be consistent with fat-tailed distributions (see Mason (1982)), and has been extensively studied for its practicalities. Furthermore, if the underlying loss distribution is Pareto, then the Hill estimator is the maximum likelihood estimate of the tail thickness parameter. The primary weakness of this index lies in the need to determine the size of the tail a priori through the determination of the number of observations k in the tail area. For a certain type of functions including the GPD, a small k helps decrease the bias because the power law is assumed to hold only in the extreme tail, but a large k reduces variance - since more data points are used - and hence yields a better precision.

A Hill plot $\{(k, \hat{\alpha_k})\}$ and the corresponding 0.90 Wald confidence interval for $\hat{\alpha_k}$ are drawn on Figure 3.

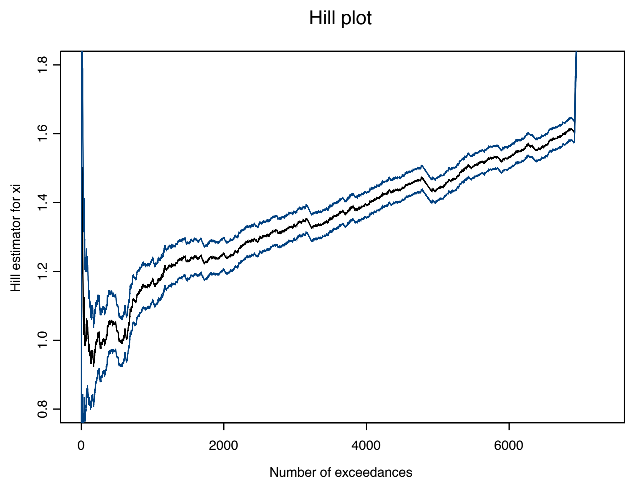

Figure 3: *Hill plot of the aggregated losses and 0.95 confidence interval*

The graph suggests that the shape parameter lies between 0.65 and 1, meaning that our data comes from a very heavy-tailed distribution. This result is confirmed by using the regression-based EVT technique proposed by Huisman et al. (2001), which corrects the bias of the Hill estimator in small samples and minimizes the role of the threshold selection.

1.2.2 Separation between the body and the tail

The selection of the threshold is a key modeling aspect. Indeed, it should be large enough to satisfy the limit law condition. However it has to leave enough observations to accurately estimate the severity of the tail. To evaluate the stability of the shape parameter, a shape plot, based on the comparison of the ξ estimates across a variety of thresholds, is used. For details on that technique we refer to Embrechts et al. (2002a), p.339. Figure 4 shows that the shape parameter is quite stable since the beginning.

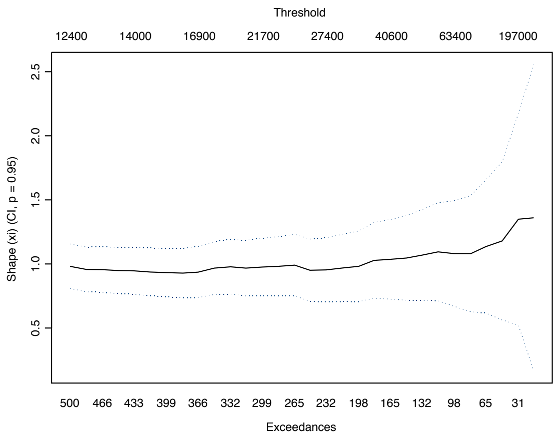

Figure 4: *Shape plot of the aggregated data*

We fix the threshold at the 90% percentile and model the data with a lognormal distribution below and a GPD for the excesses. Figure 5 displays two graphical plots illustrating the quality of the fitting: the first one represents the tail of the empirical distribution versus that of the fitted GPD, and the second one is a QQ-plot of the excesses.

