

**Anisotropic function spaces, fractals,
and spectra of some elliptic
and semi - elliptic differential operators**

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To my parents

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Introduction

The purpose of this work is to highlight some aspects concerning the close connection between fractal geometry, the theory of function spaces, Fourier analysis, and spectral theory of differential operators, and to complement the theory developed by H. Triebel in his recent book [Tri97].

The thesis has five parts: in the first three chapters we introduce the concepts and present our main results whereas the fourth chapter and the Appendix contain the details for the proofs.

The **first chapter** is based on [1] and is concerned with the distribution of eigenfrequencies of regular anisotropic fractal drums.

Let Ω be a bounded domain in \mathbb{R}^2 having C^∞ boundary $\partial\Omega$ and let $0 < d_A < 2$. An anisotropic d_A -set $\Gamma \subset \Omega$ having anisotropic deviation $0 \leq a \leq 1$ is, roughly speaking, a compact set which can be covered for any $j \in \mathbb{N}_0$ with $N_j \sim 2^{jd_A}$ disjoint rectangles R_{jl} ($l = 1, \dots, N_j$) with $\text{vol } R_{jl} \sim 2^{-2j}$, having sides parallel to the axes and side lengths $r_1^{j,l}, r_2^{j,l}$ satisfying

$$2^{-j(1+a)} \leq r_2^{j,l} \leq r_1^{j,l} \leq 2^{-j(1-a)}$$

for any $l = 1, \dots, N_j$. This concept was introduced by H. Triebel in [Tri97, 5.2]. If Γ is such an anisotropic d_A -set then there exists a uniquely determined Radon measure μ in \mathbb{R}^2 with $\text{supp } \mu = \Gamma$ and $\mu(\Gamma \cap R_{jl}) = (\text{vol } R_{jl})^{d_A/2}$ if $j \in \mathbb{N}_0$ and $l = 1, \dots, N_j$ (see [Tri97, 5.5]).

Let $(-\Delta)^{-1}$ be the inverse of the Dirichlet Laplacian in Ω . Let $W_2^1(\Omega)$ be the usual Sobolev space and let $\mathring{W}_2^1(\Omega) = \{f \in W_2^1(\Omega) : \text{tr}_{\partial\Omega} f = 0\}$. The operator tr^Γ ,

$$(\text{tr}^\Gamma f)(\varphi) = \int_\Gamma (\text{tr}_\Gamma f)(\gamma) (\varphi|_\Gamma)(\gamma) d\mu(\gamma), \quad \varphi \in D(\Omega), \quad (0.1)$$

makes sense as a mapping from $\mathring{W}_2^1(\Omega)$ into $D'(\Omega)$ and it turns out that the fractal differential operator

$$T = (-\Delta)^{-1} \circ \text{tr}^\Gamma \quad (0.2)$$

generates a compact, non-negative, self adjoint operator in $\mathring{W}_2^1(\Omega)$.

Furthermore, as proved in [Tri97, 30.7], there exist positive constants $c_1 > 0$ and $c_2 > 0$ such that for the positive eigenvalues $\lambda_k(T)$ of T :

$$c_1 k^{-(d_A+2a)/d_A} \leq \lambda_k(T) \leq c_2 k^{-d_A/(d_A+2a)} \quad , \quad k \in \mathbb{N} \quad (0.3)$$

(for the first inequality in (0.3) it is additionally required that Γ is a so-called proper anisotropic d_A -set, see [Tri97, Definition 5.11]).

If the deviation $a = 0$ then we have $\lambda_k(T) \sim k^{-1}$. This means that the Weyl exponent occurs also in the case of proper anisotropic d_A -sets in the plane with deviation zero. But this fact is not a surprise since those fractals are close to compact isotropic d_A -sets as described in [Tri97, 3.1]. On the other hand, when $a > 0$ the two exponents in (0.3) are not equal.

The study of operators of type $(-\Delta)^{-1} \circ \text{tr}^\Gamma$ is motivated in a natural way by the so-called fractal drums: the problem of finding the eigenfrequencies of a vibrating membrane

(interpreted as a bounded domain Ω in the plane \mathbb{R}^2), fixed at its boundary, having the whole mass concentrated on some fractal compact set $\Gamma \subset \Omega$, can be reduced to the study of the eigenvalues of operators of that type. We give the necessary explanations in Section 1.1.

We want to mention here that the notion of fractal drums has several meanings. The best known version is connected with fractal boundaries $\partial\Omega$ whereas the membrane itself is smoothly distributed in Ω .

More information about this subject is given in [Tri97], especially in Sections 26.2 and 30.1-30.5, where one can find a detailed discussion on these different aspects.

As for the intention here (and in [Tri97]) to study fractal membranes in smooth domains we know only a few papers in literature, see [Fuj87], [NaS94], [NaS95] and [SoV95] and the recent preprint [EdT98].

The aim of the first chapter is to discuss the sharpness of (0.3) and to shed some new light on these estimates.

We consider the class of the so-called regular anisotropic fractals (or Sierpinski carpets, see the description given in [Mul84]) which are in fact anisotropic generalisations of the Cantor set in the plane, and prove that there exist two constants $C_1, C_2 > 0$ such that for all $k \in \mathbb{N}$,

$$C_1 k^{-\omega_1} \leq \lambda_k(T) \leq C_2 k^{-\omega_2} \quad (0.4)$$

for appropriate numbers ω_1 and ω_2 satisfying

$$\frac{d_A + 2a}{d_A} > \omega_1 \geq \omega_2 > \frac{d_A}{d_A + 2a}$$

where $\lambda_k(T)$ are again the eigenvalues of the operator $T = (-\Delta)^{-1} \circ tr^\Gamma$ acting in $\mathring{W}_2^1(\Omega)$. This is an improvement of the results from [Tri97, 30.7] for the class of regular anisotropic fractals and it means that the estimates from [Tri97, 30.7], briefly presented in (0.3), are not sharp in general.

Furthermore, we indicate a large class of regular anisotropic fractals for which $\lambda_k(T) \sim k^{-1}$, the so-called strongly regular anisotropic fractals.

The main result of this chapter, containing the precise formulation of (0.4), is presented with comments in Section 1.4.

The **second chapter** deals mainly with decompositions in anisotropic function spaces of $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}(\mathbb{R}^n)$ type and is, with the exception of Section 2.4, based on [2].

If $1 < p < \infty$ and (s_1, \dots, s_n) is an n -tuple of natural numbers then

$$W_p^{(s_1, \dots, s_n)}(\mathbb{R}^n) = W_p^{s,a}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{L_p(\mathbb{R}^n)} + \sum_{k=1}^n \left\| \frac{\partial^{s_k} f}{\partial x_k^{s_k}} \right\|_{L_p(\mathbb{R}^n)} < \infty \right\}$$

is the classical anisotropic Sobolev space on \mathbb{R}^n . In contrast to the usual (isotropic) Sobolev space ($s_1 = \dots = s_n$) the smoothness properties of an element from $W_p^{s,a}(\mathbb{R}^n)$ depend on the chosen direction in \mathbb{R}^n . The number s defined by

$$\frac{1}{s} = \frac{1}{n} \left(\frac{1}{s_1} + \dots + \frac{1}{s_n} \right) \quad (0.5)$$

is usually called the "mean smoothness" and $a = (a_1, \dots, a_n)$, where

$$a_1 = \frac{s}{s_1}, \dots, a_n = \frac{s}{s_n} \quad (0.6)$$

characterises the anisotropy.

Anisotropic Bessel potential spaces, or fractional Sobolev spaces, defined by

$$H_p^{s,a}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \left\| \left(\sum_{k=1}^n (1 + \xi_k^2)^{s/(2a_k)} \widehat{f} \right)^\vee \Big|_{L_p(\mathbb{R}^n)} \right\| < \infty \right\}$$

where $1 < p < \infty$, $s \in \mathbb{R}$ and $a = (a_1, \dots, a_n)$ is a given anisotropy, generalise in a natural way the above spaces (as usual, $S'(\mathbb{R}^n)$ is the space of tempered distributions and \widehat{f} , \check{f} are respectively the Fourier and the inverse Fourier transform of f).

Similar to the isotropic case, the study of anisotropic Bessel potential spaces $H_p^{s,a}(\mathbb{R}^n)$ for a fixed anisotropy $a = (a_1, \dots, a_n)$, is a part of the more general theory of the spaces of $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ type. Spaces of that type (or on domains in \mathbb{R}^n) have been studied in great detail by S. M. Nikol'skij, [Nik77], and by O. V. Besov, V. P. Il'in and S. M. Nikol'skij, [BIN75]. It is well known that this theory has a more or less full counterpart to the basic facts (definitions, elementary properties, embeddings for different metrics, interpolation) of isotropic spaces $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$, usually known in the literature as Besov - Triebel - Lizorkin spaces, as it was presented in the books of H. Triebel, [Tri83] and [Tri92].

This second chapter starts presenting the definition of the anisotropic function spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$. They are defined in terms of Fourier analytical quasi-norms: any function $f \in S'(\mathbb{R}^n)$ is decomposed in a sum of entire analytic functions $(\varphi_j \widehat{f})^\vee$ and this decomposition is used to introduce the spaces.

Hence, as in the isotropic case, entire analytic functions may be considered as building blocks for the spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ in the following sense: let a problem be given, for example mapping properties for PDE's or ψ DE's between spaces of the above type or traces on hyperplanes etc. First one asks what happens when the problem is applied to entire analytic functions; then the rest reduces to a discussion of convergence.

In the theory of isotropic function spaces there is a well known other type of decomposition in simple building blocks, the so-called atomic decomposition. Atoms have a history of some twenty years and in [Tri92, 1.9], cf. also [AdH96], a historically report on this topic was given and we do not repeat it here. We only want to mention that the (smooth) atoms in isotropic $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$ spaces as they were defined by M. Frazier and B. Jawerth in [FrJ85], [FrJ90] (cf. also [FJW91]), proved to be a powerful tool in the theory of function spaces. We also wish to emphasise that there exist many other types of atomic decompositions in isotropic spaces but we will not discuss this point here.

More information about this subject is given in [FrJ90], [Tri92] and [AdH96] where one can find many modifications and applications as well as comprehensive references extending the subject.

Several authors were concerned in the last years with the problem of obtaining useful decompositions of anisotropic function spaces in simple building blocks. A construction of unconditional bases in $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ spaces using Meyer wavelets was done by M. Z. Berkolaiko and I. Ya. Novikov in [BeN93] (and then used in [BeN95]). In [Din95a,

Theorem 1] P. Dintelmann obtained a decomposition for anisotropic function spaces which is the counterpart of the characterisation of isotropic function spaces with the help of the φ -transform of M. Frazier and B. Jawerth (see [FrJ90] and the survey [FJW91]) and used it in connection with the theory of Fourier- multipliers for anisotropic function spaces.

Our approach is different, especially from the point of view of the localisation of the building blocks.

The main aim of this chapter is to define smooth anisotropic atoms and to obtain a decomposition theorem which extends the atomic decomposition theorem of M. Frazier and B. Jawerth, see [FrJ85] and [FrJ90], to the anisotropic function spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$. Roughly speaking, we prove that for any element $g \in F_{pq}^{s,a}(\mathbb{R}^n)$ it is possible to find a decomposition (convergence in $S'(\mathbb{R}^n)$)

$$g = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \rho_{\nu m}^a \quad , \quad (0.7)$$

where $\rho_{\nu m}^a$ are the anisotropic atoms and $\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ belongs to an appropriate sequence space f_{pq}^a such that $\|g\|_{F_{pq}^{s,a}(\mathbb{R}^n)} \sim \|\lambda\|_{f_{pq}^a}$ (and a similar assertion for $B_{pq}^{s,a}(\mathbb{R}^n)$ spaces).

Hence the study of function spaces can be done with the help of some sequence spaces in an analogous way as it is done in the isotropic case in the above cited works of M. Frazier and B. Jawerth. The necessary explanations and details are given in Section 2.3.

The essential ingredient in proving (0.7) (which is Theorem 2.6) was a theorem on local means in anisotropic function spaces, briefly presented in Section 2.2, which is of independent interest because it extends the results from the isotropic case, see [Tri92, 2.4.6, 2.5.3].

As an application of our atomic decomposition theorem we give in Section 2.4 a unified approach to the study of traces of anisotropic function spaces.

This technique allows us to re-obtain the trace theorems proved in [Nik77], [BuG79], [Gol79a], [Gol79b], [Kal79] and to extend them to the whole admissible range of parameters from the definition of the anisotropic function spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$.

However in (0.7) (and in Theorem 2.6) no information is given about the possibility of obtaining atomic decompositions in which the atoms are constructed with the help of (anisotropic) dilatations and translations from one smooth function ρ having compact support, cf. also [BeN93, Comment 2].

For isotropic function spaces this was already done by M. Frazier and B. Jawerth, see [FrJ90, 4.2], and by W. Sickel, see [Sic90]. It might be possible to extend the technique of W. Sickel, at least for large values of the smoothness parameter, using the characterisation of anisotropic function spaces via oscillation from the work of A. Seeger, [See89]. But to construct such a basic (or mother) function ρ for the atoms having all required properties seems to be not very easy, at least at the first glance, see the above cited papers.

We arrive at the subatomic (or quarkonial) decomposition theorem (Theorem 2.12) which is presented in Section 2.5. This theorem states that given $g \in F_{pq}^{s,a}(\mathbb{R}^n)$ (with s sufficiently large) it is possible to obtain the decomposition

$$g = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^{\beta} (\beta q u)_{\nu m}^a \quad , \quad (0.8)$$

convergence being in $S'(\mathbb{R}^n)$, with

$$\|g\|_{F_{pq}^{s,a}(\mathbb{R}^n)} \sim \sup_{\beta \in \mathbb{N}_0^n} 2^{ra\beta} \|\lambda^\beta\|_{f_{pq}^a}, \quad (0.9)$$

where $r > 0$ is large enough, $a\beta = a_1\beta_1 + \dots + a_n\beta_n$ if β is the multi-index $(\beta_1, \dots, \beta_n)$, $\lambda^\beta = \{\lambda_{\nu m}^\beta : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ and where each $(\beta qu)_{\nu m}^a$ is an extremely simple building block (in particular an anisotropic atom without moment conditions), called anisotropic quark, compactly supported and which can be obtained starting from one smooth function (and a corresponding assertion for $B_{pq}^{s,a}(\mathbb{R}^n)$).

One obtains a total decoupling and decomposition in elementary building blocks which resembles the Taylor expansion of analytic functions.

Of course in (0.8) there are infinitely many sums over $(\nu, m) \in \mathbb{N}_0 \times \mathbb{Z}^n$ but this is well compensated by (0.9) with r large. Furthermore, it turns out from the proof that the dependence of the coefficients $\lambda_{\nu m}^\beta$ on g is linear.

Isotropic quarks were recently introduced by H. Triebel in [Tri97] and the subatomic (quarkonial) decomposition theorem he obtained in [Tri97, Chapter 14] proved to be a very useful ingredient for the estimation of entropy numbers of compact embeddings between function spaces on fractals. Compared with the results in [Tri97, Chapter 14] our Theorem 2.12 is in fact the extension of quarkonial decompositions to anisotropic function spaces in the case of large values of the smoothness parameter, in particular for $s > 0$ if $p \geq 1$ and $q \geq 1$. Explanations about the reason why we restricted ourselves only to large values of the smoothness parameter are given in Remark 2.14.

In the **third chapter** we use the powerful tools that we have now at our disposal, the atomic and the subatomic decomposition theorems in anisotropic function spaces from Chapter 2, to study the eigenvalue distribution of some fractal semi - elliptic differential operators. This should complement the theory developed in Chapter 1 for the (fractal elliptic) operator $(-\Delta)^{-1} \circ tr^\Gamma$.

To make our plans clearer let us briefly return to what is done in Chapter 1. In Theorem 1.10 we obtain estimates of type (0.4) for the operator $(-\Delta)^{-1} \circ tr^\Gamma$, Γ being a regular anisotropic fractal. The exponents in (0.4) are not equal in general, see the precise formulation in Theorem 1.10.

This fact should be not a big surprise since one can imagine that the different nature of the (isotropic) operator $-\Delta$ and the (anisotropic) structure of Γ would cause difficulties, see also the discussion in [Tri97, 4.16].

In the third chapter our intention is to replace the Dirichlet Laplacian $-\Delta$ in (0.2) by a model semi - elliptic differential operator A which was studied by H. Triebel in [Tri83*] and V. Shevchik in [She98].

We will investigate spectral properties of the new operator $A^{-1} \circ tr^\Gamma$ acting in an appropriate anisotropic Sobolev space $\dot{W}_2^{(s_1, s_2)}(\Omega)$. Furthermore, if the anisotropy of the fractal Γ is the same with that of the Sobolev space $\dot{W}_2^{(s_1, s_2)}(\Omega)$ where the new operator is acting, we are interested in the possibility of obtaining estimates of type

$$\lambda_k(A^{-1} \circ tr^\Gamma) \sim k^{-\omega}, \quad k \in \mathbb{N}, \quad (0.10)$$

(equivalence constants independent of $k \in \mathbb{N}$) for its eigenvalues, where ω is an appropriate positive number.

The main result of Chapter 3 is presented in Section 3.3, Theorem 3.16, but there are two preparatory sections which are of independent interest.

Having the two - dimensional anisotropic Cantor sets in mind we define in Section 3.1 regular anisotropic d -sets ($0 < d < n$) in \mathbb{R}^n .

Roughly speaking, given an anisotropy $a = (a_1, \dots, a_n)$, a regular anisotropic d -set ($0 < d < n$) is a compact set $\Gamma \subset \mathbb{R}^n$ which can be covered for any $j \in \mathbb{N}_0$ with $N_j \sim 2^{jd}$ disjoint rectangles having sides parallel to the axes and with side lengths $r_1^{j,l}, \dots, r_n^{j,l}$, $l = 1, \dots, N_j$ satisfying $r_i^{j,l} \sim 2^{-ja_i}$ for any $i = 1, \dots, n$ (the equivalence constants being independent of j, l, i); this definition extends in a natural way the anisotropic Cantor sets described in the first chapter, see Example 3.8 below.

Clearly there exists a Radon measure underlying to a regular anisotropic d -set Γ .

Interpreting any $f_\Gamma \in L_p(\Gamma)$, $1 < p < \infty$, as a tempered distribution on \mathbb{R}^n in the usual way, in Section 3.2 it is shown that certain subspaces of some anisotropic function spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ can be identified with $L_p(\Gamma)$; more precisely it is proved that

$$L_p(\Gamma) = \left\{ f \in B_{p\infty}^{-\frac{n-d}{p'}, a}(\mathbb{R}^n) : f(\varphi) = 0 \text{ if } \varphi \in S(\mathbb{R}^n), \quad \varphi|_\Gamma = 0 \right\} \quad (0.11)$$

and

$$tr_\Gamma B_{p1}^{\frac{n-d}{p}, a}(\mathbb{R}^n) = L_p(\Gamma) \quad (0.12)$$

where $\varphi|_\Gamma$ is the restriction of φ to Γ and $\frac{1}{p} + \frac{1}{p'} = 1$.

The main tool in obtaining (0.11) and (0.12) is the atomic decomposition theorem in anisotropic function spaces.

The above characterisations for the spaces $L_p(\Gamma)$, (0.11) and (0.12), are not only anisotropic counterparts of the results obtained in [TrW96b] (where H. Triebel and H. Winkelvoß proved that there is a perfect link between L_p spaces on isotropic d -sets and the Fourier analytically defined - isotropic - Besov spaces B_{pq}^s on \mathbb{R}^n , see also [Tri97, 18.2, 18.6]), but they play a key role in the proof of the main result of this chapter, see Theorem 3.16 and the proof in Section 4.5.

Let $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ be the unit disc in the plane and let $\partial\Omega$ be its boundary.

In order to extend the theory of regular elliptic operators to more general classes of operators, H. Triebel considered in [Tri83*] the semi - elliptic differential operator:

$$(\mathcal{A}_r u)(x) = -\frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^4 u(x)}{\partial x_2^4} + \frac{r}{\eta^2(x_1)} u(x), \quad x = (x_1, x_2) \in \Omega,$$

where $r \in \mathbb{R}$, and η is a C^∞ function on the interval $[-1, 1]$ with $\eta(t) > 0$ if $|t| < 1$ and

$$\lim_{t \uparrow 1} \frac{\eta(t)}{1-t} = \lim_{t \downarrow -1} \frac{\eta(t)}{1+t} = 1,$$

and a corresponding boundary value problem

$$\mathcal{A}_r u(x) = f(x), \quad x \in \Omega,$$

$$u(y) = g_1(y), \quad \frac{\partial u}{\partial x_2}(y) = g_2(y), \quad y \in \partial\Omega,$$

where f , g_1 and g_2 are given functions.

In [Tri83*] he obtained a-priori estimates for the operator \mathcal{A}_r in the framework of the anisotropic Besov spaces $B_{pp}^{(s,2s)}(\Omega, x^0, x^1)$ and the anisotropic Sobolev spaces $W_p^{(s,2s)}(\Omega, x^0, x^1)$ where $1 < p < \infty$.

These two scales of spaces (our notation here is slightly different from the original one in [Tri83*] where they were denoted $\dot{B}_{pp}^{(s,2s)}(\Omega)$ and $\dot{W}_p^{(s,2s)}(\Omega)$) were considered in [Tri84] taking into account the singular points $x^0 = (-1, 0)$ and $x^1 = (1, 0)$. They are close to the anisotropic spaces on Ω but their elements show (roughly speaking) a peculiar behaviour near the singular points x^0 and x^1 .

A description of this theory may be found also in [ScT87, Section 4.8].

Spectral properties of \mathcal{A}_r acting as an unbounded operator defined in $L_p(\Omega)$, $1 < p < \infty$, having domain of definition $\mathcal{D}(\mathcal{A}_r) = \{u \in W_p^{(2,4)}(\Omega, x^0, x^1) : u|_{\partial\Omega} = \frac{\partial u}{\partial x_2}|_{\partial\Omega} = 0\}$, see (4.57), were discussed by V. Shevchik in [She98]. In particular he proved that the asymptotic behaviour of the eigenvalues of \mathcal{A}_r is "intermediate" between that one of the homogeneous boundary value problem for the Laplace operator on a domain ($\lambda_k(\Delta) \sim k$) and the same problem for the biharmonic operator ($\lambda_k(\Delta^2) \sim k^2$), namely $\lambda_k(\mathcal{A}_r) \sim k^{\frac{4}{3}}$.

Considering $\Gamma \subset \Omega$ a regular anisotropic d -set with respect to the anisotropy $a = (\frac{1}{3}, \frac{2}{3})$ and tr^Γ the trace operator in the interpretation (0.1), the main objective of this chapter is to show that for sufficiently large r the operator

$$\mathcal{A}_r^{-1} \circ tr^\Gamma$$

is compact, non-negative, self adjoint in the anisotropic Sobolev space

$$\mathring{W}_2^{(1,2)}(\Omega) = \left\{ u \in W_2^{(1,2)}(\Omega) : u|_{\partial\Omega} = \frac{\partial u}{\partial x_2}|_{\partial\Omega} = 0 \right\}$$

and that there exist constants $c, C > 0$ such that its positive eigenvalues $\lambda_k(\mathcal{A}_r^{-1} \circ tr^\Gamma)$, repeated according to multiplicity and ordered by their magnitude, can be estimated by

$$c k^{-\frac{1}{d}(d+\frac{2}{3})} \leq \lambda_k(\mathcal{A}_r^{-1} \circ tr^\Gamma) \leq C k^{-\frac{1}{d}(d+\frac{2}{3})} \quad , \quad k \in \mathbb{N}. \quad (0.13)$$

We think (0.13) is a satisfactory result since it is an estimate of type (0.10) to which we aimed at.

We also think that the proof of (0.13) is of interest for its own sake. The methods how the two sides of (0.13) are obtained are rather different.

Using the subatomic decomposition theorem we estimate (see Subsection 3.2.3) the entropy numbers of some embeddings between function spaces which generalise to the anisotropic case the results from [Tri97, 20.5,20.6] and complement Theorem 22.2 in [Tri97]; to return to the eigenvalues and to the proof of the estimate from above in (0.13) one has then only to apply Carl's inequality (see [Carl81]).

To prove the estimate from below in (0.13) we investigate the approximation numbers of the operator $(\mathcal{A}_r^{-1} \circ tr^\Gamma)^{1/2}$ (and here we use again the atomic decomposition theorem in anisotropic function spaces).

Finally we want to give here some technical explanations. As it was already mentioned, the details of the proofs are given in Chapter 4 and in the Appendix. What we called "Appendix" consists of the two papers:

- [1] W. FARKAS and H. TRIEBEL, 'The distribution of eigenfrequencies of anisotropic fractal drums', *J. London Math. Soc.*, to appear
- [2] W. FARKAS, 'Atomic and subatomic decompositions in anisotropic function spaces', *Math. Nachr.*, to appear.

We always refer to the above papers as indicated above in order to emphasise that those publications belong to the present work itself, whereas all the other literature is quoted in a different way.

Notation

As usual, \mathbb{R}^n denotes the n -dimensional real Euclidean space, \mathbb{N} are the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and \mathbb{C} stands for the complex numbers.

Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing C^∞ functions on \mathbb{R}^n equipped with the usual topology. By $S'(\mathbb{R}^n)$ we denote its topological dual, the space of all tempered distributions on \mathbb{R}^n . If $\varphi \in S(\mathbb{R}^n)$ then $\widehat{\varphi} = F\varphi$ and $\check{\varphi} = F^{-1}\varphi$ are respectively the Fourier and inverse Fourier transform of φ . One extends F and F^{-1} in the usual way from $S(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$.

For a normed or quasi-normed space X we denote by $\|x\|_X$ the norm of the vector x . Recall that X is quasi-normed when the triangle inequality is weakened to $\|x + y\|_X \leq c(\|x\|_X + \|y\|_X)$ for some $c \geq 1$ independent of x and y .

The embedding of the quasi-normed space X into the quasi-normed space Y is denoted $X \hookrightarrow Y$.

All unimportant positive constants are denoted with c , occasionally with additional subscripts within the same formulas. The equivalence "term₁ \sim term₂" means that there exist two constants $c_1, c_2 > 0$ independent of the variables in the two terms such that $c_1 \text{ term}_1 \leq \text{term}_2 \leq c_2 \text{ term}_1$.

1 The distribution of eigenfrequencies of regular anisotropic fractal drums

1.1 Motivation

Let Ω be a bounded domain in the plane \mathbb{R}^2 with C^∞ boundary $\partial\Omega$, interpreted as a membrane fixed at its boundary. Vibrations of such a membrane in \mathbb{R}^3 are measured by the deflection $u(x, t)$ where $x = (x_1, x_2) \in \Omega$ and $t \geq 0$ stands for the time. In other words, the point $(x_1, x_2, 0)$ in \mathbb{R}^3 with $(x_1, x_2) \in \Omega$ of the membrane at rest, is deflected to $(x_1, x_2, u(x, t))$. Up to constants the usual physical description is given by

$$\Delta u(x, t) = m(x) \frac{\partial^2 u(x, t)}{\partial t^2}, \quad x \in \Omega, \quad t \geq 0, \quad (1.1)$$

and

$$u(y, t) = 0 \quad \text{if } y \in \partial\Omega, \quad t \geq 0,$$

where the right-hand side of (1.1) is Newton's law with the mass density $m(x)$. Of course, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ stands for the Laplacian.

To find the eigenfrequencies one has to insert $u(x, t) = e^{i\eta t} v(x)$ with $\eta \in \mathbb{R}$ in (1.1) and obtains

$$-\Delta v(x) = \eta^2 m(x) v(x), \quad x \in \Omega; \quad v(y) = 0 \quad \text{if } y \in \partial\Omega$$

where one is interested in non-trivial solutions $v(x)$. Hence one asks for the eigenfunctions and the eigenvalues of the operator

$$T = (-\Delta)^{-1} \circ m(\cdot) \quad (1.2)$$

where $-\Delta$ stands for the Dirichlet Laplacian. If λ is such a positive eigenvalue then $\eta = \lambda^{-\frac{1}{2}}$ is the related eigenfrequency.

If the mass is evenly distributed, that means that the mass density is constant, then the eigenvalues $\lambda_k(T)$ of T satisfy

$$\lambda_k(T) \sim k^{-1}, \quad k \in \mathbb{N},$$

where " \sim " means that there exist two constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 k^{-1} \leq \lambda_k(T) \leq c_2 k^{-1}, \quad k \in \mathbb{N}.$$

If the mass is concentrated on some compact fractal set Γ with $\Gamma \subset \Omega$ the situation becomes more complicated.

Recall that a compact $\Gamma \subset \mathbb{R}^2$ is called a d -set if there exists a Radon measure μ in \mathbb{R}^2 with $\text{supp } \mu = \Gamma$ and there are two constants $c_1, c_2 > 0$ such that for all $\gamma \in \Gamma$ and all r with $0 < r < 1$ we have

$$c_1 r^d \leq \mu(B(\gamma, r) \cap \Gamma) \leq c_2 r^d,$$

see [Tri97] and references given there. It turns out that up to equivalence constants the above measure μ is $\mathcal{H}^d|_\Gamma$ where \mathcal{H}^d is the d -dimensional Hausdorff measure. Fractal geometry supplies us with a large number of d -sets such as self-similar fractals, see [Fal90].

If Γ is a compact d -set in \mathbb{R}^2 with measure μ then, according to H. Triebel [Tri97, Sections 28-30], the operator corresponding to (1.2) looks like

$$T = (-\Delta)^{-1} \circ \text{tr}^\Gamma \quad (1.3)$$

$$(AQ)^\nu = A((AQ)^{\nu-1}) = \bigcup_{1 \leq m_1, \dots, m_\nu \leq N} A_{m_1} \circ \dots \circ A_{m_\nu} Q \quad ; \quad \nu \in \mathbb{N}.$$

This sequence of sets is monotonically decreasing and by [Fal85, Theorem 8.3] its limit

$$\Gamma = (AQ)^\infty = \bigcap_{\nu \in \mathbb{N}} (AQ)^\nu = \lim_{\nu \rightarrow \infty} (AQ)^\nu$$

is the uniquely determined fractal generated by the contractions $(A_m)_{m=1}^N$.

Fractals constructed in this way are *anisotropic generalisations of the Cantor set* in \mathbb{R}^2 and were called *generalised Sierpinski carpets* in [Mul84] (since Sierpinski's universal curve is a special case of this construction) or *regular anisotropic fractals* in [Tri97, 4.18].

Let n_l denote the number of rectangles $A_m Q$ in the l th column, $l = 1, \dots, K_1$.

Throughout this work we will assume that $n_l \geq 1$ for any $l = 1, \dots, K_1$ (in each column there is at least one rectangle $A_m Q$ located).

The Hausdorff dimension (see [Fal90, 2.2] for definition) of Γ is

$$\dim_H \Gamma = \frac{1}{\log K_1} \log \left(\sum_{l=1}^{K_1} n_l^{\log K_1 / \log K_2} \right) \quad (1.6)$$

and the box-counting dimension (see [Fal90, 3.1] for definition) of Γ is

$$\dim_B \Gamma = 1 + \frac{\log(N/K_1)}{\log K_2}. \quad (1.7)$$

Proofs of (1.6) and (1.7) are given in [Mul84], see also [Fal90, Example 9.11].

Notice that in this type of examples the Hausdorff dimension depends not only on the number of rectangles selected at each stage but also on their relative position. Moreover, it is clear that $\dim_H \Gamma$ and $\dim_B \Gamma$ are not, in general, equal.

Let $(A_m)_{m=1}^N$ be the $N \geq 2$ affine maps introduced in (1.5). The *affine dimension* of $\Gamma = (AQ)^\infty$, see [Tri97, 4.12], is the uniquely determined positive number $d_A = \dim_A \Gamma$ such that

$$\sum_{m=1}^N (\text{vol } A_m Q)^{d_A/2} = 1. \quad (1.8)$$

By construction we have $N = 2^{\kappa d_A}$.

Recall we assumed $a_1 < a_2$ (which is equivalent to $K_1 < K_2$) so let

$$a = 1 - a_1 = a_2 - 1.$$

Definition 1.1 *If $n_1 = \dots = n_{K_1} = N 2^{-\kappa(1-a)}$ (in any column there is the same number of rectangles) then we call Γ a strongly regular anisotropic fractal.*

Remark 1.2 If Γ is strongly regular then $\dim_H \Gamma = \dim_B \Gamma = \frac{d_A + 2a}{1+a}$ as a simple consequence of (1.6), (1.7) and (1.8).

It will be clear from all what follows that the typical number which also appears in case of arbitrary regular anisotropic fractals is

$$d = \frac{d_A + 2a}{1+a},$$

see also Remark 1.7 below.

Theorem 1.3 ([Tri97, 4.15]) *Let Γ be the regular anisotropic fractal introduced above having the affine dimension d_A according to (1.8).*

Then there exists a Radon measure μ in \mathbb{R}^2 uniquely determined with $\text{supp } \mu = \Gamma$ and

$$\mu(\Gamma \cap A_{m_1} \circ \dots \circ A_{m_j} Q) = (\text{vol } A_{m_1} \circ \dots \circ A_{m_j} Q)^{d_A/2}$$

for all $j \in \mathbb{N}$ and all $m_1, \dots, m_j \in \{1, \dots, N\}$.

Definition 1.4 *Let $n_{max} = \max\{n_l : 1 \leq l \leq K_1\}$ and $n_{min} = \min\{n_l : 1 \leq l \leq K_1\}$.*

Then there exist two numbers $\lambda^+ \geq 0$ and $\lambda^- \leq 0$ such that

$$n_{max} = N 2^{-\kappa(1-a)} 2^{\kappa(1-a)\lambda^+} \quad \text{and} \quad n_{min} = N 2^{-\kappa(1-a)} 2^{\kappa(1-a)\lambda^-}. \quad (1.9)$$

We call λ^+ the upper mass concentration factor of Γ and λ^- the lower mass concentration factor of Γ since these numbers give information about the distribution of the rectangles in Figure 1 and about the structure of Γ .

Remark 1.5 (i) Clearly $\lambda^+ < 1$ since we assumed $n_l \geq 1$ for every $l \in \{1, \dots, K_1\}$.

(ii) We have $\lambda^+ = \lambda^- = 0$ if, and only if, Γ is strongly regular according to Definition 1.1.

It is clear that for any $j \in \mathbb{N}$ there are $N^j = 2^{j\kappa d_A}$ rectangles of type $R_j = A_{m_1} \circ \dots \circ A_{m_j} Q$, having side lengths $2^{-j\kappa(1-a)}$, $2^{-j\kappa(1+a)}$ belonging to $(AQ)^j$.

Let R_j be such a rectangle. We subdivide R_j in rectangles E_{jl} having side lengths $2^{-(j+m)\kappa(1-a)}$ and $2^{-j\kappa(1+a)}$ such that

$$2^{-j\kappa(1+a)} \sim 2^{-(j+m)\kappa(1-a)}.$$

The rectangles E_{jl} are almost squares; it is immaterial for what follows to assume that E_{jl} are squares, what means

$$2^{-j\kappa(1+a)} = 2^{-(j+m)\kappa(1-a)}.$$

The lemma below gives an information about the mass concentration in E_{jl} ($l = 1, \dots, 2^{j\kappa 2a}$) and it played a key role in our considerations.

Lemma 1.6 *There exist constants $c_1, c_2 > 0$ such that for any square E_{jl} of side length $2^{-\nu\kappa(1-a)}$ we have*

$$c_1 2^{-\nu\kappa(1-a) d(\lambda^-)} \leq \mu(\Gamma \cap E_{jl}) \leq c_2 2^{-\nu\kappa(1-a) d(\lambda^+)} \quad (1.10)$$

where

$$d(\lambda^-) = \frac{d_A + 2a(1 - \lambda^-)}{1 + a} \quad \text{and} \quad d(\lambda^+) = \frac{d_A + 2a(1 - \lambda^+)}{1 + a}.$$

The detailed proof is given in [1, 2.6, 2.7].

Remark 1.7 If Γ is strongly regular then Γ is an isotropic d -set (see the previous section for definition) where $d = \frac{d_A + 2a}{1 + a}$. This is a simple consequence of (1.10) and of Remark 1.5/(ii).

1.3 L_p - spaces on regular anisotropic fractals

Some preliminaries. We recall here the definition of Besov spaces on \mathbb{R}^2 .

Let φ_0 be a C^∞ function on \mathbb{R}^2 , $\varphi_0(x) = 1$ if $|x| \leq 1$; $\text{supp } \varphi_0 \subset \{x \in \mathbb{R}^2 : |x| \leq 2\}$ and let $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$ if $j \in \mathbb{N}$. Then $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ if $x \in \mathbb{R}^2$ and $(\varphi_j)_{j \in \mathbb{N}_0}$ is a smooth dyadic resolution of unity.

Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$; the Besov space $B_{pq}^s(\mathbb{R}^2)$ consists of all tempered distributions $f \in S'(\mathbb{R}^2)$ for which the quasi-norm

$$\|f\|_{B_{pq}^s(\mathbb{R}^2)} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^\vee\|_{L_p(\mathbb{R}^2)}^q \right)^{1/q}$$

(with the usual modification if $q = \infty$) is finite. Here $\hat{g} = Fg$ and $\check{g} = F^{-1}g$ are respectively the Fourier and inverse Fourier transform on $S'(\mathbb{R}^2)$. These are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$) which are independent of the choice of $(\varphi_j)_{j \in \mathbb{N}_0}$.

The space $H_2^s(\mathbb{R}^2) = B_{22}^s(\mathbb{R}^2)$ is the fractional Sobolev space.

Function spaces of $F_{pq}^s(\mathbb{R}^2)$ type, $0 < p < \infty$, are defined changing the roles of the spaces $L_p(\mathbb{R}^2)$ and l_q in the above definition but we do not stress this point here.

A systematic treatment of the theory of $B_{pq}^s(\mathbb{R}^2)$ and $F_{pq}^s(\mathbb{R}^2)$ spaces may be found in the books [Tri83] and [Tri92]; for a more recent account we refer also to [EdT96] and [RuS96]. These two scales of function spaces include many well-known classical spaces such as Sobolev spaces, Hölder - Zygmund spaces and inhomogeneous Hardy spaces.

The structure theorem. If Γ is a closed set with Lebesgue measure $|\Gamma| = 0$ and if $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$ then we define

$$B_{pq}^{s,\Gamma}(\mathbb{R}^2) = \{f \in B_{pq}^s(\mathbb{R}^2) : f(\varphi) = 0 \text{ if } \varphi \in S(\mathbb{R}^2), \varphi|_{\Gamma} = 0\}$$

where $\varphi|_{\Gamma}$ is the restriction of φ to Γ .

We have $\text{supp } f \subset \Gamma$ if $f \in B_{pq}^{s,\Gamma}(\mathbb{R}^2)$ in any case. Remark also that if $0 < p \leq \infty$, $0 < q \leq \infty$ and $s > 2(\frac{1}{p} - 1)_+$ (if $b \in \mathbb{R}$ then $b_+ = \max(b, 0)$) then $B_{pq}^s(\mathbb{R}^2) \hookrightarrow L_1^{loc}(\mathbb{R}^2)$ ([Tri92, Remark 2.3.2/3]) and, hence, $B_{pq}^{s,\Gamma}(\mathbb{R}^2) = \{0\}$ is trivial.

In other words, only values $s \leq 2(\frac{1}{p} - 1)_+$ (in particular $s \leq 0$ if $1 \leq p \leq \infty$) are of interest.

Let Γ be the regular anisotropic fractal constructed above. The L_p - spaces on Γ , $0 < p \leq \infty$, are introduced in the usual way with respect to the underlying Radon measure μ on Γ according to Theorem 1.3.

If $1 \leq p \leq \infty$ then any $f_{\Gamma} \in L_p(\Gamma)$ can be interpreted as a tempered distribution $f \in S'(\mathbb{R}^2)$ given by

$$f(\varphi) = \int_{\Gamma} f_{\Gamma}(\gamma) (\varphi|_{\Gamma})(\gamma) d\mu(\gamma), \quad \varphi \in S(\mathbb{R}^2). \quad (1.11)$$

Theorem 1.8 *Let Γ be a regular anisotropic fractal with upper mass concentration factor λ^+ and affine dimension d_A . If $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$ then (in the sense of (1.11))*

$$L_p(\Gamma) \hookrightarrow B_{p\infty}^{-\frac{2-d(\lambda^+)}{p'}}(\mathbb{R}^2) \quad \text{where} \quad d(\lambda^+) = \frac{d_A + 2a(1 - \lambda^+)}{1 + a}. \quad (1.12)$$

Moreover, if Γ is strongly regular and if $1 < p \leq \infty$ then (in the sense of (1.11))

$$L_p(\Gamma) = B_{p\infty}^{-\frac{2-d}{p}, \Gamma}(\mathbb{R}^2) \quad \text{where} \quad d = \frac{d_A + 2a}{1 + a}. \quad (1.13)$$

One has to consider a square $Q(x, t)$ centred at $x \in \mathbb{R}^2$ and with side length $2^{-\nu\kappa(1-a)}$; by the right-hand side of (1.10) we have $\mu(\Gamma \cap Q(x, t)) \leq c t^{d(\lambda^+)}$.

To prove (1.12) one has now to follow the lines of the proof of [Tri97, Theorem 18.15/ Step 1] making the above modification.

If Γ is strongly regular then $d(\lambda^+) = d$ and Γ becomes an isotropic d -set; then (1.13) is in fact [TrW96b, Theorem 2/(8)].

Traces. Assume that Γ is a regular anisotropic fractal; if $\varphi \in S(\mathbb{R}^2)$ then $tr_\Gamma \varphi = \varphi|_\Gamma$ makes sense pointwise. If $0 < p, q < \infty$ and $s \in \mathbb{R}$ then $tr_\Gamma B_{pq}^s(\mathbb{R}^2) \hookrightarrow L_p(\Gamma)$ must be understood as follows: there exists a positive number $c > 0$ such that for any $\varphi \in S(\mathbb{R}^2)$

$$\|tr_\Gamma \varphi | L_p(\Gamma)\| \leq c \|\varphi | B_{pq}^s(\mathbb{R}^2)\|.$$

Since $S(\mathbb{R}^2)$ is dense in $B_{pq}^s(\mathbb{R}^2)$ this inequality can be extended by completion to any $f \in B_{pq}^s(\mathbb{R}^2)$ and the resulting function is denoted $tr_\Gamma f$.

In addition, the equality $tr_\Gamma B_{pq}^s(\mathbb{R}^2) = L_p(\Gamma)$ means that any $f_\Gamma \in L_p(\Gamma)$ is the trace of a suitable $g \in B_{pq}^s(\mathbb{R}^2)$ on Γ and

$$\|f_\Gamma | L_p(\Gamma)\| \sim \inf\{\|g | B_{pq}^s(\mathbb{R}^2)\| : tr_\Gamma g = f_\Gamma\}.$$

Theorem 1.9 *Let Γ be a regular anisotropic fractal with upper mass concentration factor λ^+ and affine dimension d_A . If $1 \leq p \leq \infty$ then*

$$tr_\Gamma B_{p1}^{\frac{2-d(\lambda^+)}{p}}(\mathbb{R}^2) \hookrightarrow L_p(\Gamma) \quad \text{where} \quad d(\lambda^+) = \frac{d_A + 2a(1 - \lambda^+)}{1 + a}. \quad (1.14)$$

Moreover, if Γ is strongly regular and if $1 \leq p < \infty$ then

$$tr_\Gamma B_{p1}^{\frac{2-d}{p}}(\mathbb{R}^2) = L_p(\Gamma) \quad \text{where} \quad d = \frac{d_A + 2a}{1 + a}. \quad (1.15)$$

If $p = \infty$ we have $B_{\infty 1}^0(\mathbb{R}^2) \hookrightarrow C(\mathbb{R}^2)$ and (1.14) is obvious (here $C(\mathbb{R}^2)$ is the space of all uniformly continuous bounded functions on \mathbb{R}^2). To prove (1.14) for $p < \infty$ one has to repeat the arguments from [Tri97, Theorem 18.15/ Step 2] with $d(\lambda^+)$ instead of $\frac{d_A}{1+a}$. In addition, (1.15) is [TrW96b, Theorem 2/(9)].

Theorems 1.8 and 1.9 pave the way to the main result of this first chapter (which is presented in the next section) but we hope that they are also of independent interest. They complement the results from [Tri97, 18.15, 18.17] and are closely related to the theorems in [TrW96b], see also [Tri97, 18.2, 18.6].

1.4 The main result

As usual, Ω stands for a bounded domain in \mathbb{R}^2 with C^∞ boundary and $D'(\Omega)$ denotes the space of all complex-valued distributions on Ω .

Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$; the space $B_{pq}^s(\Omega)$ is defined as the restriction of $B_{pq}^s(\mathbb{R}^2)$ to Ω , that means $B_{pq}^s(\Omega) = \{f \in D'(\Omega) : \text{there exists a } g \in B_{pq}^s(\mathbb{R}^2) \text{ with } g|_\Omega = f\}$,

$\|f\|_{B_{pq}^s(\Omega)} = \inf \|g\|_{B_{pq}^s(\mathbb{R}^2)}$ where the infimum is taken over all $g \in B_{pq}^s(\mathbb{R}^2)$ such that its restriction to Ω , denoted by $g|_{\Omega}$, coincides in $D'(\Omega)$ with f . In particular $B_{22}^1(\Omega) = W_2^1(\Omega)$.

In the sequel $\Gamma \subset \Omega$ will be a regular anisotropic fractal and we shall not distinguish between f_{Γ} as an element of some $L_p(\Gamma)$ and as the distribution belonging to some $B_{p\infty}^{-s}(\Omega)$ according to (1.12).

To avoid any misunderstanding we emphasise that the trace operator has two different meanings which we distinguish by tr_{Γ} and tr^{Γ} if extra clarity is desirable.

