Lecture Quantitative Finance
Spring Term 2015

Prof. Dr. Erich Walter Farkas

Lecture 1: February 19, 2015
Administrative information

- **Room:** HAH E 11 at UZH
- **Thursday, 12.15 - 13.45:** no break!
- **First lecture:** Thursday, February 19, 2015
- **No lecture:** Thursday, April 09, 2015: Easter Holidays
- **No lecture:** Thursday, May 14, 2015: Ascension Day
- **Last lecture:** Thursday, May 28, 2015
- **Exam date:** Thursday, June 4, 2015, 12.00 - 14.00
- **Exam location:** Room: TBD at UZH
- **Exam details:** closed books
- **Material:** see OLAT
Target audience of this course

- UZH – MA: Pflichtmodule BF
- UZH ETH – Master of Science in Quantitative Finance (elective area: MF)
- anybody interested in an introduction to quantitative finance
Goals of this course

At the end of this course you

- will be able to understand and apply the fundamental concepts of quantitative finance;
- will have learned the (fundamental) aspects of valuing financial instruments (bonds, forwards, options, etc.) and the role of asset price sensitivities;
- will have the ability to comprehend and manage (market) risk and to use quantitative techniques to model these risks.
Selected literature

Selected literature

Lecturers

Prof. Dr. Erich Walter Farkas

- Dipl. Math., MSc. Math: University of Bucharest
- Dr. rer. nat.: Friedrich-Schiller-University of Jena
- Habilitation: Ludwig-Maximilians-University of Munich
- Since 1. Oct. 2003 at UZH & ETH: PD (reader) and wissenschaftlicher Abteilungsleiter (Director) in charge for the UZH ETH – Quantitative Finance Master first joint degree of UZH and ETH
- Since 1. Feb. 2009: Associate Professor for Quantitative Finance at UZH Program Director MSc Quantitative Finance (joint degree UZH ETH)
- Associate Faculty, Department of Mathematics, ETH Zürich
- Faculty member of the Swiss Finance Institute

PhD students

- Giada Bordogna
- Fulvia Fringuellotti
- Kevin Meyer
Guest lecturers

- Dr. Pedro Fonseca, Former Head Risk Analytics & Reporting, SIX Management AG
- Marek Krynski, Executive Director at UBS
- Robert Huitema, Associate Director at UBS
Contact details

- kevin.meyer@bf.uzh.ch
- Office: Plattenstrasse 22, 8032 Zurich
- Appointment via E-mail is kindly requested
Further lectures in Mathematical / Quantitative Finance

- **Direct continuation of this lecture**
  - **Fall 2015:**
    Mathematical Foundations of Finance
    ETH: W. Farkas, M. Schweizer
  - **Spring 2016:**
    Continuous Time Quantitative Finance
    UZH: M. Chesney

- **Related lectures**
  - **Fall 2015:** Financial Engineering
  - **Spring 2015 and Spring 2016:** Asset Management; Quantitative Risk Management
Chapter 1: Bond fundamentals
Bonds: Definition and Examples

- Bonds are financial claims which entitle the holder to receive a stream of periodic payments, known as *coupons*, as well as a final payment, known as the *principal* (or *face value*).

- In practice, depending on the nature of the issues, one distinguishes between different types of bonds; important examples include:
  - **Government or Treasury Bonds**: issued by governments, primarily to finance the shortfall between public revenues and expenditures and to pay off earlier debts;
  - **Municipal Bonds**: issued by municipalities, e.g., cities and towns, to raise the capital needed for various infrastructure works such as roads, bridges, sewer systems, etc.;
  - **Mortgage Bonds**: issued by special agencies who use the proceeds to purchase real estate loans extended by commercial banks;
  - **Corporate Bonds**: issued by large corporations to finance the purchase of property, plant and equipment.

- Among all assets, the simplest (most basic) to study are fixed-coupon bonds as their cash-flows are predetermined.

- The valuation of bonds requires a good understanding of concepts such as *compound interest, discounting, present value* and *yield*.

- For **hedging** and **risk management** of bond portfolios (risk) sensitivities such as *duration* and *convexity* are important.
A zero-coupon bond promises no coupon payments, only the repayment of the principal at maturity.

Consider an investor who wants a zero-coupon bond, which
- pays 100 CHF
- in 10 years, and
- has no default risk.

Since the payment occurs at a future date – in our case after 10 years – the value of this investment is surely less than an up-front payment of 100 CHF.

