Lecture Quantitative Finance Spring Term 2015

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Remember

- Previous chapters: introduction to the theory of options
- put-call parity
- fundamentals of option valuation
  - pricing by replication
  - risk-neutral pricing
  - last lecture: Black-Scholes PDE and formulas

In order to value an option:
a need to develop a mathematical description of how the underlying asset behaves!

Asset price movement: Assumptions

In order to value an option:
we need to develop a mathematical description of how the underlying asset behaves!

Assumptions:

- (weak form of the) Efficient market hypothesis
  - reasonable assumption: the market responds instantaneously to external influences
  - the current asset price reflects all past information
  - if one wants to predict the asset price at some future time, knowing the complete history of the asset price gives no advantage over just knowing its current price
  - from a modeling point of view, if we take on board the efficient market hypothesis, an equation to describe the evolution of the asset from time t to $t + \Delta t$ needs to involve the asset price only at time t and not at any earlier times

Assumptions

- the asset price may take any non-negative value
- buying and selling an asset may take place at any time $0 \le t \le T$
- it is possible to buy and sell any amount of the asset
- there are no transaction costs
- there are no dividend payments
- short selling is allowed: it is possible to hold a negative amount of the asset
- there is a single, constant, risk-free interest rate that applies to any amount of money borrowed from or deposited in a bank

Questions

There are many practical issues to address before a serious statistical analysis of stock market data can be performed:

- there may be missing data, if no trading took place between times t_i and t_{i+1}
- the data may require adjustments to account for dividends and stock splits
- when determining the time interval $t_{i+1} - t_i$ between price data, a decision must be made about whether to keep the clock running when the stock market has closed. Does Friday night to Monday morning count as 2.5 days, or zero days?
- for an asset that is not heavily traded, the time of the last trade may vary considerably from day to day. Consequently, daily closing prices, which pertain to the final trade for each day, may not relate to equally spaced samples in time

Motivation

- we have defined what we mean by a European call or put option on an underlying asset
- we have developed a model for the asset price movement
- key question: what is an option worth? Can we systematically determine a fair value of the option at $t = 0$?
- our basic aim is to value an option at time $t = 0$ with asset price $S(0) = S_0$ but we will look for a function $V(S, t)$ that gives the option value for any asset price $S \geq 0$ at any time $0 \leq t \leq T$
- in this setting we look for $V(S_0, 0)$
- our analysis will lead us to the celebrated Black-Scholes partial differential equation (PDE) for the function V
- the approach is quite general and the PDE is valid in particular for the cases where $V(S, t)$ corresponds to the value of a European call and put

Black-Scholes PDE

- we arrive at the famous Black-Scholes partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- it is a relationship between V, S, t and certain partial derivatives of V
- note that
    - the drift parameter μ in the asset model does not appear in the PDE
    - the PDE is satisfied for **any** option on S whose value can be expressed as some smooth function $V(S, t)$

Black-Scholes formula for a call option

- let $C(S,t)$ denote the European call option value
- we know for certain that at expiry, $t = T$, the payoff is $\max(S(T) - K, 0)$, therefore

$$C(S,T) = \max(S(T) - K, 0) \qquad (1)$$

- now from the continuous time model formula we get that if the asset price is ever zero, then $S(t)$ remains zero for all the time and hence the payoff will be zero at expiry, therefore:

$$C(0,t) = 0 \quad \text{for all} \quad 0 \leq t \leq T \qquad (2)$$

- the last two conditions are so-called boundary conditions
- one has also final conditions

$$C(S,t) \approx S, \quad \text{for large} \quad S \qquad (3)$$

Black-Scholes formula for a call option

- imposing the conditions (1), (2) and (3) from above (the boundary conditions and the final condition) it is possible to show that there exists a unique solution for the call option value
- this solution is

$$C(S,t) = S \cdot N(d_1) - K \cdot e^{-r(T-t)} \cdot N(d_2)$$

where

- $N(\cdot)$ is the $N(0,1)$ distribution function:

$$N(d) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{s^2}{2}}\, ds$$

- the number d_1 is given by

$$d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

- the number d_2 is given by $d_2 = d_1 - \sigma\sqrt{T-t}$

Black-Scholes formula for a put option

- can be derived directly imposing appropriate boundary conditions and a final condition
- simpler: derived using the put-call parity relation

$$C(S,t) - P(S,t) = S - K \cdot e^{-r(T-t)}$$

- the solution is

$$P(S,t) = K \cdot e^{-r(T-t)} \cdot N(-d_2) - S \cdot N(-d_1)$$

Black-Scholes formula: comments

- the two classic references are the paper (Black and Scholes 1973) by Fischer Black and Myron Scholes which derives the key equations and the paper (Merton, 1973) by Robert Merton which adds a rigorous mathematical analysis
- Merton and Scholes were awarded the 1997 Nobel Prize in Economic Sciences for this work (Fischer Black died in 1995)
- a heuristics discrete-time treatment of hedging can be found in the expository article of Almgren (2002)
- modern texts that give rigorous derivations of the Black-Scholes formula include Björk (1998), Duffie (2001), Karatzas & Shreve (1998), Oksendal (1998)
- it is possible to weaken the boundary conditions in the Black-Scholes PDE without sacrificing the uniqueness of the solution

Remember

- Black-Scholes partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- Black-Scholes formula to value a call option:

$$C(S,t) = S \cdot N(d_1) - K \cdot e^{-r(T-t)} \cdot N(d_2),$$

where

- $N(\cdot)$ is the $N(0,1)$ distribution function:
- the number d_1 is given by

$$\begin{aligned} d_1 &= \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\log(S/(K \cdot e^{-r(T-t)}))}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2} \end{aligned}$$

- the number d_2 is given by $d_2 = d_1 - \sigma\sqrt{T-t}$

Example: Computing the Black-Scholes value

Example. Consider an at-the money call

- on a stock worth $S = 100$
- with a strike price $K = 100$
- and maturity of six months;
- the risk free rate is $r = 5\%$
- the stock has annual volatility $\sigma = 20\%$ and pays no dividend

Solution.

- the present value factor $e^{-r(T-t)} = \exp(-5\% \times 6/12) = 0.9753$
- the value of d_1:

$$d_1 = \frac{\log(S/(K \cdot e^{-r(T-t)}))}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2} = 0.2475$$

- $d_2 = d_1 - \sigma\sqrt{T-t} = 0.1061$

Example: Computing the Black-Scholes value (2)

- using standard normal tables we find $N(d_1) = 0.5977$ and $N(d_2) = 0.5422$
- note that both values are greater than 0.5 since d_1 and d_2 are positive
- the value of the call is $c = S \cdot N(d_1) - K \cdot e^{-r(T-t)} \cdot N(d_2) = 6.89$
- the value of the call can also be viewed as an equivalent position of $N(d_1) = 59.77\%$ in the stock and some borrowing: $c = 59.77 - 52.88 = 6.89$; thus this is a leveraged position in the stock
- the value of the put is 4.42
- buying the call and selling the put costs $6.89 - 4.42 = 2.47$; this indeed equals $S - K \cdot e^{-r(T-t)} = 100 - 97.53 = 2.47$ which confirms the put-call parity

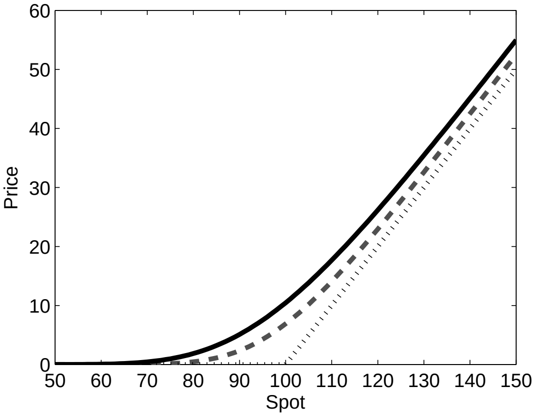

Figure: European call price with time to maturity 1 year (solid black) and 6 months (dashed gray).