If, for example, $1 < p < \infty$, then

$$tr_{\Gamma} : B_{p1}^{\frac{2-d(\lambda^+)}{p}}(\Omega) \rightarrow L_p(\Gamma) \quad (1.16)$$

by (1.14) and

$$tr^{\Gamma} : B_{p1}^{\frac{2-d(\lambda^+)}{p}}(\Omega) \rightarrow B_{p\infty}^{-\frac{2-d(\lambda^+)}{p'}}(\Omega) \quad (1.17)$$

if one applies in addition (1.12). The latter can be rephrased asking for an optimal extension of tr^{Γ} considered as a mapping from $D(\Omega)$ into $D'(\Omega)$ given by (0.1).

Recall $(-\Delta)^{-1}$ stands for the inverse of the Dirichlet Laplacian in Ω .

Theorem 1.10 *Let Ω be a bounded domain in \mathbb{R}^2 with C^∞ boundary. Let $\Gamma \subset \Omega$ be a regular anisotropic fractal having respectively upper and lower mass concentration factors λ^+ and λ^- according to (1.9) and having affine dimension d_A according to (1.8).*

Let

$$d(\lambda^-) = \frac{d_A + 2a(1 - \lambda^-)}{1 + a} \quad \text{and} \quad d(\lambda^+) = \frac{d_A + 2a(1 - \lambda^+)}{1 + a}$$

and

$$d = \frac{d_A + 2a}{1 + a}. \quad (1.18)$$

Let tr^{Γ} be the trace operator in the interpretation (1.17) and (0.1) whereas tr_{Γ} stands for the trace operator according to (1.16).

(i) *The operator $T = (-\Delta)^{-1} \circ tr^{\Gamma}$ is compact, non-negative, self adjoint in $\mathring{W}_2^1(\Omega)$, has null space $N(T) = \{f \in \mathring{W}_2^1(\Omega) : tr_{\Gamma} f = 0\}$ and is generated by the quadratic form in $\mathring{W}_2^1(\Omega)$*

$$\int_{\Gamma} f(\gamma) \overline{g(\gamma)} d\mu(\gamma) = (Tf, g)_{W_2^1(\Omega)} \quad \text{where} \quad f \in \mathring{W}_2^1(\Omega), \quad g \in \mathring{W}_2^1(\Omega),$$

and μ is the Radon measure according to Theorem 1.3.

(ii) *There exist constants $c_1, c_2 > 0$ such that the positive eigenvalues $\lambda_k(T)$ of T , repeated according to multiplicity and ordered by their magnitude, can be estimated by*

$$c_1 k^{-\frac{d(\lambda^-)}{d}} \leq \lambda_k(T) \leq c_2 k^{-\frac{d(\lambda^+)}{d}}, \quad k \in \mathbb{N}. \quad (1.19)$$

Furthermore, if Γ is strongly regular then there are constants $c_1, c_2 > 0$ such that

$$c_1 k^{-1} \leq \lambda_k(T) \leq c_2 k^{-1}.$$

We want to mention that the proof of (1.19) is splitted up into two different approaches. The estimate from above is proved using entropy numbers of some embeddings between function spaces and then applying Carl's inequality (see [Carl81]) to return to the eigenvalues whereas the estimate from below is based on the investigation of the approximation numbers of the operator \sqrt{T} .

All details (including some basic facts on entropy and approximation numbers) are given in [1], Sections 4 and 5.

Remark 1.11 According to Definition 1.4 and to Remark 1.5/(i) we have $0 \leq \lambda^+ < 1$ and this implies $\frac{d(\lambda^+)}{d} > \frac{d_A}{d_A+2a}$. Hence the estimate from above in (1.19) is an improvement of the estimate from above in [Tri97, 30.7], see (0.3).

Remark 1.12 If $-1 < \lambda^- \leq 0$ then $\frac{d(\lambda^-)}{d} < \frac{d_A+2a}{d_A}$ and the estimate from below obtained in (1.19) is better than the estimate from below in [Tri97, 30.7], see (0.3).

Remark 1.13 By [Tri97, 30.2] (isotropic fractal drum) it is not a surprise that if $\lambda^+ = \lambda^- = 0$ then the two exponents in (1.19) are both -1 since in this case the regular anisotropic fractal Γ becomes an isotropic d -set, where d is the number from (1.18).

We think that Remarks 1.11 and 1.12 give a satisfactory answer to the question posed in [Tri97, 30.11] if additional geometric information on the self affine fractals provide the possibility of an improvement of the estimates from [Tri97, 30.7], see (0.3).

2 Decompositions in anisotropic function spaces

2.1 The spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$

From now on let $n \geq 2$ and let $a = (a_1, \dots, a_n)$ be a given anisotropy (an n -tuple of strictly positive numbers with $a_1 + \dots + a_n = n$).

We put $a_{\min} = \min\{a_i : 1 \leq i \leq n\}$ and $a_{\max} = \max\{a_i : 1 \leq i \leq n\}$.

If $a = (1, \dots, 1)$ then we will speak about the "isotropic case".

The action of $t \in [0, \infty)$ on $x \in \mathbb{R}^n$ is defined by the formula:

$$t^a x = (t^{a_1} x_1, \dots, t^{a_n} x_n).$$

For $t > 0$ and $s \in \mathbb{R}$ we put $t^{sa} x = (t^s)^a x$. In particular we write $t^{-a} x = (t^{-1})^a x$ and $2^{-ja} x = (2^{-j})^a x$.

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $x \neq 0$, let $|x|_a$ be the unique positive number t such that

$$\frac{x_1^2}{t^{2a_1}} + \dots + \frac{x_n^2}{t^{2a_n}} = 1 \quad (2.1)$$

and let $|0|_a = 0$.

By M. Yamazaki, [Yam86, 1.4/3,8], $|\cdot|_a$ is an anisotropic distance function in $C^\infty(\mathbb{R}^n \setminus \{0\})$. Remark that in the isotropic case $|x|_a$ is the Euclidean distance of x to the origin.

Let φ_0 a C^∞ function on \mathbb{R}^n , $\varphi_0(x) = 1$ if $|x|_a \leq 1$, $\text{supp } \varphi_0 \subset \{x \in \mathbb{R}^n : |x|_a \leq 2\}$ and $\varphi_j(x) = \varphi_0(2^{-j} x) - \varphi_0(2^{-(j+1)} x)$ if $j \in \mathbb{N}$. Then $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ if $x \in \mathbb{R}^n$ and $(\varphi_j)_{j \in \mathbb{N}_0}$ is a smooth anisotropic dyadic resolution of unity, cf. [ScT87, 4.2].

For $f \in S'(\mathbb{R}^n)$ since $\varphi_j \hat{f}$ is compactly supported the Paley - Wiener - Schwartz theorem provides that $(\varphi_j \hat{f})^\vee$ is an entire analytic function on \mathbb{R}^n .

Definition 2.1 (i) Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$; then $B_{pq}^{s,a}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{B_{pq}^{s,a}(\mathbb{R}^n)} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^\vee\|_{L_p(\mathbb{R}^n)}^q \right)^{1/q} \quad (2.2)$$

(usual modification if $q = \infty$) is finite.

(ii) Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$; then $F_{pq}^{s,a}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{F_{pq}^{s,a}(\mathbb{R}^n)} = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \hat{f})^\vee(\cdot)|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)} \quad (2.3)$$

(usual modification if $q = \infty$) is finite.

Of course the quasi-norms in (2.2) and (2.3) depend on the chosen system $(\varphi_j)_{j \in \mathbb{N}_0}$. But this is not the case for the spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ (in the sense of equivalent quasi-norms) and that is the reason why we may omit in our notation the subscript $(\varphi_j)_{j \in \mathbb{N}_0}$.

The above spaces are denoted $B_{pq}^s(\mathbb{R}^n)$ respectively $F_{pq}^s(\mathbb{R}^n)$ in the isotropic case. As mentioned in the previous chapter, a systematic treatment of the theory of $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$ spaces may be found in the books of H. Triebel [Tri83], [Tri92]; for a more recent account of the theory we refer the reader also to [EdT96] and [RuS96].

A survey on the basic results for the (anisotropic) spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ may be found in [ScT87, 4.2.1-4.2.4] and [Joh95, 2.1-2.2]. In this context we refer to the works of S. M. Nikol'skij [Nik77], O. V. Besov, V. P. Il'in and S. M. Nikol'skij [BIN75], B. Stöckert and H. Triebel [StT79], M. Yamazaki [Yam86], A. Seeger [See89], P. Dintelmann [Din95b, 1.2.8-1.2.10] etc.

An extension of (2.3) to $p = \infty$ is not reasonable; in [Tri92, 1.5.2] this point was discussed in detail.

Both $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$).

As in the isotropic case, see [Tri83, 2.3.3], the embeddings $S(\mathbb{R}^n) \hookrightarrow B_{pq}^{s,a}(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n)$ and $S(\mathbb{R}^n) \hookrightarrow F_{pq}^{s,a}(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n)$ hold true for all admissible values of p, q, s . Furthermore, if $s \in \mathbb{R}$ and $0 < p < \infty$, $0 < q < \infty$ then $S(\mathbb{R}^n)$ is dense in $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$, see [Yam86, 3.5] and [Din95b, 1.2.10].

We want to point out that if $0 < p < \infty$ and $s \in \mathbb{R}$ then $B_{pp}^{s,a}(\mathbb{R}^n) = F_{pp}^{s,a}(\mathbb{R}^n)$.

If $1 < p < \infty$ and $s \in \mathbb{R}$ then (in the sense of equivalent quasi-norms) $F_{p2}^{s,a}(\mathbb{R}^n) = H_p^{s,a}(\mathbb{R}^n)$ where

$$H_p^{s,a}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \left\| \left(\sum_{k=1}^n (1 + \xi_k^2)^{s/(2a_k)} \widehat{f} \right)^\vee \mid L_p(\mathbb{R}^n) \right\| < \infty \right\}$$

is the anisotropic Bessel potential space (see [StT79, Remark 11], [Tri77, 2.5.2], and [Yam86, 3.11]).

Furthermore, if $1 < p < \infty$, $s \in \mathbb{R}$ and if $s_1 = s/a_1 \in \mathbb{N}, \dots, s_n = s/a_n \in \mathbb{N}$ then (in the sense of equivalent quasi-norms) $F_{p2}^{s,a}(\mathbb{R}^n) = W_p^{s,a}(\mathbb{R}^n)$ where

$$W_p^{s,a}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{L_p(\mathbb{R}^n)} + \sum_{k=1}^n \left\| \frac{\partial^{s_k} f}{\partial x_k^{s_k}} \mid L_p(\mathbb{R}^n) \right\| < \infty \right\}$$

is the classical anisotropic Sobolev space on \mathbb{R}^n .

2.2 Local means

Given the anisotropy $a = (a_1, \dots, a_n)$ if $t > 0$ and $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ we will use the notation $x + t^a y = (x_1 + t^{a_1} y_1, \dots, x_n + t^{a_n} y_n)$. If $B^a = \{y \in \mathbb{R}^n : |y|_a \leq 1\}$ is the anisotropic unit ball in \mathbb{R}^n , k is a C^∞ function on \mathbb{R}^n with $\text{supp } k \subset B^a$ then we introduce the local means (cf. [Tri92, 2.4.6/1])

$$k(t, f)(x) = \int_{\mathbb{R}^n} k(y) f(x + t^a y) dy = t^{-n} \int_{\mathbb{R}^n} k(t^{-a}(z - x)) f(z) dz \quad (2.4)$$

which make sense for any $f \in S'(\mathbb{R}^n)$ (appropriately interpreted).

If $0 < p \leq \infty$ let

$$\sigma_p = n \left(\frac{1}{p} - 1 \right)_+ .$$

Theorem 2.2 *Let k_0 and k be two C^∞ functions on \mathbb{R}^n such that*

$$\begin{aligned} \text{supp } k_0 \subset B^a, \quad |\widehat{k}_0(\xi)| > 0 & \quad \text{if} \quad |\xi|_a \leq 2, \\ \text{supp } k \subset B^a, \quad |\widehat{k}(\xi)| > 0 & \quad \text{if} \quad \frac{1}{2} \leq |\xi|_a \leq 2. \end{aligned}$$

Let $r > 0$ and assume that there exists a constant $c > 0$ such that

$$|\widehat{k}(\xi)| \leq c |\xi|_a^r \quad \text{for } \xi \text{ near zero.} \quad (2.5)$$

(i) *Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. If $r > \max(s, \sigma_p) + \sigma_p$ then*

$$\|k_0(1, f) | L_p(\mathbb{R}^n)\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \|k(2^{-j}, f) | L_p(\mathbb{R}^n)\|^q \right)^{1/q}$$

(usual modification if $q = \infty$) is an equivalent quasi-norm in $B_{pq}^{s,a}(\mathbb{R}^n)$.

(ii) *Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. If $r > \max(s, \sigma_p) + \frac{n}{\min(p,q)}$ then*

$$\|k_0(1, f) | L_p(\mathbb{R}^n)\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} |k(2^{-j}, f)(\cdot)|^q \right)^{1/q} | L_p(\mathbb{R}^n) \right\|$$

(usual modification if $q = \infty$) is an equivalent quasi-norm in $F_{pq}^{s,a}(\mathbb{R}^n)$.

The advantage of (2.4) compared with $(\varphi_j \widehat{f})^\vee$ from Definition 2.1 is its strictly local nature: in order to calculate $k(t, f)(x)$ in a given point $x \in \mathbb{R}^n$ one needs only a knowledge of $f(z)$ in an anisotropic ball $\{z \in \mathbb{R}^n : |z - x|_a \leq t\}$. This observation was of great service for us in the proof at the atomic decomposition theorem, see the next section.

The detailed proof of the above theorem is given in [2, Theorem 4.9]. The main ingredient for proving it is a rather general but highly technical characterisation of the spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ which extends to the anisotropic situation the results from [Tri97, 2.4.1, 2.5.1], see [2, Theorem 4.4].

Remark 2.3 Theorem 2.2 is the main tool in obtaining the atomic decomposition theorem for the anisotropic function spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ but it is also of independent interest. Its isotropic counterpart can be found in [Tri92, 2.4.6, 2.5.3]; instead of (2.5) it is used the representation

$$k = \Delta^N k^0 = \left(\sum_{i=1}^n \frac{\partial^2}{\partial y_i^2} \right)^N k^0(y)$$

(with $2N = r$ sufficiently large and $k^0 \in S(\mathbb{R}^n)$) which is in fact

$$\widehat{k}(\xi) = |\xi|^{2N} \widehat{k^0}(\xi).$$

But this assumption was taken only for simplicity and it can be replaced by (2.5) with the Euclidean distance $|\cdot|$ instead of $|\cdot|_a$.

Remark 2.4 Examples of functions k satisfying the assumptions of the theorem can be constructed as in [FrJ85, p.783].

2.3 Anisotropic atoms and the atomic decomposition theorem

Recall $a = (a_1, \dots, a_n)$ denotes a given anisotropy. Let \mathbb{Z}^n be the lattice of all points in \mathbb{R}^n with integer-valued components. If $\nu \in \mathbb{N}_0$ and $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ we denote $Q_{\nu m}^a$ the rectangle in \mathbb{R}^n centred at $2^{-\nu a} m = (2^{-\nu a_1} m_1, \dots, 2^{-\nu a_n} m_n)$ which has sides parallel to the axes and side lengths respectively $2^{-\nu a_1}, \dots, 2^{-\nu a_n}$. Remark that Q_{0m}^a is a cube with side length 1. If $Q_{\nu m}^a$ is such a rectangle in \mathbb{R}^n and $c > 0$ then $cQ_{\nu m}^a$ is the rectangle in \mathbb{R}^n concentric with $Q_{\nu m}^a$ and with side lengths respectively $c2^{-\nu a_1}, \dots, c2^{-\nu a_n}$.

If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$ is a multi-index the derivatives D^β have the usual meaning and if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ then $x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$. The scalar product between the anisotropy a and the multi-index β is $a\beta = a_1\beta_1 + \dots + a_n\beta_n$.

If E is a Lebesgue measurable subset of \mathbb{R}^n then $|E|$ denotes its Lebesgue measure.

We are now prepared to introduce the anisotropic atoms.

Definition 2.5 (i) Let $K \in \mathbb{R}$, $c > 1$; a function $\rho : \mathbb{R}^n \rightarrow \mathbb{C}$ for which there exist all derivatives $D^\beta \rho$ if $a\beta \leq K$ (continuous if $K \leq 0$) is called an anisotropic 1_K -atom if:

$$\text{supp } \rho \subset cQ_{0m}^a \quad \text{for some } m \in \mathbb{Z}^n \quad (2.6)$$

and

$$|D^\beta \rho(x)| \leq 1 \quad \text{if } a\beta \leq K. \quad (2.7)$$

(ii) Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $K, L \in \mathbb{R}$, $c > 1$; a function $\rho : \mathbb{R}^n \rightarrow \mathbb{C}$ for which there exist all derivatives $D^\beta \rho$ if $a\beta \leq K$ (continuous if $K \leq 0$) is called an anisotropic $(s, p)_{K,L}$ -atom, if:

$$\text{supp } \rho \subset cQ_{\nu m}^a \quad \text{for some } \nu \in \mathbb{N} \quad \text{and some } m \in \mathbb{Z}^n, \quad (2.8)$$

$$|D^\beta \rho(x)| \leq |Q_{\nu m}^a|^{\frac{s}{n} - \frac{1}{p} - \frac{a\beta}{n}} \quad \text{if } a\beta \leq K, \quad (2.9)$$

$$\int_{\mathbb{R}^n} x^\beta \rho(x) dx = 0 \quad \text{if } a\beta \leq L. \quad (2.10)$$

If the atom ρ is located at $Q_{\nu m}^a$ (that means $\text{supp } \rho_{\nu m}^a \subset cQ_{\nu m}^a$ with $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $c > 1$) then we will write it $\rho_{\nu m}^a$.

We begin with some technical explanations. The value of the number $c > 1$ in (2.6) and (2.8) is unimportant. It simply makes clear that at the level ν some controlled overlapping of the supports of $\rho_{\nu m}^a$ must be allowed.

Since $|Q_{\nu m}^a| = 2^{-\nu n}$ condition (2.9) may be written as

$$|D^\beta \rho(x)| \leq 2^{-\nu(s - \frac{n}{p})} 2^{\nu a\beta} \quad \text{if } a\beta \leq K$$

and if $K \leq 0$ then (2.9) is $|\rho(x)| \leq 2^{-\nu(s - \frac{n}{p})}$.

The moment conditions (2.10) can be reformulated as

$$D^\beta \hat{\rho}(0) = 0 \quad \text{if } a\beta \leq L,$$

which shows that a sufficiently strong decay of $\widehat{\rho}$ at the origin is required. If $L < 0$ then (2.10) simply means that there are no moment conditions.

The reason for the normalising factor in (2.7) and (2.9) is that there exists a constant $c > 0$ such that for all these atoms we have $\|\rho | B_{pq}^{s,a}(\mathbb{R}^n)\| \leq c$, $\|\rho | F_{pq}^{s,a}(\mathbb{R}^n)\| \leq c$. Hence, as in the isotropic case, atoms are normalised building blocks satisfying some moment conditions.

This construction generalises isotropic atoms as they are in the works of M. Frazier and B. Jawerth, see [FrJ85] and [FrJ90] and the survey [FJW91]. It is also slightly related to the concept of anisotropic building blocks (compactly supported and satisfying some norming and some moment conditions) used by P. Soardi in [Soa83] to define anisotropic Hardy spaces and to study the relation of these spaces to anisotropic Lipschitz and Campanato - Morrey spaces.

Before presenting the atomic decomposition theorem we introduce the sequence spaces b_{pq} and f_{pq}^a . If $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $Q_{\nu m}^a$ is a rectangle as above let $\chi_{\nu m}$ be the characteristic function of $Q_{\nu m}^a$; if $0 < p \leq \infty$ let $\chi_{\nu m}^{(p)} = 2^{\nu n/p} \chi_{\nu m}$ (obvious modification if $p = \infty$) be the $L_p(\mathbb{R}^n)$ -normalised characteristic function of $Q_{\nu m}^a$.

If $0 < p, q \leq \infty$ then b_{pq} is the collection of all sequences $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ such that

$$\|\lambda | b_{pq}\| = \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q}$$

(usual modification if $p = \infty$ and/or $q = \infty$) is finite.

If $0 < p, q \leq \infty$ then f_{pq}^a is the collection of all sequences $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ such that

$$\|\lambda | f_{pq}^a\| = \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m} \chi_{\nu m}^{(p)}(\cdot)|^q \right)^{1/q} | L_p(\mathbb{R}^n) \right\|$$

(usual modification if $p = \infty$ and/or $q = \infty$) is finite.

It is easy to see that b_{pq} and f_{pq}^a are quasi-Banach spaces and $b_{p, \min(p,q)} \hookrightarrow f_{pq}^a \hookrightarrow b_{p, \max(p,q)}$ and, in particular, $b_{pp} = f_{pp}^a$, see [Tri97, 13.6].

For $0 < p \leq \infty$ and $0 < q \leq \infty$ we will use the abbreviations

$$\sigma_p = n \left(\frac{1}{p} - 1 \right)_+ \quad \text{and} \quad \sigma_{pq} = n \left(\frac{1}{\min(p,q)} - 1 \right)_+. \quad (2.11)$$

Theorem 2.6 (i) *Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and let $K, L \in \mathbb{R}$ such that*

$$K \geq a_{max} + s \quad \text{if} \quad s \geq 0$$

and

$$L \geq \sigma_p - s.$$

Then $g \in S'(\mathbb{R}^n)$ belongs to $B_{pq}^{s,a}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$g = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \rho_{\nu m}^a, \quad \text{convergence being in } S'(\mathbb{R}^n), \quad (2.12)$$

where $\rho_{\nu m}^a$ are anisotropic 1_K -atoms ($\nu = 0$) or anisotropic $(s, p)_{K,L}$ -atoms ($\nu \in \mathbb{N}$) and $\lambda \in b_{pq}$ where $\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$.

Furthermore, $\inf \|\lambda | b_{pq}\|$ where the infimum is taken over all admissible representations (2.12), is an equivalent quasi-norm in $B_{pq}^{s,a}(\mathbb{R}^n)$.

(ii) Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and let $K, L \in \mathbb{R}$ such that

$$K \geq a_{max} + s \quad \text{if} \quad s \geq 0$$

and

$$L \geq \sigma_{pq} - s.$$

Then $g \in S'(\mathbb{R}^n)$ belongs to $F_{pq}^{s,a}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$g = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \rho_{\nu m}^a, \quad \text{convergence being in } S'(\mathbb{R}^n), \quad (2.13)$$

where $\rho_{\nu m}^a$ are anisotropic 1_K -atoms ($\nu = 0$) or anisotropic $(s,p)_{K,L}$ -atoms ($\nu \in \mathbb{N}$) and $\lambda \in f_{pq}^a$ where $\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$.

Furthermore, $\inf \|\lambda | f_{pq}^a\|$ where the infimum is taken over all admissible representations (2.13), is an equivalent quasi-norm in $F_{pq}^{s,a}(\mathbb{R}^n)$.

The convergence in $S'(\mathbb{R}^n)$ can be obtained as a by-product of the proof using the same method as in [Tri97, 13.9]. We will refer to the above theorem as to the atomic decomposition theorem in anisotropic function spaces. As mentioned in the introduction it generalises to anisotropic function spaces the results of M. Frazier and B. Jawerth from [FrJ85] and [FrJ90].

Remark 2.7 Let $d > 0$ be given, let $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ fixed and let us denote $R_{\nu m}^a$ a rectangle with sides parallel to the axes, centred at $x^{\nu m}$ where

$$|x_i^{\nu m} - 2^{-\nu a_i} m_i| \leq d 2^{-\nu a_i} \quad \text{for all } i \in \{1, \dots, n\}, \quad (2.14)$$

and with side lengths respectively $2^{-\nu a_1}, \dots, 2^{-\nu a_n}$.

Then let $c > 0$ be chosen in dependence of d such that for every choice of $\nu \in \mathbb{N}_0$ and all choices of $x^{\nu m}$ in (2.14) we have

$$\bigcup_{m \in \mathbb{Z}^n} c R_{\nu m}^a = \mathbb{R}^n. \quad (2.15)$$

It will be clear from the proof that we may replace in Definition 2.5 the rectangle $Q_{\nu m}^a$ by $R_{\nu m}^a$ where the number c is defined in (2.15). A similar remark in the isotropic case was very useful in the work of H. Triebel and H. Winkelhoß, [TrW96a], cf. also [EdT96, 2.2.3].

The detailed proof of the theorem is given in [2, Section 5.1] but we make here some comments. The first part of the proof, that in which the atoms are constructed and where it is shown that the decompositions (2.12) and (2.13) hold, is essentially based on an anisotropic version of a resolution of unity of Calderon type, see [2, Lemma 5.1] and cf. [FJW91, 5.12]; this construction is the anisotropic counterpart of what was done in [FJW91, Theorem 5.11].

To prove the second part we used the theorem on local means in anisotropic function spaces (presented in the previous section), the technique of maximal functions, see [2, Theorem 4.3], an inequality of Fefferman - Stein type, see [2, (4.2)], and an anisotropic version of Taylor's expansion theorem, see [2, Theorem 5.3].

2.4 Application to traces of anisotropic function spaces

The technique of obtaining trace theorems for isotropic function spaces of $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$ type using various type of atomic decompositions in these spaces was already used by several authors, see for example the works of M. Frazier and B. Jawerth, [FrJ85], [FrJ90] and of H. Triebel, [Tri92, 4.4.1-4.4.3], where comprehensive references to this topic are given. The aim of this section is to give a unified self contained approach to the study of traces of anisotropic function spaces of Besov - Triebel - Lizorkin type based on the atomic decomposition theorem for these spaces proved in [2] and briefly presented in the previous section.

This technique allows us to re-obtain the trace theorems proved in [Nik77], [BuG79], [Gol79a], [Gol79b], [Kal79] and to extend them to the whole admissible range of parameters from the definition of the anisotropic function spaces of $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ type.

The trace problem reads as follows. Let $x = (x', x_n) \in \mathbb{R}^n$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$ (we always assume in the sequel that $n \in \mathbb{N}$ and $n \geq 2$). We ask whether

$$tr_{\mathbb{R}^{n-1}} : f(x) \mapsto f(x', 0) \quad (2.16)$$

makes sense if f belongs to some spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ or $F_{pq}^{s,a}(\mathbb{R}^n)$.

Rather final answers are known in the isotropic case, see the above mentioned works of M. Frazier, B. Jawerth and H. Triebel and the references given there.

Starting with the n - dimensional anisotropy $a = (a_1, \dots, a_n)$ we define the $(n-1)$ - dimensional anisotropy $a^* = (a_1^*, \dots, a_{n-1}^*)$ where:

$$a_k^* = \frac{n-1}{n-a_n} a_k \quad \text{for all } 1 \leq k \leq n-1. \quad (2.17)$$

Clearly $a_1^* + \dots + a_{n-1}^* = n-1$.

If $n=2$ then $a_1^* = 1$; this corresponds to the isotropic 1- dimensional case.

The first result we will prove is the following:

Theorem 2.8 *Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and let*

$$r = \frac{n-1}{n-a_n} \left(s - \frac{a_n}{p} \right). \quad (2.18)$$

If

$$r > (n-1) \left(\frac{1}{p} - 1 \right)_+ \quad (2.19)$$

then $tr_{\mathbb{R}^{n-1}}$ is a linear and bounded operator from $B_{pq}^{s,a}(\mathbb{R}^n)$ onto $B_{pq}^{r,a^}(\mathbb{R}^{n-1})$,*

$$tr_{\mathbb{R}^{n-1}} B_{pq}^{s,a}(\mathbb{R}^n) = B_{pq}^{r,a^*}(\mathbb{R}^{n-1}) \quad (2.20)$$

and (if $0 < p < \infty$) $tr_{\mathbb{R}^{n-1}}$ is a linear and bounded operator from $F_{pq}^{s,a}(\mathbb{R}^n)$ onto $B_{pp}^{r,a^}(\mathbb{R}^{n-1})$,*

$$tr_{\mathbb{R}^{n-1}} F_{pq}^{s,a}(\mathbb{R}^n) = B_{pp}^{r,a^*}(\mathbb{R}^{n-1}). \quad (2.21)$$

The proof of this theorem is shifted to Section 4.6.

Comments. If $1 \leq p \leq \infty$ then (2.20) is well known, see the proof given by S. M. Nikol'skij in [Nik77, 6.5;6.7], see also [ScT87, 4.2.3].

But our proof in Section 4.6 is essentially different and it allows us to extend S. M. Nikol'skij's result to all $0 < p \leq \infty$.

The most interesting assertion of the theorem seems to be the independence of q in the trace of $F_{pq}^{s,a}(\mathbb{R}^n)$ on \mathbb{R}^{n-1} . If $1 < p < \infty$, $1 \leq q \leq \infty$ then (2.21) has been proved (independently) by M. L. Gol'dman in [Gol79a] and G. A. Kaljabin in [Kal79]. Furthermore, let us mention that if $1 < p < \infty$ then (2.21) was first proved for the space $H_p^{s,a}(\mathbb{R}^n) = F_{p2}^{s,a}(\mathbb{R}^n)$ by S. M. Nikol'skij in [Nik77, 9.5].

Our proof of (2.21) (which may be interpreted as the anisotropic version of the technique used in [FrJ90, 11.1]) allows us to obtain the result for all $0 < p < \infty$.

It is clear that if $n = 2$ then $B_{pq}^{r,a^*}(\mathbb{R}^{n-1})$ is the isotropic space $B_{pq}^r(\mathbb{R})$, and $B_{pp}^{r,a^*}(\mathbb{R}^{n-1})$ is the isotropic space $B_{pp}^r(\mathbb{R})$, the number r being defined from (2.18) and (2.19).

The next theorem gives information about traces in the limiting case $r = 0$ in (2.19).

Theorem 2.9 (i) *If $0 < p < \infty$ and $0 < q \leq \min(1, p)$ then $tr_{\mathbb{R}^{n-1}}$ is a linear and bounded operator from $B_{pq}^{\frac{an}{p},a}(\mathbb{R}^n)$ onto $L_p(\mathbb{R}^{n-1})$,*

$$tr_{\mathbb{R}^{n-1}} B_{pq}^{\frac{an}{p},a}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}). \quad (2.22)$$

(ii) *If $0 < p \leq 1$ and $0 < q \leq \infty$ then $tr_{\mathbb{R}^{n-1}}$ is a linear and bounded operator from $F_{pq}^{\frac{an}{p},a}(\mathbb{R}^n)$ onto $L_p(\mathbb{R}^{n-1})$,*

$$tr_{\mathbb{R}^{n-1}} F_{pq}^{\frac{an}{p},a}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}). \quad (2.23)$$

The proof of this theorem is also given in Section 4.6.

Comments. If $1 \leq p < \infty$ and $q = 1$ then (2.22) is well known, a proof was given by V. I. Burenkov and M. L. Gol'dman, see [BuG79] and [Gol79b]. They generalised to the anisotropic case

$$tr_{\mathbb{R}^{n-1}} B_{p1}^{\frac{1}{p}}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}), \quad 1 \leq p < \infty, \quad (2.24)$$

result which was proved (independently) by J. Peetre, see [P75*].

M. Frazier and B. Jawerth extended (2.24) in [FrJ85, Theorem 5.1] to

$$tr_{\mathbb{R}^{n-1}} B_{pq}^{\frac{1}{p}}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}), \quad 0 < p < \infty, \quad 0 < q \leq \min(1, p),$$

and it is clear that (2.22) is the expected extension to the anisotropic situation.

The restriction on the range of p in (2.23) will be clear from the proof we will give; only in this case the trace of $F_{pq}^{\frac{an}{p},a}(\mathbb{R}^n)$ exists and belongs to $L_p(\mathbb{R}^{n-1})$. The isotropic counterpart of (2.23) can be found in [Tri92, 4.4.3].

For the sake of completeness let us mention that if $p = \infty$ then $B_{\infty 1}^{0,a}(\mathbb{R}^n)$ is contained in the space $C(\mathbb{R}^n)$ of bounded uniformly continuous functions on \mathbb{R}^n . By [BuG79] the trace of $B_{\infty 1}^{0,a}(\mathbb{R}^n)$ coincides with $C(\mathbb{R}^{n-1})$.

2.5 Anisotropic quarks and the subatomic decomposition theorem

In this subsection we will assume that the anisotropic distance function $|\cdot|_a$ defined in (2.1) satisfies in addition

$$\{x \in \mathbb{R}^n : |x|_a \leq 2\} \subset [-\pi, \pi]^n.$$

The above restriction, which is of technical nature, was introduced by P. Dintelmann in [Din95a] and seems to be a natural one compared with the isotropic case.

For $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ let $Q_{\nu m}^a$ be the rectangles introduced in Definition 2.5. Recall Q_{00}^a is the cube with side length 1 centred at the origin and let $2^a Q_{00}^a$ be the rectangle concentric with Q_{00}^a and with side lengths respectively $2^{a_1}, \dots, 2^{a_n}$.

Definition 2.10 *Let $\psi \in S(\mathbb{R}^n)$ such that*

$$\text{supp } \psi \subset 2^a Q_{00}^a \quad \text{and} \quad \sum_{k \in \mathbb{Z}^n} \psi(x - k) = 1 \quad \text{if } x \in \mathbb{R}^n$$

and let for any multi-index $\beta \in \mathbb{N}_0^n$, $\psi^\beta(x) = x^\beta \psi(x)$. If $0 < p \leq \infty$ and $s \in \mathbb{R}$ then

$$(\beta qu)_{\nu m}^a(x) = 2^{-\nu \left(s - \frac{n}{p}\right)} \psi^\beta(2^{\nu a} x - m)$$

is called an anisotropic $(s, p) - \beta$ -quark related to $Q_{\nu m}^a$.

Remark 2.11 It is easy to see that up to normalising constants the anisotropic $(s, p) - \beta$ -quarks are anisotropic $(s, p)_{K,L}$ -atoms for any given $K \in \mathbb{R}$ and any given $L < 0$. Moreover, the normalising constants by which the anisotropic $(s, p) - \beta$ -quark must be divided to become an anisotropic $(s, p)_{K,L}$ -atom can be estimated from above by $c2^{\kappa a \beta}$ where $c > 0$ and $\kappa > 0$ are independent of β (recall the notation $a\beta = a_1\beta_1 + \dots + a_n\beta_n$ where $\beta = (\beta_1, \dots, \beta_n)$ is a multi-index).

We will use below the sequence spaces b_{pq} and f_{pq}^a with respect to the sequences

$$\lambda^\beta = \{\lambda_{\nu m}^\beta \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$$

where now $\beta \in \mathbb{N}_0^n$ is a multi-index and we will keep the notation $(\beta qu)_{\nu m}^a$ for an anisotropic $(s, p) - \beta$ -quark related to the rectangle $Q_{\nu m}^a$. The numbers σ_p and σ_{pq} have the same meaning as in (2.11).

Theorem 2.12 (i) *Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s > \sigma_p$. There exists a number $\kappa > 0$ with the following property: let $r > \kappa$; then $g \in S'(\mathbb{R}^n)$ belongs to $B_{pq}^{s,a}(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$g = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta qu)_{\nu m}^a, \quad (2.25)$$

convergence being in $S'(\mathbb{R}^n)$ (first m , then ν , then β), and

$$\sup_{\beta \in \mathbb{N}_0^n} 2^{ra\beta} \|\lambda^\beta\| b_{pq} < \infty. \quad (2.26)$$

Furthermore, the infimum in (2.26) over all admissible representations (2.25) is an equivalent quasi-norm in $B_{pq}^{s,a}(\mathbb{R}^n)$.

(ii) Let $0 < p < \infty$, $0 < q \leq \infty$ and $s > \sigma_{pq}$. There exists a number $\kappa > 0$ with the following property: let $r > \kappa$; then $g \in S'(\mathbb{R}^n)$ belongs to $F_{pq}^{s,a}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$g = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^{\beta} (\beta q u)_{\nu m}^a, \quad (2.27)$$

convergence being in $S'(\mathbb{R}^n)$ (first m , then ν , then β), and

$$\sup_{\beta \in \mathbb{N}_0^n} 2^{ra\beta} \|\lambda^{\beta} | f_{pq}^a \| < \infty. \quad (2.28)$$

Furthermore, the infimum in (2.28) over all admissible representations (2.27) is an equivalent quasi-norm in $F_{pq}^{s,a}(\mathbb{R}^n)$.

The technique developed in [2, Section 5.2] to prove the above theorem goes back to H. Triebel, see [Tri97, 14.15]. However the proof given there covered only isotropic B -spaces; the considerations in [2, Section 5.2] show that the method can be extended to F -spaces.

Remark 2.13 To prove that $g \in F_{pq}^{s,a}(\mathbb{R}^n)$ (respectively $g \in B_{pq}^{s,a}(\mathbb{R}^n)$) can be decomposed as in (2.27) with (2.28) (respectively as in (2.25) with (2.26)) we did not need the assumption $s > \sigma_{pq}$ (respectively $s > \sigma_p$). This restriction is needed only to prove the converse assertion.

Remark 2.14 If one wishes to extend the result to all $s \in \mathbb{R}$ then a lifting argument would be needed. But while the lift operator $(id - \Delta)^{\frac{L+1}{2}}$ between isotropic function spaces causes no problem in keeping the localisation of the (isotropic) quarks (and this fact was essentially used in [Tri97, 14.4]) the situation becomes difficult in case of the anisotropic lift operator. It is well known that if $\sigma \in \mathbb{R}$ then the operator

$$I_{\sigma}(f) = \left(\left(\sum_{k=1}^n (1 + \xi_k^2)^{\frac{1}{2a_k}} \right)^{\sigma} \widehat{f} \right)^{\vee}$$

maps $F_{pq}^{s,a}(\mathbb{R}^n)$ isomorphically onto $F_{pq}^{s-\sigma,a}(\mathbb{R}^n)$ and $\|I_{\sigma}(\cdot) | F_{pq}^{s-\sigma,a}(\mathbb{R}^n)\|$ is an equivalent quasi-norm on $F_{pq}^{s,a}(\mathbb{R}^n)$ (similar result for $B_{pq}^{s,a}(\mathbb{R}^n)$), see [Leo86]; it is clear that I_{σ} causes a lot of troubles in keeping the localisation of the anisotropic quarks and this is the reason why we restricted ourselves to large values of the (mean) smoothness parameter.

3 Eigenvalue distribution of some fractal semi - elliptic operators

3.1 Regular anisotropic d -sets in \mathbb{R}^n

Let again $a = (a_1, \dots, a_n)$ a given anisotropy ($a_1 + \dots + a_n = n$) and recall our abbreviations $a_{min} = \min\{a_i : 1 \leq i \leq n\}$ and $a_{max} = \max\{a_i : 1 \leq i \leq n\}$.

If $j \in \mathbb{N}_0$ and $N_j \in \mathbb{N}_0$ we deal with sets of open rectangles $\{R_{jl} : l = 1, \dots, N_j\}$ in \mathbb{R}^n having sides parallel to the axes; the side length of the rectangle R_{jl} with respect to the x_i - axis is denoted by $r_i^{j,l}$ where $i = 1, \dots, n$.

We will always assume that the side lengths of the rectangles R_{jl} are ordered in the same way, for example $r_1^{j,l} \leq \dots \leq r_n^{j,l}$ for any $j \in \mathbb{N}_0$ and any $l = 1, \dots, N_j$.

Definition 3.1 *Let Q be a cube in \mathbb{R}^n with side length 1, let $0 < d < n$, let $a = (a_1, \dots, a_n)$ a given anisotropy and let $c_1, c_2 > 0$ given numbers.*

Let $N_0 = 1$ and for any $j \in \mathbb{N}$ let N_j be a natural number satisfying

$$c_1 2^{jd} \leq N_j \leq c_2 2^{jd}.$$

A compact set $\Gamma \subset \mathbb{R}^n$ is called a regular anisotropic d -set (with respect to the anisotropy a) if for any $j \in \mathbb{N}_0$ there exists a finite sequence of open rectangles $\{R_{jl} : l = 1, \dots, N_j\}$ having sides parallel to the axes, $R_{01} = \overset{\circ}{Q}$, such that:

(i) there exists a constant $0 < c_0 \leq 1$ such that for all $i = 1, \dots, n$, all $j \in \mathbb{N}_0$ and all $l = 1, \dots, N_j$

$$(c_0 2^{-j})^{a_i} \leq r_i^{j,l} \leq 2^{-ja_i} \quad ; \quad (3.1)$$

(ii) if $l \neq l'$ then $R_{jl} \cap R_{jl'} = \emptyset$;

(iii) for any rectangle $R_{j+1,k}$ there exists a rectangle R_{jl} , $l = l(k)$, such that $R_{j+1,k} \subset R_{jl}$;

(iv) for any $j \in \mathbb{N}_0$ and any $l = 1, \dots, N_j$

$$(\text{vol } R_{jl})^{\frac{d}{n}} = \sum_{R_{j+1,k} \subset R_{jl}} (\text{vol } R_{j+1,k})^{\frac{d}{n}} \quad ; \quad (3.2)$$

(v)

$$\Gamma = \bigcap_{j=0}^{\infty} \bigcup_{l=1}^{N_j} \overline{R_{jl}}.$$

Remark 3.2 Let $n = 2$. If there exists a number $0 < c \leq 1$ such that $c 2^{-2j} \leq \text{vol } R_{jl} \leq 2^{-2j}$ for any $j \in \mathbb{N}_0$ and any $l = 1, \dots, N_j$ and if condition (i) in the above definition is replaced by:

(i) there exists a number $0 \leq \varepsilon \leq 1$ (called the anisotropic deviation) and a number $j_0 \in \mathbb{N}_0$ such that the side lengths $r_1^{j,l}$ and $r_2^{j,l}$ of the rectangle R_{jl} satisfy*

$$2^{-j(1+\varepsilon)} \leq r_1^{j,l} \leq r_2^{j,l} \leq 2^{-j(1-\varepsilon)} \quad \text{for any } j \geq j_0 \quad \text{and any } l = 1, \dots, N_j \quad (3.3)$$

then we obtain the definition of an anisotropic d -set (with anisotropic deviation ε) as it was given by H. Triebel in [Tri97, 5.2].

Corollary 3.3 *Let $n = 2$, let $a = (a_1, a_2)$ a 2- dimensional anisotropy and let $0 < d < 2$. If $\Gamma \subset \mathbb{R}^2$ is a regular anisotropic d -set in the sense of Definition 3.1 then Γ is an anisotropic d -set in the sense of [Tri97, 5.2]. Any $\varepsilon \leq 1$ with $\varepsilon > a_{max} - 1 = 1 - a_{min}$ can be chosen for the anisotropic deviation in (3.3).*

Remark 3.4 Using the normalisation $R_{01} = \overset{\circ}{Q}$ and (3.2) we have

$$\sum_{l=1}^{N_j} (\text{vol } R_{jl})^{\frac{d}{n}} = 1 \quad \text{if } j \in \mathbb{N}_0. \quad (3.4)$$

For basic facts about Radon measures on subsets of \mathbb{R}^n we refer to [Mat95] and to [Rud87].

Theorem 3.5 *Let $0 < d < n$ and let Γ be the regular anisotropic d -set (with respect to the given anisotropy $a = (a_1, \dots, a_n)$) introduced above.*

Then there exists a Radon measure μ in \mathbb{R}^n uniquely determined with $\text{supp } \mu = \Gamma$ and

$$\mu(\Gamma \cap R_{jl}) = (\text{vol } R_{jl})^{\frac{d}{n}} \quad , \quad j \in \mathbb{N}_0 \quad \text{and} \quad l = 1, \dots, N_j. \quad (3.5)$$

Proof. We use the repeated mass distribution procedure from [Fal90, Proposition 1.7].

Starting with $\mu(Q) = 1$ we distribute the mass via $\mu(R_{1l}) = (\text{vol } R_{1l})^{\frac{d}{n}}$ for $l = 1, \dots, N_1$ and, subsequently $\mu(R_{jl}) = (\text{vol } R_{jl})^{\frac{d}{n}}$ for $l = 1, \dots, N_j$.

By (3.4) and (3.2) the unit mass is finally distributed on Γ ; the result is the desired measure μ .

Without going into details since this proof is the counterpart of [Tri97, 4.15,5.5], we outline the main idea: on the set of all positive continuous functions f on Γ we construct a positive linear functional $f \mapsto Lf$ by following the above limit process in the same way as one introduces the Riemann integral via partial sums. By the Riesz representation theorem (see [Rud87, 2.14]) there is a Radon measure μ uniquely determined such that

$$Lf = \int_{\Gamma} f(\gamma) d\mu(\gamma)$$

and the measure μ has the desired properties.

By [Tri97, Definition 3.1] if Γ is an isotropic d -set with underlying measure μ and if $0 < \kappa < 1$ then

$$\mu(B(\gamma, \kappa r) \cap \Gamma) \sim \mu(B(\gamma, r) \cap \Gamma) \sim r^d \quad (3.6)$$

where the equivalence constants depend on κ but not on $\gamma \in \Gamma$ and $0 < r \leq 1$.

For a regular anisotropic d -set Γ we have (3.5) but no counterpart of (3.6). At least a weak version of (3.6) will be needed. If $0 < \kappa < 1$ then κR_{jl} denotes the rectangle concentric with R_{jl} and with side lengths respectively $\kappa r_1^{j,l}, \dots, \kappa r_n^{j,l}$.

Definition 3.6 *The regular anisotropic d -set introduced in Definition 3.1 equipped with the measure μ according to Theorem 3.5 is called proper if there exist two numbers $0 < \kappa < 1$ and $0 < c \leq 1$ such that*

$$\mu(\Gamma \cap \kappa R_{jl}) \geq c (\text{vol } R_{jl})^{\frac{d}{n}} \quad , \quad j \in \mathbb{N}_0 \quad , \quad l = 1, \dots, N_j. \quad (3.7)$$

Remark 3.7 Following the lines of the proof of [Tri97, 5.13] it turns out that if Γ is generated by linear contractions and if $\Gamma \cap \overset{\circ}{Q} \neq \emptyset$ then Γ is proper. Hence condition (3.7) is rather natural, see also the examples below.

Example 3.8 Let Γ be a regular anisotropic fractal (see Section 1.2) generated by the contractions A_m from (1.5) now with $a_1 > a_2$, see Figure 2.