To value this payment one needs two ingredients:
- the prevailing interest rate, or yield, per period
- and the tenor, denoted $T$, which gives the number of periods until maturity expressed in years.
• The **present value** (PV) of a zero-coupon bond can be computed as:

\[
PV = \frac{C_T}{(1 + y)^T},
\]

where \(C_T\) is the principal (or face value) and \(y\) is the discount rate.

• For instance, a payment of \(C_T = 100\) CHF in 10 years discounted at 6% is (only) worth 55.84 CHF.

**Note:**

• The (market) value of zero-coupon bonds decreases with longer maturities;

• keeping \(T\) fixed, the value of the zero-coupon bond decreases as the yield increases.
• Analogously to the notion of *present value*, we can define the notion of *future value* (FV) for an initial investment of amount PV:

\[ FV = PV \times (1 + y)^T \]

• For example, an investment now worth \( PV = 100 \text{ CHF} \) growing at 6% per year will have a future value of 179.08 CHF in 10 years.
• The internal rate of return of a bond, or annual growth rate, is called the *yield*, or *yield-to-maturity* (YTM).

• Yields are usually easier to deal with than CHF values.

• Rates of return are directly comparable across assets (when expressed in percentage terms and on an annual basis).

• The yield $y$ of a bond is the solution to the (non-linear) equation:

$$P = P(y),$$

where “$P$” is the (market) price of the bond and $P(\cdot)$ is the price of the bond as a function of the yield $y$; in case of a zero-coupon bond

$$P(y) := \frac{C_T}{(1 + y)^T}.$$
The yield of bonds with the same characteristics but with different maturities can differ strongly; i.e., the yield (usually) depends upon the maturity of the bond.

The *yield curve* is the set of yields as a function of maturity.

Under “normal” circumstances, the yield curve is upward sloping; i.e., the longer you lock in your money, the higher your return.
**Important:** state the method used for compounding:

- **annual compounding** (usually the norm):

\[
PV = \frac{C_T}{(1 + y)^T}.
\]

- **semi-annual compounding** (e.g. used in the U.S. Treasury bond market):

interest rate \( y_s \) is derived from:

\[
PV = \frac{C_T}{(1 + y_s/2)^{2T}}
\]

where \( 2T \) is the number of periods.

- **continuous compounding** (used ubiquitously in the quantitative finance literature) interest rate \( y_c \) is derived from:

\[
PV = \frac{C_T}{\exp(y_c T)}.
\]
**Example:** Consider our example of the zero-coupon bond, which pays 100 CHF in 10 years, once again. Recall that the PV of the bond is equal to 55.8395 CHF. Now, we can compute the 3 yields as follows:

- **annual compounding:**
  
  \[
PV = \frac{C_T}{(1 + y)^T} \Rightarrow y = 6\%\]

- **semi-annual compounding:**
  
  \[
PV = \frac{C_T}{(1 + y_s/2)^{2T}} \Rightarrow (1 + y_s/2)^{2} = 1 + y \Rightarrow y_s = 5.91\%\]

- **continuously compounding:**
  
  \[
PV = \frac{C_T}{\exp(y_c T)} \Rightarrow \exp(y_c) = 1 + y \Rightarrow y_c = 5.83\%\]

**Note:** increasing the (compounding) frequency results in a lower equivalent yield.
**Exercise L01.1:** Assume a semi-annual compounded rate of 8% (per annum). What is the equivalent annual compounded rate?

1. 9.20%
2. 8.16%
3. 7.45%
4. 8.00%.

**Exercise L01.2:** Assume a continuously compounded rate of 10% (per annum). What is the equivalent semi-annual compounded rate?

1. 10.25%
2. 9.88%
3. 9.76%
4. 10.52%.
While zero coupon bonds are a very useful (theoretical) concept, the bonds usually issued and traded are coupon bearing bonds.

Note:

- A zero coupon bond is a special case of a coupon bond (with zero coupon);
- and a coupon bond can be seen as a portfolio of zero coupon bonds.
Consider now the price (or present value) of a coupon bond with a general pattern of fixed cash-flows. We define the price-yield relationship as follows:

\[ P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}. \]

Here we have adopted the following notations:

- \( C_t \): the cash-flow (coupon or principal) in period \( t \);
- \( t \): the number of periods (e.g. half-years) to each payment;
- \( T \): the number of periods to final maturity;
- \( y \): the discounting yield.
• As indicated earlier, the typical cash-flow pattern for bonds traded in reality consists of regular coupon payments plus repayment of the principal (or face value) at the expiration.