Exercise: Computing the Black-Scholes value

Exercise 1. Using the Black-Scholes model, calculate the value of a European call option given the following information:

- spot rate 100
- strike price $K = 110$
- time to expiry 0.5 years;
- $N(d_1) = 0.457185$
- $N(d_2) = 0.374163$

The correct answer is

1. 10.90
2. 9.51
3. 6.57
4. 4.92

Evaluating options

- Notation
  - S_t = current spot price of the asset
  - F_t = current forward price of the asset
  - K = exercise price of the option contract
  - f_t = current value of the derivative instrument
  - r_t = domestic risk-free rate
  - r^* = income payment on the asset (the annual rate of dividend or coupon payments on a stock index or bond)
  - σ_t = annual volatility of the rate of change in S
  - $\tau = T - t$ = time to maturity
- for most options we can write the value of the derivative as a function

$$f_t = f(S_t, r_t, r_t^*, \sigma_t, K, T - t)$$

Evaluating options

- the contract specifications are represented by K and the time to maturity $\tau = T - t$
- the other factors are affected by market movements, creating volatility in the values of the derivative
- usually we are interested in the movements in f; the exposure profile of the derivative can be described locally by taking a Taylor expansion:

$$df = \frac{\partial f}{\partial S}\, dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\, dS^2 + \frac{\partial f}{\partial r}\, dr + \frac{\partial f}{\partial r^*}\, dr^* + \frac{\partial f}{\partial \sigma}\, d\sigma + \frac{\partial f}{\partial \tau}\, d\tau + \ldots$$

- because the value depends on S in a nonlinear fashion, we added a quadratic term for S

Evaluating options

- option pricing is about finding f
- option hedging uses the partial derivatives
- risk management is about combining those with the movements in the risk factors
- the Taylor approximation may fail for a number of reasons:
  - large movements in the underlying risk factor
  - highly nonlinear exposure (such as options near expiry or exotic options)
  - cross-partial effects (such as σ changing in relation with S)

Motivation for studying the option greeks

- the Black-Scholes option valuation formulas depend upon S, t, and the parameters K, r and σ
- we derive expressions for partial derivatives of the option values with respect to these quantities
- useful
  - traders like to know the sensitivity of the option value to changes in these quantities; the sensitivities can be measured by these partial derivatives
  - computing the partial derivatives allows us to confirm that the Black-Scholes PDE has been solved
  - examining the signs of the derivatives gives insights into the underlying formulas
  - the derivative $\partial V/\partial S$ is needed in the delta hedging process
  - the derivatives $\partial V/\partial \sigma$ plays a role later when we discuss the implied volatility
- we focus on the case of a call option

The greeks

- certain partial derivatives of the option value are so widely used that they have been assigned Greek names and symbols:

$$
\begin{aligned}
\Delta &:= \frac{\partial C}{\partial S} && \text{delta} \\
\Gamma &:= \frac{\partial^2 C}{\partial S^2} && \text{gamma} \\
\varrho &:= \frac{\partial C}{\partial r} && \text{rho} \\
\Theta &:= \frac{\partial C}{\partial t} && \text{theta} \\
\text{vega} &:= \frac{\partial C}{\partial \sigma} && \text{vega}
\end{aligned}
$$

The greeks

- **Exercise 2.** Show that

$$S\,N'(d_1) - e^{-r(T-t)}\,K\,N'(d_2) = 0,$$

- where

$$N'(x) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}.$$

- Recall

$$C(S,t) = S\cdot N(d_1) - K\cdot e^{-r(T-t)}\cdot N(d_2)$$

- taking the partial derivative with respect to S we have

$$\begin{aligned}\Delta &= N(d_1) + S\,N'(d_1)\frac{\partial d_1}{\partial S} - K\,e^{-r(T-t)}\,N'(d_2)\,\frac{\partial d_2}{\partial S}\\ &= N(d_1) + \frac{N'(d_1)}{\sigma\sqrt{T-t}} - K\,e^{-r(T-t)}\,\frac{N'(d_2)}{S\sigma\sqrt{T-t}}\end{aligned}$$