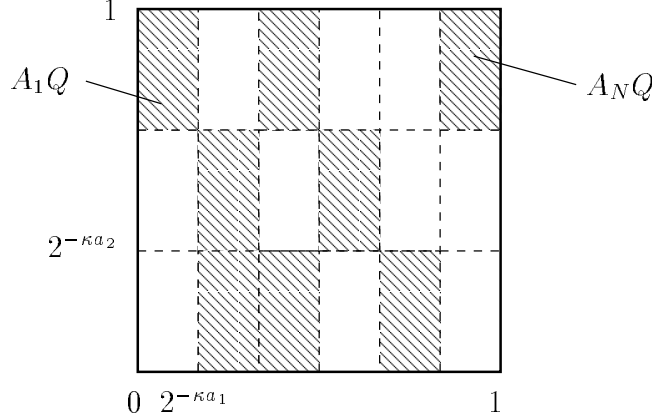


Fig. 2

We keep the same notation as in Section 1.2 and claim that Γ is a regular anisotropic d -set with respect to the anisotropy $a = (a_1, a_2)$, d being the uniquely determined positive number such that

$$\sum_{m=1}^N (\text{vol } A_m Q)^{\frac{d}{2}} = 1. \quad (3.8)$$

Indeed, let us first remark that by (3.8) we have $0 < d < 2$ and $N = 2^{\kappa d}$.

It is clear that for any $\nu \in \mathbb{N}$ and any $m_1, \dots, m_\nu \in \{1, \dots, N\}$ the set $A_{m_1} \circ \dots \circ A_{m_\nu} Q$ is a rectangle having sides parallel to the axes and having side lengths $2^{-\nu \kappa a_1}$, $2^{-\nu \kappa a_2}$; the number of such rectangles is $N^\nu = 2^{\nu \kappa d}$.

If $j \in \mathbb{N}$ let $\nu = \nu(j) = \left\lceil \frac{j}{\kappa} \right\rceil + 1$, let $N_j = N^{\nu(j)}$ and let

$$\{R_{jl} : l = 1, \dots, N_j\} = \{A_{m_1} \circ \dots \circ A_{m_\nu} \overset{\circ}{Q} : 1 \leq m_1, \dots, m_\nu \leq N\}.$$

Clearly we find two constants $c_1, c_2 > 0$ such that $c_1 2^{jd} \leq N_j \leq c_2 2^{jd}$ for any $j \in \mathbb{N}$. Furthermore, the side lengths $r_1^{j,l}$ and $r_2^{j,l}$ of the rectangle R_{jl} satisfy $r_1^{j,l} \leq r_2^{j,l}$ and

$$(c_0 2^{-j})^{a_i} \leq r_i^{j,l} \leq 2^{-ja_i} \quad \text{with } i = 1, 2$$

where $c_0 = 2^{-\kappa}$. Using (3.8) it is easy to see that all the other properties from the definition of a regular anisotropic d -set are satisfied.

We have to remark that compared with the Hausdorff dimension the number d defined from (3.8) (which is nothing else than the affine dimension of Γ , see (1.8)) depends only on the number N of the rectangles $A_m Q$ selected at the first stage of the construction and not on their relative position.

Example 3.9 In specification of the situation in Figure 2 we suppose now that in each column precisely one rectangle $A_m Q$ is located. Let

$$\frac{K_1}{K_2} = 2k + 1 \quad \text{for some } k \in \mathbb{N}$$

be an odd natural number and let the rectangles $A_m Q$ be arranged as depicted in Figure 3 where we choose in the counterpart of (1.5) always $\eta_2^m = 1$ and we choose $\eta_1^m = 1$ in the first K_2 columns, $\eta_1^m = -1$ in the second K_2 columns (additional reflection) then again $\eta_1^m = 1$ in the third K_2 columns and so on, see Figure 3.

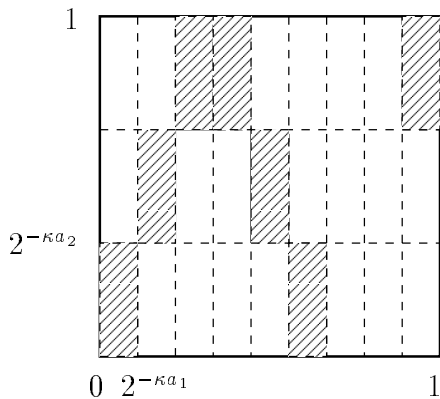


Fig. 3

Under these assumptions the resulting anisotropic fractal Γ is the graph of a continuous function, for a proof see [Tri97, 4.21]. It is clear that Γ may be interpreted as a generalisation of Hironaka's curve (briefly presented in [Mul84]).

Moreover, Γ is an isotropic d -set where $d = \dim_H \Gamma = 2 - a_2/a_1$, see [Tri97, 4.22, 4.23]. It is not difficult to see that Γ is a regular anisotropic d -set with $d = a_1$.

Example 3.10 Let A_1, A_2 be the affine contractions on \mathbb{R}^2 which map the unit square $[0, 1] \times [0, 1]$ onto the rectangles R_1 and R_2 of sides 2^{-a_1} and 2^{-a_2} where $0 < a_2 < a_1$ and $a_1 + a_2 = 2$ as in Figure 4.

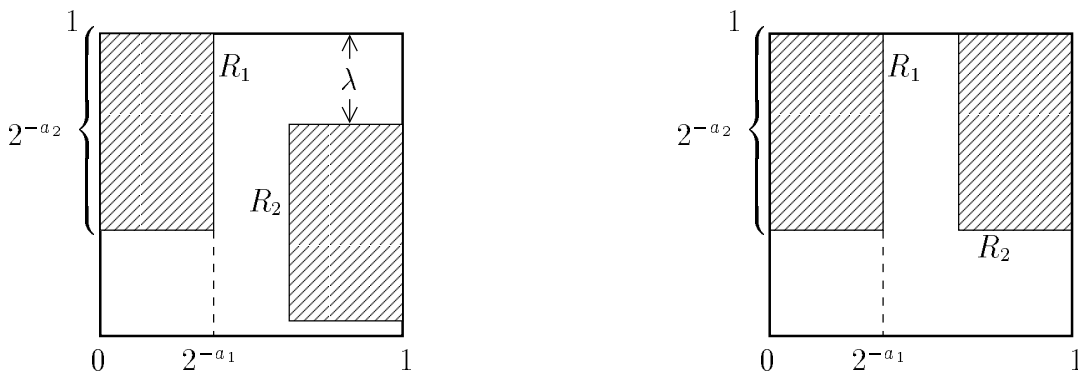


Fig. 4

The rectangle R_1 abuts the upper side of the square Q but the end of R_2 is at distance $\lambda \geq 0$ from it. If Γ is the invariant set for A_1 and A_2 then by [Fal90, Example 9.10, p.127,128] we have $\dim_H \Gamma \geq 1$ when $\lambda > 0$ and $\dim_H \Gamma < 1$ when $\lambda = 0$.

It is easy to see that in both cases Γ is a regular anisotropic 1-set with respect to the anisotropy $a = (a_1, a_2)$; as in [Tri97, 5.13] it follows that Γ is proper if, and only if, $\lambda > 0$.

Comments. It is possible to define the 'dimension' of a set in many ways, some satisfactory and other less so but it is important to realise that different definitions may give different values of dimension for the same set and may have also different properties.

First, it is clear that proper anisotropic d -sets as defined by H. Triebel in [Tri97, 5.2] with deviation $\varepsilon = 0$ in (3.3) are closely related to isotropic compact d -sets (see the definition in Section 1.1).

Secondly, if the anisotropy a is non-trivial, that means $a \neq (1, \dots, 1)$, then the number d in Definition 3.1 and the Hausdorff dimension $\dim_H \Gamma$ are completely unrelated, see also [Tri97, 4.14, 4.22].

Furthermore, it can happen that the regular anisotropic d -set Γ is also an isotropic d' -set with $d \neq d'$, see Example 3.9.

3.2 The spaces $L_p(\Gamma)$

3.2.1 The structure theorem

Let $a = (a_1, \dots, a_n)$ a given anisotropy. As in the isotropic case, if Γ is a closed set in \mathbb{R}^n with Lebesgue measure $|\Gamma| = 0$ and if $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$ we define

$$B_{pq}^{s,a;\Gamma}(\mathbb{R}^n) = \{f \in B_{pq}^{s,a}(\mathbb{R}^n) : f(\varphi) = 0 \text{ if } \varphi \in S(\mathbb{R}^n), \varphi|_{\Gamma} = 0\}$$

where $\varphi|_{\Gamma}$ is the restriction of φ to Γ .

We have $\text{supp } f \subset \Gamma$ if $f \in B_{pq}^{s,a;\Gamma}(\mathbb{R}^n)$ in any case. Remark also that if $0 < p \leq \infty$, $0 < q \leq \infty$ and $s > n(\frac{1}{p} - 1)_+$ then $B_{pq}^{s,a}(\mathbb{R}^n) \hookrightarrow L_1^{\text{loc}}(\mathbb{R}^n)$ (this can be proved as in [Tri92, Remark 2.3.2/3]) and, hence, $B_{pq}^{s,a;\Gamma}(\mathbb{R}^n) = \{0\}$ is trivial.

In other words, only values $s \leq n(\frac{1}{p} - 1)_+$ (in particular $s \leq 0$ if $1 \leq p \leq \infty$) are of interest.

Let Γ be a regular anisotropic d -set with respect to the given anisotropy $a = (a_1, \dots, a_n)$. Let μ the uniquely determined Radon measure underlying Γ according to Theorem 3.5.

The L_p -spaces on Γ , $0 < p \leq \infty$, are introduced in the usual way with respect to the underlying Radon measure μ on Γ according to Theorem 3.5.

If $1 \leq p \leq \infty$ any $f_{\Gamma} \in L_p(\Gamma)$ will be interpreted as a tempered distribution $f \in S'(\mathbb{R}^n)$ as in (1.11) with \mathbb{R}^n in place of \mathbb{R}^2 :

$$f(\varphi) = \int_{\Gamma} f_{\Gamma}(\gamma) (\varphi|_{\Gamma})(\gamma) d\mu(\gamma), \quad \varphi \in S(\mathbb{R}^n). \quad (3.9)$$

Theorem 3.11 *Let $0 < d < n$ and let Γ be a regular anisotropic d -set in \mathbb{R}^n with respect to the anisotropy $a = (a_1, \dots, a_n)$.*

If $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$ then (in the sense of (3.9))

$$L_p(\Gamma) = B_{p\infty}^{-\frac{n-d}{p'}, a; \Gamma}(\mathbb{R}^n). \quad (3.10)$$

Moreover, $\|f_{\Gamma}\|_{L_p(\Gamma)} \sim \|f\|_{B_{p\infty}^{-\frac{n-d}{p'}, a}(\mathbb{R}^n)}$.

The proof is given in Section 4.1.

3.2.2 Traces

Assume that Γ is a regular anisotropic d -set in \mathbb{R}^n with respect to the anisotropy $a = (a_1, \dots, a_n)$. If $\varphi \in S(\mathbb{R}^n)$ then $tr_\Gamma \varphi = \varphi|_\Gamma$ makes sense pointwise. If $0 < p, q < \infty$ and $s \in \mathbb{R}$ then the embedding $tr_\Gamma B_{pq}^{s,a}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma)$ must be understood as follows: there exists a positive number $c > 0$ such that for any $\varphi \in S(\mathbb{R}^n)$

$$\|tr_\Gamma \varphi|_{L_p(\Gamma)}\| \leq c \|\varphi|_{B_{pq}^{s,a}(\mathbb{R}^n)}\|.$$

Since $S(\mathbb{R}^n)$ is dense in $B_{pq}^{s,a}(\mathbb{R}^n)$ for $0 < p, q < \infty$ this inequality can be extended by completion to any $f \in B_{pq}^{s,a}(\mathbb{R}^n)$ and the resulting function is denoted $tr_\Gamma f$.

In addition, the equality $tr_\Gamma B_{pq}^{s,a}(\mathbb{R}^n) = L_p(\Gamma)$ means that any $f_\Gamma \in L_p(\Gamma)$ is the trace of a suitable $g \in B_{pq}^{s,a}(\mathbb{R}^n)$ on Γ and

$$\|f_\Gamma|_{L_p(\Gamma)}\| \sim \inf\{\|g|_{B_{pq}^{s,a}(\mathbb{R}^n)}\| : tr_\Gamma g = f_\Gamma\}.$$

Theorem 3.12 *Let $0 < d < n$ and let Γ be a regular anisotropic d -set in \mathbb{R}^n with respect to the anisotropy $a = (a_1, \dots, a_n)$. If $\frac{d}{n} < p < \infty$ and $0 < q \leq \min(1, p)$ then*

$$tr_\Gamma B_{pq}^{\frac{n-d}{p}, a}(\mathbb{R}^n) = L_p(\Gamma). \quad (3.11)$$

The proof of this theorem is shifted to Section 4.2.

It will be clear from the proof that the embedding $tr_\Gamma B_{pq}^{\frac{n-d}{p}, a}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma)$ holds for $0 < p \leq \infty$ and $0 < q \leq \min(1, p)$. Possible extensions of Theorem 3.12 to all $0 < p < \infty$ are briefly discussed in Section 4.3 where we will make also some additional remarks concerning Theorems 3.11 and 3.12.

As mentioned in the Introduction, Theorems 3.11 and 3.12 seem to be of independent interest since they are anisotropic counterparts of Theorems 2 and 3 in [TrW96b], see also [Tri97, 18.2, 18.6]. Furthermore, they complement Theorems 1.8 and 1.9.

3.2.3 Compactness of embeddings into $L_p(\Gamma)$

In this subsection we will obtain estimates for the entropy numbers of traces on regular anisotropic d -sets from \mathbb{R}^n .

Let A and B two quasi-Banach spaces. The family of all linear bounded operators $T : A \rightarrow B$ will be denoted by $L(A, B)$ or $L(A)$ if $A = B$.

We will assume that the reader is familiar with the definition of the entropy numbers $e_k(T)$, $k \in \mathbb{N}$, of a compact map $T \in L(A, B)$. This definition is given, for example, in [EdT96, 1.3.1] where one can find also comments and historical references.

The first result is an anisotropic counterpart of Theorem 20.6 in [Tri97].

Theorem 3.13 *Let $0 < d < n$ and let Γ be a regular anisotropic d -set in \mathbb{R}^n with respect to the anisotropy $a = (a_1, \dots, a_n)$.*

Let $0 < p_1 \leq \infty$, $0 < p_2 \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$ such that

$$\delta_+ = s - d \left(\frac{1}{p_1} - \frac{1}{p_2} \right)_+ > 0. \quad (3.12)$$

Then the trace operator

$$tr_\Gamma : B_{p_1 q}^{s+\frac{n-d}{p_1}, a}(\mathbb{R}^n) \rightarrow L_{p_2}(\Gamma) \quad (3.13)$$

is compact and there exists a constant $C > 0$ such that for all $k \in \mathbb{N}$,

$$e_k(tr_\Gamma : B_{p_1 q}^{s+\frac{n-d}{p_1}, a}(\mathbb{R}^n) \rightarrow L_{p_2}(\Gamma)) \leq C k^{-\frac{s}{d}}. \quad (3.14)$$

If, in addition, Γ is proper according to Definition 3.6 then there exists a constant $c > 0$ such that for all $k \in \mathbb{N}$,

$$c k^{-\frac{s}{d}} \leq e_k(tr_\Gamma : B_{p_1 q}^{s+\frac{n-d}{p_1}, a}(\mathbb{R}^n) \rightarrow L_{p_2}(\Gamma)) \quad (3.15)$$

We want to remark that the assumption (3.12) is crucial for the compactness of the operator in (3.13) and that this assumption cannot be weakened by $\delta_+ = 0$. The result below is the anisotropic version of [Tri97, 20.7].

Proposition 3.14 *Let $0 < d < n$ and let Γ be a proper regular anisotropic d -set in \mathbb{R}^n with respect to the anisotropy $a = (a_1, \dots, a_n)$.*

Let $0 < p_1 < p_2 \leq \infty$, $\bar{p}_2 = \min(1, p_2)$ and $s > 0$ such that

$$s - d \left(\frac{1}{p_1} - \frac{1}{p_2} \right) = 0. \quad (3.16)$$

Then the trace operator

$$tr_\Gamma : B_{p_1 \bar{p}_2}^{s+\frac{n-d}{p_1}, a}(\mathbb{R}^n) \rightarrow L_{p_2}(\Gamma) \quad (3.17)$$

is continuous but non-compact.

The proofs of Theorem 3.13 and of Proposition 3.14 are shifted to Section 4.4.

3.3 A model fractal semi - elliptic operator

Let

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$$

be the unit disc in the plane and let $\partial\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ be its boundary.

Let η be a C^∞ function on the interval $[-1, 1]$ with $\eta(t) > 0$ if $|t| < 1$ and

$$\lim_{t \uparrow 1} \frac{\eta(t)}{1-t} = \lim_{t \downarrow -1} \frac{\eta(t)}{1+t} = 1.$$

We consider the semi - elliptic differential operator:

$$(\mathcal{A}_r u)(x) = -\frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^4 u(x)}{\partial x_2^4} + \frac{r}{\eta^2(x_1)} u(x), \quad x = (x_1, x_2) \in \Omega, \quad (3.18)$$

where $r \in \mathbb{R}$.

We will consider the anisotropic Sobolev space

$$W_2^{(1,2)}(\Omega) = \left\{ u \in S'(\Omega) : \|u\|_{L_2(\Omega)} + \left\| \frac{\partial u}{\partial x_1} \right\|_{L_2(\Omega)} + \left\| \frac{\partial^2 u}{\partial x_2^2} \right\|_{L_2(\Omega)} < \infty \right\}. \quad (3.19)$$

Clearly, see (0.5) and (0.6), the mean smoothness is $s = \frac{4}{3}$ and $a = (\frac{4}{3}, \frac{2}{3})$ characterises the anisotropy. Using the notation from the previous sections we have

$$W_2^{\frac{4}{3}, a}(\Omega) = W_2^{(1,2)}(\Omega). \quad (3.20)$$

Remark 3.15 Let

$$\overset{\circ}{W}_2^{(1,2)}(\Omega) = \left\{ u \in W_2^{(1,2)}(\Omega) : u|_{\partial\Omega} = \frac{\partial u}{\partial x_2} |_{\partial\Omega} = 0 \right\}.$$

We know from [Tri83*, Proposition 1] that there exists an $r_0 \in \mathbb{R}$ such that for any $r > r_0$ the operator $\mathcal{A}_r^{1/2}$ maps $\overset{\circ}{W}_2^{(1,2)}(\Omega)$ isomorphically onto $L_2(\Omega)$; consequently, we may fix the norm on $\overset{\circ}{W}_2^{(1,2)}(\Omega)$ by

$$\|u\|_{\overset{\circ}{W}_2^{(1,2)}(\Omega)} = \|\mathcal{A}_r^{1/2}u\|_{L_2(\Omega)}$$

and a corresponding scalar product.

Let Γ be a regular anisotropic fractal as presented in Figure 2, now with $a_1 = \frac{4}{3}$ and $a_2 = \frac{2}{3}$. By Example 3.8 Γ can be interpreted as a regular anisotropic d -set with respect to the anisotropy $a = (\frac{4}{3}, \frac{2}{3})$ so that we may use Theorems 3.11 and 3.12.

If $1 < p < \infty$, $1 \leq q \leq \infty$ and $s \in \mathbb{R}$ then we introduce the spaces

$$\tilde{B}_{pq}^{s,a}(\Omega) = \{f \in B_{pq}^{s,a}(\mathbb{R}^2) : \text{supp } f \subset \overline{\Omega}\} \quad (3.21)$$

normed in the usual way, see also [Tri78, 4.3.2] and [Tri97, 27.11].

In the sequel we shall not distinguish between f_Γ as an element of some $L_p(\Gamma)$ and as the distribution belonging to some $\tilde{B}_{p\infty}^{-s,a}(\Omega)$ according to (3.10).

To avoid any misunderstanding, see also Section 1.4, we emphasise that the trace operator has again two different meanings which we distinguish by tr_Γ and tr^Γ if extra clarity is desirable. Let, for example, $1 < p < \infty$, then

$$tr_\Gamma : \tilde{B}_{p1}^{-\frac{2-d}{p},a}(\Omega) \rightarrow L_p(\Gamma) \quad (3.22)$$

by (3.11) and

$$tr^\Gamma : \tilde{B}_{p1}^{-\frac{2-d}{p},a}(\Omega) \rightarrow \tilde{B}_{p\infty}^{-\frac{2-d}{p},a}(\Omega) \quad (3.23)$$

if one applies in addition (3.10). The latter can be rephrased asking for an optimal extension of tr^Γ considered as a mapping from $D(\Omega)$ into $D'(\Omega)$ given by (0.1).

Theorem 3.16 *Let $0 < d < 2$ and let Ω be the unit disc in the plane. Let $\Gamma \subset \Omega$ be a regular anisotropic d -set with respect to the anisotropy $a = (\frac{4}{3}, \frac{2}{3})$.*

Let tr^Γ be the trace operator in the interpretation (3.23) and (0.1) whereas tr_Γ stands for the trace operator according to (3.22). Let \mathcal{A}_r be the operator from (3.18).

(i) *There exists $r_0 \in \mathbb{R}$ such that for any $r > r_0$ the operator*

$$T = \mathcal{A}_r^{-1} \circ tr^\Gamma \quad (3.24)$$

is compact, non-negative, self adjoint in $\overset{\circ}{W}_2^{(1,2)}(\Omega)$ and has null space

$$N(T) = \{f \in \overset{\circ}{W}_2^{(1,2)}(\Omega) : tr_\Gamma f = 0\}. \quad (3.25)$$

Furthermore, T is generated by the quadratic form in $\overset{\circ}{W}_2^{(1,2)}(\Omega)$

$$\int_{\Gamma} f(\gamma) \overline{g(\gamma)} d\mu(\gamma) = (Tf, g)_{W_2^{(1,2)}(\Omega)} \quad \text{where } f \in \overset{\circ}{W}_2^{(1,2)}(\Omega), g \in \overset{\circ}{W}_2^{(1,2)}(\Omega) \quad (3.26)$$

and μ is the Radon measure according to Theorem 3.5.

(ii) Let $r > r_0$. There exists a constant $C > 0$ such that the positive eigenvalues $\lambda_k(T)$ of T , repeated according to multiplicity and ordered by their magnitude, can be estimated by

$$\lambda_k(T) \leq C k^{-\frac{1}{d}(d+\frac{2}{3})} \quad , \quad k \in \mathbb{N}. \quad (3.27)$$

If, in addition, Γ is proper according to Definition 3.6 then there exists a constant $c > 0$ such that

$$c k^{-\frac{1}{d}(d+\frac{2}{3})} \leq \lambda_k(T) \quad , \quad k \in \mathbb{N}. \quad (3.28)$$

The proof of this theorem is given in Section 4.5.

Remark 3.17 Let Γ be a regular anisotropic d -set with respect to the anisotropy $a = (\frac{4}{3}, \frac{2}{3})$ which is generated by linear contractions.

For example one can take $K_1 = 16$, $K_2 = 4$ and let Γ be generated by the N mappings A_m defined in (1.5) with (1.4) and let the rectangles $A_m Q$ arranged as depicted in Figure 2.

The additional assumption on Γ to be proper in the estimate (3.28) excludes by Remark 3.7 only pathological cases where the whole fractal retreats in the boundary of the starting square.

Remark 3.18 We want to point out that in the proof of the estimate (3.27) the key role is played by the mapping property in the third line of (4.56), more precisely by the fact that for sufficiently large r the operator \mathcal{A}_r^{-1} maps the anisotropic Besov space $B_{2\infty}^{-\frac{2-d}{2}, a}(\Omega)$ bounded into

$$\overset{\circ}{B}_{2\infty}^{\frac{8}{3}-\frac{2-d}{2}, a}(\Omega) = \left\{ f \in B_{2\infty}^{\frac{8}{3}-\frac{2-d}{2}, a}(\Omega) : f|_{\partial\Omega} = \frac{\partial f}{\partial x_2} |_{\partial\Omega} = 0 \right\}.$$

This is obtained in Section 4.5 as a consequence of the results from [Tri83*] using some interpolation results from [Bes96] and [Bes97].

On the other hand the proof of the estimate (3.28) depends on Hilbert space techniques, in particular on Theorem 4.11 which states that for a compact, non-negative and self adjoint operator acting in a Hilbert space its approximation numbers coincide with its eigenvalues.

Comments. Let us recall a result which was briefly mentioned in the Introduction.

V. Shevchik considered in [She98] the operator \mathcal{A}_r acting as an unbounded operator defined in $L_p(\Omega)$, $1 < p < \infty$, having domain of definition

$$\mathcal{D}(\mathcal{A}_r) = \{u \in W_p^{(2,4)}(\Omega, x^0, x^1) : u|_{\partial\Omega} = \frac{\partial u}{\partial x_2} |_{\partial\Omega} = 0\},$$

see (4.57) for the precise definition of the space $W_p^{(2,4)}(\Omega, x^0, x^1)$ if $p = 2$.

He proved that the asymptotic behaviour of the eigenvalues of \mathcal{A}_r is

$$\lambda_k(\mathcal{A}_r) \sim k^{\frac{4}{3}}, \quad k \in \mathbb{N}. \quad (3.29)$$

Clearly the exponent $\frac{4}{3}$ is exactly the half of the mean smoothness parameter of the anisotropic Sobolev space considered as domain of definition.

Let us remark that comparing (3.29) with our result (3.27), complemented by (3.28), one can see that not only the half mean smoothness of the space $\overset{\circ}{W}_2^{(1,2)}(\Omega)$ (as domain of definition for the operator T) but also the anisotropic dimension d of the fractal Γ is coming in.

Of course a natural question appears: *what happens if the model operator \mathcal{A}_r is replaced by a more general (semi - elliptic) operator and if the space $\overset{\circ}{W}_2^{(1,2)}(\Omega)$ is replaced by another anisotropic function space while the fractal is still considered having the same anisotropy as the function space ?*

Nothing has been done so far in this direction. Based on the proof in Section 4.5 and on Remark 3.18 this can be a subject of further investigation.

4 Proofs

4.1 Proof of Theorem 3.11

Step 1. Let $f_\Gamma \in L_p(\Gamma)$ and let f be given by (3.9). We prove that f is an element of $B_{p\infty}^{-\frac{n-d}{p'}, a; \Gamma}(\mathbb{R}^n)$ and that for some $c > 0$

$$\|f\|_{B_{p\infty}^{-\frac{n-d}{p'}, a}(\mathbb{R}^n)} \leq c \|f_\Gamma\|_{L_p(\Gamma)}. \quad (4.1)$$

If k is a C^∞ function in \mathbb{R}^n with $\text{supp } k \subset \{y \in \mathbb{R}^n : |y|_a \leq 1\}$ and $f \in S'(\mathbb{R}^n)$ let

$$k(t, f)(x) = \int_{\mathbb{R}^n} k(y) f(x + t^a y) dy = t^{-n} \int_{\mathbb{R}^n} k(t^{-a}(z - x)) f(z) dz$$

be the local means introduced in (2.4).

In [2, Theorem 4.9] we obtained equivalent quasi-norms in anisotropic function spaces using local means, see also the brief presentation from Section 2.2. In case of negative smoothness we can weaken the assumptions used there; to do this we have only to repeat the arguments from [Win95, 3.1] and [TrW96b, Remark 6] and obtain:

$$\|f\|_{B_{p\infty}^{-\frac{n-d}{p'}, a}(\mathbb{R}^n)} \sim \sup_{j \in \mathbb{N}_0} \left(2^{-j(n-d)/p'} \|k(2^{-j}, f)\|_{L_p(\mathbb{R}^n)} \right) \quad (4.2)$$

where k is a C^∞ function on \mathbb{R}^n with $\text{supp } k \subset \{y \in \mathbb{R}^n : |y|_a \leq 1\}$ and such that $|\widehat{k}(\xi)| > 0$ if $\frac{1}{2} \leq |\xi|_a \leq 2$.

Let $p < \infty$; the modification of the following estimates are obvious if $p = \infty$. Using the definition of f and of $k(2^{-j}, f)$ and applying Hölder's inequality there exists a constant $c > 0$ such that:

$$\begin{aligned} |k(2^{-j}, f)(x)| &= \left| \int_{\mathbb{R}^n} k(y) f(x + 2^{-ja} y) dy \right| \\ &\leq 2^{jn} \int_{\Gamma} |k(2^{ja}(\gamma - x))|^{1/p} |f_\Gamma(\gamma)| \cdot |k(2^{ja}(\gamma - x))|^{1/p'} d\mu(\gamma) \\ &\leq c 2^{jn} \left(\int_{\Gamma} |f_\Gamma(\gamma)|^p |k(2^{ja}(\gamma - x))| d\mu(\gamma) \right)^{1/p} \mu(B^a(x, 2^{-j}) \cap \Gamma)^{1/p'} \end{aligned}$$

where $B^a(x, 2^{-j}) = \{y \in \mathbb{R}^n : |y - x|_a \leq 2^{-j}\} \subset \{y \in \mathbb{R}^n : |y_i - x_i| \leq c 2^{-ja_i}, i = 1, \dots, n\}$. By Definition 3.1/(i) $B^a(x, 2^{-j})$ has a nonempty intersection with at most N rectangles R_{jl} ($l = 1, \dots, N_j$) where N is independent of j so that using (3.5) we get

$$\mu(B^a(x, 2^{-j}) \cap \Gamma) \leq c' 2^{-jd}$$

where $c' > 0$ is independent of j . It follows

$$|k(2^{-j}, f)(x)| \leq c_1 2^{jn} 2^{-jd/p'} \left(\int_{\Gamma} |f_\Gamma(\gamma)|^p |k(2^{ja}(\gamma - x))| d\mu(\gamma) \right)^{1/p}.$$

Using Fubini's theorem we obtain

$$\begin{aligned} \|k(2^{-j}, f)\|_{L_p(\mathbb{R}^n)} &\leq c_1 2^{jn} 2^{-jd/p'} \left(\int_{\Gamma} |f_\Gamma(\gamma)|^p d\mu(\gamma) \int_{\mathbb{R}^n} |k(2^{ja}(\gamma - x))| dx \right)^{1/p} \\ &= c_2 2^{j(n-d)/p'} \|f_\Gamma\|_{L_p(\Gamma)} \end{aligned} \quad (4.3)$$

where c_1 and c_2 are independent of $j \in \mathbb{N}_0$. Clearly (4.3) and (4.2) lead to (4.1).

It is easy to see that we can use the above technique also for $p = 1$ and obtain $L_1(\Gamma) \hookrightarrow B_{1\infty}^{0,a}(\mathbb{R}^n)$ but this case is not especially interesting since the dimension d of Γ completely disappears in the smoothness parameter of the B -space.

Step 2. To prove the reverse inclusion we need some elementary preparations; we adapt the technique from [TrW96b] to our anisotropic situation.

Let $0 < q < \infty$ and suppose Γ is a regular anisotropic d -set with the underlying measure μ ; the space $L_q(\Gamma)$ has the above meaning with respect to μ .

Let $r > 0$ be a fixed number which will be chosen sufficiently large later on. For $\nu \in \mathbb{N}_0$ let $R_{\nu l}$ (with $l = 1, \dots, N_\nu$) be the rectangles from Definition 3.1 and let $x^{\nu,l}$ be the centre of the rectangle $R_{\nu l}$. For any $l = 1, \dots, N_\nu$ we fix a point $z_{\nu l} \in \overline{R_{\nu l}} \cap \Gamma$ such that $|x^{\nu,l} - z_{\nu l}|_a = \inf\{|x^{\nu,l} - z|_a : z \in \overline{R_{\nu l}} \cap \Gamma\}$.

From $\{z_{\nu l} : l = 1, \dots, N_\nu\}$ we extract a maximal $r 2^{-\nu}$ anisotropic distant set, that is, a set of points $\{z_{\nu t} : t \in T\}$ where $T \subset \{1, \dots, N_\nu\}$ such that

$$|z_{\nu t} - z_{\nu t'}|_a > r 2^{-\nu} \quad \text{if } t \neq t' \quad \text{and} \quad \Gamma \subset \bigcup_{t \in T} \overline{r^a R_{\nu t}}$$

and where $r^a R_{\nu t}$ is the open rectangle concentric with $R_{\nu t}$ and with side lengths respectively $r^{a_1} r_1^{\nu,t}, \dots, r^{a_n} r_n^{\nu,t}$ (recall the side lengths of the rectangle $R_{\nu t}$ are denoted $r_1^{\nu,t}, \dots, r_n^{\nu,t}$).

For any $m \in \mathbb{Z}^n$ we fix a reference point $y_{\nu m} \in \{z_{\nu t} : t \in T\}$ which minimises the anisotropic distance between $2^{-\nu a} m$ (the centre of the rectangle $Q_{\nu m}^a$ introduced in Section 2.3) and the set $\{z_{\nu t} : t \in T\}$:

$$|y_{\nu m} - 2^{-\nu a} m|_a = \min_{t \in T} |z_{\nu t} - 2^{-\nu a} m|_a.$$

We assume r to be so large such that

$$y_{\nu m} = z_{\nu t} \quad \text{if } m \in \mathbb{Z}^n \quad \text{with} \quad 2^a Q_{\nu m}^a \cap R_{\nu t} \neq \emptyset. \quad (4.4)$$

Using the maximality of $\{z_{\nu t} : t \in T\}$ one easily derives the existence of a constant $b_1 > 0$ which is independent of ν and m such that

$$|y_{\nu m} - 2^{-\nu a} m|_a \leq b_1 2^{-\nu} \quad , \quad \nu \in \mathbb{N}_0 \quad , \quad m \in \mathbb{Z}^n \quad , \quad 2^a Q_{\nu m}^a \cap \Gamma \neq \emptyset$$

(of course b_1 can depend on r). Consequently, there exists another constant $b_2 > 0$ independent of ν and m such that

$$|x - y_{\nu m}|_a \leq b_2 2^{-\nu} \quad \text{if } x \in 2^a Q_{\nu m}^a \cap \Gamma \neq \emptyset. \quad (4.5)$$

Let now ψ a C^∞ function on \mathbb{R}^n such that

$$\text{supp } \psi \subset 2^a Q_{00}^a \quad \text{and} \quad \sum_{m \in \mathbb{Z}^n} \psi(x - m) = 1 \quad \text{if } x \in \mathbb{R}^n. \quad (4.6)$$

Let g be a continuous function on Γ with compact support. Denoting

$$g_\nu = \sum_m g(y_{\nu m}) \psi(2^{\nu a} \cdot - m)|_\Gamma \quad (4.7)$$

where the sum was taken over those $m \in \mathbb{Z}^n$ such that $2^a Q_{\nu m}^a \cap \Gamma \neq \emptyset$, using (4.5) and the fact that μ is a Radon measure supported on Γ it follows that g_ν tends to g in $L_q(\Gamma)$ if $\nu \rightarrow \infty$.

Furthermore, by (4.6) and (4.4) we have $g_\nu(x) = g(z_{\nu t})$ if $x \in \Gamma \cap R_{\nu t}$.

The rectangles $\{R_{\nu t} : t \in T\}$ are in any case disjoint. Hence:

$$\begin{aligned} \|g_\nu | L_q(\Gamma)\|^q &\geq \sum_{t \in T} \int_{\Gamma \cap R_{\nu t}} |g_\nu(x)|^q d\mu(\gamma) = \sum_{t \in T} |g(z_{\nu t})|^q \mu(\Gamma \cap R_{\nu t}) \\ &\geq c 2^{-\nu d} \sum_{t \in T} |g(z_{\nu t})|^q \end{aligned}$$

where in the last inequality we used (3.5).

On the other hand $g(y_{\nu m}) = g(z_{\nu t})$ for at most N of the lattice points $m \in \mathbb{Z}^n$, where N does not depend on ν and t . Thus we arrive at:

$$2^{-\nu d/q} \left(\sum_m |g(y_{\nu m})|^q \right)^{1/q} \leq c 2^{-\nu d/q} \left(\sum_{t \in T} |g(z_{\nu t})|^q \right)^{1/q} \leq c' \|g_\nu | L_q(\Gamma)\|. \quad (4.8)$$

The functions $\psi(2^{\nu a} \cdot -m)$ overlap at most N' times where N' does not depend on ν and m . So

$$\|g_\nu | L_q(\Gamma)\|^q \leq c \sum_m |g(y_{\nu m})|^q \mu(2^a Q_{\nu m}^a \cap \Gamma) \leq c' 2^{-\nu d} \sum_m |g(y_{\nu m})|^q \quad (4.9)$$

with $c' > 0$ independent of ν , where in the last inequality we used the fact that by (3.1) the rectangle $2^a Q_{\nu m}^a$ has a non-empty intersection with at most N'' rectangles $R_{\nu l}$ ($l = 1, \dots, N_\nu$), the number N'' being independent of ν . From (4.8) and (4.9) we obtain

$$\|g_\nu | L_q(\Gamma)\| \sim 2^{-\nu d/q} \left(\sum_m |g(y_{\nu m})|^q \right)^{1/q} \quad (4.10)$$

(recall that the sum is taken over those $m \in \mathbb{Z}^n$ such that $2^a Q_{\nu m}^a \cap \Gamma \neq \emptyset$).

Step 3. If Γ is a regular anisotropic d -set with $0 < d < n$ and if $0 < p < \infty$ then $C_0(\Gamma)$, the space of all compactly supported continuous functions on Γ , is dense in $L_p(\Gamma)$.

The proof of the above assertion is well known if $1 \leq p < \infty$ see, for example, [Rud87, Theorem 3.14]. The method used in [Rud87] for $1 \leq p < \infty$ can be extended to all $0 < p < \infty$ using basic properties of the Radon measure μ , using the inequality $|u - v|^p \leq \max\{1, 2^{p-1}\}(|u|^p + |v|^p)$ and making appropriate changes at the ends of the proofs of Theorems 3.13 and 3.14 in [Rud87].

Step 4. Now we are able to prove the reverse of what has been done in Step 1.

Let $f \in B_{p\infty}^{-\frac{n-d}{p'}, a; \Gamma}(\mathbb{R}^n)$ and let $g \in C_0(\Gamma)$. Let ψ be a function chosen according to (4.6) and let $g_\nu = \psi_\nu | \Gamma$ be an approximating sequence for $g \in L_{p'}(\Gamma)$ where

$$\psi_\nu(x) = \sum_m g(y_{\nu m}) \psi(2^{\nu a} x - m) = 2^{-\nu d/p'} \sum_m g(y_{\nu m}) 2^{\nu d/p'} \psi(2^{\nu a} x - m),$$

see (4.7). Up to constants each term $2^{\nu d/p'} \psi(2^{\nu a} \cdot -m)$ is an anisotropic $(\frac{n-d}{p'}, p')$ $_{K,L}$ -atom in $B_{p'1}^{\frac{n-d}{p'}, a}(\mathbb{R}^n)$ where K is sufficiently large and $L < 0$ (since $p' \geq 1$ and $n > d$ there are no moment conditions required). Using the atomic decomposition theorem we obtain

$$\left\| \psi_\nu | B_{p'1}^{\frac{n-d}{p'}, a}(\mathbb{R}^n) \right\| \leq c 2^{-\nu d/p'} \left(\sum_m |g(y_{\nu m})|^{p'} \right)^{1/p'}$$

and using (4.10) this leads (for large values of ν) to

$$\left\| \psi_\nu | B_{p'1}^{\frac{n-d}{p'},a}(\mathbb{R}^n) \right\| \leq c \|g_\nu | L_{p'}(\Gamma)\| \leq c' \|g | L_{p'}(\Gamma)\|. \quad (4.11)$$

In particular ψ_ν belongs to the pre-dual of $B_{p\infty}^{-\frac{n-d}{p'},a}(\mathbb{R}^n)$. The

$$B_{p'1}^{\frac{n-d}{p'},a}(\mathbb{R}^n) \quad - \quad B_{p\infty}^{-\frac{n-d}{p'},a}(\mathbb{R}^n)$$

duality (see [Din95b, 2.2.3/a], cf. also [Tri83, 2.11.2(i)]) leads to

$$\begin{aligned} |f(\psi_\nu)| &\leq c \left\| f | B_{p\infty}^{-\frac{n-d}{p'},a}(\mathbb{R}^n) \right\| \cdot \left\| \psi_\nu | B_{p'1}^{\frac{n-d}{p'},a}(\mathbb{R}^n) \right\| \\ &\leq c' \left\| f | B_{p\infty}^{-\frac{n-d}{p'},a}(\mathbb{R}^n) \right\| \cdot \|g_\nu | L_{p'}(\Gamma)\|. \end{aligned} \quad (4.12)$$

Since $(\|g_\nu | L_{p'}(\Gamma)\|)_{\nu \in \mathbb{N}_0}$ converges, the last inequality implies the convergence of $(f(\psi_\nu))_{\nu \in \mathbb{N}_0}$ in \mathbb{C} . We denote the limit with $f(g)$ and have to make sure of its independence of the approximating sequence. Let ψ_1 another function satisfying (4.6). Then the limits of the traces of the corresponding approximating functions on Γ coincide and (4.12) yields the desired independence.

Hence we obtain a well defined linear functional $g \mapsto f(g)$ on $C_0(\Gamma)$ and by (4.11) and (4.12) it is also continuous with respect to the topology of $L_{p'}(\Gamma)$

$$|f(g)| \leq c \left\| f | B_{p\infty}^{-\frac{n-d}{p'},a}(\mathbb{R}^n) \right\| \cdot \|g | L_{p'}(\Gamma)\| \quad , \quad g \in C_0(\Gamma).$$

Since $p' < \infty$ by Step 3 we can extend with a standard completion argument the functional $g \mapsto f(g)$ from $C_0(\Gamma)$ in a unique way to $L_{p'}(\Gamma)$. The representation theorem for linear continuous functionals on $L_{p'}(\Gamma)$ implies the existence of an $f_\Gamma \in L_p(\Gamma)$, uniquely determined such that f is given by (3.9) and

$$\|f_\Gamma | L_p(\Gamma)\| = \|f | L_{p'}(\Gamma) \rightarrow \mathbb{C}\| \leq c \left\| f | B_{p\infty}^{-\frac{n-d}{p'},a}(\mathbb{R}^n) \right\|.$$

4.2 Proof of Theorem 3.12

Step 1. If $0 < p < \infty$ and $0 < q \leq \min(1, p)$ we will prove that $tr_\Gamma B_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma)$.

Let K, L given numbers such that $K \geq a_{max} + \frac{n-d}{p}$ and $L < 0$. Let $\varphi \in S(\mathbb{R}^n)$; using the atomic decomposition theorem (Theorem 2.6) in $B_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n)$ we represent φ as

$$\varphi = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \rho_{\nu m}^a \quad \text{convergence in } S'(\mathbb{R}^n),$$

where $\rho_{\nu m}^a$ is an anisotropic 1_K -atom ($\nu = 0$) or an anisotropic $(\frac{n-d}{p}, p)_{K,L}$ -atom ($\nu \in \mathbb{N}$) and

$$\|\lambda | b_{pq}\| \leq c \left\| \varphi | B_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n) \right\|. \quad (4.13)$$

In particular (2.9) with $s = \frac{n-d}{p}$ leads to $|\rho_{\nu m}^a(x)| \leq 2^{\nu d/p}$.

For every $\nu \in \mathbb{N}_0$ we denote $\varphi_\nu = \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \rho_{\nu m}^a$. Using the controlled overlapping of the supports at the level ν we have:

$$\begin{aligned} \|\varphi_\nu | \Gamma | L_p(\Gamma)\|^p &= \int_\Gamma \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \rho_{\nu m}^a(\gamma) \right|^p d\mu(\gamma) \\ &\leq c \int_\Gamma \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \cdot \left| 2^{\nu d/p} \tilde{\chi}_{\nu m}(\gamma) \right|^p d\mu(\gamma) \\ &\leq c' \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \end{aligned} \quad (4.14)$$

where in the calculation above we denoted $\tilde{\chi}_{\nu m}$ the characteristic function of the rectangle $cQ_{\nu m}^a$ where $\rho_{\nu m}^a$ is supported.

Applying now the triangle inequality for the L_p - norm if $p \geq 1$ or the p - triangle inequality if $0 < p < 1$ and using the restriction on q we obtain from (4.13) and (4.14):

$$\|\varphi | \Gamma | L_p(\Gamma)\| \leq c \|\lambda | b_{pq}\| \leq c' \left\| \varphi | B_{pq}^{\frac{n-d}{p}, a}(\mathbb{R}^n) \right\|$$

and this shows, after a standard completion argument, that tr_Γ is a well defined linear and bounded operator from $B_{pq}^{\frac{n-d}{p}, a}(\mathbb{R}^n)$ into $L_p(\Gamma)$.

Finally we want to remark that if $p = \infty$ then we have $B_{\infty 1}^{0, a}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$ and the embedding $tr_\Gamma B_{\infty 1}^{0, a}(\mathbb{R}^n) \hookrightarrow L_\infty(\Gamma)$ is obvious.

Step 2. Let $\frac{d}{n} < p < \infty$ and $0 < q \leq \min(1, p)$. We prove that tr_Γ is onto and that the related quasi-norms are equivalent.