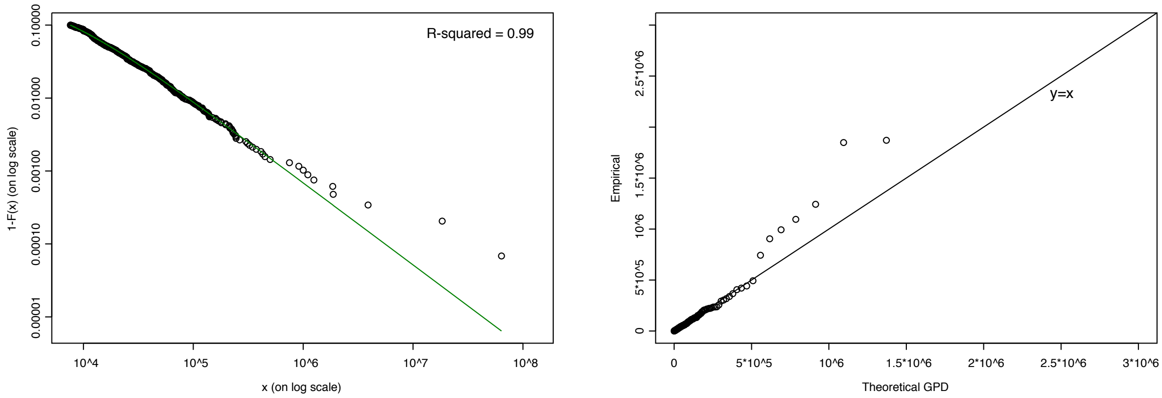

Figure 5: *Tail plot and QQ-plot of the excesses*

The GPD scale and shape estimates are respectively $\hat{\beta} = 10691.28$ and $\hat{\xi} = 0.89$. The 0.95 confidence interval $[0.75, 1.05]$ for the shape parameter is obtained through a bootstrapping procedure. This interval outlines the uncertainty about whether the model has finite mean or not.

Following Moscadelli (2004), a severity VaR performance analysis is performed to compare the different levels of accuracy of the GPD versus the lognormal distribution in representing the highest percentiles of the data. The expected and the estimated number of violations are compared for several given levels of confidence of the VaR. Table 2 provides the results of the tests.

These results confirm that the GPD model accurately reflects the extreme events compared to the lognormal model, for which the high number of violations indicates a large underestimation of the large losses.

	Number of violations		
Level of confidence	Empirical	GPD	Lognormal
0.950	36.85	37	48
0.975	18.42	18	35
0.990	7.37	10	20
0.995	3.68	5	16

Table 2: *Number of violations for different distributions*

1.2.3 Final model for the severities

The losses below the threshold are modeled by a lognormal distribution, and the excesses by a GPD. Figure 6 and Table 1[8] show that using EVT significantly improved the quality of the modeling.

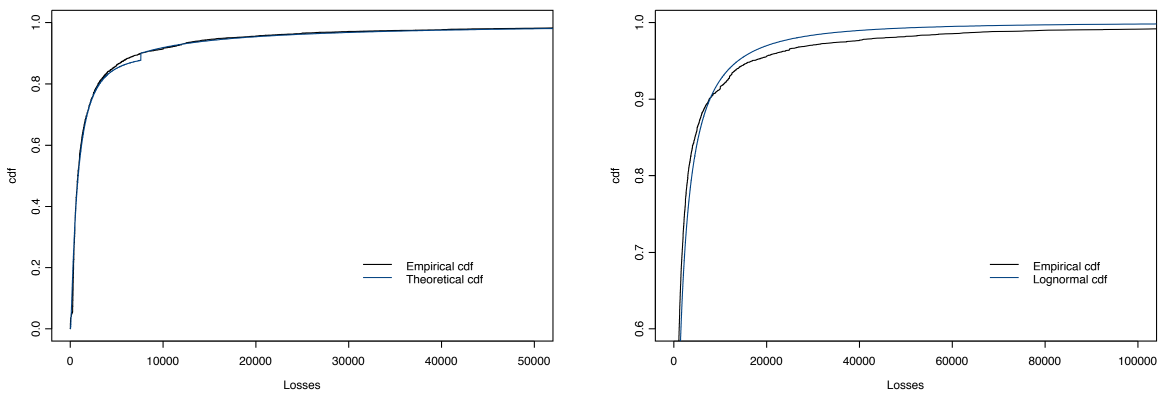

Figure 6: *Theoretical versus empirical distributions for 1- a lognormal model and 2- a log-GPD model*

[8]To be compared to the results obtained in Table 1

K-S	A-D	c.v.$_{5\%}$ K-S	c.v.$_{5\%}$ A-D
0.003	0.026	0.005	0.13

Table 3: *Goodness-of-fit test values for the lognormal-GPD approach*

1.2.4 Calculation of the capital charges for aggregated losses

Let the total claim amount be equal to $S_N = \sum_{i \leq N} X_i$, where N, the number of claims, and X_i, the loss amounts, are assumed to be independent. We estimate S_N with Monte-Carlo simulations, and model the data frequency by a homogeneous Poisson process with daily intensity 6.08. The claims' severities follow the model previously described[9]. The time horizon is chosen equal to one year, following to the Basel II requirements. The results of the simulation for the VaR are reported (in millions[10]) in Table 4.