- using the above exercise we obtain

$$\Delta = N(d_1)$$

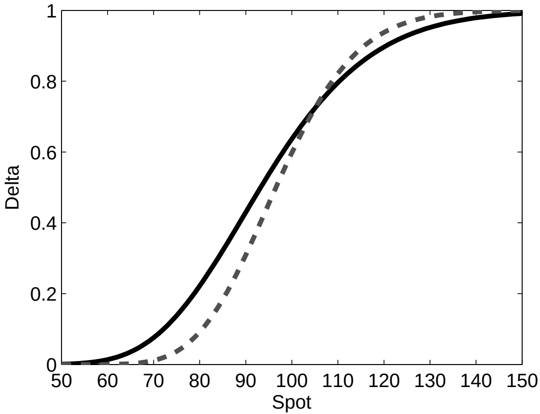

Figure: Delta of a European call option with time to maturity 1 year (solid black) and 6 months (dashed gray).

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta}{\partial S} = N'(d_1)\frac{\partial d_1}{\partial S} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

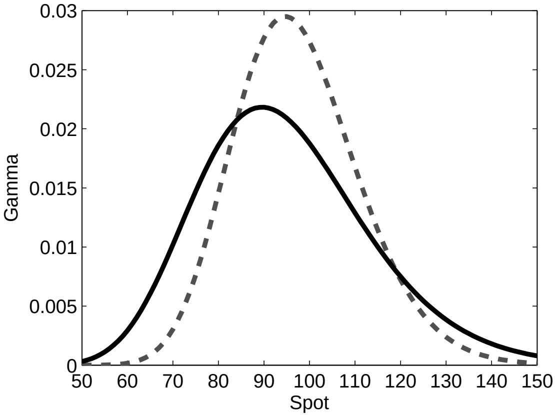

Figure: Gamma of a European call option with time to maturity 1 year (solid black) and 6 months (dashed gray).

The greeks: computation

- now differentiating C with respect to r we find that

$$
\begin{aligned}
\varrho &:= \frac{\partial C}{\partial r} \\
&= S\,N'(d_1)\,\frac{\partial d_1}{\partial r} + (T-t)K\,e^{-r(T-t)}N(d_2) \\
&\quad -K\,e^{-r(T-t)}\,N'(d_2)\frac{\partial d_2}{\partial r} \\
&= S\,N'(d_1)\,\frac{T-t}{\sigma\sqrt{T-t}} + (T-t)K\,e^{-r(T-t)}N(d_2) \\
&\quad -K\,e^{-r(T-t)}N'(d_2)\,\frac{T-t}{\sigma\sqrt{T-t}}
\end{aligned}
$$

- using an exercise from the previous lecture we get

$$\varrho = (T-t)\,K\,e^{-r(T-t)}\,N(d_2)$$

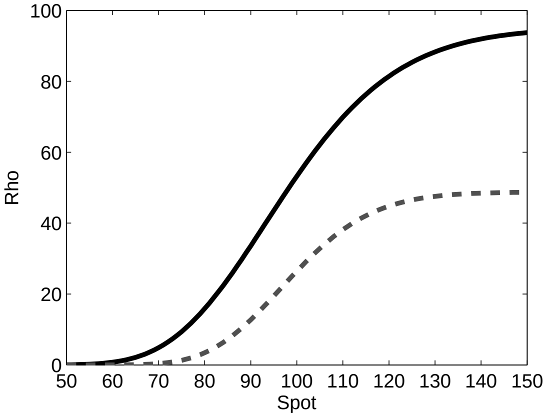

Figure: Interest rate sensitivity (Rho) of a European call option with time to maturity 1 year (solid black) and 6 months (dashed gray).

The greeks: computation

- similar analysis shows that

$$\Theta = \frac{-S\,\sigma}{2\sqrt{T-t}} N'(d_1) - rK\, e^{-r(T-t)}\, N(d_2)$$

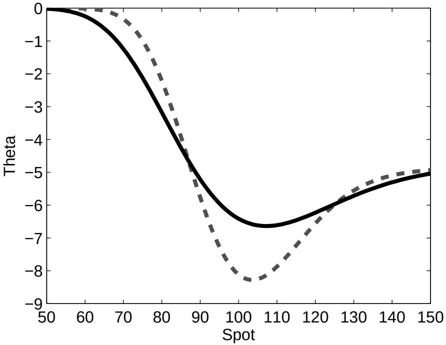

Figure: Time-decay Θ of a European call option with time to maturity 1 year (solid black) and 6 months (dashed gray).