We follow [TrW96b, p.161] making appropriate anisotropic changes. Let again ψ be a C^∞ function on \mathbb{R}^n with

$$supp \psi \subset 2^a Q_{00}^a \quad \text{and} \quad \sum_{m \in \mathbb{Z}^n} \psi(x - m) = 1 \quad \text{for all } x \in \mathbb{R}^n.$$

Using the arguments in Step 2 of the proof of Theorem 3.11 we obtain that any $h \in L_p(\Gamma)$ can be approximated in $L_p(\Gamma)$ by a sequence $(h_j)_{j \in \mathbb{N}_0}$ where $h_j = \psi_j | \Gamma$ and

$$\psi_j(x) = \sum_m \lambda_{\nu_j m} \psi(2^{\nu_j a} x - m) \quad \text{for every } j \in \mathbb{N}_0, \quad (4.15)$$

the sum being taken over those $m \in \mathbb{Z}^n$ such that $2^a Q_{\nu_j m}^a \cap \Gamma \neq \emptyset$, and where the coefficients $\lambda_{\nu_j m}$ and the numbers ν_j are determined successively such that

$$\left\| h - \sum_{j=0}^N h_j | L_p(\Gamma) \right\| \leq c 2^{-N} \|h | L_p(\Gamma)\| \quad \text{if } N \in \mathbb{N}_0. \quad (4.16)$$

In particular (4.16) implies

$$\|h_j | L_p(\Gamma)\| \leq c 2^{-j} \|h | L_p(\Gamma)\| \quad (4.17)$$

where $c > 0$ is independent of j . Furthermore, by (4.10)

$$\|h_j | L_p(\Gamma)\| \sim 2^{-\nu_j d/p} \left(\sum_m |\lambda_{\nu_j m}|^p \right)^{1/p}. \quad (4.18)$$

If we rewrite (4.15) as

$$\psi_j(x) = \sum_m 2^{-\nu_j d/p} \lambda_{\nu_j m} 2^{\nu_j d/p} \psi(2^{\nu_j a} x - m) \quad (4.19)$$

then (4.19) turns out to be an atomic decomposition of ψ_j in $B_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n)$ for every $0 < q \leq \infty$ (there are no moment conditions required since $\frac{d}{n} < p$). Hence the atomic decomposition theorem leads to

$$\|\psi_j | B_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n)\| \leq c 2^{-\nu_j d/p} \left(\sum_m |\lambda_{\nu_j m}|^p \right)^{1/p}. \quad (4.20)$$

Recall $q \leq \min(1, p)$; applying the q -triangle inequality to $\|\sum_{j=0}^N \psi_j | B_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n)\|$ and taking into account (4.17), (4.18) and (4.20) we obtain the convergence of $\{\sum_{j=0}^N \psi_j : N \in \mathbb{N}_0\}$ in $B_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n)$.

We denote the limit with *ext h* and clearly its definition and the properties of the function ψ provide that it is independent of the approximating sequence. We may write *ext h* as follows:

$$\text{ext } h = \sum_{j=0}^{\infty} \sum_m 2^{-\nu_j d/p} \lambda_{\nu_j m} 2^{\nu_j d/p} \psi(2^{\nu_j a} x - m).$$

But this is an atomic decomposition of *ext h*, the convergence being in $B_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n)$. Consequently, using the atomic decomposition theorem, (4.17) and (4.18) yield

$$\begin{aligned} \left\| \text{ext } h | B_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n) \right\| &\leq c \left(\sum_{j=0}^{\infty} \left(2^{-\nu_j d/p} \left(\sum_m |\lambda_{\nu_j m}|^p \right)^{1/p} \right)^q \right)^{1/q} \\ &\leq c' \left(\sum_{j=0}^{\infty} \|h_j | L_p(\Gamma)\|^q \right)^{1/q} \leq c'' \|h | L_p(\Gamma)\|. \end{aligned}$$

Finally, it is clear that $(tr_{\Gamma} \circ \text{ext})(h) = h$ and this completes the proof.

4.3 Comments and complements to Theorems 3.11 and 3.12

Remark 4.1 Theorems 3.11 and 3.12 are dual one to each other. To make clear what is meant we give a new proof of the embedding $tr_{\Gamma} B_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma)$ in the case $1 < p < \infty$ and $q \leq 1$ by dualising (3.10). We use the technique from [TrW96b, Remark 7], see also [Tri97, 18.8].

Let $\varphi \in S(\mathbb{R}^n)$; using the $L_p(\Gamma) - L_{p'}(\Gamma)$ duality we obtain

$$\|tr_{\Gamma} \varphi | L_p(\Gamma)\| = \sup \left\{ \left| \int_{\Gamma} (tr_{\Gamma} \varphi)(\gamma) f_{\Gamma}(\gamma) d\mu(\gamma) \right| : f_{\Gamma} \in L_{p'}(\Gamma), \|f_{\Gamma} | L_{p'}(\Gamma)\| \leq 1 \right\}.$$

Reading now (3.9) from the right to the left, where we call the generated tempered distribution f , and applying Theorem 3.11 we get:

$$\begin{aligned} \|tr_{\Gamma}\varphi | L_p(\Gamma)\| &\leq c \sup \left\{ |f(\varphi)| : f \in B_{p'\infty}^{-\frac{n-d}{p},a;\Gamma}(\mathbb{R}^n), \left\| f | B_{p'\infty}^{-\frac{n-d}{p},a}(\mathbb{R}^n) \right\| \leq c' \right\} \\ &\leq c'' \sup \left\{ \left\| f | B_{p'\infty}^{-\frac{n-d}{p},a}(\mathbb{R}^n) \right\| \cdot \left\| \varphi | B_{p1}^{\frac{n-d}{p},a}(\mathbb{R}^n) \right\| : \left\| f | B_{p'\infty}^{-\frac{n-d}{p},a}(\mathbb{R}^n) \right\| \leq c' \right\} \\ &\leq c''' \left\| \varphi | B_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n) \right\| \end{aligned}$$

where we used the $B_{p'\infty}^{-\frac{n-d}{p},a}(\mathbb{R}^n) - B_{p1}^{\frac{n-d}{p},a}(\mathbb{R}^n)$ duality (see [Din95b, 2.2.3/a]) and the restriction on q , $q \leq 1$ which enabled us to use the elementary embedding $B_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n) \hookrightarrow B_{p1}^{\frac{n-d}{p},a}(\mathbb{R}^n)$.

Since $p < \infty$ and $q \leq p$ applying the standard completion procedure we may extend tr_{Γ} to all elements of $B_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n)$.

Hence traces of anisotropic spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ on a regular anisotropic d -set Γ on the one hand and interpretations of $L_p(\Gamma)$ as spaces on \mathbb{R}^n as in Theorem 3.11 are two sides of the same coin.

Remark 4.2 The extension of Theorem 3.12 to all $p < 1$ causes some trouble if $p \leq \frac{d}{n}$; the restriction $p > \frac{d}{n}$ comes from Step 2 of the proof were we used the atomic decomposition theorem, no moment conditions being needed for the atoms. An extension to all $0 < p \leq \infty$ is presented below.

If $x \in \mathbb{R}^n$ and $t > 0$ then $B^a(x, t) = \{z \in \mathbb{R}^n : |z - x|_a \leq t\}$ denotes the (closed) anisotropic ball centred at x and of (anisotropic) radius t .

Definition 4.3 Let $a = (a_1, \dots, a_n)$ a given anisotropy. A non - empty Borel set $\Gamma \subset \mathbb{R}^n$ with $|\Gamma| = 0$ is said to satisfy the anisotropic ball condition if there exists a number $0 < \eta < 1$ with the following property:

for any anisotropic ball $B^a(x, r)$ centred at $x \in \Gamma$ and of radius $0 < r < 1$ there exists an anisotropic ball $B^a(y, \eta r)$ centred at some $y \in \mathbb{R}^n$, depending on x , and of radius ηr such that $B^a(y, \eta r) \subset B^a(x, r)$ and $B^a(y, \eta r) \cap \overline{\Gamma} = \emptyset$.

This is the anisotropic counterpart of Definition 18.10 in [Tri97] and it is clear that conditions of this type are related to the open set condition for self similar fractals, see [Fal85, p. 121] and [Tri97, 4.5] for further comments and details.

Theorem 4.4 Let $0 < d < n$ and let Γ be a regular anisotropic d -set (with respect to the anisotropy $a = (a_1, \dots, a_n)$) which satisfy the anisotropic ball condition. If $0 < p \leq \frac{d}{n}$ and $0 < q \leq \min(1, p)$ then

$$tr_{\Gamma} B_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n) = L_p(\Gamma).$$

Of course using the atomic decomposition theorem atoms with moment conditions are coming in, see Theorem 2.6. To prove Theorem 4.4 one has to use the anisotropic ball condition and

then the technique from the proof of Theorem 3/ Step 2 in [TrW96b]. But we do not go into further details.

The above results have a counterpart for anisotropic function spaces of $F_{pq}^{s,a}(\mathbb{R}^n)$ type. Without going into details we mention the following:

Theorem 4.5 *Let $0 < d < n$ and let Γ be a regular anisotropic d -set (with respect to the anisotropy $a = (a_1, \dots, a_n)$) which satisfy the anisotropic ball condition. If $0 < p \leq 1$ and $0 < q \leq \infty$ then*

$$\text{tr}_\Gamma F_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n) = L_p(\Gamma).$$

For the proof one has to use the method from [FrJ90, 11.1] to show that $\text{tr}_\Gamma F_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n)$ is independent of q ; using $F_{pp}^{\frac{n-d}{p},a}(\mathbb{R}^n) = B_{pp}^{\frac{n-d}{p},a}(\mathbb{R}^n)$ the conclusion follows from Theorem 4.4. The rest is similar to what was done in [TrW96b].

By this idea of the proof the restriction on the range of p is now clear since only in this case the trace of $F_{pq}^{\frac{n-d}{p},a}(\mathbb{R}^n)$ exists and belongs to $L_p(\Gamma)$.

4.4 Proofs of the results in Subsection 3.2.3

Some preparatory results

Lemma 4.6 *Let A, B, C be quasi-Banach spaces, let $T \in L(A, B)$ and let $V \in L(B, C)$.*

(i) $\|T\| \geq e_1(T) \geq e_2(T) \geq \dots \geq 0$; $e_1(T) = \|T\|$ if B is a Banach space.

(ii) For all $k, l \in \mathbb{N}$,

$$e_{k+l-1}(V \circ T) \leq e_k(V) e_l(T). \quad (4.21)$$

A proof is given in [EdT96, Lemma 1.3.1/1]. In case of quasi-Banach spaces it may happen that $\|T\| > e_1(T)$.

Let $M \in \mathbb{N}$ and let $0 < p \leq \infty$. By l_p^M we shall mean the linear space of all complex M -tuples $y = (y_1, \dots, y_M)$ endowed with the quasi-norm

$$\|y\|_{l_p^M} = \left(\sum_{i=1}^M |y_i|^p \right)^{1/p}$$

(modification by $\sup\{|y_i| : i = 1, \dots, M\}$ if $p = \infty$).

Proposition 4.7 ([Tri97, 7.2]) *Let $0 < p_1 \leq \infty$, $0 < p_2 \leq \infty$ and let e_k be the entropy numbers of the embedding $\text{id} : l_{p_1}^M \rightarrow l_{p_2}^M$. Then*

$$e_k \geq c \quad \text{if } 1 \leq k \leq \log(2M)$$

and

$$e_k \geq c 2^{-\frac{k}{2M}} (2M)^{\frac{1}{p_2} - \frac{1}{p_1}} \quad \text{if } k \in \mathbb{N},$$

where c is a positive constant which is independent of M and of k but may depend upon p_1 and p_2 .

Let $d > 0$, $\delta \geq 0$ and let $(M_\nu)_{\nu \in \mathbb{N}_0}$ be a sequence of natural numbers. We will assume that there exist two positive numbers c_1 and c_2 with

$$c_1 2^{\nu d} \leq M_\nu \leq c_2 2^{\nu d} \quad \text{for every } \nu \in \mathbb{N}_0. \quad (4.22)$$

Let $0 < p \leq \infty$ and $0 < q \leq \infty$. Then $l_q(2^{\nu\delta} l_p^{M_\nu})$ denotes the linear space of all complex sequences $b = \{b_{\nu i} : \nu \in \mathbb{N}_0, i = 1, \dots, M_\nu\}$ endowed with the quasi-norm

$$\|b\|_{l_q(2^{\nu\delta} l_p^{M_\nu})} = \left(\sum_{\nu=0}^{\infty} \left(\sum_{i=1}^{M_\nu} 2^{\nu\delta p} |b_{\nu i}|^p \right)^{q/p} \right)^{1/q} \quad (4.23)$$

(obvious modification if $p = \infty$ and/or $q = \infty$).

In case of $\delta = 0$ we write $l_q(l_p^{M_\nu})$ and if, in addition, $p = q$ then we have the l_p spaces with the components ordered in the given way.

Plainly, $l_q(2^{\nu\delta} l_p^{M_\nu})$ consists of dyadic blocks of spaces $l_p^{M_\nu}$ clipped together with the weights $2^{\nu\delta}$.

Let, in addition, $r \geq 0$; then by

$$l_\infty[2^{jr} l_q(2^{\nu\delta} l_p^{M_\nu})]$$

we shall mean the linear space of all $l_q(2^{\nu\delta} l_p^{M_\nu})$ valued sequences $b = \{b^j : j \in \mathbb{N}_0\}$ endowed with the quasi-norm

$$\|b\|_{l_\infty[2^{jr} l_q(2^{\nu\delta} l_p^{M_\nu})]} = \sup_{j \in \mathbb{N}_0} 2^{jr} \|b^j\|_{l_q(2^{\nu\delta} l_p^{M_\nu})}. \quad (4.24)$$

Theorem 4.8 ([Tri97, 9.2]) *Let $d > 0$, $\delta > 0$, $r_1, r_2 > 0$ and M_ν with (4.22).*

Let $0 < p_1 \leq p_2 \leq \infty$, $0 < q_1, q_2 \leq \infty$ and let e_k be the entropy numbers of the identity map

$$id : l_\infty[2^{jr_1} l_{q_1}(2^{\nu\delta} l_{p_1}^{M_\nu})] \rightarrow l_\infty[2^{jr_2} l_{q_2}(l_{p_2}^{M_\nu})].$$

Then there exist two constants $c, C > 0$ such that for all $k \in \mathbb{N}$,

$$c k^{-\frac{\delta}{d} + \frac{1}{p_2} - \frac{1}{p_1}} \leq e_k \leq C k^{-\frac{\delta}{d} + \frac{1}{p_2} - \frac{1}{p_1}}.$$

Proof of the estimate (3.14) in Theorem 3.13

Let $s_1 = s + \frac{n-d}{p_1}$.

Step 1. We first assume $0 < p_1 \leq p_2 \leq \infty$. Replacing (if necessary) $|\cdot|_a$ by an equivalent anisotropic distance function we may assume that

$$\{x \in \mathbb{R}^n : |x|_a \leq 2\} \subset [-\pi, \pi]^n.$$

So we can use the subatomic decomposition theorem in anisotropic function spaces, see [2, Theorem 3.7] and Section 2.5, see also Remark 2.13, and decompose any $f \in B_{p_1 q}^{s_1, a}(\mathbb{R}^n)$ as

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta 2^{-\nu(s_1 - \frac{n}{p})} \psi^\beta(2^{\nu a} x - m) \quad (4.25)$$

where ψ is a C^∞ function on \mathbb{R}^n with

$$\text{supp } \psi \subset 2^a Q_{00}^a \quad \text{and} \quad \sum_{m \in \mathbb{Z}^n} \psi(x - m) = 1 \quad \text{if } x \in \mathbb{R}^n,$$

$\psi^\beta(x) = x^\beta \psi(x)$ and

$$\sup_{\beta \in \mathbb{N}_0^n} 2^{r_1 a \beta} \|\lambda^\beta | b_{p_1 q} \| \leq c \|f| B_{p_1 q}^{s_1, a}(\mathbb{R}^n)\| \quad (4.26)$$

with $r_1 > 0$ large and $\lambda^\beta = \{\lambda_{\nu m}^\beta : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$.

For $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ let $Q_{\nu m}^a$ be the rectangles defined in Section 2.3. Let

$$\lambda^{\beta, \Gamma} = \{\lambda_{\nu m}^\beta : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n, CQ_{\nu m}^a \cap \Gamma \neq \emptyset\} \quad (4.27)$$

where we may assume that $C > 1$ is fixed and sufficiently large such that all what follows is justified.

For a fixed $\nu \in \mathbb{N}_0$ let M_ν be the number of the rectangles $Q_{\nu m}^a$ such that $CQ_{\nu m}^a \cap \Gamma \neq \emptyset$. By Definition 3.1/(i) it follows that there exist two constants $c_1, c_2 > 0$, independent of ν , with

$$c_1 2^{\nu d} \leq M_\nu \leq c_2 2^{\nu d} \quad (4.28)$$

and this coincides with (4.22).

Let $l_q(2^{\nu \delta} l_p^{M_\nu})$ and $l_\infty[2^{r_1 a \beta} l_q(2^{\nu \delta} l_p^{M_\nu})]$ be the sequence spaces introduced in (4.23) and (4.24) adapted to our present situation where $\delta = s - d \left(\frac{1}{p_1} - \frac{1}{p_2} \right)$ and r_1 and M_ν have the same meaning as in (4.26) and (4.28) and the l_∞ norm is now modified by $\sup_{\beta \in \mathbb{N}_0^n}$.

Let us define the (non-linear) operator

$$U : B_{p_1 q}^{s_1, a}(\mathbb{R}^n) \rightarrow l_\infty[2^{r_1 a \beta} l_q(2^{\nu \delta} l_p^{M_\nu})] \quad \text{by} \quad Uf = \{\theta^{\beta, \Gamma} : \beta \in \mathbb{N}_0^n\}$$

with

$$\theta^{\beta, \Gamma} = \{2^{-\nu \delta} \lambda_{\nu m}^\beta : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n, CQ_{\nu m}^a \cap \Gamma \neq \emptyset\}$$

where f is given by (4.25) with (4.26). By (4.26) it follows that U is a bounded map.

Let $r_2 > 0$ at our disposal. We define

$$V : l_\infty[2^{r_2 a \beta} l_q(2^{\nu \delta} l_p^{M_\nu})] \rightarrow L_{p_2}(\Gamma)$$

by

$$V(\eta)(\gamma) = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_m \eta_{\nu m}^\beta 2^{\nu d/p_2} \psi^\beta(2^{\nu a} \gamma - m) \quad \text{where} \quad \gamma \in \Gamma, \quad (4.29)$$

and the sum over m in (4.29) is taken according to (4.27) now with $\eta_{\nu m}^\beta$ in place of $\lambda_{\nu m}^\beta$.

By Definition 3.1/(i) and (3.5) there exists a constant $c > 0$ independent of $\nu \in \mathbb{N}_0$ with $\mu(CQ_{\nu m}^a \cap \Gamma) \leq c 2^{-\nu d}$. Hence we have for fixed $\beta \in \mathbb{N}_0^n$ and $\nu \in \mathbb{N}_0$:

$$\left\| \sum_m \eta_{\nu m}^\beta 2^{\nu d/p_2} \psi^\beta(2^{\nu a} \cdot - m) \right\|_{L_{p_2}(\Gamma)} \leq c^{a\beta} \left(\sum_m |\eta_{\nu m}^\beta|^{p_2} \right)^{1/p_2}$$

where $c > 1$ is independent of β . Let $\overline{p_2} = \min(1, p_2)$; then

$$\begin{aligned} & \left\| \sum_{\nu=0}^{\infty} \sum_m \eta_{\nu m}^\beta 2^{\nu d/p_2} \psi^\beta(2^{\nu a} \cdot - m) \right\|_{L_{p_2}(\Gamma)}^{\overline{p_2}} \\ & \leq \sum_{\nu=0}^{\infty} \left(c^{a\beta} \left(\sum_m |\eta_{\nu m}^\beta|^{p_2} \right)^{1/p_2} \right)^{\overline{p_2}} = c^{a\beta \overline{p_2}} \left\| \eta^\beta \right\|_{l_{\overline{p_2}}(l_{p_2}^{M_\nu})}^{\overline{p_2}} \end{aligned}$$

where $\eta^\beta = \{\eta_{\nu m}^\beta : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ and $c > 1$ is independent of β . Since $r_2 > 0$ is at our disposal it follows from

$$\begin{aligned} \|V(\eta) | L_{p_2}(\Gamma)\|^{\overline{p_2}} &\leq \sum_{\beta \in \mathbb{N}_0^n} c^{a\beta\overline{p_2}} 2^{-r_2 a \beta \overline{p_2}} \sup_{\beta \in \mathbb{N}_0^n} \left(2^{r_2 a \beta} \|\eta^\beta | l_{\overline{p_2}}(l_{p_2}^{M_\nu})\| \right)^{\overline{p_2}} \\ &\leq c' \|\eta | l_\infty[2^{r_2 a \beta} l_{\overline{p_2}}(l_{p_2}^{M_\nu})]\|^{\overline{p_2}} \end{aligned}$$

that V is linear and bounded. With $\delta = s - d \left(\frac{1}{p_1} - \frac{1}{p_2} \right)$ we have the decomposition

$$tr_\Gamma = V \circ id \circ U : B_{p_1 q}^{s_1, a}(\mathbb{R}^n) \rightarrow L_{p_2}(\Gamma) \quad (4.30)$$

where

$$id : l_\infty[2^{r_1 a \beta} l_q(2^{\nu \delta} l_{p_1}^{M_\nu})] \rightarrow l_\infty[2^{r_2 a \beta} l_{\overline{p_2}}(l_{p_2}^{M_\nu})]$$

is the identity operator and the final outcome is independent of ambiguities in the non-linear construction of U . Using now the multiplication property (4.21) for entropy numbers and Theorem 4.8 it follows from (4.30) that there exists a constant $c > 0$ such that for all $k \in \mathbb{N}$

$$e_k(tr_\Gamma : B_{p_1 q}^{s_1, a}(\mathbb{R}^n) \rightarrow L_{p_2}(\Gamma)) \leq c k^{-\frac{\delta}{d} + \frac{1}{p_2} - \frac{1}{p_1}} \quad (4.31)$$

and this completes the proof if $0 < p_1 \leq p_2 \leq \infty$ since we have only to insert (3.12) in (4.31).

Step 2. Let $0 < p_2 < p_1 \leq \infty$; then we have

$$L_{p_1}(\Gamma) \hookrightarrow L_{p_2}(\Gamma). \quad (4.32)$$

The desired estimate follows now from Step 1 with $p_1 = p_2$ and (4.32).

Proof of the estimate (3.15) in Theorem 3.13

Let Γ be a proper regular anisotropic d -set in \mathbb{R}^n according to Definition 3.6.

Step 1. We first assume $0 < p_1 \leq \infty$ and $1 \leq p_2 \leq \infty$. Let $x^{j,l}$ be the centre of the rectangle R_{jl} of side lengths $r_1^{j,l}, \dots, r_n^{j,l}$ and let $N_j \sim 2^{jd}$ having the same meaning as in Definition 3.1. We use the abbreviation

$$\frac{2(\gamma - x^{j,l})}{r^{j,l}} = \left(\frac{2(\gamma_1 - x_1^{j,l})}{r_1^{j,l}}, \dots, \frac{2(\gamma_n - x_n^{j,l})}{r_n^{j,l}} \right). \quad (4.33)$$

Let φ and ψ be two non-negative C^∞ functions on \mathbb{R}^n having supports in $\{x \in \mathbb{R}^n : |x_i| < 1, i = 1, \dots, n\}$ and

$$c_{j,l} 2^{jd} \int_\Gamma \varphi \left(\frac{2(\gamma - x^{j,l})}{r^{j,l}} \right) \psi \left(\frac{2(\gamma - x^{j,l})}{r^{j,l}} \right) d\mu(\gamma) = 1. \quad (4.34)$$

Of course the integral above can be restricted to $\Gamma \cap R_{jl}$. We may assume that

$$\varphi(x) \psi(x) \geq \eta > 0 \quad \text{if } x \in \{y \in \mathbb{R}^n : |y_i| \leq \kappa, i = 1, \dots, n\}$$

where $0 < \kappa < 1$ is the number from Definition 3.6 of a proper set. Then

$$\begin{aligned} &2^{jd} \int_\Gamma \varphi \left(\frac{2(\gamma - x^{j,l})}{r^{j,l}} \right) \psi \left(\frac{2(\gamma - x^{j,l})}{r^{j,l}} \right) d\mu(\gamma) \\ &\geq 2^{jd} \int_{\Gamma \cap \kappa R_{jl}} \varphi \left(\frac{2(\gamma - x^{j,l})}{r^{j,l}} \right) \psi \left(\frac{2(\gamma - x^{j,l})}{r^{j,l}} \right) d\mu(\gamma) \\ &\geq c \eta 2^{jd} (\text{vol } R_{jl})^{\frac{d}{n}} \geq c' > 0 \end{aligned} \quad (4.35)$$

c' being independent of j and l . So, using (4.35) and (3.5), we may assume that there are two constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 \leq c_{j,l} \leq c_2 \quad \text{for any } j \in \mathbb{N}_0 \quad \text{and any } l = 1, \dots, N_j. \quad (4.36)$$

We define the operator $U : l_{p_1}^{N_j} \rightarrow B_{p_1 q}^{s + \frac{n-d}{p_1}, a}(\mathbb{R}^n)$ by

$$U(\lambda = \{\lambda_l : l = 1, \dots, N_j\}) = \sum_{l=1}^{N_j} \lambda_l \left[2^{-j(s - \frac{d}{p_1})} \varphi \left(\frac{2(\gamma - x^{j,l})}{r^{j,l}} \right) \right] \quad (4.37)$$

and may interpret the term in brackets as an anisotropic $(s + \frac{n-d}{p_1}, p_1)_{K,L}$ -atom where K is sufficiently large and $L < 0$ (it follows from (3.12) and $p_2 \geq 1$ that

$$s + \frac{n-d}{p_1} > n \left(\frac{1}{p_1} - 1 \right)_+$$

so there are no moment conditions required for the atoms). Hence the right-hand side of (4.37) is an atomic decomposition in $B_{p_1 q}^{s + \frac{n-d}{p_1}, a}(\mathbb{R}^n)$. Consequently, there exists a constant $c > 0$ (independent of j) such that

$$\left\| U(\lambda) \mid B_{p_1 q}^{s + \frac{n-d}{p_1}, a}(\mathbb{R}^n) \right\| \leq c \|\lambda\|_{l_{p_1}^{N_j}}. \quad (4.38)$$

We define now $V : L_{p_2}(\Gamma) \rightarrow l_{p_2}^{N_j}$ by

$$V(g) = \left\{ c_{j,l} 2^{-jd/p_2} 2^{jd} \int_{\Gamma} g(\gamma) \psi \left(\frac{2(\gamma - x^{j,l})}{r^{j,l}} \right) d\mu(\gamma) : l = 1, \dots, N_j \right\} \quad (4.39)$$

(with the usual modification if $p_2 = \infty$); again the integral can be restricted to $\Gamma \cap R_{j,l}$. Let $b_{j,l}$ be the numbers in (4.39). Applying Hölder's inequality with $\frac{1}{p_2} + \frac{1}{p_2'} = 1$, using the fact that for a fixed j the rectangles $R_{j,l}$ are disjoint and using (3.5) there exists a constant $c > 0$ (independent of j) such that:

$$\begin{aligned} |b_{j,l}|^{p_2} &\leq c_{j,l}^{p_2} 2^{-jd/p_2} 2^{jd} \int_{\Gamma \cap R_{j,l}} |g(\gamma)|^{p_2} d\mu(\gamma) \cdot c (\mu(\Gamma \cap R_{j,l}))^{p_2/p_2'} \\ &\leq c c_{j,l}^{p_2} \int_{\Gamma \cap R_{j,l}} |g(\gamma)|^{p_2} d\mu(\gamma). \end{aligned} \quad (4.40)$$

From (4.36) and (4.40) it follows that there exists a constant $c > 0$ independent of j such that

$$\left(\sum_{l=1}^{N_j} |b_{j,l}|^{p_2} \right)^{1/p_2} \leq c \|g\|_{L_{p_2}(\Gamma)}$$

(usual modification if $p_2 = \infty$). In other words, both U and V are well defined linear and bounded operators and the corresponding norms can be estimated independently of j .

Denoting $id^j : l_{p_1}^{N_j} \rightarrow l_{p_2}^{N_j}$ the identity operator, by (4.34) we have

$$V \circ tr_{\Gamma} \circ U : l_{p_1}^{N_j} \rightarrow l_{p_2}^{N_j} \quad \text{and} \quad V \circ tr_{\Gamma} \circ U = 2^{-j(s - \frac{d}{p_1})} 2^{-jd/p_2} id^j. \quad (4.41)$$

Using the multiplication property (4.21) for entropy numbers we obtain

$$e_k(tr_\Gamma : B_{p_1 q}^{s+\frac{n-d}{p_1}, a}(\mathbb{R}^n)) \rightarrow L_{p_2}(\Gamma) \geq c 2^{-j(s-\frac{d}{p_1})} 2^{-jd/p_2} e_k(id^j) \quad (4.42)$$

for some $c > 0$. By Proposition 4.7 with $k = 2N_j \sim c 2^{jd}$ we have

$$e_{c 2^{jd}}(id^j) \geq c' 2^{jd\left(\frac{1}{p_2} - \frac{1}{p_1}\right)}.$$

Inserting this last inequality in (4.42) we obtain the desired estimate for $k = c 2^{jd}$; using Lemma 4.6 this is sufficient to prove the assertion for all $k \in \mathbb{N}$.

Step 2. Let now $0 < p_1 \leq \infty$ and $0 < p_2 < 1$. Assume that there is no $c > 0$ such that (3.15) is satisfied. Then we find a sequence $k_j \rightarrow \infty$ such that

$$k_j^{\frac{s}{d}} \cdot e_{k_j}(tr_\Gamma : B_{p_1 q}^{s+\frac{n-d}{p_1}, a}(\mathbb{R}^n)) \rightarrow L_{p_2}(\Gamma)) \rightarrow 0 \quad \text{if } j \rightarrow \infty. \quad (4.43)$$

Using the multiplication property (4.21) for entropy numbers and elementary embeddings between anisotropic Besov spaces it is clear that we may assume in (4.43) $p_1 > 1$.

By Theorem 3.12 the operator

$$tr_\Gamma : B_{p_1 q}^{s+\frac{n-d}{p_1}, a}(\mathbb{R}^n) \rightarrow L_{p_1}(\Gamma)$$

is bounded and by the previous considerations we know the behaviour of its entropy numbers. If now

$$0 < \theta < 1 \quad \text{and} \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2} \quad (4.44)$$

then for any $f \in L_p(\Gamma)$ we have $\|f\|_{L_p(\Gamma)} \leq \|f\|_{L_{p_1}(\Gamma)}^{1-\theta} \cdot \|f\|_{L_{p_2}(\Gamma)}^\theta$.

Using the interpolation property for entropy numbers from [EdT96, 1.3.2] and the estimate (3.14) we find constants $c, c' > 0$ independent of j with

$$\begin{aligned} & e_{2k_j}(tr_\Gamma : B_{p_1 q}^{s+\frac{n-d}{p_1}, a}(\mathbb{R}^n) \rightarrow L_p(\Gamma)) \\ & \leq c [e_{k_j}(tr_\Gamma : B_{p_1 q}^{s+\frac{n-d}{p_1}, a}(\mathbb{R}^n) \rightarrow L_{p_1}(\Gamma))]^{1-\theta} \cdot [e_{k_j}(tr_\Gamma : B_{p_1 q}^{s+\frac{n-d}{p_1}, a}(\mathbb{R}^n) \rightarrow L_{p_2}(\Gamma))]^\theta \\ & \leq c' k_j^{-\frac{s}{d}} \left(k_j^{\frac{s}{d}} \cdot e_{k_j}(tr_\Gamma : B_{p_1 q}^{s+\frac{n-d}{p_1}, a}(\mathbb{R}^n) \rightarrow L_{p_2}(\Gamma)) \right)^\theta. \end{aligned}$$

Hence

$$k_j^{\frac{s}{d}} \cdot e_{2k_j}(tr_\Gamma : B_{p_1 q}^{s+\frac{n-d}{p_1}, a}(\mathbb{R}^n) \rightarrow L_p(\Gamma)) \rightarrow 0. \quad (4.45)$$

By (4.44) we may assume $p \geq 1$ so that (4.45) contradicts Step 1.

The proof is complete.

Proof of Proposition 3.14

The continuity of the operator in (3.17) is a simple consequence of the embedding from [Joh95, (2.13)] and of Theorem 3.12/Step 1.

As in the proof of Theorem 3.13 we choose φ a non-negative C^∞ function on \mathbb{R}^n having support in $\{x \in \mathbb{R}^n : |x_i| < 1, i = 1, \dots, n\}$. Let $x^{j,l}$ be the same points as there and let

$$\Phi_j(x) = \sum_{l=1}^{N_j} \lambda_l 2^{-j\left(s-\frac{d}{p_1}\right)} \varphi\left(\frac{2(x-x^{j,l})}{r^{j,l}}\right) \quad , \quad x \in \mathbb{R}^n$$

as in (4.37) where we used again the abbreviation (4.33).

Assuming that φ satisfies the necessary moment conditions, as in the proof of (4.38), we find a constant $c > 0$ independent of j such that

$$\left\| \Phi_j | B_{\frac{s+\frac{n-d}{p_1}, a}}{\frac{p_1 p_2}}(\mathbb{R}^n) \right\| \leq c \left(\sum_{l=1}^{N_j} |\lambda_l|^{p_1} \right)^{1/p_2} .$$

On the other hand, we find two constants $c_1, c_2 > 0$ independent of j such that

$$c_1 \left(\sum_{l=1}^{N_j} |\lambda_l|^{p_2} \right)^{1/p_2} \leq \|\Phi_j | L_{p_2}(\Gamma)\| \leq c_2 \left(\sum_{l=1}^{N_j} |\lambda_l|^{p_2} \right)^{1/p_2} . \quad (4.46)$$

Indeed, we may assume $\varphi(x) \geq \eta > 0$ if $|x_i| \leq \kappa, i = 1, \dots, n$ where $0 < \kappa < 1$ is the number from Definition 3.6 of a proper set. Then

$$\begin{aligned} \|\Phi_j | L_{p_2}(\Gamma)\| &= 2^{-j\left(s-\frac{d}{p_1}\right)} \left(\sum_{l=1}^{N_j} |\lambda_l|^{p_2} \int_{\Gamma \cap R_{jl}} \left| \varphi\left(\frac{2(\gamma-x^{j,l})}{r^{j,l}}\right) \right|^{p_2} d\mu(\gamma) \right)^{1/p_2} \\ &\geq 2^{-j\left(s-\frac{d}{p_1}\right)} \left(\sum_{l=1}^{N_j} |\lambda_l|^{p_2} \eta^{p_2} \mu(\Gamma \cap \kappa R_{jl}) \right)^{1/p_2} \geq c_1 \left(\sum_{l=1}^{N_j} |\lambda_l|^{p_2} \right)^{1/p_2} \end{aligned}$$

where we used (3.16). The second part of (4.46) is a simple consequence of (3.5).

Hence by the analogue of (4.41) we find a constant $c > 0$ independent of j with

$$e_k(id : l_{p_1}^{N_j} \rightarrow l_{p_2}^{N_j}) \leq c e_k(tr_\Gamma : B_{\frac{s+\frac{n-d}{p_1}, a}}{\frac{p_1 p_2}}(\mathbb{R}^n) \rightarrow L_{p_2}(\Gamma)) . \quad (4.47)$$

Assuming that the operator from (3.17) is compact it follows from (4.47) that

$$\sup_{j \geq 0} e_k(id : l_{p_1}^{N_j} \rightarrow l_{p_2}^{N_j}) \rightarrow 0 \quad \text{for } k \rightarrow \infty .$$

But this contradicts Proposition 4.7 and proves our assertion.

4.5 Proof of Theorem 3.16

Step 0. Prerequisites.

We start recalling some basic facts about entropy numbers and approximation numbers of operators and about their relation to eigenvalues.

Let A be a complex quasi-Banach space and $T \in L(A)$ a compact map. We know from [EdT96, Theorem 1.2] that the spectrum of T , apart from the point 0, consists solely of eigenvalues of finite algebraic multiplicity: let $\{\lambda_k(T) : k \in \mathbb{N}\}$ be the sequence of all non-zero eigenvalues of T , repeated according to algebraic multiplicity and ordered so that

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \dots \geq 0. \quad (4.48)$$

If T has only $m (< \infty)$ distinct eigenvalues and M is the sum of their algebraic multiplicities, we put $\lambda_k(T) = 0$ for $k > M$.

Perhaps the most useful connection for our purposes between the eigenvalues of the operator T and its entropy numbers is the following:

Theorem 4.9 *Let T and $\{\lambda_k(T) : k \in \mathbb{N}\}$ as above. Then*

$$|\lambda_k(T)| \leq \sqrt{2} e_k(T). \quad (4.49)$$

A proof of this result, originally proved by B. Carl in [Carl81], see also [CaT80], is given in [EdT96, Theorem 1.3.4].

Definition 4.10 *Let A, B be two quasi-Banach spaces and let $T \in L(A, B)$. Then given any $k \in \mathbb{N}$, the k th approximation number $\alpha_k(T)$ of T is defined by*

$$\alpha_k(T) = \inf \{\|T - L\| : L \in L(A, B), \text{rank } L < k\}$$

where $\text{rank } L$ is the dimension of the range of L .

Usually the approximation numbers are denoted $a_k(T)$. The above notation is used only to avoid any possible confusion between these numbers and the anisotropy $a = (a_1, \dots, a_n)$.

These numbers have various properties similar to those of the entropy numbers: we have $\|T\| = \alpha_1(T) \geq \alpha_2(T) \geq \dots \geq 0$ and a counterpart of the multiplication property (4.21) for entropy numbers, see [EdT96, Lemma 1.3.1/2].

On the other hand there are radical differences between entropy numbers and approximation numbers, see [EdT96, Remark 1.3.2/6] and [EEv87, II.2.3], but we do not go into details here.

The approximation numbers have important connections with eigenvalues, the picture being clearest in a Hilbert space setting.

Theorem 4.11 *Let H be a Hilbert space and let $T \in L(H)$ be a compact, non-negative and self adjoint operator. Then the approximation numbers $\alpha_k(T)$ of T coincide with its eigenvalues (ordered as in (4.48)).*

A proof can be found in [EEv87, II.5.10], see also [EdT96, p.21].

Step 1. We prove now part (i) of Theorem 3.16.

Using (3.11) and the elementary embedding

$$W_2^{(1,2)}(\Omega) = W_2^{\frac{4}{3},a}(\Omega) \hookrightarrow B_{21}^{\frac{2-d}{2},a}(\Omega),$$

see [Tri83, Proposition 2.3.2/2] and [Joh95, (2.11)] for the anisotropic counterpart, there exists a constant $c > 0$ such that

$$\|tr_{\Gamma} f | L_2(\Gamma)\| \leq c \|f | \mathring{W}_2^{(1,2)}(\Omega)\| \quad \text{for any } f \in \mathring{W}_2^{(1,2)}(\Omega). \quad (4.50)$$

Defining

$$q(f, g) = \int_{\Gamma} f(\gamma) \overline{g(\gamma)} d\mu(\gamma) \quad \text{for any } f, g \in \mathring{W}_2^{(1,2)}(\Omega),$$

it is clear that $q(\cdot, \cdot)$ is a non-negative quadratic form in $\mathring{W}_2^{(1,2)}(\Omega)$. By [Tri92*, p. 91] there exists a non-negative and self adjoint operator T uniquely determined such that

$$q(f, g) = (Tf, g)_{W_2^{(1,2)}(\Omega)} \quad \text{for any } f, g \in \mathring{W}_2^{(1,2)}(\Omega).$$

Furthermore,

$$\|tr_{\Gamma} f | L_2(\Gamma)\| = \|\sqrt{T}f | \mathring{W}_2^{(1,2)}(\Omega)\| \quad (4.51)$$

where $\sqrt{T} = T^{1/2}$ and this proves (3.25).

So it remains to prove that the above operator is the same as in (3.24). Let $f \in \mathring{W}_2^{(1,2)}(\Omega)$ and $\varphi \in D(\Omega)$. Then

$$\begin{aligned} \int_{\Gamma} f(\gamma) \overline{\varphi(\gamma)} d\mu(\gamma) &= (Tf, \varphi)_{W_2^{(1,2)}(\Omega)} = (\mathcal{A}_r^{1/2}Tf, \mathcal{A}_r^{1/2}\varphi)_{L_2(\Omega)} \\ &= (\mathcal{A}_r Tf, \varphi)_{L_2(\Omega)} \end{aligned} \quad (4.52)$$

the second equality in (4.52) being justified by the fact that we fixed the norm in $\mathring{W}_2^{(1,2)}(\Omega)$ by $\|f | \mathring{W}_2^{(1,2)}(\Omega)\| = \|\mathcal{A}_r^{1/2}f | L_2(\Omega)\|$ for $r > r_0$, see Remark 3.15.

Considered as a dual pairing in $(D(\Omega), D'(\Omega))$ we obtain $\mathcal{A}_r Tf = tr^{\Gamma} f$ and (3.26) follows by the same arguments as in [Tri97, Theorem 27.15/Step 1].

This completes the first part of the proof of Theorem 3.16.

Step 2. We prove now (3.27).

Step 2.1. Anisotropic function spaces on domains.

If $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq q \leq \infty$ then $B_{pq}^{s,a}(\Omega)$ is the restriction of $B_{pq}^{s,a}(\mathbb{R}^2)$ to Ω normed by

$$\|f | B_{pq}^{s,a}(\Omega)\| = \inf \|g | B_{pq}^{s,a}(\mathbb{R}^2)\| \quad (4.53)$$

where the infimum is taken over all $g \in B_{pq}^{s,a}(\mathbb{R}^2)$ with $g|_{\Omega} = f|_{\Omega}$ (in the sense of distributions on Ω). Of course this definition works for all (bounded or unbounded) domains Ω but we are interested here only in the unit disc.

Let us recall that

$$W_2^{s,a}(\Omega) = B_{22}^{s,a}(\Omega) \quad \text{if } s > 0. \quad (4.54)$$

For $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq q \leq \infty$ we will use the (non-standard) notation

$$\mathring{B}_{pq}^{s,a}(\Omega) = \left\{ f \in B_{pq}^{s,a}(\Omega) : f|_{\partial\Omega} = \frac{\partial f}{\partial x_2} |_{\partial\Omega} = 0 \right\}. \quad (4.55)$$

Step 2.2. Let $a = (\frac{4}{3}, \frac{2}{3})$ and let, according to (3.20),

$$\overset{\circ}{W}_2^{\frac{4}{3},a}(\Omega) = \overset{\circ}{W}_2^{(1,2)}(\Omega).$$

We prove now that the operator $T = \mathcal{A}_r^{-1} \circ tr^\Gamma$ can be factorised by $T = id_2 \circ \mathcal{A}_r^{-1} \circ id_1 \circ tr_\Gamma$ where

$$\begin{aligned} tr_\Gamma &: \overset{\circ}{W}_2^{\frac{4}{3},a}(\Omega) \rightarrow L_2(\Gamma) \\ id_1 &: L_2(\Gamma) \rightarrow B_{2\infty}^{-\frac{2-d}{2},a}(\Omega) \\ \mathcal{A}_r^{-1} &: B_{2\infty}^{-\frac{2-d}{2},a}(\Omega) \rightarrow \overset{\circ}{B}_{2\infty}^{\frac{8}{3}-\frac{2-d}{2},a}(\Omega) \\ id_2 &: \overset{\circ}{B}_{2\infty}^{\frac{8}{3}-\frac{2-d}{2},a}(\Omega) \rightarrow \overset{\circ}{W}_2^{\frac{4}{3},a}(\Omega). \end{aligned} \quad (4.56)$$

The boundedness of tr_Γ in the first line of (4.56) was discussed in Step 1, see (4.50).

According to (3.21) and (4.53) the embedding id_1 is (3.10) whereas the embedding id_2 is a simple consequence of the inequality

$$\frac{8}{3} - \frac{2-d}{2} > \frac{4}{3}$$

and of the elementary embedding between anisotropic function spaces from [Joh95, (2.11)] (see [Tri83, Proposition 2.3.2/2] for the isotropic counterpart).

So it remains only to justify the boundedness of \mathcal{A}_r^{-1} as indicated in the third line of (4.56). Let us denote $x^0 = (-1, 0)$ and $x^1 = (1, 0)$ and

$$\begin{aligned} &W_2^{\frac{8}{3},a}(\Omega, x^0, x^1) = \\ &= \left\{ f \in W_2^{\frac{8}{3},a}(\Omega) : \frac{\partial^{m_1+m_2} f}{\partial x_1^{m_1} \partial x_2^{m_2}}(x^0) = \frac{\partial^{m_1+m_2} f}{\partial x_1^{m_1} \partial x_2^{m_2}}(x^1) = 0 \quad \text{if } 2m_1 + m_2 + \frac{3}{2} < 4 \right\}, \end{aligned} \quad (4.57)$$

see [Tri83*, (17)], and let

$$\overset{\circ}{W}_2^{\frac{8}{3},a}(\Omega, x^0, x^1) = \left\{ f \in W_2^{\frac{8}{3},a}(\Omega, x^0, x^1) : f|_{\partial\Omega} = \frac{\partial f}{\partial x_2}|_{\partial\Omega} = 0 \right\}. \quad (4.58)$$

Recall that there exists a number $r_0 \in \mathbb{R}$ such that for any $r > r_0$ the operator

$$\mathcal{A}_r \text{ maps } \overset{\circ}{W}_2^{\frac{8}{3},a}(\Omega, x^0, x^1) \text{ isomorphically onto } L_2(\Omega); \quad (4.59)$$

this was proved by H. Triebel in [Tri83*, Theorem 4], see also [She98, Theorem 2.1].