• Specifically, if we denote \( c \) the coupon rate and \( F \) the face value, then the bond will generate the following stream of cash flows:

\[
\begin{align*}
C_t &= cF & \text{prior to expiration} \\
C_T &= cF + F & \text{at expiration}.
\end{align*}
\]

• Using this particular cash-flow pattern, we can arrive (with the use of the geometric series formula) arrive at a more compact formula for the price of a coupon bond:

\[
P = \frac{cF}{1+y} + \frac{cF}{(1+y)^2} + \cdots + \frac{cF}{(1+y)^{T-1}} + \frac{cF + F}{(1+y)^T} \\
= cF \cdot \frac{\frac{1}{1+y} - \frac{1}{(1+y)^{T+1}}}{1 - \frac{1}{1+y}} + \frac{F}{(1+y)^T} \\
= \frac{cF}{y} \cdot \left(1 - \frac{1}{(1+y)^T}\right) + \frac{F}{(1+y)^T}.
\]
**Remark.** If the coupon rate matches the yield \((c = y)\) (using the same compounding frequency) then the price of the bond equals its face value; such a bond is said to be priced *at par*.

**Example:** Consider a bond that pays 100 CHF in 10 years and has a 6% annual coupon.

a.) What is the market value of the bond if the yield is 6%?

b.) What is the market value of the bond if the yield falls to 5%?

**Solution:** The cash flows are \(C_1 = 6, C_2 = 6, \ldots, C_{10} = 106\). Discounting at 6% gives PVs of 5.66, 5.34, ..., 59.19, which sum up to 100 CHF; so the bond is selling at par. Alternatively, discounting at 5% leads to a price of 107.72 CHF.
Exercise L01.3: Consider a 1-year fixed-rate bond currently priced at 102.9 CHF and paying a 8% coupon (semi-annually). What is the yield of the bond?

1. 8%
2. 7%
3. 6%
4. 5%.
Another special case of a general coupon bond is the so-called \emph{perpetual bond}, or \emph{consol}.

These are bonds with regular coupon payments of $C_t = cF$ and with infinite maturity.

The price of a consol is given by:

$$ P = \frac{c}{y} F. $$
Derivation:

\[ P = \frac{cF}{1+y} + \frac{cF}{(1+y)^2} + \frac{cF}{(1+y)^3} + \cdots \]

\[ = cF \left[ \frac{1}{1+y} + \frac{1}{(1+y)^2} + \frac{1}{(1+y)^3} + \cdots \right] \]

\[ = cF \frac{1}{1+y} \left[ 1 + \frac{1}{(1+y)} + \frac{1}{(1+y)^2} + \cdots \right] \]

\[ = cF \frac{1}{1+y} \left[ \frac{1}{1 - (1/(1+y))} \right] \]

\[ = cF \frac{1}{1+y} \frac{1+y}{y} \]

\[ = \frac{cF}{y} \]
We will now address the question: what happens to the price of the bond when the yield changes from its initial value say, $y_0$, to a new value $y_1 = y_0 + \Delta y$, where $\Delta y$ is assumed to be ‘small’.

Assessing the effect of changes in risk factors (in our case, the yield) on the price of assets is of key importance for hedging and risk management.

We start from the price-yield relationship $P = P(y)$. We now have an initial value of the bond $P_0 = P(y_0)$, and a new value of the bond $P_1 = P(y_1)$.

For a ‘small’ yield change $\Delta y$, we can approximate $P_1$ from a Taylor expansion,

$$P_1 = P_0 + P'(y_0)\Delta y + \frac{1}{2} P''(y_0)(\Delta y)^2 + \ldots.$$

This is an infinite expansion with increasing powers of $\Delta y$; only the first two terms (linear and quadratic) are usually used by finance practitioners.
The first- and second-order derivative of the bond price w.r.t. yield are very important, so they have been given special names.

The negative of the first-order derivative is the *dollar duration* (DD):

\[ DD = -P'(y) = -\frac{dP}{dy} = D^* \times P, \]

where \( D^* \) is the *modified duration*.