$conf$	0.90	0.95	0.99	0.999
VaR$_{conf\%}$	26	36	104	639

Table 4: *1 year Value-at-Risk*

These results are coherent with regards to the yearly quantiles of our data. Nevertheless, the capital charges estimate grows very fast with the chosen quantile: the 0.999 VaR is more than 6 times larger than the 0.99 VaR. Therefore it makes sense to perform a sensitivity analysis with respect to the key parameters.

1.2.5 Sensitivity analysis of the capital charges

In this part we study the sensitivity of the VaR to the shape parameter, the level of confidence and the frequency model.

[9]The losses which are below the threshold are modelled using a lognormal distribution and the excesses over the threshold by a GPD

[10]Results are rounded due to the imprecision inherent to the use of Monte-Carlo simulations

• *Sensitivity to the shape parameter*

The Hill plot and the confidence interval for the shape parameter ξ have suggested a quite strong uncertainty. Figure 7 illustrates the impact of the shape parameter on the VaR, for different levels of confidence.

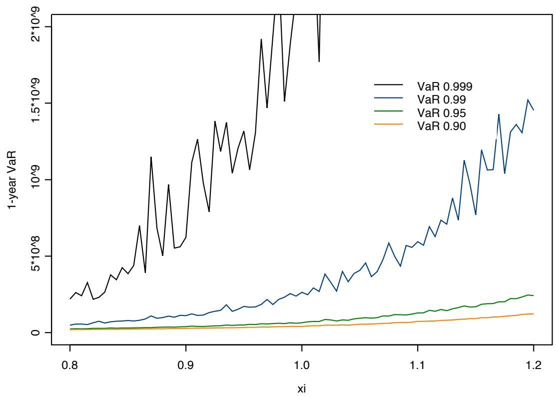

Figure 7: *1 year VaR versus shape parameter*

The 99.9% VaR is very unstable and requires a higher number of simulations. In a reasonable amount of time, it cannot be relied upon for a precise estimate of capital charges. Furthermore, the larger the shape parameter, the less stable the VaR, especially for $\xi > 1$. This is in line with Neslehova et al. (2006), who show that in an infinite mean model, the VaR grows with the tail index parameter at an exponential rate. That is one reason why the heavy-tailedness of our data represents a challenge to precisely calculate the capital charges.

• *Sensitivity to the level of confidence*

The Basel II Accord specifies that the capital charges must be calculated at the 0.999 quantile of the loss distribution. Figure 8 illustrates the relationship between the 1-year

VaR and the quantile it is computed at.

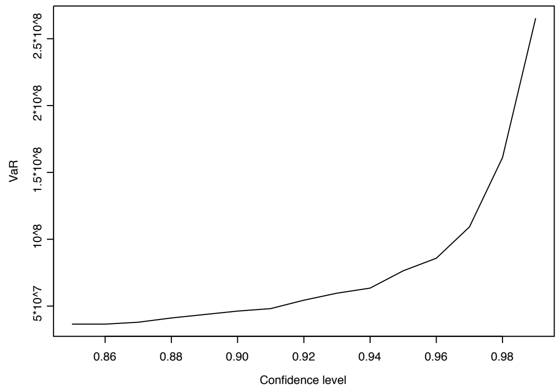

Figure 8: *1 year VaR versus confidence level*

The graph is quite stable between the 0.85% and 0.92% quantiles, then it increases exponentially. So the VaR is not very robust with respect to the confidence level.

• *Importance of the frequency model*

We chose a Poisson process to simulate the loss frequency, which does not reflect the over-dispersion of our data, (see also Evans et al. (2007)). The Negative Binomial distribution provides a way to solve this issue. However, Table 5 shows that the differences in the computed capital charges are quite small (around 5%) before the 0.999 level of confidence. After this level, the VaR is higher than using the Poisson model, which already seems to overestimate the capital charges. This motivates the use of the Poisson model as a good approximation for capital charges in the rest of this study.