Furthermore, by Proposition 2 and Remark 6 in [Tri83*] the operator

$$\mathcal{A}_r \text{ maps } \overset{\circ}{W}_2^{\frac{4}{3},a}(\Omega) \text{ isomorphically onto } B_{22}^{-\frac{4}{3},a}(\Omega). \quad (4.60)$$

Let

$$\theta = \frac{3(2-d)}{8}. \quad (4.61)$$

Then clearly we have $0 < \theta < 1$ and

$$(1-\theta) \cdot 0 + \theta \cdot \left(-\frac{4}{3}\right) = -\frac{2-d}{2}.$$

Since the unit disc Ω is a domain in \mathbb{R}^2 having the so-called C^m flexible $(\frac{4}{3}, \frac{2}{3})$ horn condition for any $m \in \mathbb{N}_0^2$, (see [Bes96, Definition 1.2]) by [Bes97] with the above θ we have the following interpolation result:

$$\left(L_2(\Omega), B_{22}^{-\frac{4}{3}, a}(\Omega) \right)_{\theta, \infty} = B_{2\infty}^{-\frac{2-d}{2}, a}(\Omega). \quad (4.62)$$

From (4.59), (4.60) and (4.62) it follows that for $r > r_0$ sufficiently large the operator

$$\mathcal{A}_r^{-1} \text{ maps } B_{2\infty}^{-\frac{2-d}{2}, a}(\Omega) \text{ bounded into } \left(\overset{\circ}{W}_2^{\frac{8}{3}, a}(\Omega, x^0, x^1), \overset{\circ}{W}_2^{\frac{4}{3}, a}(\Omega) \right)_{\theta, \infty}. \quad (4.63)$$

Since

$$\left\{ f \in W_2^{\frac{8}{3}, a}(\Omega, x^0, x^1) + W_2^{\frac{4}{3}, a}(\Omega) : f|_{\partial\Omega} = \frac{\partial f}{\partial x_2} |_{\partial\Omega} = 0 \right\}$$

is a complemented subspace of $W_2^{\frac{8}{3}, a}(\Omega, x^0, x^1) + W_2^{\frac{4}{3}, a}(\Omega)$ with the same projection operator we may use [Tri78, Theorem 1.17.1/1] and have

$$\begin{aligned} & \left(\overset{\circ}{W}_2^{\frac{8}{3}, a}(\Omega, x^0, x^1), \overset{\circ}{W}_2^{\frac{4}{3}, a}(\Omega) \right)_{\theta, \infty} = \\ & = \left\{ f \in \left(W_2^{\frac{8}{3}, a}(\Omega, x^0, x^1), W_2^{\frac{4}{3}, a}(\Omega) \right)_{\theta, \infty} : f|_{\partial\Omega} = \frac{\partial f}{\partial x_2} |_{\partial\Omega} = 0 \right\}. \end{aligned} \quad (4.64)$$

On the other hand, using (4.54), by [Bes96, Theorem 4.2/a] we have

$$\left(W_2^{\frac{8}{3}, a}(\Omega), W_2^{\frac{4}{3}, a}(\Omega) \right)_{\theta, \infty} = B_{2\infty}^{\frac{8}{3} - \frac{2-d}{2}, a}(\Omega) \quad (4.65)$$

where θ is the number defined in (4.61). It follows from (4.63), (4.64), (4.65) and elementary properties of real interpolation that

$$\mathcal{A}_r^{-1} \text{ maps } B_{2\infty}^{-\frac{2-d}{2}, a}(\Omega) \text{ bounded into } \overset{\circ}{B}_{2\infty}^{\frac{8}{3} - \frac{2-d}{2}, a}(\Omega)$$

and this completes the proof of the boundedness from the third line of (4.56).

Step 2.3. Let $f \in \overset{\circ}{W}_2^{(1,2)}(\Omega)$ be an eigenfunction of T . Then it follows from (4.56) that f belongs also to

$$\overset{\circ}{B}_{2\infty}^{\frac{8}{3} - \frac{2-d}{2}, a}(\Omega)$$

and so it is an eigenfunction of the operator T restricted to this space. Obviously the converse is also true.

Hence the root systems considered in $\overset{\circ}{W}_2^{(1,2)}(\Omega)$ and in $\overset{\circ}{B}_{2\infty}^{\frac{8}{3} - \frac{2-d}{2}, a}(\Omega)$ coincide. Then the eigenvalues of T considered in these spaces also coincide, inclusively their multiplicities.

Using the multiplicity property for entropy numbers and (4.56) there exists a constant $c > 0$ such that for all $k \in \mathbb{N}$,

$$e_k(T : \overset{\circ}{B}_{2\infty}^{\frac{8}{3} - \frac{2-d}{2}, a}(\Omega) \rightarrow \overset{\circ}{B}_{2\infty}^{\frac{8}{3} - \frac{2-d}{2}, a}(\Omega)) \leq c e_k(\text{tr}_\Gamma : \overset{\circ}{B}_{2\infty}^{\frac{8}{3} - \frac{2-d}{2}, a}(\Omega) \rightarrow L_2(\Gamma)). \quad (4.66)$$

Inserting in Theorem 3.13 $n = 2$, $a = (\frac{4}{3}, \frac{2}{3})$, $p_1 = p_2 = 2$, $q = \infty$ and

$$s + \frac{2-d}{2} = \frac{8}{3} - \frac{2-d}{2}$$

we have from (4.66)

$$e_k(T : B_{2\infty}^{\circ, \frac{8}{3} - \frac{2-d}{2}, a}(\Omega) \rightarrow B_{2\infty}^{\circ, \frac{8}{3} - \frac{2-d}{2}, a}(\Omega)) \leq c k^{-\frac{1}{d}(d + \frac{2}{3})}. \quad (4.67)$$

The estimate (3.27) is now a simple consequence of (4.67) using Carl's inequality (4.49).

Step 3. If one applies Theorem 4.11 then the estimate (3.28) is covered by the next Proposition.

Proposition 4.12 *Let Ω , d as above, let Γ be proper and let T be the compact, non-negative, self adjoint operator in $W_2^{\circ(1,2)}(\Omega)$ defined in (3.24).*

There exists a constant $c > 0$ such that the approximation numbers $\alpha_k(\sqrt{T})$ of $\sqrt{T} = T^{1/2}$ can be estimated by

$$\alpha_k(\sqrt{T}) \geq c k^{-\frac{1}{d}(\frac{d}{2} + \frac{1}{3})}, \quad k \in \mathbb{N}. \quad (4.68)$$

Proof of Proposition 4.12.

Let $x^{j,l}$ be the centre of the rectangle R_{jl} of side lengths $r_1^{j,l}$, $r_2^{j,l}$ and let $N_j \sim 2^{jd}$ having the same meaning as in Definition 3.1. Since $\Gamma \subset \Omega$ it is clear that there exists a $j_0 \in \mathbb{N}_0$ such that for any $j \geq j_0$ the rectangles R_{jl} ($l = 1, \dots, N_j$) are contained in Ω .

Let φ a non-negative C^∞ function on \mathbb{R}^2 with support in $\{x \in \mathbb{R}^2 : |x_1| < 1, |x_2| < 1\}$.

We may assume $|\varphi(x)| \geq \delta > 0$ if $|x_1| \leq \kappa$, $|x_2| \leq \kappa$ where $0 < \kappa < 1$ is the number from Definition 3.6 of a proper set.

If

$$\varphi_{jl}(x) = \varphi\left(\frac{2(x_1 - x_1^{j,l})}{r_1^{j,l}}, \frac{2(x_2 - x_2^{j,l})}{r_2^{j,l}}\right)$$

then $\text{supp } \varphi_{jl} \subset R_{jl}$. Furthermore, there exist two constants $c_1, c_2 > 0$ such that

$$c_1 2^{-jd/2} \left(\sum_{l=1}^{N_j} |c_{jl}|^2 \right)^{1/2} \leq \left\| \sum_{l=1}^{N_j} c_{jl} \varphi_{jl} \mid L_2(\Gamma) \right\| \leq c_2 2^{-jd/2} \left(\sum_{l=1}^{N_j} |c_{jl}|^2 \right)^{1/2} \quad (4.69)$$

for any complex numbers c_{jl} and for any $j \geq j_0$ and $l = 1, \dots, N_j$.

Indeed, using (3.7) we have

$$\begin{aligned} \left\| \sum_{l=1}^{N_j} c_{jl} \varphi_{jl} \mid L_2(\Gamma) \right\|^2 &= \sum_{l=1}^{N_j} \int_{\Gamma \cap R_{jl}} |c_{jl}|^2 |\varphi_{jl}(\gamma)|^2 d\mu(\gamma) \\ &\geq \sum_{l=1}^{N_j} |c_{jl}|^2 \delta^2 \mu(\Gamma \cap \kappa R_{jl}) \geq c 2^{-jd} \sum_{l=1}^{N_j} |c_{jl}|^2 \end{aligned}$$

and this is the first inequality in (4.69). The second part of (4.69) is a simple consequence of (3.5).

Remark now that if $j \geq j_0$ is fixed and $l \in \{1, \dots, N_j\}$ then

$$2^{-j\frac{1}{3}} \varphi_{jl}$$

is an anisotropic $(\frac{4}{3}, 2)$ atom in $\mathring{W}_2^{(1,2)}(\Omega)$. Hence, using (3.20) and the atomic decomposition theorem with $n = 2$, $s = \frac{4}{3}$ and $p = q = 2$ we have

$$\|g_j | \mathring{W}_2^{(1,2)}(\Omega)\| \leq c 2^{j\frac{1}{3}} \left(\sum_{l=1}^{N_j} |c_{jl}|^2 \right)^{1/2} \quad (4.70)$$

for any function g_j of type

$$g_j = \sum_{l=1}^{N_j} c_{jl} \varphi_{jl} = 2^{j\frac{1}{3}} \sum_{l=1}^{N_j} c_{jl} \left(2^{-j\frac{1}{3}} \varphi_{jl} \right) \quad , \quad (4.71)$$

the constant $c > 0$ in (4.70) being independent of j , l and of the complex numbers c_{jl} .

By (4.51), (4.69) and (4.70) we find a constant c_0 such that

$$\begin{aligned} \|\sqrt{T}g_j | \mathring{W}_2^{(1,2)}(\Omega)\| &\sim 2^{-j\frac{d}{2}} \left(\sum_{l=1}^{N_j} |c_{jl}|^2 \right)^{1/2} \\ &\geq c_0 2^{-j(\frac{d}{2} + \frac{1}{3})} \|g_j | \mathring{W}_2^{(1,2)}(\Omega)\| \end{aligned} \quad (4.72)$$

There exists an operator $L = L(N_j)$ in $\mathring{W}_2^{(1,2)}(\Omega)$ with $rank L < N_j$ such that

$$\alpha_{N_j}(\sqrt{T}) \geq \|\sqrt{T} - L\| - \frac{1}{2} c_0 2^{-j(\frac{d}{2} + \frac{1}{3})}$$

where c_0 is the number from the last line in (4.72).

We may assume that the dimension of the span of the admitted functions g_j in (4.71) is larger than $N_j \sim 2^{jd}$. Then we find a function g_j of type (4.71) in $\mathring{W}_2^{(1,2)}(\Omega)$ such that $\|g_j | \mathring{W}_2^{(1,2)}(\Omega)\| = 1$ and $Lg_j = 0$. It follows

$$\begin{aligned} \alpha_{N_j}(\sqrt{T}) &\geq \|\sqrt{T}g_j - Lg_j | \mathring{W}_2^{(1,2)}(\Omega)\| - \frac{1}{2} c_0 2^{-j(\frac{d}{2} + \frac{1}{3})} \\ &\geq c_0 2^{-j(\frac{d}{2} + \frac{1}{3})} \end{aligned} \quad (4.73)$$

where $c_0 > 0$ is independent of j . Using elementary properties of approximation numbers it is easy to see that (4.73) implies (4.68).

4.6 Proofs of the results in Section 2.4

We begin with some remarks on the notation and terminology which we will use in the proofs.

If $n \geq 2$ then $a = (a_1, \dots, a_n)$ will be a given anisotropy. We call a an n - dimensional anisotropy and write it $a = (a', a_n)$ where, of course, $a' = (a_1, \dots, a_{n-1})$.

In analogy to our notation in the previous sections, $a_{max} = \max\{a_k : 1 \leq k \leq n\}$, let $a_{max}^* = \max\{a_k^* : 1 \leq k \leq n-1\}$ where any a_k is defined from (2.18).

We will use the notation $a^* \alpha' = a_1^* \alpha_1 + \dots + a_{n-1}^* \alpha_{n-1}$ for the scalar product of a^* with the multi-index $\alpha' = (\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{N}_0^{n-1}$.

Throughout the proofs we will assume that the involved functions are smooth enough, for example f in (2.16). The necessary approximation procedures and Fatou arguments have been discussed with details in [Tri92, Remark 4.2.2/1] and will not be repeated here.

Proof of Theorem 2.8

Step 1. We first prove (2.20). By (2.19) there are no moment conditions required for the atoms in the atomic decomposition theorem for the space $B_{pq}^{r,a^*}(\mathbb{R}^{n-1})$. Using both (2.18) and (2.19) one can see that $s > \sigma_p$ and so the same assertion is valid for the atomic decomposition theorem for the space $B_{pq}^{s,a}(\mathbb{R}^n)$.

We will assume $p < \infty$; for $p = \infty$ one has to make obvious changes in the proof below.

Let K and L be fixed numbers such that $K \geq a_{max} + s$ and $L < 0$.

Let $g \in B_{pq}^{s,a}(\mathbb{R}^n)$ decomposed as in (2.12) with $\|\lambda\| b_{pq} \leq c \|g\| B_{pq}^{s,a}(\mathbb{R}^n)$. Recall that each atom $\rho_{\nu m}^a$ is supported in some rectangle $cQ_{\nu m}^a$ with $c > 1$, see (2.8).

For the trace problem only rectangles $Q_{\nu m}^a$ are of interest for which $cQ_{\nu m}^a$ has a non-empty intersection with the hyperplane $\{x = (x', x_n) \in \mathbb{R}^n : x_n = 0\}$. Let us denote

$$A = \{(\nu, m) \in \mathbb{N}_0 \times \mathbb{Z}^n : cQ_{\nu m}^a \cap \{x = (x', x_n) \in \mathbb{R}^n : x_n = 0\} \neq \emptyset\}. \quad (4.74)$$

Fix $(\nu, m) \in A$ and let Q' be the projection of $Q = Q_{\nu m}^a$ on that hyperplane, now being identified with \mathbb{R}^{n-1} . Clearly Q' is an $(n-1)$ -dimensional rectangle with side lengths respectively $2^{-\nu a_1}, \dots, 2^{-\nu a_{n-1}}$ and $|Q| = |Q'|^{n/(n-a_n)}$.

If $\alpha = (\alpha', 0) \in \mathbb{N}_0^n$ such that $a'\alpha' \leq K$ then using (2.18) and (2.9) we get:

$$|D^{\alpha'} \rho_{\nu m}^a(x', 0)| \leq |Q|^{\frac{s}{n} - \frac{1}{p} - \frac{a'\alpha'}{n}} = |Q'|^{\frac{s}{n-a_n} - \frac{n}{(n-a_n)p} - \frac{a'\alpha'}{n-a_n}} = |Q'|^{\frac{r}{n-1} - \frac{1}{p} - \frac{a'\alpha'}{n-1}}. \quad (4.75)$$

If $K' = \frac{n-1}{n-a_n}K$ then $K' \geq a_{max}^* + r$ and by (4.75) it follows that $\rho_{\nu m}^a(x', 0)$ is an $(n-1)$ -dimensional a^* -anisotropic $(r, p)_{K', L'}$ -atom located at some rectangle cQ' , where L' can be chosen such that $L' < 0$ (recall that by (2.18) and (2.19) there are no moment conditions required for the atomic decomposition theorem in $B_{pq}^{r,a^*}(\mathbb{R}^{n-1})$).

From $g(x', 0) = \sum_{(\nu, m) \in A} \lambda_{\nu m} \rho_{\nu m}^a(x', 0)$ we get $tr_{\mathbb{R}^{n-1}} g \in B_{pq}^{r,a^*}(\mathbb{R}^{n-1})$. Furthermore,

$$\|g(\cdot, 0) | B_{pq}^{r,a^*}(\mathbb{R}^{n-1})\| \leq c \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q} \leq c' \|g | B_{pq}^{s,a}(\mathbb{R}^n)\|$$

(usual modification if $q = \infty$) which shows that $tr_{\mathbb{R}^{n-1}}$ is a linear bounded operator from $B_{pq}^{s,a}(\mathbb{R}^n)$ into $B_{pq}^{r,a^*}(\mathbb{R}^{n-1})$.

It remains to show that if $h \in B_{pq}^{r,a^*}(\mathbb{R}^{n-1})$ then there exists a function $g \in B_{pq}^{s,a}(\mathbb{R}^n)$ with $g(x', 0) = h(x')$ and $\|g | B_{pq}^{s,a}(\mathbb{R}^n)\| \leq c \|h | B_{pq}^{r,a^*}(\mathbb{R}^{n-1})\|$.

We start choosing a real number K' such that

$$K' \geq a_{max}^* + r + \frac{(n-1)a_n}{(n-a_n)p} \geq a_{max}^* + r \quad (4.76)$$

and a number $L' < 0$.

We decompose $h \in B_{pq}^{r,a^*}(\mathbb{R}^{n-1})$ via the atomic decomposition theorem in $S'(\mathbb{R}^{n-1})$ as

$$h = \sum_{\nu=0}^{\infty} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu m'} \rho_{\nu m'}^{a^*}$$

where $\rho_{\nu m'}^{a^*}$ is an a^* -anisotropic $1_{K'}$ -atom ($\nu = 0$) or an a^* -anisotropic $(r, p)_{K', L'}$ -atom ($\nu \in \mathbb{N}$). More precisely, let us denote $Q_{\nu m'}^{a^*}$ the $(n-1)$ -dimensional rectangle with sides

parallel to the axes, centred at $(2^{-\nu a_1^*} m_1, \dots, 2^{-\nu a_{n-1}^*} m_{n-1})$ and with side lengths respectively $2^{-\nu a_1^*}, \dots, 2^{-\nu a_{n-1}^*}$. We have:

$$\begin{aligned} \text{supp } \rho_{\nu m'}^{a^*} &\subset cQ_{\nu m'}^{a^*} \\ |D^{\alpha'} \rho_{\nu m'}^{a^*}(x')| &\leq 2^{-\nu \left(r - \frac{n-1}{p}\right)} 2^{\nu a^* \alpha'} \quad \text{if } a^* \alpha' \leq K' \end{aligned}$$

if $\nu \in \mathbb{N}$ and the standard modification if $\nu = 0$.

We will extend the $(n-1)$ -dimensional atoms to atoms on \mathbb{R}^n . To do this we consider a function $\psi \in C_0^\infty(\mathbb{R})$,

$$\psi(0) = 1, \quad |\psi(t)| \leq 1, \quad \text{and} \quad \text{supp } \psi \subset [-1, 1],$$

and define

$$g(x) = \text{ext } h(x) = \sum_{\nu=0}^{\infty} \sum_{m' \in \mathbb{Z}^{n-1}} \lambda_{\nu m'} \rho_{\nu m'}^{a^*}(x') \psi(2^{\nu(n-1)a_n/(n-a_n)} x_n). \quad (4.77)$$

Let $K = \frac{n-a_n}{n-1} K'$ and remark that by (4.76) we have $K \geq a_{max} + s$; choose also $L < 0$. Denoting

$$\rho_{\nu m'}^a(x', x_n) = \rho_{\nu m'}^{a^*}(x') \psi(2^{\nu(n-1)a_n/(n-a_n)} x_n),$$

one can see that $\rho_{\nu m'}^a$ is, besides a unimportant constant (independent of ν and m'), an n -dimensional anisotropic $(s, p)_{K, L}$ -atom supported in a rectangle with sides parallel to the axes and with side lengths respectively $c2^{-\nu(n-1)a_1/(n-a_n)}, \dots, c2^{-\nu(n-1)a_{n-1}/(n-a_n)}, c2^{-\nu(n-1)a_n/(n-a_n)}$ for an appropriate constant $c > 0$.

Hence (4.77) yields an atomic decomposition for g in the sense of Remark 2.7. So

$$\|g\|_{B_{pq}^{s,a}(\mathbb{R}^n)} \leq c \left(\sum_{\nu=0}^{\infty} \left(\sum_{m' \in \mathbb{Z}^{n-1}} |\lambda_{\nu m'}|^p \right)^{q/p} \right)^{1/q} \leq c' \|h\|_{B_{pq}^{r,a^*}(\mathbb{R}^{n-1})}$$

(usual modification if $q = \infty$) and this completes the proof.

Step 2. To prove (2.21) we use the method from [FrJ90, 11.1] and show that $\text{tr}_{\mathbb{R}^{n-1}} F_{pq}^{s,a}(\mathbb{R}^n)$ is independent of q . Then we have $\text{tr}_{\mathbb{R}^{n-1}} F_{pq}^{s,a}(\mathbb{R}^n) = \text{tr}_{\mathbb{R}^{n-1}} F_{pp}^{s,a}(\mathbb{R}^n)$ and using $F_{pp}^{s,a}(\mathbb{R}^n) = B_{pp}^{s,a}(\mathbb{R}^n)$ the conclusion follows from (2.20).

Let $0 < q < t \leq \infty$. The elementary embedding $F_{pq}^{s,a}(\mathbb{R}^n) \hookrightarrow F_{pt}^{s,a}(\mathbb{R}^n)$ (see [Joh95, (2.10)]) implies $\text{tr}_{\mathbb{R}^{n-1}} F_{pq}^{s,a}(\mathbb{R}^n) \hookrightarrow \text{tr}_{\mathbb{R}^{n-1}} F_{pt}^{s,a}(\mathbb{R}^n)$.

To prove the converse inclusion let K and L be real numbers such that $K \geq a_{max} + s$ and $L \geq \sigma_{pt} - s$ and let $g \in F_{pt}^{s,a}(\mathbb{R}^n)$ decomposed as

$$g = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \rho_{\nu m}^{a,t}$$

(convergence in $S'(\mathbb{R}^n)$) with $\rho_{\nu m}^{a,t}$ anisotropic 1_K -atoms ($\nu = 0$) or anisotropic $(s, p)_{K, L}$ -atoms ($\nu \in \mathbb{N}$) and with $\|\lambda\|_{f_{pt}^a} \leq c \|g\|_{F_{pt}^{s,a}(\mathbb{R}^n)}$.

We claim that there exists a function $\tilde{g} \in F_{pq}^{s,a}(\mathbb{R}^n)$ with $\text{tr}_{\mathbb{R}^{n-1}} g = \text{tr}_{\mathbb{R}^{n-1}} \tilde{g}$.

Again only rectangles $Q_{\nu m}^a$ are of interest for which $cQ_{\nu m}^a$ has a non-empty intersection with the hyperplane $\{x = (x', x_n) \in \mathbb{R}^n : x_n = 0\}$; let A be the same set of pairs (ν, m) as in (4.74).

We define $\tilde{\lambda}_{\nu m} = \lambda_{\nu m}$ if $(\nu, m) \in A$ and $\tilde{\lambda}_{\nu m} = 0$ else and put $\tilde{\lambda} = \{\tilde{\lambda}_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$. Let now

$$\psi \in S(\mathbb{R}), \quad \text{supp } \psi \subset \left[-\frac{1}{2}, \frac{1}{2}\right], \quad \psi(0) = 1,$$

and

$$\int_{\mathbb{R}} z^{\beta_n} \psi(z) dz = 0 \quad \text{for all } \beta_n \in \mathbb{N}_0 \quad \text{such that } a_n \beta_n \leq L.$$

We define:

$$\tilde{\rho}_{\nu m}^{a,q}(x', x_n) = \rho_{\nu m}^{a,t}(x', 0) \psi(2^{\nu a_n} x_n)$$

and remark that $\tilde{\rho}_{\nu m}^{a,q}$ is supported in a rectangle $c\tilde{Q}_{\nu m}^a$ where $\tilde{Q}_{\nu m}^a$ has sides parallel to the axes, is centred at $(2^{-\nu a_1} m_1, \dots, 2^{-\nu a_{n-1}} m_{n-1}, 0)$ and its side lengths are respectively $2^{-\nu a_1}, \dots, 2^{-\nu a_{n-1}}, 2^{-\nu a_n}$.

Furthermore, if $\alpha = (\alpha', \alpha_n) \in \mathbb{N}_0^n$ such that $a\alpha \leq K$ and if $\nu \in \mathbb{N}$ then

$$|D^\alpha \tilde{\rho}_{\nu m}^{a,q}(x)| \leq c |D^{\alpha'} \rho_{\nu m}^{a,t}(x', 0)| \cdot 2^{\nu a_n \alpha_n} \leq c' 2^{-\nu \left(s - \frac{n}{p}\right)} 2^{\nu a \alpha}.$$

It follows that each $\tilde{\rho}_{\nu m}^{a,q}$ is, besides an unimportant constant, an anisotropic 1_K -atom for $\nu = 0$ or an anisotropic $(s, p)_{K,L}$ -atom (due to its product structure and to the assumptions on the function ψ there are no problems to check the moment conditions, too).

Defining

$$\tilde{g} = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \tilde{\lambda}_{\nu m} \tilde{\rho}_{\nu m}^{a,q} = \sum_{(\nu, m) \in A} \lambda_{\nu m} \tilde{\rho}_{\nu m}^{a,q} \quad (4.78)$$

clearly $tr_{\mathbb{R}^{n-1}} g = tr_{\mathbb{R}^{n-1}} \tilde{g}$.

For $(\nu, m) \in A$ let

$$\tilde{E}_{\nu m}^a = \left\{ x \in \tilde{Q}_{\nu m}^a : \frac{1}{2} 2^{-(\nu+1)a_n} < x_n \leq \frac{1}{2} 2^{-\nu a_n} \right\}.$$

Obviously, $\frac{|\tilde{E}_{\nu m}^a|}{|\tilde{Q}_{\nu m}^a|} = \frac{1-2^{-a_n}}{2} > 0$. Then

$$\|\tilde{\lambda} | f_{pq}^a \| \sim \left\| \left(\sum_{(\nu, m) \in A} |\lambda_{\nu m} \tilde{\chi}_{\nu m}^{(p)}(\cdot)|^q \right)^{1/q} | L_p(\mathbb{R}^n) \right\| \quad (4.79)$$

with the usual modification if $q = \infty$, where $\tilde{\chi}_{\nu m}^{(p)}$ denotes the $L_p(\mathbb{R}^n)$ -normalised characteristic function of the rectangle $\tilde{E}_{\nu m}^a$.

This is a consequence of an inequality of Fefferman - Stein type for the anisotropic Hardy - Littlewood maximal function and the proof is a simple anisotropic counterpart of [FrJ90, 2.7].

For $(\nu, m) \in A$ the sets $\tilde{E}_{\nu m}^a$ are pairwise disjoint and so at most one term in the sum on the right-hand side from above is nonzero. Hence q and $1/q$ can be canceled in (4.79) and can be replaced by t and $1/t$. It follows

$$\|\tilde{\lambda} | f_{pq}^a \| \sim \left\| \left(\sum_{(\nu, m) \in A} |\lambda_{\nu m} \tilde{\chi}_{\nu m}^{(p)}(\cdot)|^t \right)^{1/t} | L_p(\mathbb{R}^n) \right\| \leq c \|\lambda | f_{pt}^a \|\|$$

(with the usual modification if $t = \infty$).

The last relation together with (4.78) prove the fact that $\tilde{g} \in F_{pq}^{s,a}(\mathbb{R}^n)$ and

$$\|\tilde{g} | F_{pq}^{s,a}(\mathbb{R}^n)\| \leq c \|\tilde{\lambda} | f_{pq}^a\| \leq c' \|\lambda | f_{pt}^a\| \leq c'' \|g | F_{pt}^{s,a}(\mathbb{R}^n)\|.$$

This verifies our claim and shows that $tr_{\mathbb{R}^{n-1}} F_{pt}^{s,a}(\mathbb{R}^n) \hookrightarrow tr_{\mathbb{R}^{n-1}} F_{pq}^{s,a}(\mathbb{R}^n)$ and, consequently, the trace of $F_{pq}^{s,a}(\mathbb{R}^n)$ is independent of q .

Proof of Theorem 2.9

To prove Theorem 2.9 we will use the technique developed in Section 4.1. But since Theorem 2.9 is of independent interest we give all details.

Step 1. Let K, L given numbers such that

$$K \geq a_{max} + \frac{a_n}{p} \quad \text{and} \quad L \geq \sigma_p - \frac{a_n}{p}$$

and let $g \in B_{pq}^{\frac{a_n}{p},a}(\mathbb{R}^n)$.

We decompose g via the atomic decomposition theorem as in (2.12) where now $\rho_{\nu m}^a$ is an 1_K -atom ($\nu = 0$) or an $\left(\frac{a_n}{p}, p\right)_{K,L}$ -atom ($\nu \in \mathbb{N}$) and $\|\lambda | b_{pq}\| \leq c \|g | B_{pq}^{\frac{a_n}{p},a}(\mathbb{R}^n)\|$.

Clearly $\rho_{\nu m}^a(x', 0)$ is supported in an $(n-1)$ -dimensional rectangle $cQ'_{\nu m}$ where $Q'_{\nu m}$ has side lengths respectively $2^{-\nu a_1}, \dots, 2^{-\nu a_{n-1}}$; by (2.9) with $s = \frac{a_n}{p}$ we have also

$$|\rho_{\nu m}^a(x', 0)| \leq 2^{\nu(n-a_n)/p}. \quad (4.80)$$

For every $\nu \in \mathbb{N}_0$ we denote $g_\nu(x', 0) = \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \rho_{\nu m}^a(x', 0)$. Using the controlled overlapping of the supports at the level ν and (4.80) we have:

$$\begin{aligned} \|g_\nu(\cdot, 0) | L_p(\mathbb{R}^{n-1})\|^p &= \int_{\mathbb{R}^{n-1}} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \rho_{\nu m}^a(x', 0) \right|^p dx' \\ &\leq c \int_{\mathbb{R}^{n-1}} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \cdot \left| 2^{\nu(n-a_n)/p} \tilde{\chi}_{\nu m}(x') \right|^p dx' \\ &\leq c' \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \end{aligned} \quad (4.81)$$

where in the calculation above we denoted $\tilde{\chi}_{\nu m}$ the characteristic function of the rectangle $cQ'_{\nu m}$ where $\rho_{\nu m}^a(x', 0)$ is supported.

Applying now the triangle inequality for the $L_p(\mathbb{R}^{n-1})$ -norm if $p \geq 1$ and the p -triangle inequality if $0 < p < 1$, using the restriction on q we obtain from (4.81):

$$\|g(\cdot, 0) | L_p(\mathbb{R}^{n-1})\| \leq c \|\lambda | b_{pq}\|$$

and this shows that $tr_{\mathbb{R}^{n-1}}$ is a linear bounded operator from $B_{pq}^{\frac{a_n}{p},a}(\mathbb{R}^n)$ into $L_p(\mathbb{R}^{n-1})$.

Step 2. To prove that $tr_{\mathbb{R}^{n-1}}$ is onto we begin with a preparation. Let us consider φ a C^∞ function on \mathbb{R}^n with $supp \varphi \subset 2^a Q_{00}^a$ (the rectangle concentric with Q_{00}^a and side lengths respectively $2^{a_1}, \dots, 2^{a_n}$) which satisfies $\sum_{m \in \mathbb{Z}^n} \varphi(x - m) = 1$ for all $x \in \mathbb{R}^n$.

As in Step 2 of the proof of Theorem 3.11 (see Section 4.1), see also [TrW96b, Theorem 2/(i), p.159] and [Win95, 5.4.1], one can show that

$$\bigcup_{\nu=0}^{\infty} \left\{ \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \varphi(2^{\nu a} x - m) \mid \mathbb{R}^{n-1} : \lambda_{\nu m} \in \mathbb{C} \right\}$$

is dense in $L_p(\mathbb{R}^{n-1})$ for $0 < p < \infty$ (of course $2^{\nu a} x - m = (2^{\nu a_1} x_1 - m_1, \dots, 2^{\nu a_n} x_n - m_n)$).

Now we return to our proof and let $h \in L_p(\mathbb{R}^{n-1})$.

Step 2.1. We first assume $\frac{n-a_n}{n} < p$.

By the above mentioned approximation result we may construct an approximating sequence $(h_j)_{j \in \mathbb{N}_0}$ for h in $L_p(\mathbb{R}^{n-1})$ where $h_j = \psi_j \mid \mathbb{R}^{n-1}$ and

$$\psi_j(x) = \sum_m \lambda_{\nu_j m} \varphi(2^{\nu_j a} x - m) \quad \text{for every } j \in \mathbb{N}_0 \quad (4.82)$$

the sum being taken over those $m \in \mathbb{Z}^n$ such that $2^a Q_{\nu_j m}^a \cap \mathbb{R}^{n-1} \neq \emptyset$ and where the coefficients $\lambda_{\nu_j m}$ and the numbers ν_j are determined successively such that

$$\left\| h - \sum_{j=0}^N h_j \mid L_p(\mathbb{R}^{n-1}) \right\| \leq c 2^{-N} \|h \mid L_p(\mathbb{R}^{n-1})\| \quad \text{if } N \in \mathbb{N}_0. \quad (4.83)$$

In particular (4.83) implies

$$\|h_j \mid L_p(\mathbb{R}^{n-1})\| \leq c 2^{-j} \|h \mid L_p(\mathbb{R}^{n-1})\| \quad (4.84)$$

where $c > 0$ is independent of j .

Using now the same technique as in the proof of Theorem 3.11 (see Section 4.1) in particular formula (4.10), see also [TrW96b, Theorem 2/(ii), p.161] and [Win95, Corollary 5.4.1/2], we get:

$$\|h_j \mid L_p(\mathbb{R}^{n-1})\| \sim 2^{-\frac{\nu_j(n-a_n)}{p}} \left(\sum_m |\lambda_{\nu_j m}|^p \right)^{1/p}. \quad (4.85)$$

If we rewrite (4.82) as

$$\psi_j(x) = \sum_m 2^{-\frac{\nu_j(n-a_n)}{p}} \lambda_{\nu_j m} 2^{\frac{\nu_j(n-a_n)}{p}} \varphi(2^{\nu_j a} x - m) \quad (4.86)$$

then (4.86) turns out to be an atomic decomposition of ψ_j in $B_{pq}^{\frac{a_n}{p}, a}(\mathbb{R}^n)$ for every $0 < q \leq \infty$; there are no moment conditions required since $\frac{n-a_n}{n} < p$. Hence the atomic decomposition theorem leads to

$$\|\psi_j \mid B_{pq}^{\frac{a_n}{p}, a}(\mathbb{R}^n)\| \leq c 2^{-\frac{\nu_j(n-a_n)}{p}} \left(\sum_m |\lambda_{\nu_j m}|^p \right)^{1/p}. \quad (4.87)$$

Recall $q \leq \min(1, p)$; applying the q -triangle inequality to $\left\| \sum_{j=0}^N \psi_j \mid B_{pq}^{\frac{a_n}{p}, a}(\mathbb{R}^n) \right\|$ and taking

into account (4.84), (4.85) and (4.87) we obtain the convergence of $\left\{ \sum_{j=0}^N \psi_j : N \in \mathbb{N}_0 \right\}$ in $B_{pq}^{\frac{a_n}{p}, a}(\mathbb{R}^n)$.

We denote the limit with *ext h* and clearly its definition and the properties of the function φ provide that it is independent of the approximating sequence. We may write *ext h* as follows:

$$\text{ext } h = \sum_{j=0}^{\infty} \sum_m 2^{-\frac{\nu_j(n-a_n)}{p}} \lambda_{\nu_j m} 2^{\frac{\nu_j(n-a_n)}{p}} \varphi(2^{\nu_j a} x - m).$$

But this is an atomic decomposition of *ext h*, the convergence being in $B_{pq}^{\frac{a_n}{p}, a}(\mathbb{R}^n)$.

Consequently, Theorem 2.6, (4.84) and (4.85) yield

$$\begin{aligned} \left\| \text{ext } h \mid B_{pq}^{\frac{a_n}{p}, a}(\mathbb{R}^n) \right\| &\leq c \left(\sum_{j=0}^{\infty} \left(2^{-\frac{\nu_j(n-a_n)}{p}} \left(\sum_m |\lambda_{\nu_j m}|^p \right)^{1/p} \right)^q \right)^{1/q} \\ &\leq c' \left(\sum_{j=0}^{\infty} \|h_j \mid L_p(\mathbb{R}^{n-1})\|^q \right)^{1/q} \leq c'' \|h \mid L_p(\mathbb{R}^{n-1})\|. \end{aligned}$$

Finally, it is clear that $(tr_{\mathbb{R}^{n-1}} \circ \text{ext})(h) = h$ and this completes the proof.

Step 2.2. Let now $0 < p \leq \frac{n-a_n}{n}$.

Here we will use the same technique as in [TrW96b, Theorem 3, p. 164]. If $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ let $2^a Q_{\nu m}^a$ be the rectangle concentric with $Q_{\nu m}^a$ having side lengths respectively $2^{(1-\nu)a_1}, \dots, 2^{(1-\nu)a_n}$. It is clear that \mathbb{R}^{n-1} , identified in the usual way with the hyperplane $\{x = (x', x_n) : x_n = 0\}$, satisfies an anisotropic ball condition, *i.e.* there exists a $k \in \mathbb{N}$ such that for every rectangle $Q_{\nu m}^a$ with $2^a Q_{\nu m}^a \cap \{x = (x', x_n) : x_n = 0\} \neq \emptyset$ one of the 2^{kn} congruent sub-rectangles of $Q_{\nu m}^a$ with side lengths $2^{(1-\nu-k)a_1}, \dots, 2^{(1-\nu-k)a_n}$ does not intersect $\{x = (x', x_n) : x_n = 0\}$.

We proceed as in Step 2.1 and obtain (4.86). Then $2^{\frac{\nu_j(n-a_n)}{p}} \varphi(2^{\nu_j a} x - m)$ is almost an anisotropic $(\frac{a_n}{p}, p)$ -atom. It may be furnished with the necessary moment conditions and for this purpose we make a similar construction to that one in the proof of [TrW96a, 3.6], in other words we put

$$\varphi_{\nu_j m} = \varphi(2^{\nu_j a} \cdot -m) - \eta_{\nu_j m}$$

where $\eta_{\nu_j m}$ is a C^∞ function supported in the proportional rectangle related to $2^a Q_{\nu m}^a$ via the anisotropic ball condition such that

$$\int_{\mathbb{R}^n} x^\beta \varphi_{\nu_j m}(x) dx = 0 \quad \text{if } a\beta \leq L$$

where $L \geq \sigma_p - \frac{a_n}{p}$.

Since the proportional rectangles do not intersect $\{x = (x', x_n) : x_n = 0\}$ we have

$$\sum_{m \in \mathbb{Z}^n} \varphi_{\nu_j m} \equiv 1$$

on that hyperplane which is sufficient for the rest of the proof. Moreover, the construction in [TrW96a, 3.6] guarantees that the sequence $\{\varphi_{\nu_j m} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ is uniformly bounded and so we may proceed as in Step 2.1; this completes the proof of (2.22).

Step 3. To prove (2.23) we may repeat the arguments in the proof of (2.21) and obtain that $tr_{\mathbb{R}^{n-1}} F_{pq}^{\frac{a_n}{p}, a}(\mathbb{R}^n)$ is independent of the parameter q . Hence $tr_{\mathbb{R}^{n-1}} F_{pq}^{\frac{a_n}{p}, a}(\mathbb{R}^n) = tr_{\mathbb{R}^{n-1}} F_{pp}^{\frac{a_n}{p}, a}(\mathbb{R}^n)$ and (2.23) follows from $F_{pp}^{\frac{a_n}{p}, a}(\mathbb{R}^n) = B_{pp}^{\frac{a_n}{p}, a}(\mathbb{R}^n)$ applying (2.22).

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Appendix

Appendix 1: The distribution of eigenfrequencies of anisotropic fractal drums

Walter Farkas and Hans Triebel

Abstract. Let Γ be an anisotropic fractal as it was defined in [Tri97, 5.2]. The aim of the paper is to investigate the distribution of the eigenvalues of the fractal differential operator

$$(-\Delta)^{-1} \circ tr^\Gamma$$

acting in the classical Sobolev space $\mathring{W}_2^1(\Omega)$ where Ω is a bounded C^∞ domain in the plane \mathbb{R}^2 with $\Gamma \subset \Omega$. Here $-\Delta$ is the Dirichlet Laplacian in Ω and tr^Γ is closely related to the trace operator tr_Γ . Our results shed new light on [Tri97, 30.7].

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Key words: regular anisotropic fractals, distribution of eigenvalues, entropy numbers

1 Introduction

Motivated by some aspects of boundary value problems for partial differential equations, several authors were concerned in the last years with the study of function spaces on and of fractals. We refer mainly to the works by A. Jonsson and H. Wallin [JoW84], [Jon93], [Jon94], [Jow95] and to the recent book [Tri97] where complete references to this topic are given.

Let Ω be a bounded domain in \mathbb{R}^2 having C^∞ boundary $\partial\Omega$ and let $0 < d_A < 2$. An anisotropic d_A -set $\Gamma \subset \Omega$ having anisotropic deviation $0 \leq a \leq 1$ is, roughly speaking, a compact set which can be covered for any $j \in \mathbb{N}_0$ with $N_j \sim 2^{jd_A}$ disjoint rectangles R_{jl} ($l = 1, \dots, N_j$) with $vol R_{jl} \sim 2^{-2j}$, having sides parallel to the axes and side lengths $r_1^{j,l}, r_2^{j,l}$ satisfying

$$2^{-j(1+a)} \leq r_2^{j,l} \leq r_1^{j,l} \leq 2^{-j(1-a)}$$

for any $l = 1, \dots, N_j$. This concept was introduced in [Tri97, 5.2].

If Γ is such an anisotropic d_A -set then there exists an uniquely determined Radon measure μ in \mathbb{R}^2 with $supp \mu = \Gamma$ and $\mu(\Gamma \cap R_{jl}) = (vol R_{jl})^{d_A/2}$ if $j \in \mathbb{N}_0$ and $l = 1, \dots, N_j$ (see [Tri97, 5.5]).

Let $(-\Delta)^{-1}$ be the inverse of the Dirichlet Laplacian in Ω . Let $W_2^1(\Omega)$ be the usual Sobolev space and let $\mathring{W}_2^1(\Omega) = \{f \in W_2^1(\Omega) : tr_{\partial\Omega} f = 0\}$. The operator tr^Γ ,

$$(tr^\Gamma f)(\varphi) = \int_\Gamma (tr_\Gamma f)(\gamma) (\varphi|_\Gamma)(\gamma) d\mu(\gamma), \quad \varphi \in D(\Omega), \quad (1.1)$$

makes sense as a mapping from $\mathring{W}_2^1(\Omega)$ into $D'(\Omega)$ and it turns out that the fractal differential operator $T = (-\Delta)^{-1} \circ tr^\Gamma$ generates a compact, non-negative, self adjoint operator in $\mathring{W}_2^1(\Omega)$.

Furthermore, as proved in [Tri97, 30.7], there exist positive constants $c_1 > 0$ and $c_2 > 0$ such that for the positive eigenvalues $\lambda_k(T)$ of T

$$c_1 k^{-(d_A+2a)/d_A} \leq \lambda_k(T) \leq c_2 k^{-d_A/(d_A+2a)} \quad , \quad k \in \mathbb{N} \quad (1.2)$$

(for the first inequality in (1.2) it is additionally required that Γ is a so-called proper anisotropic d_A -set, see [Tri97, Definition 5.11]).

If the deviation $a = 0$ then we have $\lambda_k(T) \sim k^{-1}$. This means that the Weyl exponent occurs also in the case of proper anisotropic d_A -sets in the plane with deviation zero. But this fact is not a surprise since those fractals are close to compact isotropic d_A -sets as described in [Tri97, 3.1]. On the other hand, when $a > 0$ the two exponents in (1.2) are not equal.

The study of operators of type $(-\Delta)^{-1} \circ tr^\Gamma$ is motivated in a natural way by the so-called fractal drums: the problem of finding the eigenfrequencies of a vibrating membrane (interpreted as a bounded domain Ω in the plane \mathbb{R}^2), fixed at its boundary, having the whole mass concentrated on some fractal compact set $\Gamma \subset \Omega$, can be reduced to the study of eigenvalues of operators of that type.

More information about this subject is given in [Tri97], especially in Sections 26.2 and 30.1-30.5, where one can find some modifications and a detailed discussion about this topic as well as further references extending the problem.

The aim of this paper is to discuss the sharpness of (1.2) and to shed some new light on these estimates.

Restricting ourselves to the class of regular anisotropic fractals (anisotropic generalisations of the Cantor set in the plane) we prove that there exist two constants $C_1, C_2 > 0$ such that for all $k \in \mathbb{N}$,

$$C_1 k^{-\rho_1} \leq \lambda_k(T) \leq C_2 k^{-\rho_2} \quad (1.3)$$

for appropriate numbers ρ_1 and ρ_2 satisfying

$$(d_A + 2a)/d_A > \rho_1 \geq \rho_2 > d_A/(d_A + 2a)$$

where $\lambda_k(T)$ are, again, the eigenvalues of the operator $T = (-\Delta)^{-1} \circ tr^\Gamma$ acting in $\mathring{W}_2^1(\Omega)$. This means, in particular, that the estimates in (1.2) are not sharp in general.

Furthermore, we will indicate a large class of regular anisotropic fractals for which $\lambda_k(T) \sim k^{-1}$, the so-called strongly regular anisotropic fractals.

Briefly about the organising of the manuscript. In Section 2 we present the basic facts concerning regular anisotropic fractals. The main result, containing the precise formulation of (1.3), is presented with comments in Section 3. The proof is shifted to Section 5, whereas Section 4 contains some preparatory facts for the proof.

All unimportant positive constants are denoted with c , occasionally with additional subscripts within the same formulas. The equivalence "term₁ \sim term₂" means that there exist two constants $c_1, c_2 > 0$ independent of the variables in the two terms such that $c_1 \text{term}_1 \leq \text{term}_2 \leq c_2 \text{term}_1$.

2 Preliminaries

2.1 Regular anisotropic fractals

Let \mathbb{N} denote the natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $Q = [0, 1] \times [0, 1]$ and let \log be taken with respect to the base 2, let $1 < K_1 < K_2$ be natural numbers, let

$$a_1 = \frac{2 \log K_1}{\log(K_1 K_2)} \quad , \quad a_2 = \frac{2 \log K_2}{\log(K_1 K_2)} \quad , \quad \kappa = \frac{1}{2} \log(K_1 K_2)$$

and let $a = 1 - a_1 = a_2 - 1$.

Let $(A_m)_{m=1}^N$ be $N \geq 2$ contractions of \mathbb{R}^2 into itself specified by

$$A_m \quad : \quad x = (x_1, x_2) \mapsto (\eta_1^m 2^{-\kappa(1-a)} x_1, \eta_2^m 2^{-\kappa(1+a)} x_2) + x^m \quad (2.1)$$

for every $m = 1, \dots, N$ where $\eta_1^m, \eta_2^m \in \{-1, +1\}$ (including possible reflections).

We assume $A_m Q \subset Q$ for all $m = 1, \dots, N$; $A_m \overset{\circ}{Q} \cap A_{m'} \overset{\circ}{Q} = \emptyset$ if $m \neq m'$ and $\sum_{m=1}^N \text{vol } A_m Q < 1$.