$$
\begin{aligned}
\text{Vega} &= \frac{\partial C}{\partial \sigma} \\
&= S \cdot N'(d_1) \cdot \frac{\partial d_1}{\partial \sigma} - K \cdot e^{-r(T-t)} \cdot N'(d_2) \cdot \frac{\partial d_2}{\partial \sigma} \\
&= S \cdot \phi(d_1) \left(\frac{\partial d_2}{\partial \sigma} + \sqrt{T-t} \right) - \\
&\quad K \cdot e^{-r(T-t)} \cdot \phi(d_2) \cdot \frac{\partial d_2}{\partial \sigma} \\
&= S \cdot \phi(d_1)\sqrt{T-t} + \\
&\quad \frac{\partial d_2}{\partial \sigma} \cdot \left(S \cdot \phi(d_1) - K \cdot e^{-r(T-t)} \cdot \phi(d_2) \right) \\
&= S \cdot \phi(d_1)\sqrt{T-t}.
\end{aligned}
$$

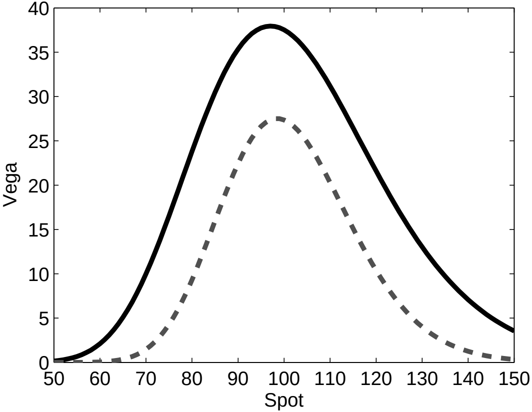

Figure: Vega of a European call option with time to maturity 1 year (solid black) and 6 months (dashed gray).

Interpreting the greeks: delta and gamma

- the delta of an at-the-money call option is close to 0.5. Delta moves to 1 as the call goes deep in the money. It moves to zero as the call goes deep out of the money.
- the delta of an at-the-money put option is close to -0.5. Delta moves to -1 as the put goes deep in the money. It moves to zero as the call goes deep out of the money.
- the parameter Γ measures the instability of Δ with respect to S; note that Γ is identical for a call and put with identical characteristics
- at-the-money options have the highest gamma, which indicates that Δ changes very fast as S changes
- both in-the money and out-of-the-money options have low gammas because their delta is constant, close to one or zero, respectively
- as maturity nears the option gamma increases

Interpreting the greeks: Example 1

A bank has sold 300'000 call options on 100'000 equities. The equities trade at 50, the option price is 49, the maturity is in three months, volatility is 20% and the interest rate is 5%. How does the bank hedge (round to the nearest thousand share)

- buy 65'000 shares?
- buy 100'000 shares?
- buy 21'000 shares?
- sell 100'000 shares?

Solution. This is an at-the-money option with a delta of about 0.5. Since the bank sold calls, it needs to delta-hedge by buying the shares. With a delta of 0.54 it would need to buy approximatively 50'000 shares. Therefore the first answer is the closest. Note also that most other information is superfluous.

Interpreting the greeks: Example 2

Which of the following IBM options has the highest gamma with the current market price of IBM common stock at 68:

- call option expiring in 10 days with strike 70?
- call option expiring in 10 days with strike 50?
- put option expiring in 10 days with strike 50?
- put option expiring in 2 months with strike 70?

Solution. Gamma is highest for short-term at-the-money options. The first answer has strike price close to 68 and short maturity.