	0.90	0.95	0.99	0.999
VaR Neg. Bin.	28	38	99	724
VaR Neg. Bin./Poisson	×1.04	×1.05	×0.95	×1.13

Table 5: *Capital charges using the Negative Binomial distribution for frequencies*

1.3 Analysis of the losses by business line using EVT

The analysis at the enterprise level highlights some general properties of our data. However, we also conduct an analysis at the business line level, since it is likely to outline inhomogeneities from one business line to another. Due to a lack of data, we restrict ourselves to the business lines (BL) for which we have more than 250 data points: BL1, BL3, BL6, BL7 and BL9. We now assume that the losses belonging to BL_i are i.i.d. realizations of random variables with distribution F_i.

1.3.1 Final individual models

Based on the results of conventional inference and EVT and following the approach described in Section 1.2, the models reported in Table 6 are inferred:

	Distribution	Threshold	$\hat{\xi}$	confidence interval
BL1	lognormal-GPD	0.85	0.87	[0.55,1.23]
BL3	lognormal-GPD	0.65	1.70	[1.13,2.25]
BL6	GPD	-	0.85	[0.77,1.01]
BL7	lognormal-GPD	0.90	1.05	[0.85,1.20]
BL9	lognormal-GPD	0.75	0.37	[0.13,0.60]

Table 6: *Final models for each business line*

We can notice that BL9 is much lighter-tailed than the other BL; BL1, BL6 and BL7 are quite heavy-tailed with shape parameters very close to 1, and BL3 is the heaviest-tailed BL, with a shape parameter close to 2.

1.3.2 Analysis of the tail severity

Using the properties of the GPD described in McNeil et al. (2005), p.283, we calculate the VaR of the business lines' severities, to give an idea of the magnitude of the tail. Table 7 reports the results.

Business line	0.95	0.99	0.999
BL1	11 396	47 549	330 467
BL3	45 309	733 620	39 735 356
BL6	23 104	127 463	1 408 266
BL7	10 297	43 150	321 986
BL9	50 973	118 403	322 985

Table 7: *VaR of the severity distribution of the business lines modelled with GPD distributions*

Owing to the heavy-tailedness of the data, the VaR grows very rapidly with the quantile it is calculated at. Furthermore, this table confirms the importance of BL3 regarding the amount of money the bank has to put aside. At the 99.9% percentile, the VaR corresponding to BL3 is 17 times larger than the sum of all the VaRs of other business lines modeled by the GPD distribution. The biggest losses in BL3 have therefore a fundamental impact on the overall capital charges.

This is in line with the concept of subexponentiality[11], referred to in Embrechts and Puccetti (2006) as the "one loss causes ruin problem". This issue arises for certain distributions including the GPD and lognormal distribution. It is strongly related to the tail of the distribution. For instance, for a tail index close to 1 - which is our case for BL1 and BL7 - 0.1% of the losses produce 95% of the total portfolio loss. Intuitively, that means that one single loss is responsible for a large part of the sum.

[11] See Mandelbrot and Hudson (2004), p.232, for a discussion of the consequences of subexponentiality.

1.3.3 Capital charges at the business line level

The capital charges are computed and reported (in thousands) in Table 8, with regards to the confidence level of the VaR.

	0.90	0.95	0.99	0.999
BL1	1 846	2 696	7 963	48 430
BL3	43 865	138 513	2 134 864	107 031 623
BL6	3 905	6 197	23 118	200 134
BL7	25 546	42 798	184 316	1 967 351
BL9	2 691	2 897	3 396	4 486

Table 8: *1 year VaR for each business line*

The confidence intervals for the VaR calculation are very large for the highest percentiles, as illustrated by Table 9 for BL1. This means that the VaR cannot be accurately determined at a high level of precision given the data we have.

	BL1
0.90	[2 073, 2 220]
0.95	[3 120, 3 430]
0.99	[9 347 , 12 225]
0.999	[38 325 , 121 738]

Table 9: *Confidence intervals of the VaR for BL1*

Furthermore, the total capital charges mainly reflect the big losses that have occurred in BL3. Very few events drive the dynamics of the whole system, as described in Section 1.3.2.