We suppose, in addition, that the rectangles $A_m Q$ are located in the columns as indicated in Figure 1.

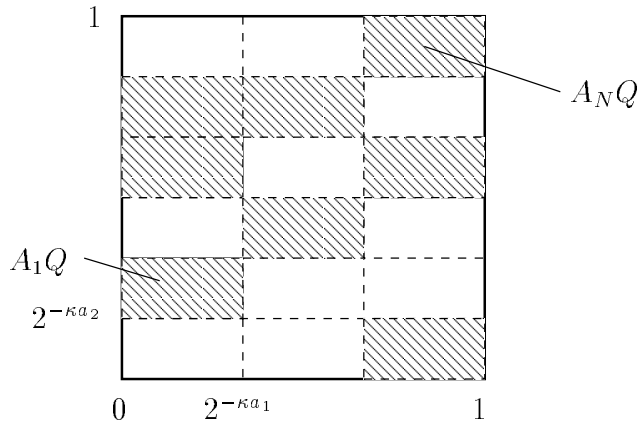


Fig. 1

Let

$$AQ = (AQ)^1 = \bigcup_{m=1}^N A_m Q \quad ; \quad (AQ)^0 = Q \quad ;$$

$$(AQ)^\nu = A((AQ)^{\nu-1}) = \bigcup_{1 \leq m_1, \dots, m_\nu \leq N} A_{m_1} \circ \dots \circ A_{m_\nu} Q \quad ; \quad \nu \in \mathbb{N}.$$

This sequence of sets is monotonically decreasing and by [Fal85, Theorem 8.3] its limit

$$\Gamma = (AQ)^\infty = \bigcap_{\nu \in \mathbb{N}} (AQ)^\nu = \lim_{\nu \rightarrow \infty} (AQ)^\nu$$

is the uniquely determined fractal generated by the contractions $(A_m)_{m=1}^N$.

Fractals constructed in this way are anisotropic generalisations of the Cantor set in \mathbb{R}^2 and were called generalised Sierpinski carpets in [Mul84] (since Sierpinski's universal curve is a special case of this construction) or regular anisotropic fractals in [Tri97, 4.18].

Let n_l denote the number of rectangles $A_m Q$ in the l th column, $l = 1, \dots, K_1$.

Throughout the paper we will assume that $n_l \geq 1$ for any $l = 1, \dots, K_1$ (in each column there is at least one rectangle $A_m Q$ located).

The Hausdorff dimension (see [Fal90, 2.2] for definition) of Γ is

$$\dim_H \Gamma = \frac{1}{\log K_1} \log \left(\sum_{l=1}^{K_1} n_l^{\log K_1 / \log K_2} \right) \quad (2.2)$$

and the box-counting dimension (see [Fal90, 3.1] for definition) of Γ is

$$\dim_B \Gamma = 1 + \frac{\log(N/K_1)}{\log K_2}. \quad (2.3)$$

Proofs of (2.2) and (2.3) are given in [Mul84], see also [Fal90, Example 9.11].

Notice that in this type of examples the Hausdorff dimension depends not only on the number of rectangles selected at each stage but also on their relative position. Moreover, it is clear that $\dim_H \Gamma$ and $\dim_B \Gamma$ are not, in general, equal.

Let $(A_m)_{m=1}^N$ be the $N \geq 2$ affine maps introduced in (2.1). The affine dimension of $\Gamma = (AQ)^\infty$, see [Tri97, 4.12], is the uniquely determined positive number $d_A = \dim_A \Gamma$ such that

$$\sum_{m=1}^N (\text{vol } A_m Q)^{d_A/2} = 1. \quad (2.4)$$

By construction we have $N = 2^{\kappa d_A}$.

Definition 2.1 *If $n_1 = \dots = n_{K_1} = N 2^{-\kappa(1-a)}$ (in any column there is the same number of rectangles) then we call Γ a strongly regular anisotropic fractal.*

Remark 2.2 If Γ is strongly regular then

$$\dim_H \Gamma = \dim_B \Gamma = \frac{d_A + 2a}{1 + a}$$

as a simple consequence of (2.2), (2.3) and (2.4).

It will be clear from all what follows that the typical number which also appears in case of arbitrary regular anisotropic fractals is

$$d = \frac{d_A + 2a}{1 + a}.$$

Theorem 2.3 ([Tri97, 4.15]) *Let Γ be the regular anisotropic fractal introduced above having the affine dimension d_A according to (2.4).*

Then there exists a Radon measure μ in \mathbb{R}^2 uniquely determined with $\text{supp } \mu = \Gamma$ and

$$\mu(\Gamma \cap A_{m_1} \circ \dots \circ A_{m_j} Q) = (\text{vol } A_{m_1} \circ \dots \circ A_{m_j} Q)^{d_A/2} \quad (2.5)$$

for all $j \in \mathbb{N}$ and all $m_1, \dots, m_j \in \{1, \dots, N\}$.

Definition 2.4 Let $n_{max} = \max\{n_l : 1 \leq l \leq K_1\}$ and $n_{min} = \min\{n_l : 1 \leq l \leq K_1\}$. Then there exist two numbers $\lambda^+ \geq 0$ and $\lambda^- \leq 0$ such that

$$n_{max} = N 2^{-\kappa(1-a)} 2^{\kappa(1-a)\lambda^+} \quad \text{and} \quad n_{min} = N 2^{-\kappa(1-a)} 2^{\kappa(1-a)\lambda^-}. \quad (2.6)$$

We call λ^+ the upper mass concentration factor of Γ and λ^- the lower mass concentration factor of Γ since these numbers give information about the distribution of the rectangles in Figure 1 and about the structure of Γ .

Remark 2.5 Clearly $\lambda^+ < 1$ since we assumed $n_l \geq 1$ for every $l \in \{1, \dots, K_1\}$.

Remark also that $\lambda^+ = \lambda^- = 0$ if, and only if, Γ is strongly regular according to Definition 2.1.

It is clear that for any $j \in \mathbb{N}$ there are $N^j = 2^{j\kappa d_A}$ rectangles of type $R_j = A_{m_1} \circ \dots \circ A_{m_j} Q$, having side lengths $2^{-j\kappa(1-a)}$, $2^{-j\kappa(1+a)}$ belonging to $(AQ)^j$.

Let R_j be such a rectangle. We subdivide R_j in rectangles E_{jl} having side lengths $2^{-(j+m)\kappa(1-a)}$ and $2^{-j\kappa(1+a)}$ such that

$$2^{-j\kappa(1+a)} \sim 2^{-(j+m)\kappa(1-a)}.$$

The rectangles E_{jl} are almost squares; it is immaterial for what follows to assume that E_{jl} are squares, what means

$$2^{-j\kappa(1+a)} = 2^{-(j+m)\kappa(1-a)}. \quad (2.7)$$

The lemma below gives an information about the mass concentration in E_{jl} ($l = 1, \dots, 2^{j\kappa 2a}$).

Lemma 2.6 There exist constants $c_1, c_2 > 0$ such that the measure of any square E_{jl} can be estimated by

$$c_1 2^{-j\kappa(d_A + 2a(1-\lambda^-))} \leq \mu(\Gamma \cap E_{jl}) \leq c_2 2^{-j\kappa(d_A + 2a(1-\lambda^+))}. \quad (2.8)$$

Proof. For $j \in \mathbb{N}$ let $\nu = j + m$ in (2.7). Then the rectangle R_j contains $N^{\nu-j}$ rectangles R_ν with side lengths $2^{-\nu\kappa(1-a)}$, $2^{-\nu\kappa(1+a)}$ belonging to $(AQ)^\nu$ which are obtained from R_j after $\nu - j$ steps of iteration.

Hence any square E_{jl} contains at most $n_{max}^{\nu-j}$ of those rectangles R_ν , see Figure 2.

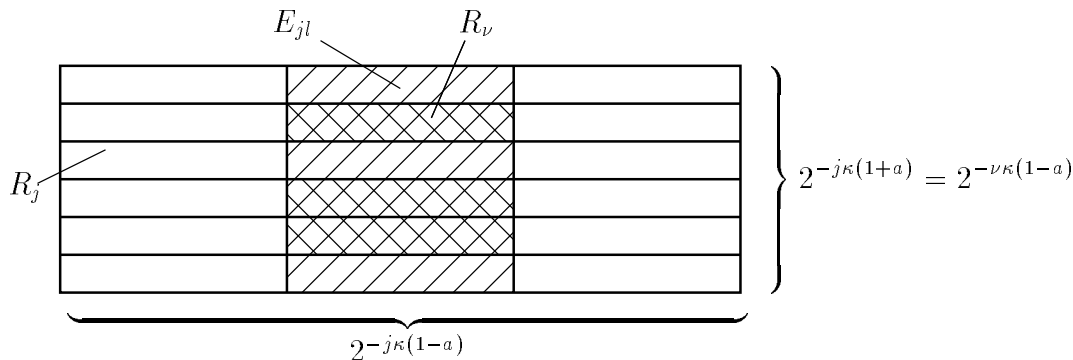


Fig. 2

Clearly $\mu(\Gamma \cap E_{ji}) \leq c n_{max}^{\nu-j} \mu(\Gamma \cap R_\nu)$. By (2.4), (2.5), (2.6) and $\nu(1-a) = j(1+a)$ the estimate from above in (2.8) follows from

$$\mu(\Gamma \cap E_{ji}) \leq c N^{\nu-j} 2^{-(\nu-j)\kappa(1-a)(1-\lambda^+)} 2^{-\nu\kappa d_A} = c 2^{-j\kappa d_A} 2^{-j\kappa 2a(1-\lambda^+)}.$$

The estimate from below can be obtained in the same way.

As a simple consequence of $\nu(1-a) = j(1+a)$ we obtain:

Corollary 2.7 *There exist constants $c_1, c_2 > 0$ such that for any square E_{ji} of side length $2^{-\nu\kappa(1-a)}$ we have*

$$c_1 2^{-\nu\kappa(1-a)d(\lambda^-)} \leq \mu(\Gamma \cap E_{ji}) \leq c_2 2^{-\nu\kappa(1-a)d(\lambda^+)} \quad (2.9)$$

where

$$d(\lambda^-) = \frac{d_A + 2a(1-\lambda^-)}{1+a} \quad \text{and} \quad d(\lambda^+) = \frac{d_A + 2a(1-\lambda^+)}{1+a}.$$

We want to remark that if Γ is strongly regular then Γ is an isotropic d -set (see [Tri97, 3.1] for definition) where

$$d = \frac{d_A + 2a}{1+a}.$$

This is a simple consequence of (2.9) and of Remark 2.5.

Remark 2.8 In specification of the situation in Figure 1 and with changing the roles of columns and rows we suppose now that in each column precisely one rectangle $A_m Q$ is located; then $1 < a_1 < 2$ (which is equivalent to $K_1 > K_2$). Let

$$\frac{K_1}{K_2} = 2k + 1 \quad \text{for some } k \in \mathbb{N}$$

be an odd natural number and let the rectangles $A_m Q$ be arranged as depicted in Figure 3 where we choose in the counterpart of (2.1) always $\eta_2^m = 1$ and we choose $\eta_1^m = 1$ in the first K_2 columns, $\eta_1^m = -1$ in the second K_2 columns (additional reflection) then again $\eta_1^m = 1$ in the third K_2 columns and so on, see Figure 3.

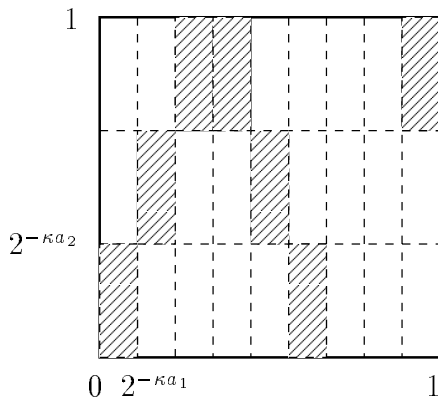


Fig. 3

Under these assumptions the resulting anisotropic fractal Γ is the graph of a continuous function, for a proof see [Tri97, 4.21]. It is clear that Γ may be interpreted as a generalisation of Hironaka's curve (briefly presented in [Mul84]).

The Hausdorff dimension of Γ is $\dim_H \Gamma = 2 - a_2/a_1$, see [Tri97, 4.22]. It is not difficult to see that Γ is a strongly regular anisotropic fractal with affine dimension $d_A = a_1$.

2.2 The spaces $L_p(\Gamma)$

Some preliminaries. We recall here the definition of Besov spaces on \mathbb{R}^2 .

Let φ_0 be a C^∞ function on \mathbb{R}^2 , $\varphi_0(x) = 1$ if $|x| \leq 1$; $\text{supp } \varphi_0 \subset \{x \in \mathbb{R}^2 : |x| \leq 2\}$ and let $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$ if $j \in \mathbb{N}$. Then $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ if $x \in \mathbb{R}^2$ and $(\varphi_j)_{j \in \mathbb{N}_0}$ is a smooth dyadic resolution of unity.

Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$; the Besov space $B_{pq}^s(\mathbb{R}^2)$ consists of all tempered distributions $f \in S'(\mathbb{R}^2)$ for which the quasi-norm

$$\|f\|_{B_{pq}^s(\mathbb{R}^2)} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^\vee\|_{L_p(\mathbb{R}^2)}^q \right)^{1/q}$$

(with the usual modification if $q = \infty$) is finite. Here $\hat{g} = Fg$ and $\check{g} = F^{-1}g$ are respectively the Fourier and inverse Fourier transform on $S'(\mathbb{R}^2)$. These are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$) which are independent of the choice of $(\varphi_j)_{j \in \mathbb{N}_0}$.

The space $H_2^s(\mathbb{R}^2) = B_{22}^s(\mathbb{R}^2)$ is the fractional Sobolev space.

Function spaces of $F_{pq}^s(\mathbb{R}^2)$ type, $0 < p < \infty$, are defined changing the roles of the spaces $L_p(\mathbb{R}^2)$ and l_q in the definition above but we do not stress this point here.

A systematic treatment of the theory of $B_{pq}^s(\mathbb{R}^2)$ and $F_{pq}^s(\mathbb{R}^2)$ spaces may be found in the books [Tri83] and [Tri92]; for a more recent account we refer also to [EdT96] and [RuS96]. These two scales of function spaces include many well-known classical spaces such as Sobolev spaces, Hölder - Zygmund spaces and inhomogeneous Hardy spaces.

The structure theorem. If Γ is a closed set with Lebesgue measure $|\Gamma| = 0$ and if $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$ then we define

$$B_{pq}^{s,\Gamma}(\mathbb{R}^2) = \{f \in B_{pq}^s(\mathbb{R}^2) : f(\varphi) = 0 \text{ if } \varphi \in S(\mathbb{R}^2), \varphi|_\Gamma = 0\}$$

where $\varphi|_\Gamma$ is the restriction of φ to Γ .

We have $\text{supp } f \subset \Gamma$ if $f \in B_{pq}^{s,\Gamma}(\mathbb{R}^2)$ in any case. Remark also that if $0 < p \leq \infty$, $0 < q \leq \infty$ and $s > 2(\frac{1}{p} - 1)_+$ (if $b \in \mathbb{R}$ then $b_+ = \max(b, 0)$) then $B_{pq}^s(\mathbb{R}^2) \hookrightarrow L_1^{loc}(\mathbb{R}^2)$ ([Tri92, Remark 2.3.2/3]) and, hence, $B_{pq}^{s,\Gamma}(\mathbb{R}^2) = \{0\}$ is trivial.

In other words, only values $s \leq 2(\frac{1}{p} - 1)_+$ (in particular $s \leq 0$ if $1 \leq p \leq \infty$) are of interest.

Let Γ be the regular anisotropic fractal constructed above. The L_p -spaces on Γ , $0 < p \leq \infty$, are introduced in the usual way with respect to the underlying Radon measure μ on Γ according to Theorem 2.3.

If $1 \leq p \leq \infty$ then any $f_\Gamma \in L_p(\Gamma)$ can be interpreted as a tempered distribution $f \in S'(\mathbb{R}^2)$ given by

$$f(\varphi) = \int_\Gamma f_\Gamma(\gamma) (\varphi|_\Gamma)(\gamma) d\mu(\gamma), \quad \varphi \in S(\mathbb{R}^2). \quad (2.10)$$

Theorem 2.9 *Let Γ be a regular anisotropic fractal with upper mass concentration factor λ^+ and affine dimension d_A . If $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$ then (in the sense of (2.10))*

$$L_p(\Gamma) \hookrightarrow B_{p\infty}^{-\frac{2-d(\lambda^+)}{p'}}(\mathbb{R}^2) \quad \text{where} \quad d(\lambda^+) = \frac{d_A + 2a(1 - \lambda^+)}{1 + a}. \quad (2.11)$$

Moreover, if Γ is strongly regular and if $1 < p \leq \infty$ then (in the sense of (2.10))

$$L_p(\Gamma) = B_{p\infty}^{-\frac{2-d}{p}, \Gamma}(\mathbb{R}^2) \quad \text{where} \quad d = \frac{d_A + 2a}{1+a}. \quad (2.12)$$

Proof. We consider a square $Q(x, t)$ centred at $x \in \mathbb{R}^2$ and with side length $2^{-\nu\kappa(1-a)}$; by the right-hand side of (2.9) we have $\mu(\Gamma \cap Q(x, t)) \leq c t^{d(\lambda^+)}$.

To prove (2.11) one has now to follow the lines of the proof of [Tri97, Theorem 18.15/ Step 1] making the above modification.

If Γ is strongly regular then $d(\lambda^+) = d$ and Γ becomes an isotropic d -set; then (2.12) is in fact [TrW96, Theorem 2/(8)].

Traces. Assume that Γ is a regular anisotropic fractal; if $\varphi \in S(\mathbb{R}^2)$ then $tr_\Gamma \varphi = \varphi|_\Gamma$ makes sense pointwise. If $0 < p, q < \infty$ and $s \in \mathbb{R}$ then $tr_\Gamma B_{pq}^s(\mathbb{R}^2) \hookrightarrow L_p(\Gamma)$ must be understood as follows: there exists a positive number $c > 0$ such that for any $\varphi \in S(\mathbb{R}^2)$

$$\|tr_\Gamma \varphi\|_{L_p(\Gamma)} \leq c \|\varphi\|_{B_{pq}^s(\mathbb{R}^2)}.$$

Since $S(\mathbb{R}^2)$ is dense in $B_{pq}^s(\mathbb{R}^2)$ this inequality can be extended by completion to any $f \in B_{pq}^s(\mathbb{R}^2)$ and the resulting function is denoted $tr_\Gamma f$.

In addition, the equality $tr_\Gamma B_{pq}^s(\mathbb{R}^2) = L_p(\Gamma)$ means that any $f_\Gamma \in L_p(\Gamma)$ is the trace of a suitable $g \in B_{pq}^s(\mathbb{R}^2)$ on Γ and

$$\|f_\Gamma\|_{L_p(\Gamma)} \sim \inf \{\|g\|_{B_{pq}^s(\mathbb{R}^2)} : tr_\Gamma g = f_\Gamma\}.$$

Theorem 2.10 *Let Γ be a regular anisotropic fractal with upper mass concentration factor λ^+ and affine dimension d_A . If $1 \leq p \leq \infty$ then*

$$tr_\Gamma B_{p1}^{\frac{2-d(\lambda^+)}{p}}(\mathbb{R}^2) \hookrightarrow L_p(\Gamma) \quad \text{where} \quad d(\lambda^+) = \frac{d_A + 2a(1 - \lambda^+)}{1+a}. \quad (2.13)$$

Moreover, if Γ is strongly regular and if $1 \leq p < \infty$ then

$$tr_\Gamma B_{p1}^{\frac{2-d}{p}}(\mathbb{R}^2) = L_p(\Gamma) \quad \text{where} \quad d = \frac{d_A + 2a}{1+a}. \quad (2.14)$$

Proof. If $p = \infty$ we have $B_{\infty 1}^0(\mathbb{R}^2) \hookrightarrow C(\mathbb{R}^2)$ and (2.13) is obvious (here $C(\mathbb{R}^2)$ is the space of all uniformly continuous bounded functions on \mathbb{R}^2). To prove (2.13) for $p < \infty$ one has to repeat the arguments from [Tri97, Theorem 18.15/ Step 2] with $d(\lambda^+)$ instead of $\frac{d_A}{1+a}$. In addition, (2.14) is [TrW96, Theorem 2/(9)].

Theorems 2.9 and 2.10 pave the way to our main result (which is presented in the next Section) but we hope they are also of independent interest. They are the anisotropic counterparts of Theorems 2 and 3 in [TrW96], see also [Tri97, 18.2, 18.6], and complement the results from [Tri97, 18.15, 18.17].

3 The main result

As usual, Ω stands for a bounded domain in \mathbb{R}^2 with C^∞ boundary and $D'(\Omega)$ denotes the space of all complex-valued distributions on Ω .

Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$; the space $B_{pq}^s(\Omega)$ is defined as the restriction of $B_{pq}^s(\mathbb{R}^2)$ to Ω , that means

$$B_{pq}^s(\Omega) = \{f \in D'(\Omega) : \text{there exists a } g \in B_{pq}^s(\mathbb{R}^2) \text{ with } g|_\Omega = f\},$$

$$\|f|_{B_{pq}^s(\Omega)}\| = \inf \|g|_{B_{pq}^s(\mathbb{R}^2)}\|$$

where the infimum is taken over all $g \in B_{pq}^s(\mathbb{R}^2)$ such that its restriction to Ω , denoted by $g|_\Omega$, coincides in $D'(\Omega)$ with f . In particular $B_{22}^1(\Omega) = W_2^1(\Omega)$.

In the sequel $\Gamma \subset \Omega$ will be a regular anisotropic fractal and we shall not distinguish between f_Γ as an element of some $L_p(\Gamma)$ and as the distribution belonging to some $B_{p\infty}^{-s}(\Omega)$ according to (2.11).

To avoid any misunderstanding we emphasise that the trace operator has two different meanings which we distinguish by tr_Γ and tr^Γ if extra clarity is desirable. If, for example, $1 < p < \infty$, then

$$tr_\Gamma : B_{p1}^{\frac{2-d(\lambda^+)}{p}}(\Omega) \rightarrow L_p(\Gamma) \quad (3.1)$$

by (2.13) and

$$tr^\Gamma : B_{p1}^{\frac{2-d(\lambda^+)}{p}}(\Omega) \rightarrow B_{p\infty}^{-\frac{2-d(\lambda^+)}{p'}}(\Omega) \quad (3.2)$$

if one applies in addition (2.11). The latter can be rephrased asking for an optimal extension of tr^Γ considered as a mapping from $D(\Omega)$ into $D'(\Omega)$ given by (1.1).

Recall $(-\Delta)^{-1}$ stands for the inverse of the Dirichlet Laplacian in Ω .

Theorem 3.1 *Let Ω be a bounded domain in \mathbb{R}^2 with C^∞ boundary. Let $\Gamma \subset \Omega$ be a regular anisotropic fractal having respectively upper and lower mass concentration factors λ^+ and λ^- according to (2.6) and having affine dimension d_A according to (2.4).*

Let

$$d(\lambda^-) = \frac{d_A + 2a(1 - \lambda^-)}{1 + a} \quad \text{and} \quad d(\lambda^+) = \frac{d_A + 2a(1 - \lambda^+)}{1 + a}$$

and

$$d = \frac{d_A + 2a}{1 + a}. \quad (3.3)$$

Let tr^Γ be the trace operator in the interpretation (3.2) and (1.1) whereas tr_Γ stands for the trace operator according to (3.1).

(i) *The operator $T = (-\Delta)^{-1} \circ tr^\Gamma$ is compact, non-negative, self adjoint in $\mathring{W}_2^1(\Omega)$, has null space $N(T) = \{f \in \mathring{W}_2^1(\Omega) : tr_\Gamma f = 0\}$ and is generated by the quadratic form in $\mathring{W}_2^1(\Omega)$*

$$\int_\Gamma f(\gamma) \overline{g(\gamma)} d\mu(\gamma) = (Tf, g)_{W_2^1(\Omega)} \quad \text{where } f \in \mathring{W}_2^1(\Omega), g \in \mathring{W}_2^1(\Omega) \quad (3.4)$$

and μ is the Radon measure according to Theorem 2.3.

(ii) There exist constants $c_1, c_2 > 0$ such that the positive eigenvalues $\lambda_k(T)$ of T , repeated according to multiplicity and ordered by their magnitude, can be estimated by

$$c_1 k^{-\frac{d(\lambda^-)}{d}} \leq \lambda_k(T) \leq c_2 k^{-\frac{d(\lambda^+)}{d}}, \quad k \in \mathbb{N}. \quad (3.5)$$

Furthermore, if Γ is strongly regular then there are constants $c_1, c_2 > 0$ such that $c_1 k^{-1} \leq \lambda_k(T) \leq c_2 k^{-1}$.

The proof of the theorem is shifted to Section 5 but we make here some comments.

Remark 3.2 According to Definition 2.4 and to Remark 2.5 we have $0 \leq \lambda^+ < 1$ and this implies $\frac{d(\lambda^+)}{d} > \frac{d_A}{d_A+2a}$. Hence the estimate from above in (3.5) is an improvement of the estimate from above in [Tri97, 30.7], see (1.2).

Remark 3.3 If $-1 < \lambda^- \leq 0$ then $\frac{d(\lambda^-)}{d} < \frac{d_A+2a}{d_A}$ and so the estimate from below obtained in (3.5) is better than the estimate from below in [Tri97, 30.7], see (1.2).

Remark 3.4 By [Tri97, 30.2] (isotropic fractal drum) it is not a surprise that if $\lambda^+ = \lambda^- = 0$ then the two exponents in (3.5) are both -1 since in this case the regular anisotropic fractal Γ becomes an isotropic d -set, where d is the number from (3.3).

4 Prerequisites for the proof

4.1 Basic facts about entropy numbers and approximation numbers

Let A and B two quasi-Banach spaces and let $T : A \rightarrow B$ be linear. Just as for the Banach space case, T will be called bounded or continuous if

$$\|T\| := \sup \{ \|Tx\|_B : x \in A, \|x\|_A \leq 1 \} < \infty.$$

The family of all such T will be denoted by $L(A, B)$ or $L(A)$ if $A = B$.

Otherwise terminology which is standard in the context of Banach spaces will be taken over without special comments to the quasi-Banach situation.

If B is a quasi-Banach space then $U_B = \{b \in B : \|b\|_B \leq 1\}$ stands for the unit ball in B .

Definition 4.1 Let A, B be two quasi-Banach spaces and let $T \in L(A, B)$. Then for all $k \in \mathbb{N}$, the k th entropy number $e_k(T)$ of T is defined by

$$e_k(T) = \inf \left\{ \varepsilon > 0 : T(U_A) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon U_B) \text{ for some } b_1, \dots, b_{2^{k-1}} \in B \right\}.$$

This formulation, which simply generalises to quasi-Banach spaces what has been done before for Banach spaces, coincides with the definition given in [EdT96, 1.3.1] where the reader can find further comments and historical references.

Lemma 4.2 Let A, B, C be quasi-Banach spaces, let $T \in L(A, B)$ and let $V \in L(B, C)$.

(i) $\|T\| \geq e_1(T) \geq e_2(T) \geq \dots \geq 0$; $e_1(T) = \|T\|$ if B is a Banach space.

(ii) For all $k, l \in \mathbb{N}$,

$$e_{k+l-1}(V \circ T) \leq e_k(V) e_l(T). \quad (4.1)$$

A proof can be found in [EdT96, Lemma 1.3.1/1]. In case of quasi-Banach spaces it may happen that $\|T\| > e_1(T)$.

Remark 4.3 Since the numbers $e_k(T)$ decrease as k increases and are non-negative,

$\lim_{k \rightarrow \infty} e_k(T)$ exists and plainly equals

$$\inf \{ \varepsilon > 0 : T(U_A) \text{ can be covered by finitely } B\text{-balls of radius } \varepsilon \}.$$

Recall that $T \in L(A, B)$ is compact if, and only if, for every $\varepsilon > 0$ there is a finite ε -net in B covering $T(U_A)$. Hence $T \in L(A, B)$ is compact if, and only if, $\lim_{k \rightarrow \infty} e_k(T) = 0$.

Let A be a complex quasi-Banach space and $T \in L(A)$ a compact map. We know from [EdT96, Theorem 1.2] that the spectrum of T , apart from the point 0, consists solely of eigenvalues of finite algebraic multiplicity: let $\{\lambda_k(T) : k \in \mathbb{N}\}$ be the sequence of all non-zero eigenvalues of T , repeated according to algebraic multiplicity and ordered so that

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \dots \geq 0. \quad (4.2)$$

If T has only $m (< \infty)$ distinct eigenvalues and M is the sum of their algebraic multiplicities, we put $\lambda_k(T) = 0$ for $k > M$.

Perhaps the most useful connection for our purposes between the eigenvalues of the operator T and its entropy numbers is the following:

Theorem 4.4 *Let T and $\{\lambda_k(T) : k \in \mathbb{N}\}$ as above. Then*

$$|\lambda_k(T)| \leq \sqrt{2} e_k(T). \quad (4.3)$$

A proof of this result, originally proved by B. Carl in [Carl81], see also [CaT80], is given in [EdT96, Theorem 1.3.4].

Definition 4.5 *Let A, B be two quasi-Banach spaces and let $T \in L(A, B)$. Then given any $k \in \mathbb{N}$, the k th approximation number $a_k(T)$ of T is defined by*

$$a_k(T) = \inf \{ \|T - L\| : L \in L(A, B), \text{rank } L < k \}$$

where $\text{rank } L$ is the dimension of the range of L .

These numbers have various properties similar to those of the entropy numbers: we have $\|T\| = a_1(T) \geq a_2(T) \geq \dots \geq 0$ and a counterpart of the multiplication property (4.1) for entropy numbers, see [EdT96, Lemma 1.3.1/2].

On the other hand there are radical differences between entropy numbers and approximation numbers, see [EdT96, Remark 1.3.2/6] and [EEv87, II.2.3], but we do not go into further details.

The approximation numbers have important connections with eigenvalues, the picture being clearest in a Hilbert space setting.

Theorem 4.6 *Let H be a Hilbert space and let $T \in L(H)$ be a compact, non-negative and self adjoint operator. Then the approximation numbers $a_k(T)$ of T coincide with its eigenvalues (ordered as in (4.2)).*

A proof can be found in [EEv87, II.5.10], see also [EdT96, p.21].

4.2 Compactness of embeddings into $L_p(\Gamma)$

We will use the following improvement of [Tri97, 22.2].

Theorem 4.7 *Let Γ be a regular anisotropic fractal having upper mass concentration factor λ^+ , affine dimension d_A and let $d = \frac{d_A+2a}{1+a}$.*

Let $0 < p_1 \leq p_2 \leq \infty$, $0 < q \leq \infty$ and

$$s(p_1, p_2) = \frac{2}{p_1} - \frac{d(\lambda^+)}{p_2}. \quad (4.4)$$

If $s > 0$ then the trace operator

$$tr_\Gamma : B_{p_1 q}^{s(p_1, p_2)+s}(\mathbb{R}^2) \rightarrow L_{p_2}(\Gamma)$$

is compact and there exists a constant $c > 0$ such that for any $k \in \mathbb{N}$,

$$e_k(tr_\Gamma : B_{p_1 q}^{s(p_1, p_2)+s}(\mathbb{R}^2) \rightarrow L_{p_2}(\Gamma)) \leq c k^{-\frac{s}{d} - \frac{1}{p_1} + \frac{1}{p_2}}.$$

Proof. One has only to repeat the arguments from [Tri97, 22.2] using (4.4) instead of [Tri97, (22.1)].

5 Proof of Theorem 3.1

Step 1. Part (i) of the theorem is covered by [Tri97, Theorem 30.7]. In particular from [Tri97, (30.25)] it follows that there exists a constant $c > 0$ such that

$$\|tr_\Gamma f\|_{L_2(\Gamma)} \leq c \|f\|_{W_2^1(\Omega)} \quad \text{for any } f \in \mathring{W}_2^1(\Omega) \quad (5.1)$$

and from (3.4) we have

$$\|tr_\Gamma f\|_{L_2(\Gamma)} = \|\sqrt{T}f\|_{W_2^1(\Omega)} \quad (5.2)$$

where $\sqrt{T} = T^{1/2}$.

Step 2. To prove the estimate from above in (3.5) we factorise the operator T by $T = id_2 \circ (-\Delta)^{-1} \circ id_1 \circ tr_\Gamma$ with

$$\begin{aligned} tr_\Gamma & : \mathring{W}_2^1(\Omega) & \rightarrow & L_2(\Gamma) \\ id_1 & : L_2(\Gamma) & \rightarrow & B_{2\infty}^{-\frac{2-d(\lambda^+)}{2}}(\Omega) \\ (-\Delta)^{-1} & : B_{2\infty}^{-\frac{2-d(\lambda^+)}{2}}(\Omega) & \rightarrow & B_{2\infty}^{2-\frac{2-d(\lambda^+)}{2}}(\Omega) \\ id_2 & : B_{2\infty}^{2-\frac{2-d(\lambda^+)}{2}}(\Omega) & \rightarrow & \mathring{W}_2^1(\Omega). \end{aligned} \quad (5.3)$$

The boundedness of tr_Γ in the first line of (5.3) is (5.1) and the embedding id_1 is (2.11).

Recall $(-\Delta)$ maps any space $B_{pq,0}^s(\Omega) = \{g \in B_{pq}^s(\Omega) : tr_{\partial\Omega} g = 0\}$ onto $B_{pq}^{s-2}(\Omega)$ provided $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $s > \frac{1}{p}$. This is a consequence of [Tri78, 5.7.1, Remark 1] complemented by [Tri83, 4.3.3-4.3.4].

By this mapping property the boundedness of $(-\Delta)^{-1}$ as indicated in the third line in (5.3) is justified.

Finally, the embedding id_2 is a consequence of the elementary embedding from [Tri83, Proposition 2.3.2/2].

Let $f \in \overset{\circ}{W}_2^1(\Omega)$ be an eigenfunction of T . Then it follows that f belongs also to $B_{2\infty}^{2-\frac{2-d(\lambda^+)}{2}}(\Omega)$ and so it is also an eigenfunction of the operator T restricted to this space. Obviously the converse is also true.

Hence the root systems of T considered in $W_2^1(\Omega)$ (or $\overset{\circ}{W}_2^1(\Omega)$ which is the same in our context) and in $B_{2\infty}^{2-\frac{2-d(\lambda^+)}{2}}(\Omega)$ coincide. Then the eigenvalues of T considered in these spaces also coincide, inclusively their multiplicities.

Using the multiplication property (4.1) for entropy numbers and (5.3) there exists a constant $c > 0$ such that for all $k \in \mathbb{N}$,

$$e_k(T : B_{2\infty}^{2-\frac{2-d(\lambda^+)}{2}}(\Omega) \rightarrow B_{2\infty}^{2-\frac{2-d(\lambda^+)}{2}}(\Omega)) \leq c e_k(\text{tr}_\Gamma : B_{2\infty}^{2-\frac{2-d(\lambda^+)}{2}}(\Omega) \rightarrow L_2(\Gamma)). \quad (5.4)$$

From Theorem 4.7 (with $p_1 = p_2 = 2$) if $s > 0$ then there exists a constant $c > 0$ such that for all $k \in \mathbb{N}$,

$$e_k(\text{tr}_\Gamma : B_{2\infty}^{s+\frac{2-d(\lambda^+)}{2}}(\mathbb{R}^2) \rightarrow L_2(\Gamma)) \leq c k^{-\frac{s}{d}}. \quad (5.5)$$

Inserting $s = d(\lambda^+)$ in (5.5) and using Carl's inequality (4.3), from (5.4) we obtain the estimate from above in (3.5).

Step 3. We prove that there exists a constant $c > 0$ such that the approximation numbers $a_k(\sqrt{T})$ of \sqrt{T} can be estimated by

$$a_k(\sqrt{T}) \geq c k^{-\frac{d(\lambda^-)}{2d}}, \quad k \in \mathbb{N}. \quad (5.6)$$

The estimate from below in (3.5) is then a simple application of Theorem 4.6.

We rely on Lemma 2.6 and Figure 2 assuming again (2.7) without restriction of generality. By (2.4) we have $N = 2^{\kappa d_A}$ and hence by (3.3) for any $j \in \mathbb{N}$ there are

$$2^{2j\kappa a} N^j = 2^{2j\kappa a + j\kappa d_A} = 2^{j\kappa(1+a)d}$$

squares E_{jl} (of side length $2^{-j\kappa(1+a)}$). We put $m = m(j) = j\kappa(1+a)$ and denote the corresponding squares E_l^m . Of course $m = m(j)$ need not to be a natural number but this is immaterial for what follows. In other words, the disjoint squares E_l^m ($l = 1, \dots, 2^{m/d}$) of side length 2^{-m} cover Γ and originate from the squares E_{jl} in Lemma 2.6.

Let $x^{m,l}$ be the centre of the square E_l^m .

Let φ be an appropriately chosen non-trivial C^∞ function on \mathbb{R}^2 supported near the origin. If

$$\varphi_{ml}(x) = \varphi\left(2^m(x - x^{m,l})\right)$$

then $\text{supp } \varphi_{ml} \subset E_l^m$. Furthermore, as a simple consequence of (2.9), there exists a constant $c > 0$ such that

$$\left\| \sum_{l=1}^{2^{m/d}} c_{ml} \varphi_{ml} \mid L_2(\Gamma) \right\| \geq c 2^{-md(\lambda^-)/2} \left(\sum_{l=1}^{2^{m/d}} |c_{ml}|^2 \right)^{1/2} \quad (5.7)$$

for any complex numbers c_{ml} and for any m and $l = 1, \dots, 2^{md}$.

By (5.2) and (5.7) there exists a constant $c > 0$ such that

$$\|\sqrt{T}g_m | W_2^1(\Omega)\| \geq c 2^{-md(\lambda^-)/2} \left(\sum_{l=1}^{2^{md}} |c_{ml}|^2 \right)^{1/2} \quad (5.8)$$

for any function g_m of type

$$g_m = \sum_{l=1}^{2^{md}} c_{ml} \varphi_{ml} \quad ,$$

the constant $c > 0$ in (5.8) being independent of m , l and of the complex numbers c_{ml} .

Then there exists a number $c > 0$ which is independent of m such that

$$a_{2^{md}}(\sqrt{T}) \geq c 2^{-md(\lambda^-)/2}. \quad (5.9)$$

Using elementary properties of approximation numbers it is easy to see that (5.9) implies (5.6) and this completes the proof of the estimate from below in (3.5).

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Appendix 2: Atomic and subatomic decompositions in anisotropic function spaces

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Abstract. This work deals with decompositions in anisotropic function spaces. Defining anisotropic atoms as smooth building blocks which are the counterpart of the atoms from the works of M. Frazier and B. Jawerth, it is shown that the study of anisotropic function spaces can be done with the help of some sequence spaces in a similar way as it is done in the isotropic case. It is also shown that the subatomic decomposition theorem for isotropic function spaces, recently proved by H. Triebel, can be extended to the anisotropic case if the mean smoothness parameter is sufficiently large.

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1 Introduction

If $1 < p < \infty$ and (s_1, \dots, s_n) is an n -tuple of natural numbers then

$$W_p^{s,a}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{L_p(\mathbb{R}^n)} + \sum_{k=1}^n \left\| \frac{\partial^{s_k} f}{\partial x_k^{s_k}} \right\|_{L_p(\mathbb{R}^n)} < \infty \right\}$$

is the classical anisotropic Sobolev space on \mathbb{R}^n . In contrast to the usual (isotropic) Sobolev space ($s_1 = \dots = s_n$) the smoothness properties of an element from $W_p^{s,a}(\mathbb{R}^n)$ depend on the chosen direction in \mathbb{R}^n . The number s defined by

$$\frac{1}{s} = \frac{1}{n} \left(\frac{1}{s_1} + \dots + \frac{1}{s_n} \right)$$

is usually called the "mean smoothness" and $a = (a_1, \dots, a_n)$, where $a_1 = s/s_1, \dots, a_n = s/s_n$, characterises the anisotropy.

Anisotropic Bessel potential spaces, or fractional Sobolev spaces, defined by

$$H_p^{s,a}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \left\| \left(\sum_{k=1}^n (1 + \xi_k^2)^{s/(2a_k)} \widehat{f} \right)^\vee \right\|_{L_p(\mathbb{R}^n)} < \infty \right\}$$

where $1 < p < \infty$, $s \in \mathbb{R}$ and $a = (a_1, \dots, a_n)$ is a given anisotropy, generalise in a natural way the above spaces (as usual, $S'(\mathbb{R}^n)$ is the space of tempered distributions and \widehat{f} , \check{f} are respectively the Fourier and the inverse Fourier transform of f).

Similar to the isotropic case, the study of anisotropic Bessel potential spaces $H_p^{s,a}(\mathbb{R}^n)$ for a fixed anisotropy $a = (a_1, \dots, a_n)$, is a part of the more general theory of the spaces of $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ type. Spaces of that type (or on domains in \mathbb{R}^n) have been studied in great detail by S. M. Nikol'skij, [Nik77], and by O. V. Besov, V. P. Il'in and S. M. Nikol'skij, [BIN75], and it is well known that this theory has a more or less full counterpart to the basic facts (definitions, elementary properties, embeddings for different metrics, interpolation) of isotropic spaces $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$ as it was presented in the works of H. Triebel, [Tri83] and [Tri92].

The anisotropic function spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ are defined in terms of Fourier analytical quasi-norms: any function $f \in S'(\mathbb{R}^n)$ is decomposed in a sum of entire analytic functions $(\varphi_j \widehat{f})^\vee$ and this decomposition is used to introduce the spaces.

Hence, as in the isotropic case, entire analytic functions may be considered as building blocks for the spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ in the following sense: let a problem be given, for example mapping properties for PDE's or ψ DE's between spaces of the above type or traces on hyperplanes etc. First one asks what happens when the problem is applied to entire analytic functions; then the rest reduces to a discussion of convergence.

In the theory of isotropic function spaces there is a well known other type of decomposition in simple building blocks, the so-called atoms. They have a history of some twenty years and in [Tri92, 1.9], cf. also [AdH96], a historical report was given on this topic; we do not repeat it here. We only want to mention that the (smooth) atoms in isotropic $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$ spaces as they were defined by M. Frazier and B. Jawerth in [FrJ85], [FrJ90] (cf. also [FJW91]), proved to be a powerful tool in the theory of function spaces. We also wish to emphasise that there exist many other types of atomic decompositions in isotropic spaces but we will not discuss this point here.

More information about this subject is given in [FrJ90], [Tri92] and [AdH96] where one can find many modifications and applications as well as comprehensive references extending the subject.

Several authors were concerned in the last years with the problem of obtaining useful decompositions of anisotropic function spaces in simple building blocks: a construction of unconditional bases in $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$ spaces using Meyer wavelets was done by M. Z. Berkolaiko and I. Ya. Novikov in [BeN93] (and used then in [BeN95]). P. Dintelmann obtained in [Din95a, Theorem 1] a decomposition for anisotropic function spaces which is the counterpart of the characterisation of isotropic function spaces with the help of the φ -transform of M. Frazier and B. Jawerth (see [FrJ90] and the survey [FJW91]) and used it in connection with the theory of Fourier- multipliers for anisotropic function spaces (we will return to his result in Section 5). Our approach will be different, especially from the point of view of the localisation of the building blocks.

The first aim of this paper is to introduce smooth anisotropic atoms and to obtain a decomposition theorem which extends the atomic decomposition theorem of M. Frazier and B. Jawerth, see [FrJ85] and [FrJ90], to the anisotropic function spaces $B_{pq}^{s,a}(\mathbb{R}^n)$ and $F_{pq}^{s,a}(\mathbb{R}^n)$. Roughly speaking, we will show that for any $g \in F_{pq}^{s,a}(\mathbb{R}^n)$ it is possible to find a decomposition

(convergence in $S'(\mathbb{R}^n)$)

$$g = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \rho_{\nu m}^a, \quad (1.10)$$

where $\rho_{\nu m}^a$ are the anisotropic atoms and $\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ belongs to an appropriate sequence space f_{pq}^a , such that

$$\|g | F_{pq}^{s,a}(\mathbb{R}^n)\| \sim \|\lambda | f_{pq}^a\| \quad (1.11)$$

(and a similar assertion for $B_{pq}^{s,a}(\mathbb{R}^n)$ spaces).

Hence the study of function spaces can be done with the help of some sequence spaces in an analogous way as it is done in the isotropic case in the above cited works of M. Frazier and B. Jawerth. The necessary explanations and details are given in Section 3.

However in (1.10) (and in Theorem 3.3) no information is given about the possibility to obtain atomic decompositions in which the atoms are constructed with the help of (anisotropic) dilatations and translations from one smooth function ρ having compact support, cf. also [BeN93, Comment 2].

For isotropic function spaces this was already done by M. Frazier and B. Jawerth, see [FrJ90, 4.2], and W. Sickel, see [Sic90]. It might be possible to extend the technique of W. Sickel, at least for large values of the smoothness parameter, using the characterisation of anisotropic function spaces via oscillation from the work of A. Seeger, [See89]. But to construct such a basic (or mother) function ρ for the atoms having all required properties seems to be not very easy, at least at the first glance, see the above cited papers.

We arrive at the second aim of this paper, the subatomic (or quarkonial) decomposition theorem (Theorem 3.7) which states that given $g \in F_{pq}^{s,a}(\mathbb{R}^n)$ (with s sufficiently large) it is possible to obtain the decomposition

$$g = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^{\beta} (\beta q u)_{\nu m}^a, \quad (1.12)$$

convergence being in $S'(\mathbb{R}^n)$, with

$$\|g | F_{pq}^{s,a}(\mathbb{R}^n)\| \sim \sup_{\beta \in \mathbb{N}_0^n} 2^{r a \beta} \|\lambda^{\beta} | f_{pq}^a\|, \quad (1.13)$$

where $r > 0$ is large enough, $a\beta = a_1\beta_1 + \dots + a_n\beta_n$ if β is the multi-index $(\beta_1, \dots, \beta_n)$, $\lambda^{\beta} = \{\lambda_{\nu m}^{\beta} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ and where each $(\beta q u)_{\nu m}^a$ is an extremely simple building block (in particular an anisotropic atom without moment conditions), called anisotropic quark, compactly supported and which can be obtained starting from one smooth function (and a corresponding assertion for $B_{pq}^{s,a}(\mathbb{R}^n)$).