Interpreting the greeks: rho and vega

- the sensitivity of an option to the interest rate is called the option rho
- an increase in the rate of interest increases the value of the call as the underlying asset grows at a higher rate, which increases the probability of exercising the call, with a fixed strike price K
- in the limit, for an infinite interest rate, the probability of exercise is one and the call option is equivalent to the stock itself (the reasoning is opposite for a put option)

- the sensitivity of an option to volatility is called the option vega and (similarly to gamma) vega is identical for similar call and puts
- vega must be positive for long option positions
- at-the-money options are the most sensitive to volatility
- vega is highest for long term at the money options

Interpreting the greeks: theta

- the variation in option value due to the passage of time; this is also the time decay
- unlike other factors, however, the movement in remaining maturity is perfectly predictable; time is not a risk factor
- theta is generally negative for long positions in both calls and puts (this means that the option loses value as time goes by)
- for American options, however, theta is always negative: because they give their holder the choice to exercise early, shorter-term American options are unambiguously less valuable than longer-term options
- like gamma, theta is greatest for short-term-at-the money options, when measured in absolute value

Interpreting the greeks: Black-Scholes PDE solution

- having worked out the partial derivatives, we are in a position to confirm that $C(S,t)$ satisfies the Black-Scholes PDE
- using our expressions for Δ, Γ, ϱ, and Θ we have

$$
\begin{aligned}
& \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC \\
= \quad & \frac{-S\,\sigma}{2\sqrt{T-t}} N'(d_1) - rK\,e^{-r(T-t)}\,N(d_2) \\
& + \frac{1}{2}\sigma^2 S^2\,\frac{N'(d_1)}{S\sigma\sqrt{T-t}} + rSN(d_1) \\
& - r\,\left(S\,N(d_1) - K\,e^{-r(T-t)}\,N(d_2)\right) \\
& = 0
\end{aligned}
$$

Answers to the Exercises

Exercise 1 The third answer is correct. Assuming that there is no income payment on the asset, applying the formula of Black-Scholes we get $c = 6.568$.

Exercise 2 This can be shown with direct computation.

Numerical methods: generalities

- some options have analytical solutions (such as Black-Scholes models for European vanilla options); for more general options, however, we need to use numerical methods
- most popular: binomial trees
- the method consists of chopping up the time horizon into n intervals $\Delta t = T/n$ and setting up the tree so that the characteristics of price movements fit the log-normal distribution
- asset prices will be considered at times $t_i = i\Delta t$, for $0 \leq i \leq n$
- key assumption: between successive time levels the asset price moves either up by a factor u or down by a factor d
- an upward movement occurs with probability p and a downward movement occurs with probability $1 - p$
- the standard choice is

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = 1/u, \quad p = \frac{e^{\mu \Delta t} - d}{u - d}$$

Numerical methods: generalities

- since this is a risk-neutral process, the total expected return must be equal to the risk-free rate r
- allowing for an income payment of r^* this gives $\mu = r - r^*$
- the tree is built starting from the current time to maturity, from the left to the right
- the derivative is valued by starting at the end of the tree and working backward to the initial time, from the right to the left

Numerical methods: European call option

- consider first a European call option
- at time T (maturity) and node j, the call option is worth $\max(S_{T_j} - K, 0)$
- at time $T-1$ and node j, the call option is the discounted value of the option at time T and nodes j and $j+1$:

$$c_{T-1,j} = e^{-r\Delta t}[p\, c_{T,j+1} + (1-p)\, c_{T,j}]$$

- we then work backward through the tree until the current time

Numerical methods: American call option

- the procedure is slightly different
- at each point in time, the holder compares the value of the option alive and dead (=exercised)
- the American call option value at node $T-1, j$ is

$$C_{T-1,j} = \max[(S_{T-1,j} - K, c_{T-1,j})]$$

Example: computing an American option value

- an at-the-money call on a foreign currency with a spot price 100
- strike price is $K = 100$
- maturity is six months
- the annualized volatility is $\sigma = 20\%$
- the domestic interest rate is $r = 5\%$; the foreign rate is $r^* = 8\%$
- note that we require an income payment for the American feature to be valuable
- if $r^* = 0$ we know that the American option is worth the same as a European option, which can be priced with the Black-Scholes model (there would be no point in using a numerical method)