Let us point out that the VaR for BL3 is higher than the global VaR for aggregated losses, as calculated in Table 4. This is probably due to the uncertainty of both models: the first one, at the enterprise level, does not take into account the inhomogeneity of the data, and the latter, at the individual level, suffers from a much smaller amount of data in each business line examined.

1.3.4 Accuracy of the capital charges by business line

A fundamental question is raised at this level of the study: are our calculations of VaR accurate to determine the capital requirements? It is well known that the VaR does not satisfy the sub-additivity axiom described in Delbaen (2002), and is therefore a non-coherent risk measure. The comparison of the sum of the VaR for all business lines (in millions) modeled with a lognormal-GPD distribution, to the VaR of their sum is summarized in Table 10. Let us point out that business line 3, which satisfies $\xi > 1$, is part of the current calculation.

	Var($\sum_{i=1,3,7,9} BL_i$)	$\sum_{i=1,3,7,9}$ (Var BL_i)
0.90	87	74
0.95	214	187
0.99	2 461	2 331
0.999	126 007	109 052

Table 10: *Coherence of the Value-at-Risk*

The results show that the VaR is super-additive, meaning that one has to be careful when choosing VaR as the risk measure to compute capital charges when models have infinite mean as it is the case for BL3. Following Neslehova et al. (2006), these kinds of models cause serious problems regarding diversification. Indeed, they yield higher capital charges for uncorrelated losses than for correlated losses, which goes against the diversification principle.

We make the same calculations without BL3, i.e. only with business lines which have shape parameters smaller than 1 or around one (BL7). We get the results reported in Table 11.

	$\text{Var}(\sum_{i=1,7,9} BL_i)$	$\sum_{i=1,7,9} (\text{Var } BL_i)$
0.90	30	30
0.95	47	48
0.99	196	195
0.999	1 922	2 020

Table 11: *Coherence of the Value-at-Risk for less heavy-tailed distributions*

The differences between the sum of the VaR and the VaR of the sum are much less significant here. Therefore, the results suggest that distributions satisfying $\xi \leq 1$ allow approximating the capital charges by the sum of the business lines' individual VaR with more realism. The same calculations when removing BL7 confirm this result. The heavier-tailed the data, the more dangerous and unrealistic the computation of the capital charges.

In theory, Proposition 4.1 of Degen et al. (2007) states that when $\xi < 1$, the VaR_α is subadditive for α sufficiently large. This has critical implications in terms of risk management. Indeed, we have shown that VaR was neither stable nor reliable at very high quantiles. In practice, this is often dealt with by calculating it at smaller quantiles, for instance 0.90, and scaling it up. Our study shows that this is not accurate since the VaR might get super-additive and lead to underestimated capital charges, if infinite-mean distributions are involved.

1.3.5 Concluding remarks on the application of the VaR to heavy-tailed data

What should be remembered from this is that in the case of extremely heavy-tailed loss distributions, standard economic thinking about diversification and the VaR are inappropriate. The analysis of our data is in line with the literature and emphasizes the pitfalls of using the VaR as a risk measure. However, Delbaen (2002) shows that any coherent risk measure ρ which is dependent on the distribution function of the risks and for which $\rho \geq VaR_\alpha$ has to satisfy $\rho \geq ES_\alpha$, where ES (Expected Shortfall) requires the existence of the first moment (i.e. $\xi < 1$). This means that ES is the smallest coherent risk measure to be greater than the VaR, and, hence that there does not exist any coherent risk measure larger than the VaR which yield finite capital charges, if the losses are GPD with $\xi > 1$.

2 Modeling of the dependency structure of operational risks with copulas

Calculating the minimum capital requirements as the sum of the VaR over the different business lines assumes a perfect dependence between them. However, Frachot et al. (2004b) argue on the one side that operational risk models cannot, by construction, show high levels of correlation between losses from different business lines, suggesting that the capital charges are highly overestimated. On the other side, Embrechts and Puccetti (2006) point out that when VaR is not sub-additive, dependence can lead to an underestimation of the capital requirements. In this section, we see how copulas can be used to describe dependencies and we analyze the variations of capital they lead to.

Copulas express dependence on a quantile scale and facilitate a bottom-up approach for building multivariate models[12]. Many successful applications have been developed recently, particularly in actuarial science, survival analysis and hydrology. In finance, the methodology is extensively studied in the books of Cherubini et al. (2004) and McNeil

[12]The copula approach provides a way to describe the joint distribution of a random vector of risk factors through the individual behaviors of each of the risk factors and the dependence structure which links them.

et al. (2005).