Another duration measure is the so-called *Macaulay duration* (D), which is defined as:

\[ D = \frac{1}{P} \left( \sum_{t=1}^{T} \frac{t \times cF}{(1+y)^t} + \frac{T \times F}{(1+y)^T} \right). \]

Often risk is measured as the *dollar value of a basis point* (DVBP) (also known as DV01):

\[ DV01 = (D^* \times P) \times BP, \]

where BP stands for *basis point* (= 0.01%).
The second-order derivative is the dollar convexity (DC):

\[ DC = P''(y) = \frac{d^2 P}{dy^2} = \kappa \times P, \]

where \( \kappa \) is called the convexity.

For fixed-coupon bonds, the cash-flow pattern is known and we have an explicit price-yield function; therefore one can compute analytically the first- and second-order derivatives.
Example: Recall that a zero-coupon bond has only payment at maturity equal to the face value \( C_T = F \):

\[
P(y) = \frac{F}{(1 + y)^T}.
\]

Then, we have \( D = T \) and

\[
\frac{dP}{dy} = (-T) \times \frac{F}{(1 + y)^{T+1}} = -\frac{T}{1 + y} \times P,
\]

so the modified duration is \( D^* = \frac{T}{1 + y} \). Additionally, we have

\[
\frac{d^2 P}{dy^2} = -(T + 1) \times (-T) \times \frac{F}{(1 + y)^{T+2}} = \frac{(T + 1)T}{(1 + y)^2} \times P,
\]

so that the convexity is \( \kappa = \frac{(T+1)T}{(1+y)^2} \).
Remarks:

- note the difference between the modified duration \( D^* = T/(1 + y) \) and the Macaulay duration \( D = T \);
- duration is measured in periods, like \( T \);
- considering annual compounding, duration is measured in years, whereas with semi-annual compounding duration is in half-years and has to be divided by two for conversion to years;
- dimension of convexity is expressed in periods squared;
- considering semi-annual compounding, convexity is measured in half-years squared and has to be divided by four for conversion to years squared.
Summary: Using the duration-convexity terminology developed so far, we can rewrite the Taylor expansion for the change in the price of a bond, as follows:

\[ \Delta P = - (D^* \times P_0) (\Delta y) + \frac{1}{2} (\kappa \times P_0) (\Delta y)^2 + \cdots, \]

where

- duration measures the first-order (linear) effect of changes in yield,
- convexity measures the second-order (quadratic) term,

and recall that \( P_0 = P(y_0) \).
Example: Consider a zero-coupon bond with $T = 10$ years to maturity and a yield of $y_0 = 6\%$ (semi-annually). The (initial) price of this bond is $P_0 = 55.368$ CHF as obtained from:

$$P_0 = \frac{100}{(1 + \frac{6\%}{2})^{2 \cdot 10}} = 55.368.$$  

We now compute the various sensitivities of this bond:

- Macaulay duration $D = T = 10$ years;
- Modified duration is given by $\frac{dP_0}{dy/2} = -D^* \times P_0$:
  $$D^* = \frac{2 \cdot 10}{1 + \frac{6\%}{2}} = 19.42 \text{ half-years,}$$
  or $D^* = 9.71$ years;
- Dollar duration $DD = D^* \times P_0 = 9.71 \times 55.37 = 537.55$;
- Dollar value of a basis point is $DVBP = DD \times 0.0001 = 0.0538$;
- Convexity is
  $$\frac{21 \times 20}{(1 + \frac{6\%}{2})^2} = 395.89 \text{ half-years squared,}$$
  or $\kappa = 98.97$ years squared.
Finally, we can now turn to the problem of estimating the change in the value of the bond if the yield goes from $y_0 = 6\%$ to, say, $y_1 = 7\%$, i.e., $\Delta y = 1\%$:

$$
\Delta P \sim - (D^* \times P_0) (\Delta y) + \frac{1}{2} (\kappa \times P_0) (\Delta y)^2 \\
= - (9.71 \times 55.37) \cdot 1\% + \frac{1}{2} (98.97 \times 55.37) \cdot (1\%)^2 \\
= -5.101.
$$

Note that the exact value for the yield $y_1 = 7\%$ is 50.257 CHF. Thus:

- using only the first term in the expansion, the predicted price is $55.368 - 5.375 = 49.992$ CHF, and
- the linear approximation has a pricing error of $-0.53\%$ (not bad given the large change in the yield).
- Using the first two terms in the expansion, the predicted price is $55.368 - 5.101 = 50.266$ CHF,
- thus adding the second term reduces the approximation error to 0.02%.
Exercise L01.4: What is the price impact of a 10-BP increase in yield on a 10-year zero-coupon bond whose price, duration and convexity are $P = 100$ CHF, $D = 7$ and $\kappa = 50$, respectively.