Of course in (1.12) there are infinitely many sums over $(\nu, m) \in \mathbb{N}_0 \times \mathbb{Z}^n$ but this is well compensated by (1.13) with r large. Furthermore, it turns out from the proof that the dependence of the coefficients $\lambda_{\nu m}^{\beta}$ on g is linear.

Isotropic quarks were recently introduced by H. Triebel in [Tri97] and the subatomic (quarkonial) decomposition theorem he obtained in [Tri97, Chapter 14] proved to be a very useful ingredient for the estimation of entropy numbers of compact embeddings between function

spaces on fractals. Compared with the results in [Tri97, Chapter 14] our Theorem 3.7 is in fact the extension of quarkonial decompositions to anisotropic function spaces in the case of large values of the smoothness parameter, in particular for $s > 0$ if $p \geq 1$ and $q \geq 1$.

If one wishes to extend the result to all $s \in \mathbb{R}$ then a lifting argument would be needed. But while the lift operator $(id - \Delta)^{\frac{L+1}{2}}$ between isotropic function spaces causes no problem in keeping the localisation of the (isotropic) quarks (and this fact was essentially used in [Tri97, 14.4]) the situation becomes difficult in case of the anisotropic lift operator. It is well known that if $\sigma \in \mathbb{R}$ then the operator

$$I_\sigma(f) = \left(\left(\sum_{k=1}^n (1 + \xi_k^2)^{\frac{1}{2a_k}} \right)^\sigma \widehat{f} \right)^\vee$$

maps $F_{pq}^{s,a}(\mathbb{R}^n)$ isomorphically onto $F_{pq}^{s-\sigma,a}(\mathbb{R}^n)$ and $\|I_\sigma(\cdot) | F_{pq}^{s-\sigma,a}(\mathbb{R}^n)\|$ is an equivalent quasi-norm on $F_{pq}^{s,a}(\mathbb{R}^n)$ (similar result for $B_{pq}^{s,a}(\mathbb{R}^n)$), see [Leo86]; it is clear that I_σ causes a lot of troubles in keeping the localisation of the anisotropic quarks and this is the reason why we will restrict ourselves to large values of the smoothness parameter.

Briefly about the organising of the manuscript. In Section 2 we set up notation and terminology and summarise some basic facts on anisotropic function spaces. In Section 3 the main results are presented with comments but without proofs. Section 4 will be concerned with the extension to anisotropic function spaces of some powerful tools (especially a theorem on local means) from the isotropic case. All these results are used in Section 5 where we prove the results announced in Section 3.

2 Definitions and basic facts

2.1 Notation

As usual, \mathbb{R}^n denotes the n -dimensional real euclidean space, \mathbb{N} are the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and \mathbb{C} stands for the complex numbers.

If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is a multi-index its length is $|\alpha| = \alpha_1 + \dots + \alpha_n$, the derivatives D^α have the usual meaning and if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ then $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing C^∞ functions on \mathbb{R}^n equipped with the usual topology. By $S'(\mathbb{R}^n)$ we denote its topological dual, the space of all tempered distributions on \mathbb{R}^n . If $\varphi \in S(\mathbb{R}^n)$ then $\widehat{\varphi} = F\varphi$ and $\check{\varphi} = F^{-1}\varphi$ are respectively the Fourier and inverse Fourier transform of φ . One extends F and F^{-1} in the usual way from $S(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$.

We adopt here and in the sequel the following convention: if there is no danger of confusion we omit \mathbb{R}^n in $S(\mathbb{R}^n)$ and in the other spaces below.

For a normed or quasi-normed space X we denote by $\|x | X\|$ the norm of the vector x . Recall that X is quasi-normed when the triangle inequality is weakened to $\|x + y | X\| \leq c(\|x | X\| + \|y | X\|)$ for some $c \geq 1$ independent of x and y .

If $0 < p \leq \infty$ then L_p denotes the usual Lebesgue space on \mathbb{R}^n quasi-normed by $\|\cdot | L_p\|$.

All unimportant positive constants are denoted with c , occasionally with additional subscripts within the same formulas. The equivalence "term₁ \sim term₂" means that there exist two constants $c_1, c_2 > 0$ such that $c_1 \text{ term}_1 \leq \text{term}_2 \leq c_2 \text{ term}_1$.

2.2 Anisotropic distance functions

Through the whole work $n \geq 2$ and $a = (a_1, \dots, a_n)$ will designate a given anisotropy, that is a fixed n -tuple of positive numbers with $a_1 + \dots + a_n = n$. We will denote $a_{\min} = \min\{a_i : 1 \leq i \leq n\}$ and $a_{\max} = \max\{a_i : 1 \leq i \leq n\}$. If $a = (1, \dots, 1)$ we speak about the "isotropic case".

The action of $t \in [0, \infty)$ on $x \in \mathbb{R}^n$ is defined by the formula:

$$t^a x = (t^{a_1} x_1, \dots, t^{a_n} x_n). \quad (2.1)$$

For $t > 0$ and $s \in \mathbb{R}$ we put $t^{sa} x = (t^s)^a x$. In particular we write $t^{-a} x = (t^{-1})^a x$ and $2^{-ja} x = (2^{-j})^a x$.

Definition 2.1 *An anisotropic distance function is a continuous function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ with the properties $u(x) > 0$ if $x \neq 0$ and $u(t^a x) = tu(x)$ for all $t > 0$ and all $x \in \mathbb{R}^n$.*

It is easy to see that $u_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$u_\lambda(x) = \left(\sum_{i=1}^n |x_i|^{\lambda/a_i} \right)^{1/\lambda} \quad (2.2)$$

is an anisotropic distance function for every $0 < \lambda < \infty$. If $x \in \mathbb{R}^n$ then $u_2(x)$ is usually called the anisotropic distance of x to the origin, see [ScT87, 4.2.1].

It is well known, see [Din95b, 1.2.3] and [Yam86, 1.4], that any two anisotropic distance functions u and u' are equivalent (in the sense that there exist constants $c, c' > 0$ such that $c u(x) \leq u'(x) \leq c' u(x)$ for all $x \in \mathbb{R}^n$) and that if u is an anisotropic distance function then there exists a constant $c > 0$ such that $u(x+y) \leq c(u(x) + u(y))$ for all $x, y \in \mathbb{R}^n$.

We are interested to use smooth anisotropic distance functions. Remark that for appropriate values of λ we can obtain arbitrary (finite) smoothness of the function u_λ from above (cf. [Din95b, 1.2.4]). A standard method concerning the construction of anisotropic distance functions in $C^\infty(\mathbb{R}^n \setminus \{0\})$ was given by E. M. Stein and S. Wainger in [StW78].

The lemma below will play an essential role in our considerations. Given the anisotropy $a = (a_1, \dots, a_n)$ and the multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ we use the notation $a\alpha = a_1\alpha_1 + \dots + a_n\alpha_n$.

Lemma 2.2 *There exists an anisotropic distance function $|\cdot|_a \in C^\infty(\mathbb{R}^n \setminus \{0\})$ with the following property: for any real number s and for any multi-index α there exists a constant $c = c(s, \alpha) > 0$ such that*

$$|D^\alpha (|x|_a^s)| \leq c |x|_a^{s-a\alpha} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \quad (2.3)$$

Proof. We have only to recall the construction of M. Yamazaki from [Yam86, 1.4/3,8]: for $x \neq 0$ one can define $|x|_a$ as the unique positive number t such that

$$\frac{x_1^2}{t^{2a_1}} + \dots + \frac{x_n^2}{t^{2a_n}} = 1 \quad (2.4)$$

and then put $|0|_a = 0$ for $x = 0$.

Given the anisotropy $a = (a_1, \dots, a_n)$ through the whole work we will keep the notation $|\cdot|_a$ for a fixed anisotropic distance function as in the lemma above.

Let us remark that to work with an anisotropic distance function $|\cdot|_a$ satisfying (2.3) is natural since denoting $|\cdot|$ the euclidean distance in \mathbb{R}^n , for every real number s and for any multi-index α there exists a constant $c > 0$ such that $|D^\alpha(|x|^s)| \leq c|x|^{s-|\alpha|}$ for all $x \in \mathbb{R}^n \setminus \{0\}$ (see [RuS96, Lemma 2.3.1/(20)]).

2.3 Anisotropic function spaces

Let φ_0 a C^∞ function on \mathbb{R}^n , $\varphi_0(x) = 1$ if $|x|_a \leq 1$, $\text{supp } \varphi_0 \subset \{x \in \mathbb{R}^n : |x|_a \leq 2\}$ and $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-(j+1)}x)$ if $j \in \mathbb{N}$. Then $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ if $x \in \mathbb{R}^n$ and $(\varphi_j)_{j \in \mathbb{N}_0}$ is a smooth anisotropic dyadic resolution of unity, cf. [ScT87, 4.2].

For $f \in S'$ since $\varphi_j \widehat{f}$ is compactly supported the Paley - Wiener - Schwartz theorem provides that $(\varphi_j \widehat{f})^\vee$ is an entire analytic function on \mathbb{R}^n .

Definition 2.3 (i) Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$; then

$$B_{pq}^{s,a} = \left\{ f \in S' : \|f\|_{B_{pq}^{s,a}} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \widehat{f})^\vee\|_{L_p}^q \right)^{1/q} < \infty \right\} \quad (2.5)$$

(with the usual modification if $q = \infty$).

(ii) Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$; then

$$F_{pq}^{s,a} = \left\{ f \in S' : \|f\|_{F_{pq}^{s,a}} = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q \right)^{1/q} \right\|_{L_p} < \infty \right\} \quad (2.6)$$

(with the usual modification if $q = \infty$).

Of course the quasi-norms in (2.5) and (2.6) depend on the chosen system $(\varphi_j)_{j \in \mathbb{N}_0}$. But this is not the case for the spaces $B_{pq}^{s,a}$ and $F_{pq}^{s,a}$ (in the sense of equivalent quasi-norms) and that is the reason why we omit in our notation the subscript $(\varphi_j)_{j \in \mathbb{N}_0}$.

A systematic treatment of the theory of (isotropic) B_{pq}^s and F_{pq}^s spaces may be found in the works of H. Triebel [Tri83], [Tri92], for a more recent account of the theory we refer the reader also to [EdT96] and [RuS96]. A survey on the basic results for the (anisotropic) spaces $B_{pq}^{s,a}$ and $F_{pq}^{s,a}$ may be found in [ScT87, 4.2.1-4.2.4] and [Joh95, 2.1-2.2]. In this context we refer to the works of S. M. Nikol'skij [Nik77], O. V. Besov, V. P. Il'in and S. M. Nikol'skij [BIN75], B. Stöckert and H. Triebel [StT79], M. Yamazaki [Yam86], A. Seeger [See89], P. Dintelmann [Din95b, 1.2.8-1.2.10] etc.

An extension of (2.6) to $p = \infty$ is not reasonable; in [Tri92, 1.5.2] this point was discussed in detail.

Both $B_{pq}^{s,a}$ and $F_{pq}^{s,a}$ are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$).

As in the isotropic case, see [Tri83, 2.3.3]tri1, the embeddings $S \hookrightarrow B_{pq}^{s,a} \hookrightarrow S'$ and $S \hookrightarrow F_{pq}^{s,a} \hookrightarrow S'$ hold true for all admissible values of p, q, s . If $s \in \mathbb{R}$ and $0 < p < \infty$, $0 < q < \infty$ then S is dense in $B_{pq}^{s,a}$ and $F_{pq}^{s,a}$, see [Yam86, 3.5] and [Din95b, 1.2.10].

We want to point out that if $1 < p < \infty$, $s \in \mathbb{R}$ then $F_{p2}^{s,a}$ is the anisotropic Bessel potential space $H_p^{s,a}$; a proof can be found in [StT79, Remark 11] (see also [Tri77, 2.5.2]) and in [Yam86, 3.11].

It will be very useful to remark that if denoting for each $k \in \{1, \dots, n\}$, $s_k = s/a_k$ and

$$H_{x_k, p}^{s, a} = \left\{ f \in S' : \|f\|_{H_{x_k, p}^{s, a}} = \left\| \left((1 + \xi_k^2)^{s_k/2} \widehat{f} \right)^\vee \Big| L_p \right\| < \infty \right\} \quad (2.7)$$

by [Nik77, 9.1] we have (in the sense of equivalent quasi-norms):

$$H_p^{s, a} = \bigcap_{k=1}^n H_{x_k, p}^{s, a} \quad \text{and} \quad \|f\|_{H_p^{s, a}} = \sum_{k=1}^n \|f\|_{H_{x_k, p}^{s, a}} \quad (2.8)$$

and if $s_k \in \mathbb{N}$ then (in the sense of equivalent quasi-norms)

$$H_{x_k, p}^{s, a} = \left\{ f \in S' : \|f\|_{H_{x_k, p}^{s, a}} = \|f\|_{L_p} + \left\| \frac{\partial^{s_k} f}{\partial x_k^{s_k}} \Big| L_p \right\| < \infty \right\}. \quad (2.9)$$

In particular, if $s_1, \dots, s_n \in \mathbb{N}$ then (in the sense of equivalent quasi-norms) $H_p^{s, a} = W_p^{s, a}$ is the classical anisotropic Sobolev space.

3 The main results

3.1 Anisotropic atoms and the atomic decomposition theorem

Recall $a = (a_1, \dots, a_n)$ is a given anisotropy and let \mathbb{Z}^n be the lattice of all points in \mathbb{R}^n with integer-valued components. If $\nu \in \mathbb{N}_0$ and $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ we denote $Q_{\nu m}^a$ the rectangle in \mathbb{R}^n centred at $2^{-\nu a} m = (2^{-\nu a_1} m_1, \dots, 2^{-\nu a_n} m_n)$ which has sides parallel to the axes and side lengths respectively $2^{-\nu a_1}, \dots, 2^{-\nu a_n}$. Remark that Q_{0m}^a is a cube with side length 1. If $Q_{\nu m}^a$ is such a rectangle in \mathbb{R}^n and $c > 0$ then $cQ_{\nu m}^a$ is the rectangle in \mathbb{R}^n concentric with $Q_{\nu m}^a$ and with side lengths respectively $c2^{-\nu a_1}, \dots, c2^{-\nu a_n}$.

If E is a Lebesgue measurable subset of \mathbb{R}^n then $|E|$ denotes its Lebesgue measure; recall our notation: $a\alpha = a_1\alpha_1 + \dots + a_n\alpha_n$ where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index.

We are now prepared to introduce the anisotropic atoms.

Definition 3.1 (i) Let $K \in \mathbb{R}$, $c > 1$; a function $\rho : \mathbb{R}^n \rightarrow \mathbb{C}$ for which there exist all derivatives $D^\alpha \rho$ if $a\alpha \leq K$ (continuous if $K \leq 0$) is called an anisotropic 1_K -atom if:

$$\text{supp } \rho \subset cQ_{0m}^a \quad \text{for some } m \in \mathbb{Z}^n, \quad (3.1)$$

$$|D^\alpha \rho(x)| \leq 1 \quad \text{if } a\alpha \leq K. \quad (3.2)$$

(ii) Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $K, L \in \mathbb{R}$, $c > 1$; a function $\rho : \mathbb{R}^n \rightarrow \mathbb{C}$ for which there exist all derivatives $D^\alpha \rho$ if $a\alpha \leq K$ (continuous if $K \leq 0$) is called an anisotropic $(s, p)_{K, L}$ -atom if:

$$\text{supp } \rho \subset cQ_{\nu m}^a \quad \text{for some } \nu \in \mathbb{N}, m \in \mathbb{Z}^n, \quad (3.3)$$

$$|D^\alpha \rho(x)| \leq |Q_{\nu m}^a|^{\frac{s}{n} - \frac{1}{p} - \frac{a\alpha}{n}} \quad \text{if } a\alpha \leq K, \quad (3.4)$$

$$\int_{\mathbb{R}^n} x^\beta \rho(x) dx = 0 \quad \text{if } a\beta \leq L. \quad (3.5)$$

If the atom ρ is located at $Q_{\nu m}^a$ (that means $\text{supp } \rho_{\nu m}^a \subset cQ_{\nu m}^a$ with $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $c > 1$) then we will write it $\rho_{\nu m}^a$.

We give some technical explanations.

The value of the number $c > 1$ in (3.1) and (3.3) is unimportant. It simply makes clear that at the level ν some controlled overlapping of the supports of $\rho_{\nu m}^a$ must be allowed.

Since $|Q_{\nu m}^a| = 2^{-\nu n}$ condition (3.4) may be written as

$$|D^\alpha \rho(x)| \leq 2^{-\nu(s-\frac{n}{p})} 2^{\nu a \alpha} \quad \text{if } a\alpha \leq K \quad (3.6)$$

and if $K \leq 0$ then (3.4) is $|\rho(x)| \leq 2^{-\nu(s-\frac{n}{p})}$.

The moment conditions (3.5) can be reformulated as $D^\beta \hat{\rho}(0) = 0$ if $a\beta \leq L$, which shows that a sufficiently strong decay of $\hat{\rho}$ at the origin is required. If $L < 0$ then (3.5) simply means that there are no moment conditions.

The reason for the normalising factor in (3.2) and (3.4) is that there exists a constant $c > 0$ such that for all these atoms we have $\|\rho | B_{pq}^{s,a}\| \leq c$, $\|\rho | F_{pq}^{s,a}\| \leq c$. Hence, as in the isotropic case, atoms are normalised building blocks satisfying some moment conditions.

Our construction of anisotropic atoms which generalise isotropic atoms as they are in the works of M. Frazier and B. Jawerth, is slightly related to the concept of anisotropic building blocks (compactly supported and satisfying some norming and some moment conditions) used by P. Soardi in [Soa83] to define anisotropic Hardy spaces and to study the relations of these spaces to anisotropic Lipschitz and Campanato - Morrey spaces.

We introduce now the sequence spaces b_{pq} and f_{pq}^a .

If $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $Q_{\nu m}^a$ is a rectangle as above let $\chi_{\nu m}$ be the characteristic function of $Q_{\nu m}^a$; if $0 < p \leq \infty$ let $\chi_{\nu m}^{(p)} = 2^{\nu n/p} \chi_{\nu m}$ (obvious modification if $p = \infty$) be the L_p -normalised characteristic function of $Q_{\nu m}^a$.

Definition 3.2 *Let $0 < p \leq \infty$, $0 < q \leq \infty$. Then:*

(i) b_{pq} is the collection of all sequences $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ such that

$$\|\lambda | b_{pq}\| = \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q} \quad (3.7)$$

(with the usual modification if $p = \infty$ and/or $q = \infty$) is finite;

(ii) f_{pq}^a is the collection of all sequences $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ such that

$$\|\lambda | f_{pq}^a\| = \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m} \chi_{\nu m}^{(p)}(\cdot)|^q \right)^{1/q} | L_p \right\| \quad (3.8)$$

(with the usual modification if $p = \infty$ and/or $q = \infty$) is finite.

One can easily see that b_{pq} and f_{pq}^a are quasi-Banach spaces and using $\|\chi_{\nu m}^{(p)} | L_p\| = 1$ it is clear that comparing $\|\lambda | b_{pq}\|$ and $\|\lambda | f_{pq}^a\|$ the roles of the quasi-norms in L_p and l_q are interchanged.

Let $d_+ = \max(d, 0)$. For $0 < p \leq \infty$ and $0 < q \leq \infty$ we introduce the abbreviations

$$\sigma_p = n \left(\frac{1}{p} - 1 \right)_+ \quad \text{and} \quad \sigma_{pq} = n \left(\frac{1}{\min(p, q)} - 1 \right)_+. \quad (3.9)$$

Theorem 3.3 *Let $0 < p < \infty$ (respectively $0 < p \leq \infty$), $0 < q \leq \infty$, $s \in \mathbb{R}$ and let $K, L \in \mathbb{R}$ such that*

$$K \geq a_{max} + s \quad \text{if } s \geq 0, \quad (3.10)$$

$$L \geq \sigma_{pq} - s \quad (\text{respectively } L \geq \sigma_p - s). \quad (3.11)$$

Then $g \in S'$ belongs to $F_{pq}^{s,a}$ (respectively $B_{pq}^{s,a}$) if, and only if, it can be represented as

$$g = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \rho_{\nu m}^a, \quad (3.12)$$

convergence being in S' , where $\rho_{\nu m}^a$ are anisotropic 1_K -atoms ($\nu = 0$) or anisotropic $(s, p)_{K,L}$ -atoms ($\nu \in \mathbb{N}$) and $\lambda \in f_{pq}^a$ (respectively $\lambda \in b_{pq}$) where $\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$.

Furthermore, $\inf \|\lambda\|_{f_{pq}^a}$ (respectively $\inf \|\lambda\|_{b_{pq}}$), where the infimum is taken over all admissible representations (3.12), is an equivalent quasi-norm in $F_{pq}^{s,a}$ (respectively $B_{pq}^{s,a}$).

The convergence in S' can be obtained as a by-product of the proof using the same method as in [Tri97, 13.9] so we will not stress this point. We refer to the above theorem as to the atomic decomposition theorem in anisotropic function spaces.

Remark 3.4 Let $d > 0$ be given, let $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ fixed and let us denote $R_{\nu m}^a$ a rectangle with sides parallel to the axes, centred at $x^{\nu m}$ where

$$|x_i^{\nu m} - 2^{-\nu a_i} m_i| \leq d 2^{-\nu a_i} \quad \text{for all } i \in \{1, \dots, n\}, \quad (3.13)$$

and with side lengths respectively $2^{-\nu a_1}, \dots, 2^{-\nu a_n}$.

Then let $c > 0$ be chosen in dependence of d such that for every choice of $\nu \in \mathbb{N}_0$ and all choices of $x^{\nu m}$ in (3.13) we have

$$\bigcup_{m \in \mathbb{Z}^n} c R_{\nu m}^a = \mathbb{R}^n. \quad (3.14)$$

It will be clear from the proof that we may replace in Definition 3.1 the rectangle $Q_{\nu m}^a$ by $R_{\nu m}^a$, the number c being that from (3.14). A similar remark in the isotropic case was very useful in the work of H. Triebel and H. Winkelvoß, [TrW96a], cf. also [EdT96, 2.2.3].

We shift the proof of the theorem to Section 5 but let us made here some remarks. The first part of the proof, that in which the atoms are constructed and where it is shown that the decomposition (3.12) holds, is essentially based on an anisotropic version of a resolution of unity of Calderon type, cf. [FJW91, 5.12]; this construction is the anisotropic counterpart of what was done in [FJW91, Theorem 5.11].

To prove the second part we will use a theorem on local means in anisotropic function spaces, the technique of maximal functions and an inequality of Fefferman - Stein type. All needed results are presented in the next section.

3.2 Anisotropic quarks and the subatomic decomposition theorem

In this subsection we will assume that $|\cdot|_a \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is an anisotropic distance function according to (2.3) satisfying in addition

$$\{x \in \mathbb{R}^n : |x|_a \leq 2\} \subset [-\pi, \pi]^n. \quad (3.15)$$

The above restriction, which is of technical nature, was introduced by P. Dintelmann in [Din95a] and seems to be a natural one compared with the isotropic case.

For $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ let $Q_{\nu m}^a$ be the rectangles introduced in Definition 3.1. Let in particular Q_{00}^a be the cube with side length 1 centred at the origin and let $2^a Q_{00}^a$ be the rectangle concentric with Q_{00}^a and with side lengths respectively $2^{a_1}, \dots, 2^{a_n}$.

Definition 3.5 *Let $\psi \in S$ such that*

$$\text{supp } \psi \subset 2^a Q_{00}^a \quad \text{and} \quad \sum_{k \in \mathbb{Z}^n} \psi(x - k) = 1 \quad \text{if } x \in \mathbb{R}^n \quad (3.16)$$

and let for any $\beta \in \mathbb{N}_0^n$, $\psi^\beta(x) = x^\beta \psi(x)$. If $0 < p \leq \infty$ and $s \in \mathbb{R}$ then

$$(\beta qu)_{\nu m}^a(x) = 2^{-\nu \left(s - \frac{n}{p}\right)} \psi^\beta(2^{\nu a} x - m) \quad (3.17)$$

is called an anisotropic $(s, p) - \beta$ -quark related to $Q_{\nu m}^a$.

Remark 3.6 It is easy to see that up to normalising constants the anisotropic $(s, p) - \beta$ -quarks are anisotropic $(s, p)_{K,L}$ -atoms for any given $K \in \mathbb{R}$ and any given $L < 0$. Moreover, the normalising constants by which the anisotropic $(s, p) - \beta$ -quark must be divided to become an anisotropic $(s, p)_{K,L}$ -atom can be estimated from above by $c 2^{\kappa a \beta}$ where $c > 0$ and $\kappa > 0$ are independent of β (recall the notation $a\beta = a_1\beta_1 + \dots + a_n\beta_n$ where $\beta = (\beta_1, \dots, \beta_n)$ is a multi-index).

We will use below the sequence spaces b_{pq} and f_{pq}^a with respect to the sequences $\lambda^\beta = \{\lambda_{\nu m}^\beta \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ where now $\beta \in \mathbb{N}_0^n$ is a multi-index and we will keep the notation $(\beta qu)_{\nu m}^a$ for an anisotropic $(s, p) - \beta$ -quark related to the rectangle $Q_{\nu m}^a$. The numbers σ_p and σ_{pq} have the same meaning as in (3.9).

Theorem 3.7 *Let $0 < p < \infty$ (respectively $0 < p \leq \infty$), $0 < q \leq \infty$ and $s > \sigma_{pq}$ (respectively $s > \sigma_p$). There exists a number $\kappa > 0$ with the following property: let $r > \kappa$; then $g \in S'$ belongs to $F_{pq}^{s,a}$ (respectively $B_{pq}^{s,a}$) if, and only if, it can be represented as*

$$g = \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta qu)_{\nu m}^a, \quad (3.18)$$

convergence being in S' (first m , then ν , then β), and

$$\sup_{\beta \in \mathbb{N}_0^n} 2^{r a \beta} \|\lambda^\beta | f_{pq}^a\| < \infty \quad (\text{respectively} \quad \sup_{\beta \in \mathbb{N}_0^n} 2^{r a \beta} \|\lambda^\beta | b_{pq}\| < \infty). \quad (3.19)$$

Furthermore, the infimum in (3.19) over all admissible representations (3.18) is an equivalent quasi-norm in $F_{pq}^{s,a}$ (respectively $B_{pq}^{s,a}$).

The technique developed in Section 5 to prove the above theorem is that of H. Triebel from [Tri97, 14.15]. However the proof given there covered only isotropic B - spaces; the considerations in Section 5 show that the method can be extended to F - spaces.

To show that $g \in F_{pq}^{s,a}$ (respectively $g \in B_{pq}^{s,a}$) can be decomposed as in (3.18) with (3.19) we need not the assumption $s > \sigma_{pq}$ (respectively $s > \sigma_p$). This restriction is needed only to prove the converse assertion.

4 Results on anisotropic function spaces

4.1 Prerequisites

If $z \in \mathbb{R}^n$ and $t > 0$ then the set $\Omega^a(z, t) = \{y \in \mathbb{R}^n : |y - z|_a \leq t\}$ is the (closed) anisotropic ball centred at z with (anisotropic) radius t . The Lebesgue measure of such an anisotropic ball is $|\Omega^a(z, t)| = ct^n$ with c independent of t .

If f is a complex-valued locally integrable function on \mathbb{R}^n then

$$M^a f(x) = \sup \frac{1}{|\Omega^a|} \int_{\Omega^a} |f(y)| dy \quad (4.1)$$

is the anisotropic Hardy - Littlewood maximal function, where the supremum is taken over all anisotropic balls Ω^a containing x .

Let $1 < p < \infty$, $1 < q \leq \infty$. There exists a constant $c > 0$ such that:

$$\|(M^a f_j)_{j \in \mathbb{N}_0} | L_p(l_q)\| \leq c \|f | L_p(l_q)\| \quad (4.2)$$

for all sequences $f = (f_j)_{j \in \mathbb{N}_0}$ of complex-valued locally Lebesgue integrable functions on \mathbb{R}^n .

Comments and further information to the isotropic version of (4.2) which essentially goes back to C. Fefferman and E. M. Stein, see [FeS71], may be found in [Tri92, 2.2.2] and [Tri83, 1.2.3].

B. Stöckert and H. Triebel remarked in [StT79, p.257] that the maximal function in (4.1) is equivalent to a maximal function where in (4.1) we can take rectangles with sides parallel to the axes (containing x) and that the iterative application of the isotropic one dimensional case ($n = 1$) leads to (4.2). Different proofs of (4.2) can be found in the works of M. Yamazaki [Yam86, 2.2], A. Seeger [See89] and P. Dintelmann [Din95b, A.1.3-A.1.4].

For $0 < p \leq \infty$ let again $\sigma_p = n \left(\frac{1}{p} - 1\right)_+$.

To prove the theorem below one has only to adapt the method from [Tri83, 1.5.2].

Theorem 4.1 *Let Ω be a compact subset of \mathbb{R}^n , $0 < p \leq \infty$. Let $r \geq 0$ and let w be a weight function for which there exists a constant $c > 0$ such that*

$$0 < w(x) \leq c w(y)(1 + |x - y|_a)^r, \quad \text{for all } x, y \in \mathbb{R}^n. \quad (4.3)$$

If $s > r + \frac{n}{2} + \sigma_p$ then there exists a constant $c > 0$ such that

$$\|w(m\hat{f})^\vee | L_p\| \leq c \|m | H_2^{s,a}\| \cdot \|wf | L_p\| \quad (4.4)$$

for all $m \in H_2^{s,a}$ and all $f \in S'$, $\text{supp } \hat{f} \subset \Omega$ with $wf \in L_p$.

If $0 < p \leq \infty$, $0 < q \leq \infty$ and $\Omega = (\Omega_j)_{j \in \mathbb{N}_0}$ is a collection of compact subsets in \mathbb{R}^n we define the space $L_p^\Omega(l_q)$ as the collection of all systems $f = (f_j)_{j \in \mathbb{N}_0} \subset S'$ such that $\text{supp } \widehat{f}_j \subset \Omega_j$ if $j \in \mathbb{N}_0$ which satisfy $\|f\|_{L_p(l_q)} < \infty$. By [Tri83, 1.6.1] $L_p^\Omega(l_q)$ is a quasi-Banach space with quasi-norm $\|f\|_{L_p(l_q)}$ if $f = (f_j)_{j \in \mathbb{N}_0}$.

Theorem 4.2 *Let $0 < p < \infty$, $0 < q \leq \infty$. For every $j \in \mathbb{N}_0$ let $R_j > 0$ be a given number, let $\Omega_j = \{\xi \in \mathbb{R}^n : |\xi|_a \leq R_j\}$ and let $\Omega = (\Omega_j)_{j \in \mathbb{N}_0}$.*

(i) *If $0 < t < \min(p, q)$ then there exists a constant $c > 0$ such that*

$$\left\| \left(\sup_{z \in \mathbb{R}^n} \frac{|f_j(\cdot - z)|}{1 + |R_j^a z|_a^{n/t}} \right)_{j \in \mathbb{N}_0} \middle| L_p(l_q) \right\| \leq c \|f\|_{L_p(l_q)} \quad (4.5)$$

for all $f = (f_j)_{j \in \mathbb{N}_0} \in L_p^\Omega(l_q)$.

(ii) *If $s > \frac{n}{2} + \frac{n}{\min(p, q)}$ then there exists a constant $c > 0$ such that*

$$\left\| \left((m_j \widehat{f}_j)^\vee \right)_{j \in \mathbb{N}_0} \middle| L_p(l_q) \right\| \leq c \sup_{j \in \mathbb{N}_0} \|m_j(R_j^a \cdot) \middle| H_2^{s, a}\| \cdot \|f\|_{L_p(l_q)} \quad (4.6)$$

for all $(m_j)_{j \in \mathbb{N}_0}$ such that $m_j(R_j^a \cdot) \in H_2^{s, a}$ if $j \in \mathbb{N}_0$ and all $f = (f_j)_{j \in \mathbb{N}_0} \in L_p^\Omega(l_q)$.

The above result is the anisotropic counterpart of [Tri83, 1.6.2; 1.6.3] and the proof can be done in the same manner as there making standard anisotropic changes, see also [Tri92, 2.2.4], [BeN93, Proposition 1]be-no1 and [Din95b, A.1.4].

4.2 Equivalent quasi-norms

If $(\psi_j)_{j \in \mathbb{N}_0} \subset S$ and $r > 0$ we define the maximal functions

$$(\psi_j^* f)_r(x) = \sup_{z \in \mathbb{R}^n} \frac{|(\psi_j \widehat{f})^\vee(x - z)|}{1 + |2^{ja} z|_a^r} \quad (4.7)$$

for $f \in S'$ where $x \in \mathbb{R}^n$ and $j \in \mathbb{N}_0$. The above maximal function essentially goes back to J. Peetre, see [P75] and [P76]; see also [Tri83, 2.3.6/Remark 2] and [Yam86].

The result below is the anisotropic version of [Tri92, 2.3.2] and it can be obtained as a simple consequence of (4.5), cf. also [Din95b, A.1.5].

Theorem 4.3 *Let $(\varphi_j)_{j \in \mathbb{N}_0}$ be a smooth anisotropic dyadic resolution of unity.*

(i) *Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $r > \frac{n}{p}$. Then:*

$$B_{pq}^{s, a} = \left\{ f \in S' : \|f\|_{B_{pq}^{s, a}} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j^* f)_r\|_{L_p}^q \right)^{1/q} < \infty \right\} \quad (4.8)$$

(modification if $q = \infty$) in the sense of equivalent quasinorms.

(ii) *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $r > \frac{n}{\min(p, q)}$. Then:*

$$F_{pq}^{s, a} = \left\{ f \in S' : \|f\|_{F_{pq}^{s, a}} = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j^* f)_r(\cdot)|^q \right)^{1/q} \middle| L_p \right\| < \infty \right\} \quad (4.9)$$

(modification if $q = \infty$) in the sense of equivalent quasi-norms.

4.3 General characterisations

We present now a rather general, but highly technical, characterisation of the space $F_{pq}^{s,a}$ (respectively of $B_{pq}^{s,a}$). This characterisation is the anisotropic counterpart of [Tri92, 2.4.1] (respectively [Tri92, 2.5.1]).

In the theorem below φ need not to be an element of S and it is not immediately clear what is meant by $(\varphi(2^{-ja}\cdot)\widehat{f})^\vee$. It is defined via limiting procedures as in Step 3 of the proof in [Tri92, 2.4.1].

The numbers σ_p and σ_{pq} have the same meaning as in (3.9).

Given the anisotropy $a = (a_1, \dots, a_n)$ recall our notations $2^{ka}x = (2^{ka_1}x_1, \dots, 2^{ka_n}x_n)$ and $2^{-ja}x = (2^{-ja_1}x_1, \dots, 2^{-ja_n}x_n)$.

Let $h \in S$ and $G \in S$ with

$$\text{supp } h \subset \{x \in \mathbb{R}^n : |x|_a \leq 2\}, \quad h(x) = 1 \quad \text{if } |x|_a \leq 1, \quad (4.10)$$

$$\text{supp } G \subset \left\{x \in \mathbb{R}^n : \frac{1}{4} \leq |x|_a \leq 4\right\}, \quad G(x) = 1 \quad \text{if } \frac{1}{2} \leq |x|_a \leq 2. \quad (4.11)$$

Theorem 4.4 *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. Let s_0, s_1 be two real numbers with*

$$s_0 + \sigma_{pq} < s < s_1 \quad \text{and} \quad s_1 > \sigma_p. \quad (4.12)$$

Let φ_0 and φ be two complex-valued C^∞ functions on \mathbb{R}^n and $\mathbb{R}^n \setminus \{0\}$, respectively, which satisfy the following Tauberian conditions:

$$|\varphi_0(x)| > 0 \quad \text{if} \quad |x|_a \leq 2, \quad (4.13)$$

$$|\varphi(x)| > 0 \quad \text{if} \quad \frac{1}{2} \leq |x|_a \leq 2. \quad (4.14)$$

Let $r > \frac{n}{\min(p,q)}$ and assume:

$$\int_{\mathbb{R}^n} \left| \left(\frac{\varphi(\cdot)h(\cdot)}{|\cdot|_a^{s_1}} \right)^\vee(y) \right| (1 + |y|_a)^r dy < \infty, \quad (4.15)$$

$$\sup_{k \in \mathbb{N}} 2^{-ks_0} \int_{\mathbb{R}^n} |(\varphi(2^{ka}\cdot)G(\cdot))^\vee(y)| (1 + |y|_a)^r dy < \infty, \quad (4.16)$$

$$\sup_{k \in \mathbb{N}} 2^{-ks_0} \int_{\mathbb{R}^n} |(\varphi_0(2^{ka}\cdot)G(\cdot))^\vee(y)| (1 + |y|_a)^r dy < \infty. \quad (4.17)$$

Let $\varphi_j(x) = \varphi(2^{-ja}x)$ if $x \in \mathbb{R}^n \setminus \{0\}$ and $j \in \mathbb{N}$.

Then

$$\left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q \right)^{1/q} \right\|_{L_p} \quad (4.18)$$

(usual modification if $q = \infty$) is an equivalent quasi-norm in $F_{pq}^{s,a}$.

Proof. The proof is following the lines of the proof of [Tri92, 2.4.1]. We indicate briefly the necessary modifications.

Starting with a smooth anisotropic dyadic resolution of unity we have to make usual anisotropic changes and to use the maximal function from (4.7) to obtain counterparts of (14-19) in [Tri92, 2.4.1].

Since $r > \frac{n}{\min(p,q)}$ we may use the maximal inequality (4.5); then we use the vector valued multiplier theorem (4.6) and obtain counterparts of (21-23) in [Tri92, 2.4.1]. The term with $j = 0$ is critical but it may be incorporated by the lemma below.

Using again the maximal function (4.7) and the multiplier theorem (4.4) with weight $w(y) = (1 + |y|_a)^r$ we obtain anisotropic counterparts of (24-29) in [Tri92, 2.4.1].

Counterparts of (30-33) in [Tri92, 2.4.1] can be obtained in the same manner using in addition the embedding for different metrics $F_{pq}^{s,a} \hookrightarrow F_{1q}^{\sigma,a}$ which holds for $0 < p < 1$, $\sigma = s - \sigma_p$, see [Joh95, (2.14)].

To obtain counterparts of (36-45) in [Tri92, 2.4.1] we have to use the anisotropic version of the Plancherel - Polya - Nikol'skij inequality from [Yam86, 2.13] and the Fefferman - Stein inequality (4.2).

Finally, to complete the proof, we have to replace Remark 1 from the end of the proof of [Tri92, 2.4.1] by the lemma below.

Lemma 4.5 *Let $0 < p < \infty$ and $\sigma \in S$, $\sigma(x) = 1$ if $|x|_a \leq 1$, $\text{supp } \sigma \subset \{x \in \mathbb{R}^n : |x|_a \leq 2\}$. If $s_1 > \sigma_p$ then there exists a constant $c > 0$ such that*

$$\|(| \cdot |_a^{s_1} \sigma(\cdot) \widehat{g}(\cdot))^\vee | L_p\| \leq c \|g | L_p\| \quad (4.19)$$

for all $g \in S' \cap L_p$ with $\text{supp } \widehat{g} \subset \{\xi \in \mathbb{R}^n : |\xi|_a \leq 1\}$.

Proof. By the unweighted version ($r = 0$, $w(x) \equiv 1$) of the multiplier theorem (4.4) the estimate is valid if $\psi(x) = |x|_a^{s_1} \sigma(x) \in H_2^{v,a}$ where $v > \frac{n}{2} + \sigma_p$. Since $s_1 > \sigma_p$ we may choose $\frac{n}{2} + s_1 > v > \frac{n}{2} + \sigma_p$.

Let now χ a C^∞ function on \mathbb{R}^n such that $\chi(x) = 0$ if $|x|_a \leq 1$ and $\chi(x) = 1$ if $|x|_a \geq 2$ and let $\psi_j(x) = |x|_a^{s_1} \sigma(x) \chi(2^{j_a} x)$ for every $j \in \mathbb{N}$.

The sequence $(\psi_j)_{j \in \mathbb{N}}$ is fundamental in $H_2^{v,a}$; by (2.8) it is sufficient to prove that it is fundamental in each $H_{x_k, 2}^{v,a}$ where $1 \leq k \leq n$.

If $v_k = \frac{v}{a_k} \in \mathbb{N}$ this can be done by straightforward calculations using (2.9) and the estimate $|D^\alpha(|x|_a^{s_1})| \leq c |x|_a^{s_1 - a\alpha}$ for any $|x|_a > 0$.

If $v_k = \frac{v}{a_k} \notin \mathbb{N}$ it is a matter of interpolation: we write $s_k = \theta m_k$ with $\theta \in (0, 1)$, $m_k \in \mathbb{N}$ and use the interpolation result $(L_2, L_2(w_k))_{\theta, 2} = L_2(w)$ with $w_k(\xi) = (1 + \xi_k^2)^{m_k/2}$ and $w = w_k^\theta$ which is a simple consequence of [Tri78, 1.18.4] where $L_2(w_k)$ and $L_2(w)$ are the weighted L_2 spaces with weights respectively w_k and w .

Remark that an isotropic version of the assertion $\psi(x) = |x|_a^{s_1} \sigma(x) \in H_2^{v,a}$ where $v > \frac{n}{2} + \sigma_p$ is presented in a more general context in [RuS96, Lemma 2.3.1/1].

It is clear that following the lines of the above proof we can obtain an anisotropic counterpart of [Tri92, 2.5.1] for $B_{pq}^{s,a}$ spaces.

As in [Tri92] conditions (4.15-4.17) can be reformulated.

Corollary 4.6 *Let p, q, s, s_0, s_1 and r be the same numbers as in the theorem. Let φ_0, φ be two complex-valued C^∞ functions on \mathbb{R}^n and $\mathbb{R}^n \setminus \{0\}$, respectively, which satisfy the Tauberian conditions (4.13), (4.14) and let $v > r + \frac{n}{2}$. Assume:*

$$\left\| \frac{\varphi(\cdot)h(\cdot)}{|\cdot|_a^{s_1}} \Big| H_2^{v,a} \right\| < \infty, \quad (4.20)$$

$$\sup_{k \in \mathbb{N}} 2^{-ks_0} \|\varphi(2^{ka}\cdot)G(\cdot) \Big| H_2^{v,a}\| < \infty, \quad (4.21)$$

$$\sup_{k \in \mathbb{N}} 2^{-ks_0} \|\varphi_0(2^{ka}\cdot)G(\cdot) \Big| H_2^{v,a}\| < \infty, \quad (4.22)$$

where h, G have the same meaning as in (4.10), (4.11).

Let $\varphi_j(x) = \varphi(2^{-ja}x)$ if $x \in \mathbb{R}^n \setminus \{0\}$ and $j \in \mathbb{N}$. Then (4.18) is an equivalent quasi-norm in $F_{pq}^{s,a}$.

Proof. The result is a simple consequence of the inequality $\|(1 + |\cdot|_a)^r \widehat{\psi} \Big| L_1\| \leq c \|\psi \Big| H_2^{v,a}\|$ for all $\psi \in H_2^{v,a}$ where $v > r + \frac{n}{2}$ which can be proved as in [ScT87, 1.7.5].

The counterpart of the above corollary for $B_{pq}^{s,a}$ spaces reads as follows:

Corollary 4.7 *Let $0 < p \leq \infty, 0 < q \leq \infty, s \in \mathbb{R}$ and let s_0, s_1 two real numbers with $s_0 + \sigma_p < s < s_1$ and $s_1 > \sigma_p$. Let φ_0, φ be two complex-valued C^∞ functions on \mathbb{R}^n and $\mathbb{R}^n \setminus \{0\}$, respectively, which satisfy the Tauberian conditions (4.13), (4.14); let $v > \sigma_p + \frac{n}{2}$ and assume (4.20), (4.21) and (4.22) are satisfied. Let $\varphi_j(x) = \varphi(2^{-ja}x)$ if $x \in \mathbb{R}^n \setminus \{0\}$ and $j \in \mathbb{N}$.*

Then $(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \widehat{f})^\vee \Big| L_p\|^q)^{1/q}$ (usual modification if $q = \infty$) is an equivalent quasi-norm in $B_{pq}^{s,a}$.

4.4 Local means

For the anisotropy $a = (a_1, \dots, a_n)$ we will use the notation $x + t^a y = (x_1 + t^{a_1} y_1, \dots, x_n + t^{a_n} y_n)$. If $B^a = \{y \in \mathbb{R}^n : |y|_a \leq 1\}$ is the anisotropic unit ball in \mathbb{R}^n , k is a C^∞ function on \mathbb{R}^n , $\text{supp } k \subset B^a$ then we introduce the local means (cf. [Tri92, 2.4.6/1])

$$k(t, f)(x) = \int_{\mathbb{R}^n} k(y) f(x + t^a y) dy = t^{-n} \int_{\mathbb{R}^n} k(t^{-a}(z - x)) f(z) dz \quad (4.23)$$

which make sense for any $f \in S'$ (appropriately interpreted).

Lemma 4.8 *Let $s_1 > 0$ a given number and assume $k \in S$ such that there exists a constant $c > 0$ with $|\widehat{k}(\xi)| \leq c |\xi|_a^{s_1}$ for ξ near zero. Then $\int_{\mathbb{R}^n} x^\alpha k(x) dx = 0$ for all $\alpha \in \mathbb{N}_0^n$ with $a\alpha < s_1$.*

Proof. We have to prove that $D^\alpha \widehat{k}(0) = 0$ for all $\alpha \in \mathbb{N}_0^n$ with $a\alpha < s_1$. If $s_1 < a_{\min}$ the assertion is clear so let us assume that $s_1^* = [s_1/a_{\min}] \geq 1$.