2.1 Preliminary analysis of the dependence structure

Coefficients of tail dependence and rank correlations are used to detect dependence between the different business lines. They measure pairwise extremal dependence and only depend on the copula of a pair of random variables. A formal definition as well as further explanations on the intuition behind can be found in McNeil et al. (2005), p.206-209.

To evaluate these coefficients, we calculate monthly total losses for each business line and assume that the dependence structure on a monthly time horizon is identical to that on a yearly basis. Empirical pairwise coefficients of upper tail dependence are around 0.5 between BL1, BL6, BL7 and BL9. All pairs of business lines present lower tail dependence, which is not surprising considering the large amount of small losses in the database.

The computation of Spearman and Kendall rank correlation matrices: $\rho_{S/\tau}(BL_i, BL_j)$, for the 5 business lines that we considered in the last section, clearly shows that no BL is perfectly correlated with another BL. This goes along with Frachot et al. (2004b). The rank correlations are in most of the cases quite weak, except BL1/BL9. As in Genest and Favre (2007), we performed a test of independence based on Kendall's τ. The hypothesis H_0 that two business lines are independent is only rejected for BL1/BL9.

Many copulas are compatible with this dependence structure. However, very few of them can be used in a n-dimension framework, with $n > 2$. We will now examine the capital charges resulting from the most standard copulas.

2.2 Copula modeling

First, we use meta-Gaussian and meta-t copulas to be able to use the models inferred in the first part as marginal distribution functions of the yearly losses. The parameters are fitted using the maximum of likelihood approach, which requires the existence of the density of the copula. The pseudo-likelihood $\sum_{t=1}^{n} ln\ c(\widehat{\mathbf{U}}_{\mathbf{t}})$, where $\widehat{\mathbf{U}}_{\mathbf{t}}$ is the vector of pseudo-observations, is maximized, then we perform Monte-Carlo simulations to calculate the capital charges with copulas. We use the procedure described in Clemente and Romano (2003).

The adjusted capital charges are compared to the sum of the capital charges over business lines - comonotonicity case -. Table 12 presents the VaR (in millions) at different levels of confidence, and in the case where all business lines would be independent.

	0.90	0.95	0.99	0.999
VaR comonotonicity	65	163	2 076	98 669
VaR_P meta-Gaussian copula	72	172	2 023	103 805
VaR_{Id} meta-Gaussian copula	72	172	2 145	97 217
VaR meta-Student copula	69	171	2 133	126 914

Table 12: *Variation of capital charges when using a meta-Gaussian copula*

The meta-Gaussian and meta-t copula yield capital charges which are in most of the cases a little bit higher than if the business lines were comonotonic. The difference is around 5%. This result is theoretically consistent with the fact that VaR is superadditive for heavy-tailed data. Furthermore, the third row of the table, which reports the value of the VaR if the data were independent, shows that until the 0.999 percentile, VaR is higher in the independent case than in the comonotonic case. This is in line with Böcker and Klüppelberg (2006) - see Figure 3.13 - and proves, one more time, that risk management in the case of heavy-tailed data can be problematic.

We evaluate the impact of the heaviest-tailed BL, BL3, by repeating the same procedure without including it in the computation. The results (in millions) are reported in Table 13.

	0.90	0.95	0.99	0.999
VaR comonotonicity	24	35	110	868
VaR meta-Gaussian copula	24	34	103	742

Table 13: *Variation of capital charges when excluding BL3*

Without BL3, using copulas allows to decrease the capital charges. The gain increases with the percentile and reaches 14% for the 0.999 level of confidence. This confirms that the heavy-tailedness of BL3 was at the origin of the increase of the VaR when using copulas.

The Gaussian and t-copulas are copulas of elliptical distributions. This implies, as pointed out in Mikosch (2005), that all the realizations of the loss variables are given the same importance in the statistical exercise. This goes against the basic principles of risk management, which gives more weight to the highest and hence most dangerous losses. Furthermore, the dependence structure of elliptical distributions is mainly determined by the covariance matrices. Embrechts et al. (2002b) argue that it is not an accurate tool for risk modeling purposes.