1. $-0.705$
2. $-0.700$
3. $-0.698$
4. $-0.690$. 
Having done the numerical calculations, it is now helpful to have a graphical representation of the duration-convexity approximation. The graph (on the next slide) compares the following three curves:

1. the actual, exact price-yield relationship:

   \[ P = P(y); \]

2. the duration based estimate (first-order approximation):

   \[ P = P_0 - D^* P_0 \cdot \Delta y; \]

3. the duration and convexity estimate (second-order approximation):

   \[ P = P_0 - D^* \times P_0 \cdot \Delta y + \frac{1}{2} \kappa \times P_0 \times (\Delta y)^2. \]
1. Bond fundamentals

Bonds:
- Definition and Examples
- Zero-Coupon Bonds
- Coupon Bonds
- Price-yield relationship

Yield Sensitivity and Duration

Answers to the Exercises

Figure: The price-yield relationship and the duration-convexity based approximation. **Solid black**: The exact price-yield relationship, **Dashed gray**: Linear and Second order approximations.
Conclusions:

- for small movements in the yield, the duration-based linear approximation provides a reasonable fit to the exact price; including the convexity term, increases the range of yields over which the approximation remains reasonable;

- Dollar duration measures the (negative) slope of the tangent to the price-yield curve at the starting point $y_0$;

- when the yield rises, the price drops but less than predicted by the tangent; if the yield falls, the price increases faster than the duration model. In other words, the quadratic term is always beneficial.
Notes:

- In economic terms, duration is the average time to wait for each payment weighted by their present values.

- For the standard bonds considered so far, we have been able to compute duration and convexity analytically. However, in practice there exist bonds with more complicated features (such as mortgage-backed securities with an embedded prepayment option), for which it is not possible to compute duration and convexity in closed form.

- Instead, we need to resort to numerical method, in particular, approximating the bond price sensitivities with finite differences.
Choose a change in the yield, $\Delta y$, and reprice the bond under an up-move scenario $P_+ = P(y_0 + \Delta y)$ and a down-move scenario $P_- = P(y_0 - \Delta y)$.

Then approximate the first-order derivative with a centered finite difference. From

$$D^* = -\frac{1}{P} \frac{dP}{dy}$$

effective duration is estimated as:

$$D^* \approx -\frac{1}{P_0} \times \frac{P_+ - P_-}{2\Delta y} = \frac{1}{P_0} \times \frac{P(y_0 - \Delta y) - P(y_0 + \Delta y)}{2\Delta y}.$$  

Similarly, from

$$\kappa = \frac{1}{P} \frac{d^2P}{dy^2}$$

effective convexity is estimated as:

$$\kappa \approx \frac{1}{P_0} \times \left[ \frac{P(y_0 - \Delta y) - P_0}{\Delta y} - \frac{P_0 - P_0(y_0 + \Delta y)}{\Delta y} \right] \times \frac{1}{\Delta y}.$$
Exercise L01.1: b) This is derived from \((1 + \frac{y_s}{2})^2 = (1 + y)\) or, equivalently, 
\((1 + 0.08/2)^2 = (1 + y)\), which gives \(y = 8.16\%\). compounding.

Exercise L01.2: a) This is derived from \((1 + \frac{y_s}{2})^2 = \exp(y_c)\) or, equivalently, 
\((1 + y_s/2)^2 = 1.1056\), which gives \(y_s = 10.25\%\).

Exercise L01.3: d) We need to find \(y\) such that 
\[
\frac{4}{(1 + y_s/2)} + \frac{104}{(1 + y_s/2)^2} = 102.9.
\] Solving, we find \(y_s = 5\%\).

Exercise L01.4: c) The initial price is \(P_0 = 100\). The yield increase is 10-BP, which means \(\Delta y = 10 \cdot 0.0001 = 0.001\). The price impact is
\[
\Delta P = -(D^* \times P_0)(\Delta y) + \frac{1}{2} (\kappa \times P_0)(\Delta y)^2
\]
\[
= -7 \cdot 100 \cdot (0.001) + \frac{1}{2} \cdot 50 \cdot 100 \cdot (0.001)^2
\]
\[
= -0.6975.
\]