By Taylor's expansion theorem we get for ξ near zero

$$\widehat{k}(\xi) = \sum_{|\alpha| < s_1^*} \frac{D^\alpha \widehat{k}(0)}{\alpha!} \xi^\alpha + \sum_{|\alpha| = s_1^*} R_\alpha(\xi) \quad (4.24)$$

where $|R_\alpha(\xi)| \leq c_\alpha |\xi^\alpha| \leq c'_\alpha |\xi|_a^{\alpha}$ for ξ small and for some constants $c_\alpha, c'_\alpha > 0$.

By standard limit arguments we arrive at $D^\alpha \widehat{k}(0) = 0$ for all α with $|\alpha| < s_1^*$ and this leads immediately to the conclusion.

Theorem 4.9 *Let k_0 and k be two C^∞ functions on \mathbb{R}^n such that*

$$\text{supp } k_0 \subset B^a, \quad |\widehat{k}_0(\xi)| > 0 \quad \text{if} \quad |\xi|_a \leq 2, \quad (4.25)$$

$$\text{supp } k \subset B^a, \quad |\widehat{k}(\xi)| > 0 \quad \text{if} \quad \frac{1}{2} \leq |\xi|_a \leq 2. \quad (4.26)$$

Let $s_1 > 0$ and assume that there exists a constant $c > 0$ such that

$$|\widehat{k}(\xi)| \leq c |\xi|_a^{s_1} \quad \text{for} \quad \xi \quad \text{near zero.} \quad (4.27)$$

(i) Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. If $s_1 > \max(s, \sigma_p) + \sigma_p$ then

$$\|k_0(1, f) | L_p\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \|k(2^{-j}, f) | L_p\|^q \right)^{1/q} \quad (4.28)$$

(usual modification if $q = \infty$) is an equivalent quasi-norm in $B_{pq}^{s,a}$.

(ii) Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. If $s_1 > \max(s, \sigma_p) + \frac{n}{\min(p,q)}$ then

$$\|k_0(1, f) | L_p\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} |k(2^{-j}, f)(\cdot)|^q \right)^{1/q} \right\| | L_p \quad (4.29)$$

(usual modification if $q = \infty$) is an equivalent quasi-norm in $F_{pq}^{s,a}$.

Proof. We sketch the proof for F -spaces. By our assumption on s_1 we find numbers s_2, v and r such that $s_2 > \max(s, \sigma_p)$, $r > \frac{n}{\min(p,q)}$ and $v > \frac{n}{2} + r$ such that

$$s_1 > s_2 + v - \frac{n}{2}. \quad (4.30)$$

The functions \check{k}_0 and \check{k} fulfill the Tauberian conditions (4.13), (4.14) and we will identify them respectively with φ_0 and φ in Corollary 4.6, now with s_2 in place of s_1 .

To prove (4.20) we use (2.8). Fix $i \in \{1, \dots, n\}$.

If $v_i = \frac{v}{a_i} \in \mathbb{N}$ then we recall (2.9); clearly $\widehat{k}(\cdot)h(\cdot) \cdot |_a^{-s_2} \in L_2$ where h is the function from (4.10). Let $l_i \in \{0, \dots, v_i\}$; then by Lemma 2.2 there exists a constant $c > 0$ such that for ξ near zero

$$\left| D_i^{v_i - l_i} (|\xi|_a^{-s_2}) \right| \leq c |\xi|_a^{-s_2 - a_i(v_i - l_i)}. \quad (4.31)$$

Using the above lemma we find also a constant $c > 0$ such that for ξ near zero

$$\left| D_i^{l_i} \left(\widehat{k}(\xi)h(\xi) \right) \right| \leq c |\xi|_a^{s_1 - a_i l_i}. \quad (4.32)$$

By (4.31) and (4.32) using (4.30) we get $D_i^{v_i} \left(\widehat{k}(\cdot)h(\cdot) \cdot |_a^{-s_2} \right) \in L_2$ which is just what we want.

If $v_i = \frac{v}{a_i} \notin \mathbb{N}$ we have to use the interpolation result which was mentioned at the end of the proof of Lemma 4.5 and this completes the proof of (4.20).

To prove (4.21) let $s_0 \in \mathbb{R}$ with $s_0 + \sigma_{pq} < s$ and let G be the function from (4.11). If $m \in \mathbb{N}$ then $\psi_m(\cdot) = \check{k}(2^{ma}\cdot)G(\cdot)$ has a compact support which is at most $\Omega = \{x \in \mathbb{R}^n : \frac{1}{4} \leq |x|_a \leq 4\}$.

If $j \in \mathbb{N}$ such that $j > 1 + \left\lceil \frac{v}{a_i} \right\rceil$ for all $i \in \{1, \dots, n\}$ then there exists a constant $c > 0$ such that

$$\|\psi_m\|_{H_2^{v,a}} \leq c \sum_{a\alpha \leq j} \|D^\alpha \psi_m\|_{L_\infty}. \quad (4.33)$$

To prove (4.33) we have only to recall (2.8) and to distinguish between $v_i = \frac{v}{a_i} \in \mathbb{N}$ and $v_i = \frac{v}{a_i} \notin \mathbb{N}$.

Let $l \in \mathbb{N}$ such that $j - l \leq s_0$ and such that $|D^\alpha \check{k}(x)| \leq c(1 + |x|_a)^{-l}$ for all $x \in \mathbb{R}^n$ and any $\alpha \in \mathbb{N}_0^n$ with $a\alpha \leq j$.

Then there exist constants $c, c' > 0$ independent of $m \in \mathbb{N}$ such that for any $x \in \Omega$ we have $|D^\alpha \psi_m(x)| \leq c 2^{ma\alpha} (1 + |2^{ma}x|_a)^{-l} \leq c' 2^{m(j-l)} \leq c' 2^{ms_0}$ if $\alpha \in \mathbb{N}_0^n$ with $a\alpha \leq j$.

It follows that there exists a constant $c > 0$ independent of $m \in \mathbb{N}$ such that

$$\sum_{a\alpha \leq j} \|D^\alpha \psi_m\|_{L_\infty} \leq c 2^{ms_0}. \quad (4.34)$$

The condition (4.21) is now a simple consequence of (4.33) and (4.34). A similar argument can be used to check (4.22).

For B -spaces one has to use Corollary 4.7 and to make obvious changes above.

Remark 4.10 The isotropic counterpart of the above result can be found in [Tri92, 2.4.6, 2.5.3]; instead of (4.27) it is used the representation $k = \Delta^N k^0$ (with $2N = s_1$ sufficiently large and $k^0 \in S$) which is in fact $\widehat{k}(\xi) = |\xi|^{2N} \widehat{k^0}(\xi)$. But this assumption was taken only for simplicity and it can be replaced by (4.27) with the euclidean distance $|\cdot|$ instead of $|\cdot|_a$.

Remark 4.11 Examples of functions k as in the theorem can be constructed as in [FrJ85, p.783].

The advantage of (4.23) compared with $(\varphi_j \widehat{f})^\vee$ from Definition 2.3 is its strictly local nature: in order to calculate $k(t, f)(x)$ in a given point $x \in \mathbb{R}^n$ one needs only a knowledge of $f(z)$ in an anisotropic ball $\Omega^a(x, t)$. This observation will be of great service for us in the proof of the atomic decomposition theorem.

5 Proofs

5.1 Proof of the atomic decomposition theorem

We begin with some preparations. Our theorem is based on the following lemma which provides the existence of an anisotropic resolution of unity of Calderon type and which is a generalisation of the result from [FJW91, 5.12].

Lemma 5.1 *Let $\theta_0, \theta \in S$ functions with:*

$$|\widehat{\theta}_0(\xi)| > 0 \quad \text{if} \quad |\xi|_a \leq 2, \quad (5.1)$$

$$|\widehat{\theta}(\xi)| > 0 \quad \text{if} \quad \frac{1}{2} \leq |\xi|_a \leq 2. \quad (5.2)$$

Then there exist functions $\varphi_0, \varphi \in S$ and a positive number $\delta \leq 2$ such that

$$\text{supp } \varphi_0 \subset \{\xi \in \mathbb{R}^n : |\xi|_a \leq 2\} \quad \text{and} \quad |\varphi_0(\xi)| > 0 \quad \text{if} \quad |\xi|_a \leq \delta, \quad (5.3)$$

$$\text{supp } \varphi \subset \left\{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi|_a \leq 2 \right\} \quad \text{and} \quad |\varphi(\xi)| > 0 \quad \text{if} \quad \frac{3}{5} \leq |\xi|_a \leq \frac{5}{3}, \quad (5.4)$$

and

$$\widehat{\theta}_0(\xi)\varphi_0(\xi) + \sum_{\nu=1}^{\infty} \widehat{\theta}(2^{-\nu a}\xi)\varphi(2^{-\nu a}\xi) = 1 \quad \text{for all} \quad \xi \in \mathbb{R}^n. \quad (5.5)$$

The proof is classical and it can be done using the technique of M. Frazier, B. Jawerth, G. Weiss from [FJW91, Lemma 6.9] adapted to our purpose so we do not go into details.

Remark 5.2 A construction as in [FrJ85, p.783] can be used to prove that if $L \geq 0$ is a given number there exists a function $\theta \in S$ satisfying (5.2) such that, in addition,

$$\text{supp } \theta \subset \{x \in \mathbb{R}^n : |x|_a \leq 1\}, \quad (5.6)$$

$$\int_{\mathbb{R}^n} x^\beta \theta(x) dx = 0 \quad \text{if} \quad a\beta \leq L \quad (5.7)$$

(cf. also Remark 4.11).

An instrument which will be of considerable use is an anisotropic version of Taylor's expansion theorem. Recall the binomial notation $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ if $x \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index.

Theorem 5.3 *Let $A \geq 0$ be a given number. Let U be an open convex subset of \mathbb{R}^n and assume $f : U \rightarrow \mathbb{C}$ is a function such that $D^\alpha f$ exists for all $\alpha \in \mathbb{N}_0^n$ with $a\alpha \leq A + a_{max}$.*

Let $y \in U$ and $t > 0$ such that $\Omega^a(y, t) = \{z \in \mathbb{R}^n : |z - y|_a \leq t\} \subset U$.

Then there exists a constant $c > 0$ such that for all $x \in \Omega^a(y, t)$:

$$f(x) = \sum_{a\alpha \leq A} \frac{1}{\alpha!} D^\alpha f(y) (x - y)^\alpha + R_A(x) \quad (5.8)$$

with

$$|R_A(x)| \leq c \sum_{a\alpha > A} t^{a\alpha} \sup_{z \in \Omega^a(y, t)} |D^\alpha f(z)| \quad (5.9)$$

where the notation $\sum_{a\alpha > A}^{A+a_{max}}$ means that the sum is taken over all $\alpha \in \mathbb{N}_0^n$ such that $A < a\alpha \leq$

$A + a_{max}$.

We will refer to this result as to the anisotropic Taylor expansion theorem of (anisotropic) order A on the set $\Omega^a(y, t)$.

The main idea in proving the above theorem is an iterative application of the one dimensional classical Taylor expansion theorem. In lack of a convincing reference we sketch a proof in the case $n = 2$.

Applying Taylor's classical expansion theorem with Lagrange remainder there exists $\xi_1 = \xi_1(x_1)$ between x_1 and y_1 such that

$$f(x_1, x_2) = \sum_{\alpha_1=0}^{[A/a_1]} \frac{1}{\alpha_1!} D_1^{\alpha_1} f(y_1, x_2) (x_1 - y_1)^{\alpha_1} + R_{1+[A/a_1]}(x)$$

where

$$R_{1+[A/a_1]}(x) = c_1 D_1^{1+[A/a_1]} f(\xi_1, x_2) (x_1 - y_1)^{1+[A/a_1]}.$$

Applying now Taylor's classical expansion theorem with Lagrange remainder to each $D_1^{\alpha_1} f(y_1, \cdot)$ there exists $\xi_2^{\alpha_1} = \xi_2^{\alpha_1}(x_2)$ between x_2 and y_2 such that

$$D_1^{\alpha_1} f(y_1, x_2) = \sum_{\alpha_2=0}^{[(A-a_1\alpha_1)/a_2]} \frac{1}{\alpha_2!} D_1^{\alpha_1} D_2^{\alpha_2} f(y_1, y_2) (x_2 - y_2)^{\alpha_2} + R_{\alpha_1, 1+[(A-a_1\alpha_1)/a_2]}(x)$$

where the remainder $R_{\alpha_1, 1+[(A-a_1\alpha_1)/a_2]}(x)$ is

$$c_2(\alpha_1) D_1^{\alpha_1} D_2^{1+[(A-a_1\alpha_1)/a_2]} f(y_1, \xi_2^{\alpha_1}) (x_2 - y_2)^{1+[(A-a_1\alpha_1)/a_2]}.$$

If $n = 2$ the expansion (5.8) with (5.9) is now a simple consequence of the last four relations and of

$$R_A(x) = R_{1+[A/a_1]}(x) + \sum_{\alpha_1=0}^{[A/a_1]} \frac{1}{\alpha_1!} (x_1 - y_1)^{\alpha_1} R_{\alpha_1, 1+[(A-a_1\alpha_1)/a_2]}(x).$$

The general case can be treated in a similar manner.

Proof. (Atomic decomposition theorem)

We present here the proof for $F_{pq}^{s,a}$ spaces; the proof for $B_{pq}^{s,a}$ spaces is simpler and is obtained essentially by interchanging the roles of the L_p and l_q quasi-norms in the proof below.

Part I. Let $g \in F_{pq}^{s,a}$; we use the method of M. Frazier, B. Jawerth and G. Weiss from [FJW91, Theorem 5.11] to construct atoms and to decompose g as in (3.12).

Let $\theta_0, \theta, \varphi_0$ and φ functions in S satisfying (5.1-5.7). Then $(\varphi_0 \hat{g})^\vee$ and $(\varphi(2^{-\nu a} \cdot) \hat{g})^\vee$ are entire analytic functions; using $\hat{\theta}(2^{-\nu a} \xi) = 2^{\nu n} F(\theta(2^{\nu a} \cdot))(\xi)$ we obtain the following equality in S' :

$$\begin{aligned} g(x) &= \sum_{m \in \mathbb{Z}^n} \int_{Q_{0m}^a} \theta_0(x-y) (\varphi_0 \hat{g})^\vee(y) dy \\ &+ \sum_{\nu=1}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{\nu n} \int_{Q_{\nu m}^a} \theta(2^{\nu a}(x-y)) (\varphi(2^{-\nu a} \cdot) \hat{g})^\vee(y) dy. \end{aligned} \quad (5.10)$$

We define for every $\nu \in \mathbb{N}$ and all $m \in \mathbb{Z}^n$

$$\lambda_{\nu m} = C 2^{\nu(s-\frac{n}{p})} \sup_{y \in Q_{\nu m}^a} |(\varphi(2^{-\nu a} \cdot) \hat{g})^\vee(y)| \quad (5.11)$$

where $C = \max_{a \leq K} \sup_{|x|_a \leq 1} |D^\alpha \theta(x)|$ and

$$\rho_{\nu m}^a(x) = \frac{1}{\lambda_{\nu m}} 2^{\nu n} \int_{Q_{\nu m}^a} \theta(2^{\nu a}(x-y)) (\varphi(2^{-\nu a} \cdot) \widehat{g})^\vee(y) dy. \quad (5.12)$$

Similarly we define for every $m \in \mathbb{Z}^n$ the numbers λ_{0m} and the functions ρ_{0m}^a taking in (5.11) and (5.12) $\nu = 0$ and replacing φ and θ by φ_0 and θ_0 , respectively.

It is obvious that (3.12) is satisfied and it follows by straightforward calculations, using the properties of the functions θ_0 , θ , φ_0 and φ , that ρ_{0m}^a are anisotropic 1_K -atoms and that $\rho_{\nu m}^a$ are anisotropic $(s, p)_{K,L}$ -atoms for $\nu \in \mathbb{N}$.

Finally, we will show that there exists a constant $c > 0$ such that $\|\lambda |f_{pq}^a|\| \leq c \|g |F_{pq}^{s,a}|\|$. We have for a fixed $\nu \in \mathbb{N}$:

$$\begin{aligned} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{\nu m}^{(p)}(x) &= C 2^{\nu(s-\frac{n}{p})} \sum_{m \in \mathbb{Z}^n} \sup_{y \in Q_{\nu m}^a} |(\varphi(2^{-\nu a} \cdot) \widehat{g})^\vee(y)| \cdot 2^{\nu \frac{n}{p}} \chi_{\nu m}(x) \\ &\leq c' 2^{\nu s} \left(\sup_{|z|_a \leq c 2^{-\nu}} \frac{|(\varphi(2^{-\nu a} \cdot) \widehat{g})^\vee(x-z)|}{(1+|2^{\nu a} z|_a)^r} (1+|2^{\nu a} z|_a)^r \right) \\ &\leq c'' 2^{\nu s} (\varphi_\nu^* g)_r(x) \end{aligned} \quad (5.13)$$

since $|x-y|_a \leq c 2^{-\nu}$ for $x, y \in Q_{\nu m}^a$ and $\sum_{m \in \mathbb{Z}^n} \chi_{\nu m}(x) = 1$. Here $\varphi_\nu = \varphi(2^{-\nu a} \cdot)$, $r > \frac{n}{\min(p,q)}$ and $(\varphi_\nu^* g)_r$ is the maximal function from (4.7). It follows

$$\sum_{\nu=1}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m} \chi_{\nu m}^{(p)}(\cdot)|^q \leq c \sum_{\nu=1}^{\infty} 2^{\nu s q} (\varphi_\nu^* g)_r(\cdot)^q \quad (5.14)$$

(with the usual modification if $q = \infty$) where c is a positive constant.

Now we have to use (5.14), its counterpart for $\nu = 0$ (which can be obtained by a similar calculation), Theorem 4.3 and get

$$\|\lambda |f_{pq}^a|\| \leq c \left\| \left(\sum_{\nu=0}^{\infty} 2^{\nu s q} (\varphi_\nu^* g)_r(\cdot)^q \right)^{1/q} |L_p|\right\| \leq c' \|g |F_{pq}^{s,a}|\| \quad (5.15)$$

(with the usual modification if $q = \infty$) and this completes the proof of the first part of the theorem.

Part II. Reciprocally, assume now g can be represented by (3.12), with K and L satisfying respectively (3.10) and (3.11). We will show that $g \in F_{pq}^{s,a}$ and that $\|g |F_{pq}^{s,a}|\| \leq c \|\lambda |f_{pq}^a|\|$ for some constant $c > 0$.

Let k_0 and k two C^∞ functions on \mathbb{R}^n as in Theorem 4.9 and let $s_1 > 0$ in that theorem enough large such that we have also $s_1 > K$.

Temporarily let $\nu, j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$ be fixed; we start finding convenient estimates for $2^{js} k(2^{-j}, \rho_{\nu m}^a)(x)$.

Step II.1 Let $j \geq \nu$ and let

$$2^{js} k(2^{-j}, \rho_{\nu m}^a)(x) = 2^{js} \int_{|y|_a \leq 1} k(y) \rho_{\nu m}^a(x + 2^{-ja} y) dy. \quad (5.16)$$

Let us remark that in this case x is located in some $cQ_{\nu m}^a$ (else, the integral above would be certainly zero by (3.3)).

Suppose first $s \geq 0$ and let $A = K - a_{max} \geq s \geq 0$. The derivatives $D^\alpha \rho_{\nu m}^a$ exist if $a\alpha \leq K$ so we can use the anisotropic Taylor expansion theorem of order A for the function $w \mapsto \rho(w)$ on the set $\Omega^a(x, 2^{-j})$. We put then $w = x + 2^{-j}y$ and get the expansion

$$\rho_{\nu m}^a(x + 2^{-j}y) = \sum_{a\alpha \leq A} c_\alpha (x + 2^{-j}y - z)^\alpha D^\alpha \rho_{\nu m}^a(z) + R_K(x, y) \quad (5.17)$$

where

$$|R_K(x, y)| \leq c \sum_{a\alpha > A}^K 2^{-j a \alpha} \sup_{z \in c' Q_{\nu m}^a} |D^\alpha \rho_{\nu m}^a(z)| \quad (5.18)$$

for some $c, c' > 0$. We may choose $A' > A$ such that for all α with $A < a\alpha \leq K$ we have $a\alpha > A'$. Using this remark and the estimate (3.6) the inequality (5.18) becomes

$$|R_K(x, y)| \leq c \sum_{a\alpha > A'}^K 2^{-j a \alpha} 2^{-\nu(s - \frac{n}{p})} 2^{\nu a \alpha} \tilde{\chi}_{\nu m}(x) \leq c' 2^{-\nu s} 2^{(\nu-j)A'} \tilde{\chi}_{\nu m}^{(p)}(x) \quad (5.19)$$

where $\tilde{\chi}_{\nu m}^{(p)}$ is the p -normalised characteristic function of some rectangle $cQ_{\nu m}^a$.

Recall $s_1 > K$ so we may use Lemma 4.8 and obtain $\int_{\mathbb{R}^n} (x + 2^{-j}y - z)^\alpha k(y) dy = 0$ for all α such that $a\alpha \leq A < s_1$. Hence (5.17) and (5.19) yield

$$|2^{js} k(2^{-j}, \rho_{\nu m}^a)(x)| \leq c 2^{-(j-\nu)(A'-s)} \tilde{\chi}_{\nu m}^{(p)}(x) \quad \text{where } A' > s. \quad (5.20)$$

If now $s < 0$ we have to use only the estimate $|\rho_{\nu m}^a(x)| \leq 2^{-\nu(s - \frac{n}{p})} \tilde{\chi}_{\nu m}(x)$ from (3.6) and get

$$|2^{js} k(2^{-j}, \rho_{\nu m}^a)(x)| \leq c 2^{-(j-\nu)(-s)} \tilde{\chi}_{\nu m}^{(p)}(x). \quad (5.21)$$

So, by (5.20) and (5.21), it is clear that we arrive in any case to:

$$|2^{js} k(2^{-j}, \rho_{\nu m}^a)(x)| \leq c 2^{-(j-\nu)\delta} \tilde{\chi}_{\nu m}^{(p)}(x) \quad \text{for some } \delta > 0. \quad (5.22)$$

Step II.2 Let now $j < \nu$; by a change of variables we have

$$2^{js} k(2^{-j}, \rho_{\nu m}^a)(x) = 2^{js} 2^{jn} \int_{\mathbb{R}^n} k(2^{ja}y) \rho_{\nu m}^a(x + y) dy. \quad (5.23)$$

Clearly the integration above can be restricted to the set $\{y \in \mathbb{R}^n : |y|_a \leq 2^{-j}\}$; we remark also that by our assumption on j and ν , x is located in some $c\Omega_{jm}$ where $\Omega_{jm} = \{z \in \mathbb{R}^n : |z - 2^{-\nu a}m|_a \leq 2^{-j}\}$ is the anisotropic ball centred at $2^{-\nu a}m$ and radius 2^{-j} (else the integral above would be certainly zero). This can be easily proved if we recall the definition of the rectangle $Q_{\nu m}^a$, the assumption (3.3) and use the generalised triangle inequality for $|\cdot|_a$.

Since k is a smooth function on \mathbb{R}^n we may use the anisotropic Taylor expansion theorem of order L for the function $w \mapsto k(w)$ on the set $\Omega^a(z_x, 2^{j-\nu})$, where $z_x = z(j, \nu, m, x) = 2^{ja}(2^{-\nu a}m - x)$. After that we let $w = 2^{ja}y$ and get

$$k(2^{ja}y) = \sum_{a\alpha \leq L} c_\alpha (2^{ja}y - z_x)^\alpha D^\alpha k(z_x) + R_L(y, x) \quad (5.24)$$

where

$$|R_L(y, x)| \leq c \sum_{a\alpha > L}^{L+a_{max}} 2^{(j-\nu)a\alpha} \quad (5.25)$$

for some positive constant c .

By the moment conditions (3.5) we have $\int_{\mathbb{R}^n} (2^{ja}y - z_x)^\alpha \rho_{\nu m}^a(x+y) dy = 0$ if $a\alpha \leq L$; using (5.24) we may replace (5.23) by:

$$2^{js} k(2^{-j}, \rho_{\nu m}^a)(x) = 2^{js} 2^{jn} \int_{\mathbb{R}^n} R_L(y, x) \rho_{\nu m}^a(x+y) dy \quad (5.26)$$

where the integration can be restricted to the set $\{y \in \mathbb{R}^n : |y|_a \leq 2^{-j}\}$. As in the first step we may choose $L' > L$ such that for all α with $L < a\alpha \leq L + a_{max}$ we have $a\alpha > L'$. Hence by (5.25) and (3.6) we get:

$$\begin{aligned} |2^{js} k(2^{-j}, \rho_{\nu m}^a)(x)| &\leq c 2^{js} 2^{jn} \sum_{a\alpha > L'}^{L+a_{max}} 2^{(j-\nu)a\alpha} \int_{|y|_a \leq 2^{-j}} |\rho_{\nu m}^a(x+y)| dy \\ &\leq c' 2^{js} 2^{jn} 2^{(j-\nu)L'} 2^{-\nu(s-\frac{n}{p})} \int_{|y|_a \leq 2^{-j}} \tilde{\chi}_{\nu m}(x+y) dy \\ &= c' 2^{(j-\nu)(L'+s)} 2^{jn} 2^{\nu\frac{n}{p}} \int_{|y|_a \leq 2^{-j}} \tilde{\chi}_{\nu m}(x+y) dy \end{aligned} \quad (5.27)$$

where $\tilde{\chi}_{\nu m}$ is the characteristic function of some rectangle $cQ_{\nu m}^a$.

Let now χ^{jm} be the characteristic function of the anisotropic ball $c\Omega_{jm}$ where x is located; by straightforward computation we have:

$$\int_{|y|_a \leq 2^{-j}} \tilde{\chi}_{\nu m}(x+y) dy \leq 2^{-\nu n} \chi^{jm}(x). \quad (5.28)$$

Recall $L' > L \geq \sigma_{pq} - s$; we may choose an $\omega < \min(1, p, q)$ such that $L'+s > n(\frac{1}{\omega} - 1) > \sigma_{pq}$. Denoting, as usual, $M^a \chi_{\nu m}^\omega$ the anisotropic Hardy - Littlewood maximal function of $\chi_{\nu m}^\omega$ we get

$$\chi^{jm}(\cdot) \leq c 2^{(\nu-j)\frac{n}{\omega}} (M^a \chi_{\nu m}^\omega(\cdot))^{1/\omega}. \quad (5.29)$$

Finally, using (5.28) and (5.29), the estimate (5.27) becomes:

$$|2^{js} k(2^{-j}, \rho_{\nu m}^a)(x)| \leq c 2^{(j-\nu)(L'+s-n(\frac{1}{\omega}-1))} \left(M^a \chi_{\nu m}^{(p)\omega}(x) \right)^{1/\omega} \quad (5.30)$$

which is in fact

$$|2^{js} k(2^{-j}, \rho_{\nu m}^a)(x)| \leq c 2^{-(\nu-j)\varepsilon} \left(M^a \chi_{\nu m}^{(p)\omega}(x) \right)^{1/\omega} \quad \text{for some } \varepsilon > 0. \quad (5.31)$$

Remark that the terms with $j = 0$ and/or $\nu = 0$ can also be covered by the technique in steps II.1-2.

Step II.3 Using (5.22) and (5.31) we get for $0 < q \leq 1$:

$$\begin{aligned} \left| 2^{js} k(2^{-j}, \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \rho_{\nu m}^a)(x) \right|^q &\leq c \sum_{\nu \leq j} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^q 2^{-\delta(j-\nu)q} \tilde{\chi}_{\nu m}^{(p)q}(x) + \\ &+ c' \sum_{\nu > j} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^q 2^{-\varepsilon(\nu-j)q} \left(M^a \chi_{\nu m}^{(p)\omega}(x) \right)^{q/\omega} \end{aligned} \quad (5.32)$$

for some $\delta, \varepsilon > 0$, with the usual modification if $1 < q \leq \infty$.

We sum over j , take the $\frac{1}{q}$ -th power and then the L_p - quasi-norm and obtain that

$$\left\| \left(\sum_{j=1}^{\infty} 2^{jsq} |k(2^{-j}, \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \rho_{\nu m}^a(\cdot)|^q) \right)^{1/q} \right\|_{L_p}$$

can be estimated from above by

$$\begin{aligned} & c \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^q \tilde{\chi}_{\nu m}^{(p)}(\cdot)^q \right)^{1/q} \right\|_{L_p} \\ & + c' \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^q \left(M^a \chi_{\nu m}^{(p)\omega}(\cdot) \right)^{q/\omega} \right)^{1/q} \right\|_{L_p} \end{aligned} \quad (5.33)$$

with the usual modification if $q = \infty$.

The first term of (5.33) is just what we want since $\tilde{\chi}_{\nu m}^{(p)}$ can be replaced by $\chi_{\nu m}^{(p)}$. With $h_{\nu m} = \lambda_{\nu m} \chi_{\nu m}^{(p)}$ the second term of (5.33) can be written as:

$$c'' \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} M^a h_{\nu m}^{\omega}(\cdot)^{q/\omega} \right)^{\omega/q} \right\|_{L_{p/\omega}}^{\frac{1}{\omega}} \quad (5.34)$$

(usual modification if $q = \infty$). Recall $1 < \frac{p}{\omega} < \infty$ and $1 < \frac{q}{\omega} \leq \infty$ so that we can apply the Fefferman - Stein inequality (4.2) and obtain again what we want.

The term with $j = 0$ can be incorporated by the same technique.

5.2 Proof of the subatomic decomposition theorem

For the given anisotropy $a = (a_1, \dots, a_n)$ let $|\cdot|_a \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be an anisotropic distance function according to (3.15). We begin with a preparation.

Theorem 5.4 *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, let $(\varphi_j)_{j \in \mathbb{N}_0}$ be a smooth anisotropic dyadic resolution of unity and let $\rho \in S$ with $\rho(x) = 1$ if $|x|_a \leq 2$ and $\text{supp } \rho \subset [-\pi, \pi]^n$. The operators $U_\varphi : F_{pq}^{s,a} \rightarrow f_{pq}^a$ and $T_\rho : f_{pq}^a \rightarrow F_{pq}^{s,a}$ defined by:*

$$U_\varphi(g) = \left\{ (2\pi)^{-n/2} 2^{\nu(s-\frac{n}{p})} (\varphi_\nu \hat{g})^\vee (2^{-\nu a} m) : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n \right\} \quad (5.35)$$

if $g \in F_{pq}^{s,a}$ and

$$T_\rho(\lambda) = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} 2^{-\nu(s-\frac{n}{p})} \check{\rho}(2^{\nu a} \cdot -m) \quad (5.36)$$

if $\lambda \in f_{pq}^a$, $\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ are bounded.

Furthermore, $(T_\rho \circ U_\varphi)(g) = g$ for any $g \in F_{pq}^{s,a}$ and $\|U_\varphi(\cdot)\|_{f_{pq}^a}$ is an equivalent quasi-norm on $F_{pq}^{s,a}$.

The same result holds for $B_{pq}^{s,a}$ spaces with $0 < p \leq \infty$ and with b_{pq} in place of f_{pq}^a .

Remark 5.5 This theorem is due to P. Dintelmann, see [Din95a, Theorem 1], and is the anisotropic counterpart of the characterisation of isotropic function spaces by the φ -transform of M. Frazier and B. Jawerth, see [FrJ90] and [FJW91]; he considered in the cited paper more general distance functions but for our purpose the above form of his result will be sufficient. We have to remark that he made the proof using the density of S in $B_{pq}^{s,a}$ and $F_{pq}^{s,a}$ (see 2.3) and, in consequence, restricted to $0 < p < \infty$, $0 < q < \infty$. There is no problem to obtain his result for all admissible values of parameters; this can be done, for example, as in [Tri97, 14.15].

Proof. (Subatomic decomposition theorem)

As usually we present here the proof for $F_{pq}^{s,a}$ spaces; of course the same can be done for $B_{pq}^{s,a}$.

Step 1. Assume that $g \in S'$ is given by (3.18) with (3.19). We show that $g \in F_{pq}^{s,a}$ and that there exists a constant $c > 0$ (independent of g) such that

$$\|g | F_{pq}^{s,a}\| \leq c \sup_{\beta \in \mathbb{N}_0^n} 2^{r\alpha\beta} \|\lambda^\beta | f_{pq}^a\|. \quad (5.37)$$

Since for any given $K \in \mathbb{R}$ and any given $L < 0$ the anisotropic $(s, p) - \beta$ -quarks are anisotropic $(s, p)_{K,L}$ -atoms multiplied with normalising constants which can be estimated from above by $c2^{\kappa\alpha\beta}$ where $c > 0$ and $\kappa > 0$ are independent of β , it follows from the atomic decomposition theorem that for any fixed $\beta \in \mathbb{N}_0^n$:

$$g^\beta = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}^\beta (\beta q u)_{\nu m}^a \quad (5.38)$$

converges in S' , $g^\beta \in F_{pq}^{s,a}$ and $\|g^\beta | F_{pq}^{s,a}\| \leq c 2^{\kappa\alpha\beta} \|\lambda^\beta | f_{pq}^a\|$, where $c > 0$ and $\kappa > 0$ are independent of β (and of course of g). So, for $r > \kappa$

$$\|g^\beta | F_{pq}^{s,a}\| \leq c 2^{-(r-\kappa)\alpha\beta} \sup_{\beta \in \mathbb{N}_0^n} 2^{r\alpha\beta} \|\lambda^\beta | f_{pq}^a\| \quad (5.39)$$

where $c > 0$ is independent of β . Applying now the t -triangle inequality, where $t = \min(1, p, q)$ to $\|\sum_{\beta \in \mathbb{N}_0^n} g^\beta | F_{pq}^{s,a}\|$ we obtain that $g = \sum_{\beta \in \mathbb{N}_0^n} g^\beta$ converges in $F_{pq}^{s,a}$ and that for some $c > 0$ we have (5.37) and this completes the first part of the proof.

Remark that in this step the restriction $s > \sigma_{pq}$ was essentially for the using of the atomic decomposition theorem with no moment conditions required for the atoms.

Step 2. If now $g \in F_{pq}^{s,a}$ we will show that we can decompose it as in (3.18) with (3.19). Let $\rho \in S$ with $\rho(x) = 1$ if $|x|_a \leq 2$ and $\text{supp } \rho \subset [-\pi, \pi]^n$ and let ψ be the function from (3.16). We may assume $r \in \mathbb{N}$.

By the above theorem there exists a sequence $\lambda \in f_{pq}^a$, $\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ such that

$$g(x) = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} 2^{-\nu \left(s - \frac{n}{p}\right)} \sum_{k \in \mathbb{Z}^n} \check{\rho}(2^{\nu a} x - m) \psi(2^{(\nu+r)a} x - k) \quad (5.40)$$

if $x \in \mathbb{R}^n$ where

$$\|\lambda | f_{pq}^a\| \sim \|g | F_{pq}^{s,a}\|. \quad (5.41)$$

The entire analytic function $\check{\rho} \in S$ can be extended from \mathbb{R}^n to \mathbb{C}^n .

Using $c_1(1 + |\xi|)^{1/a_{max}} \leq 1 + |\xi|_a \leq c_2(1 + |\xi|)^{1/a_{min}}$ for some constants $c_1, c_2 > 0$ which depend only on the anisotropy a (see [Leo86]), by the Paley - Wiener - Schwartz theorem we have for any $\varepsilon > 0$ and an appropriate $c_\varepsilon > 0$

$$|\check{\rho}(x + iy)| \leq c_\varepsilon \exp(c|y|) (1 + |x|_a)^{-\varepsilon} \quad (5.42)$$

(see [Tri83, 1.2.1]) where $x \in \mathbb{R}^n, y \in \mathbb{R}^n$.

Iterative application of Cauchy's representation formula in the complex plane yields:

$$\check{\rho}(z) = (2\pi i)^{-n} \int_{|\zeta_1 - z_1|=1} \cdots \int_{|\zeta_n - z_n|=1} \frac{\check{\rho}(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \dots d\zeta_n \quad (5.43)$$

where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. By (5.42) we obtain from (5.43) in particular

$$\left| D^\beta \check{\rho}(x) \right| \leq c'_\varepsilon \beta! (1 + |x|_a)^{-\varepsilon} \quad \text{for } x \in \mathbb{R}^n \quad (5.44)$$

where c'_ε does not depend on $x \in \mathbb{R}^n$ and on the multi-index β .

For every fixed $k \in \mathbb{Z}^n$ we expand $\check{\rho}(2^{\nu a} \cdot - m)$ at the point $2^{-(\nu+r)a}k$ and obtain

$$\check{\rho}(2^{\nu a}x - m) = \sum_{\beta \in \mathbb{N}_0^n} \frac{2^{\nu a \beta} D^\beta \check{\rho}(2^{-ra}k - m)}{\beta!} (x - 2^{-(\nu+r)a}k)^\beta \quad (5.45)$$

and so $\check{\rho}(2^{\nu a}x - m) \psi(2^{(\nu+r)a}x - k)$ can be replaced in (5.40) by

$$\sum_{\beta \in \mathbb{N}_0^n} \frac{D^\beta \check{\rho}(2^{-ra}k - m)}{\beta!} 2^{-ra\beta} \psi^\beta(2^{(\nu+r)a}x - k). \quad (5.46)$$

We get

$$\begin{aligned} g(x) &= \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} 2^{-\nu(s-\frac{n}{p})} \sum_{k \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{N}_0^n} \frac{D^\beta \check{\rho}(2^{-ra}k - m)}{\beta!} 2^{-ra\beta} \psi^\beta(2^{(\nu+r)a}x - k) = \\ &= \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} 2^{-\nu(s-\frac{n}{p})} \psi^\beta(2^{(\nu+r)a}x - k) \left(\sum_{m \in \mathbb{Z}^n} \frac{D^\beta \check{\rho}(2^{-ra}k - m)}{\beta!} 2^{-ra\beta} \lambda_{\nu m} \right) = \\ &= \sum_{\beta \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \lambda_{\nu+r,k}^\beta 2^{r(s-\frac{n}{p})} (\beta q u)_{\nu+r,k}^a(x) \end{aligned} \quad (5.47)$$

where $(\beta q u)_{\nu k}^a(x) = 2^{-\nu(s-\frac{n}{p})} \psi^\beta(2^{\nu a}x - k)$ are the anisotropic $(s, p) - \beta$ -quarks (related to the rectangles $Q_{\nu k}^a$) and

$$\lambda_{\nu+r,k}^\beta = 2^{-ra\beta} \sum_{m \in \mathbb{Z}^n} \frac{D^\beta \check{\rho}(2^{-ra}k - m)}{\beta!} \lambda_{\nu m} = 2^{-ra\beta} \theta_{\nu+r,k}^\beta. \quad (5.48)$$

We may replace in (5.47) $\nu + r$ by ν and obtain (3.18) (recall we assumed $r \in \mathbb{N}$).

Let $\theta^\beta = \{\theta_{\nu k}^\beta : \nu \in \mathbb{N}_0, k \in \mathbb{Z}^n\}$; if we would prove now that there exists a constant $c > 0$ independent of β such that

$$\|\theta^\beta | f_{pq}^a\| \leq c \|\lambda | f_{pq}^a\| \quad (5.49)$$

we would obtain from (5.48) and (5.41)

$$2^{ra\beta} \|\lambda^\beta | f_{pq}^a \| \leq c \|g | F_{pq}^{s,a} \| \quad (5.50)$$

for some $c > 0$ independent of β and this is the counterpart of (3.19).

So it remains to prove (5.49) where the numbers $\theta_{\nu k}^\beta$ are defined from (5.48) by

$$\theta_{\nu+r,k}^\beta = \sum_{m \in \mathbb{Z}^n} \frac{D^\beta \tilde{\rho}(2^{-ra}k - m)}{\beta!} \lambda_{\nu m} \quad \text{if } \nu \in \mathbb{N}_0 \quad \text{and} \quad k \in \mathbb{Z}^n. \quad (5.51)$$

Let now $\nu \in \mathbb{N}_0$ and $k \in \mathbb{Z}^n$ be fixed. By (5.44) there exists a constant $c > 0$ independent of β with

$$\begin{aligned} |\theta_{\nu+r,k}^\beta| &\leq c \sum_{m \in \mathbb{Z}^n} (1 + |2^{-ra}k - m|_a)^{-\varepsilon} |\lambda_{\nu m}| \\ &= c \sum_{m \in \mathbb{Z}^n} (1 + |2^{\nu a}(2^{-(\nu+r)a}k) - m|_a)^{-\varepsilon} |\lambda_{\nu m}|. \end{aligned} \quad (5.52)$$

We denote $x_k = 2^{-(\nu+r)a}k$ and let $m_k \in \mathbb{Z}^n$ such that $x_k \in Q_{\nu, m_k}^a$; then clearly

$$|2^{\nu a}x_k - m_k|_a \leq d$$

for some $d > 0$, where d depends only on the anisotropy and is independent of ν , k and m_k . We decompose \mathbb{Z}^n in the sets $E_j = \{m \in \mathbb{Z}^n : 2^j - 1 \leq |m - m_k|_a < 2^{j+1} - 1\}$ where $j \in \mathbb{N}_0$. If j is fixed, for $m \in E_j$ we have on the one hand

$$\begin{aligned} 2^j \leq 1 + |m - m_k|_a &\leq c (1 + |2^{\nu a}x_k - m_k|_a + |2^{\nu a}x_k - m|_a) \\ &\leq c' (1 + |2^{\nu a}x_k - m|_a) \end{aligned} \quad (5.53)$$

where $c' > 0$ is independent of ν , k , m and so

$$(1 + |2^{\nu a}x_k - m|_a)^{-\varepsilon} \leq c 2^{-j\varepsilon}. \quad (5.54)$$

On the other hand, if $x \in Q_{\nu+r,k}^a$ and $y \in Q_{\nu m}^a$ using

$$|y - x|_a \leq c (|y - 2^{-\nu a}m|_a + |2^{-\nu a}m - 2^{-\nu a}m_k|_a + |2^{-\nu a}m_k - x_k|_a + |x_k - x|_a)$$

we get

$$|y - x|_a \leq c' 2^{-\nu} (1 + |m - m_k|_a) \leq C 2^{j-\nu} \quad (5.55)$$

where $C > 0$ is independent of ν , k , m but may depend on r .

Choose now $0 < \omega < \min(1, p, q)$; for a fixed ν the rectangles $Q_{\nu m}^a$ have the volume $2^{-\nu n}$ and are disjoint so that using the embedding $l_\omega \hookrightarrow l_1$ and (5.55) we obtain:

$$\begin{aligned} \sum_{m \in E_j} |\lambda_{\nu m}| &\leq \left(\sum_{m \in E_j} |\lambda_{\nu m}|^\omega \right)^{1/\omega} \\ &= \left(2^{\nu n} \int_{|y-x|_a \leq C 2^{j-\nu}} \left(\sum_{m \in E_j} |\lambda_{\nu m}| \chi_{\nu m}(y) \right)^\omega dy \right)^{1/\omega} \\ &\leq c \left(2^{jn} M^a \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| \chi_{\nu m} \right)^\omega(x) \right)^{1/\omega} \end{aligned} \quad (5.56)$$

for $x \in Q_{\nu+r,k}^a$ and where M^a is the anisotropic Hardy - Littlewood maximal function and all the constants are independent of ν, k, m .

Let $\chi_{\nu+r,k}$ be the characteristic function of the rectangle $Q_{\nu+r,k}^a$ and $\chi_{\nu+r,k}^{(p)} = 2^{(\nu+r)\frac{n}{p}} \chi_{\nu+r,k}$. Using (5.54) and (5.56) in (5.52) and assuming that $\varepsilon > n/\omega$ is sufficiently large we have

$$\begin{aligned}
\left| \theta_{\nu+r,k}^\beta \chi_{\nu+r,k}^{(p)}(x) \right| &\leq c_1 \sum_{m \in \mathbb{Z}^n} (1 + |2^{\nu a} x_k - m|_a)^{-\varepsilon} |\lambda_{\nu m}| 2^{(\nu+r)\frac{n}{p}} \chi_{\nu+r,k}(x) \\
&\leq c_2 \sum_{j=0}^{\infty} 2^{-j\varepsilon} \sum_{m \in E_j} |\lambda_{\nu m}| 2^{\nu \frac{n}{p}} \chi_{\nu+r,k}(x) \\
&\leq c_3 \sum_{j=0}^{\infty} 2^{-j(\varepsilon - \frac{n}{\omega})} \left(M^a \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| \chi_{\nu m}^{(p)} \right) (x) \right)^{\omega} \chi_{\nu+r,k}(x) \\
&\leq c \left(M^a \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| \chi_{\nu m}^{(p)} \right) (x) \right)^{1/\omega} \chi_{\nu+r,k}(x) \tag{5.57}
\end{aligned}$$

where the constants above do not depend on ν and k but may depend on r .

In (5.57) we take the q -th power, sum over $k \in \mathbb{Z}^n$ and then over $\nu \in \mathbb{N}_0$ and get

$$\sum_{\nu=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| \theta_{\nu k}^\beta \chi_{\nu k}^{(p)}(x) \right|^q \leq c \sum_{\nu=0}^{\infty} (M^a h_\nu^\omega(x))^{q/\omega} \tag{5.58}$$

(with the usual modification if $q = \infty$) where $h_\nu = \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| \chi_{\nu m}^{(p)}$.

Taking the $\frac{1}{q}$ -th power and the L_p - quasi-norm we obtain that $\|\theta^\beta \| f_{pq}^a \|$ can be estimated from above by

$$c \left\| \left((M^a h_\nu^\omega(\cdot))^{1/\omega} \right)_{\nu \in \mathbb{N}_0} \middle| L_p(l_q) \right\| = c \left\| (M^a h_\nu^\omega(\cdot))_{\nu \in \mathbb{N}_0} \middle| L_{p/\omega}(l_{q/\omega}) \right\|^{1/\omega} \tag{5.59}$$

(with the usual modification if $q = \infty$). To obtain (5.49) we have now only to apply the Fefferman - Stein inequality (4.2) to the right-hand side of (5.59); this can be done since $1 < \frac{p}{\omega} < \infty$ and $1 < \frac{q}{\omega} \leq \infty$ and so the proof is finished.

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Selbständigkeitserklärung

Ich erkläre, daß ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Hilfsmittel und Literatur angefertigt habe.

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