Archimedean copulas allow for a greater variety of dependence structures. Embrechts et al. (2001) give a comprehensive overview of their mathematical properties. To fit the parameters using maximum of likelihood, we use the close form expressions provided in Savu and Trede (2008). We calculate the capital charges resulting from Frank and Cook-Johnson copulas. We do not fit Gumbel copula since it is an extreme-value copula, and hence, as explained in Mikosch (2005), it is not relevant for the structure of our data.

Table 14 shows that there is not much difference between the two copulas. They both yield results which are very close to that of the meta-Gaussian copula, and do not reduce the capital charges for most of the percentiles.

	0.90	0.95	0.99	0.999
without copula	65	163	2 076	98 669
Frank copula	76	180	2 119	87 438
Cook-Johnson copula	68	177	2 131	92 789

Table 14: *Variation of capital charges with Frank and Cook-Johnson copulas*

If we exclude business line 3, the capital charges remain unchanged for the 0.90 and 0.95 percentiles, and then are reduced by 20 to 30%. So in general, using copulas seems to reduce capital charges at the highest percentiles, but increase them for lower quantiles.

More flexible Archimedean copulas have been developed, allowing for more than one parameter, and hence for a better flexibility in the modeling of the dependence structure. In particular, a very promising alternative to standard copulas has been proposed by Joe (1993). He presents three types of nested Archimedean copulas: the fully, partially and hierarchical nested Archimedean copulas as well as the pair copula construction. We refer to the paper of Berg and Aas (2007) for further explanation on the benefits of such multi-dimensional copula constructions. Having more parameters requires much more data than the amount we have at our disposal, but might be worth exploring when more data is available.

2.3 Evaluating the accuracy of a copula

As it is noticed by Mikosch (2005), it is quite hard to see which copula should be used for a given set of data in more than two dimensions. Following Berg and Aas (2007) (section 3.1. p.15), we choose the goodness-of-fit test suggested by Genest and Rémillard (2005) and Genest et al. (2007), which is based on the Euclidean distance between the hypothetical and empirical copula distributions, applied to the pseudo-observations.

Large values of the statistic S lead to the rejection of the null hypothesis. Table 15 report the test values and 95% critical values for the different copulas considered above.

	Gaussian	t	Frank	Cook-Johnson
GoF value	0.027	0.025	0.107	0.832
$c.v._{95\%}$	0.083	0.131	0.218	0.414

Table 15: *Goodness-of-fit test values for the choice of the copula*

The t-copula is the only copula we considered which has more than one parameter. This can explain why it yields the best test results.

2.4 Concluding remarks on the modeling of the dependencies using copulas

Using copulas to model the dependence structure allows for a better realism but does not provide a solution to the calculation of capital charges when the model involves GP distributions with $\xi > 1$. Indeed, the super-additivity of the VaR avoids capital charges to be decreased as expected. Our results are in line with the already existing literature and show that the standard copula approach presented yields results which contradict standard economic thinking about diversification.

Conclusion

Several issues have been empirically highlighted in this study. First we showed that heavy-tailed data is hard to model, and thus requires much caution when interpreting the resulting capital charges. In the presence of an infinite-mean model, a few losses in a business line may have a huge impact at the enterprise level and make drastically increase the capital charges. This leads to an obvious overestimation of the charges and to a lack of stability growing with the percentile the VaR is calculated at. We emphasized the sensitivity of the overall analysis to the shape parameter of the distribution when the excesses are GPD, as well as that to the level of confidence. Furthermore, we showed that for heavy-tailed distributions, summing the VaR over business lines as recommended in the Basel II agreements is likely to lead to a underestimation of the capital charges due to the super-additivity of the VaR.

By modeling the dependence structure of the data using copulas, we showed that infinite-mean models lead to an increase of the capital charges which goes against the diversification principle. The nature of the copula does not have a high impact on the overall capital charges compared to that of the parameters previously mentioned. We emphasized some theoretical limitations of the copulas considered, and introduced more sophisticated copulas allowing for a larger flexibility.

Finally, we think that the key message of this study is that heavy-tailed distributions with shape parameters bigger than one are at the origin of many theoretical problems: high impact on the overall capital charges, instability and high uncertainty of the results, incoherence of the VaR leading to a possible mis-estimation of the capital charges, etc. Therefore one needs to keep in mind the shortfalls of the current modeling techniques. However, this is not a lost cause, and the studies on operational risk keep uncovering parts of the iceberg.

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