Example: computing an American option value

- first we divide the period into four intervals, for instance so that $\Delta t = 0.5/4 = 0.125$
- the discounting factor over one interval is $e^{-r\Delta t} = 0.9938$
- we then compute
  - $u = e^{\sigma\sqrt{\Delta t}} = e^{0.20\sqrt{0.125}} = 1.0733$
  - $d = 1/u = 0.9317$
  - $a = e^{(r-r^*)\Delta t} = e^{(-0.03)0.125} = 0.9963$
  - $$p = \frac{a-d}{u-d} = \frac{0.9963 - 0.9317}{1.0733 - 0.9317} = 0.4559$$
- first we lay out the tree for the spot price, starting with $S_0 = 100$ at time $t = 0$ then $uS_0 = 107.33$ and $dS_0 = 93.17$ at time $t = 1$, and so on

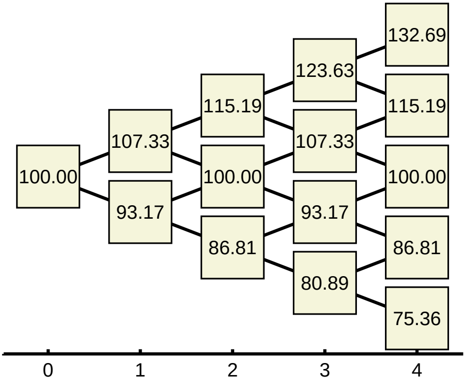

Figure: Binomial tree for spot price, with parameters $S_0 = 100$, $\sigma = 0.2$, $r = 0.05$, $r^* = 0.08$, $T = 0.5$ and $n = 4$.

Example: computing an American option value

- this allows us to value the European call; we start from the end, at time $t = 4$ and set the call price to $c = S - K = 132.69 - 100 = 32.69$ for the highest spot price,
- 15.19 for the next price and so on, down to $c = 0$ if the spot price is below $K = 100$,
- at the previous step and highest node, the value of the call is

$$c = 0.9938\,[0.4559 \times 32.69 + (1 - 0.4559) \times 15.19] = 23.02$$

- continuing through the tree to time 0, yields a European call value of 4.43
- the Black-Scholes formula gives an exact value of 4.76; note how close the binomial approximation is, with just four steps

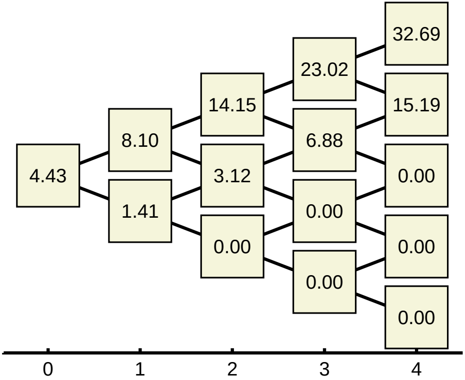

Figure: Binomial tree European call price, with parameters $S_0 = 100$, $K = 100$, $\sigma = 0.2$, $r = 0.05$, $r^* = 0.08$, $T = 0.5$ and $n = 4$.

Example: computing an American option value

- next we examine the American call
- at time $t = 4$ the values are the same as above since the call expires
- at time $t = 3$ and node $j = 4$, the option holder can either keep the call (in which case the value is still 23.02) or exercise
- when exercised, the option payoff is $S - K = 123.63 - 100 = 23.63$
- since this is greater than the value of the option alive, the holder should optimally exercise the option
- we replace the European option value by 23.63 at that node
- continuing through the tree in the same fashion, we find a starting value of 4.74
- the value of the American call is slightly greater than the European call price, as expected

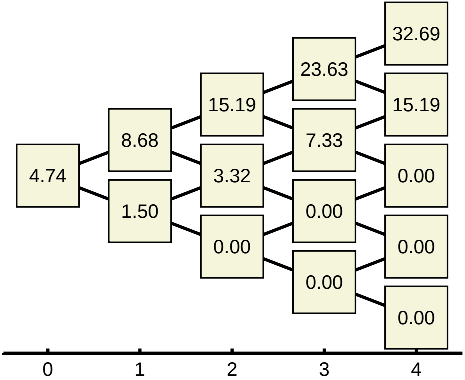

Figure: Binomial tree for American call price, with parameters $S_0 = 100$, $K = 100$, $\sigma = 0.2$, $r = 0.05$, $r^* = 0.08$, $T = 0.5$ and $n = 4$.