Lecture Quantitative Finance Spring Term 2015

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Outline

Introduction

- key words: modeling dependencies and copulas
- the study of copulas and their applications (in risk management, in option pricing) is a rather modern phenomenon!
- several international conferences in the last 15 years!
- why are copulas of interest to students?
  Fisher, Encyclopedia of Statistical Sciences (1997):
  - firstly: as a way of studying scale-free measures of dependence
  - secondly: as a starting point for constructing families of bivariate distributions, sometimes with a view to simulation

Introduction

- the word *copula* is a Latin noun that means "a link, tie, bond" (Casell's Latin Dictionary)
- is used in grammar and logic to describe "that part of a proposition which connects the subject and predicate" (Oxford English Dictionary)
- aim of Quantitative Risk Management: find good joint models $F(x_1, ..., x_n)$, e.g. $N_n(\mu, \Sigma)$; $t_k^n(\mu, \Sigma)$
- the whole idea of copulas is to go from individual models to the joint model
- if we don't have a basic joint model then we can try to use copulas!

Generalities on bivariate copulas: Outline

- Grounded functions
- Margins
- 2-increasing functions
- Definition of copulas
- Frechet bounds for copulas

Grounded functions

- let S_1 and S_2 be nonempty subsets of $[-\infty, \infty]$
- **Definition:**
  - suppose S_1 has a least element a_1 and S_2 has a least element a_2
  - a function $H : S_1 \times S_2 \to \mathbb{R}$ is grounded if

$$H(x, a_2) = 0 = H(a_1, y) \quad \text{for all} \quad (x, y) \in S_1 \times S_2.$$

- **Example:**

  $H : [-1, 1] \times [0, \infty] \to \mathbb{R}$

$$H(x, y) = \frac{(x + 1)(e^y - 1)}{x + 2e^y - 1}$$

is grounded!

Margins

- let S_1 and S_2 be nonempty subsets of $[-\infty, \infty]$
- **Definition:**
  - suppose S_1 has a greatest element b_1 and b_2 has a greatest element b_2
  - a function $H : S_1 \times S_2 \to \mathbb{R}$ has margins and the margins of H are given by:

$$F(x) = H(x, b_2) \quad \text{for all} \quad x \in S_1$$

$$G(y) = H(b_1, y) \quad \text{for all} \quad y \in S_2.$$

Margins: Example

- **Example:**
  - the function $H : [-1,1] \times [0,\infty] \to \mathbb{R}$

$$H(x,y) = \frac{(x+1)(e^y-1)}{x+2e^y-1}$$

has margins:

- $$F(x) = H(x,\infty) = \frac{x+1}{2}$$

- $$G(y) = H(1,y) = 1 - e^{-y}$$

2-increasing functions

- let S_1 and S_2 be nonempty subsets of $[-\infty, \infty]$
- **Definition:**
    - let $B = [x_1, x_2] \times [y_1, y_2] \subset S_1 \times S_2$
    - the H-volume of the rectangle B is

$$V_H(B) = \Delta_{x_1}^{x_2} \Delta_{y_1}^{y_2} H(x, y)$$

    - or

$$V_H(B) = H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1)$$

- a function $H : S_1 \times S_2 \to \mathbb{R}$ is 2-increasing if $V_H(B) \geq 0$ for all rectangles $B \subset S_1 \times S_2$

2-increasing functions: Examples

- **Example 1:**
  - $H : [0,1] \times [0,1] \to \mathbb{R}$

    $$H(x,y) = (2x-1)(2y-1) \quad \text{is 2-increasing!}$$
  - however it is decreasing in x for any $y \in (0, 1/2)$ and decreasing in y for any $x \in (0, 1/2)$!

Grounded and 2-increasing functions

- **Example 2:**
  - $H : [0,1] \times [0,1] \to \mathbb{R}$

    $$H(x,y) = \max(x,y)$$

    is a nondecreasing function of x and a nondecreasing function of y
  - however

    $$V_H([0,1] \times [0,1]) = -1$$

    thus H is NOT 2-increasing
- grounded and 2-increasing implies non-decreasing in each argument!

(Bivariate) Copulas: Definition

- a function $C : [0,1] \times [0,1] \to [0,1]$ is a copula if
  1. C is grounded,
  2. for every $u, v \in [0,1]$
     $$C(u,1) = u \quad \text{and} \quad C(1,v) = v$$
  3. C is 2-increasing.

Bivariate copulas: Examples

- fundamental examples

$$
\begin{aligned}
\Pi(x, y) &= xy \\
W(x, y) &= \max(x + y - 1, 0) \\
M(x, y) &= \min(x, y)
\end{aligned}
$$

- let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$; then

$$
C_{\alpha,\beta}(x, y) = \alpha M(x, y) + (1 - \alpha - \beta)\Pi(x, y) + \beta W(x, y)
$$

is a copula (Frechet-Mardia)!

Bivariate copulas: Frechet bounds

- lower Frechet-Hoeffding bound (copula only for $n = 2$):

$$W(u, v) = \max(u + v - 1, 0)$$

- upper Frechet-Hoeffding bound (a copula also for $n \geq 2$):

$$M(u, v) = \min(u, v)$$

- note that for any copula C

$$W(u, v) \leq C(u, v) \leq M(u, v)$$

Distribution functions

- **Definition:**

  a distribution function is a function $F : [-\infty, \infty] \to [0, 1]$ such that

  - F is nondecreasing
  - $F(-\infty) = 0$ and $F(+\infty) = 1$

- **Example:** (unit step at a)

$$\varepsilon_a(x) = \begin{cases} 0 & , \quad x \in [-\infty, a) \\ 1 & , \quad x \in [a, \infty] \end{cases}$$

Distribution functions: Example

- the uniform distribution on $[a, b]$:

$$U_{ab}(x) = \begin{cases} 0 & , \quad x \in [-\infty, a) \\ \frac{x-a}{b-a} & , \quad x \in [a, b] \\ 1 & , \quad x \in (b, \infty] \end{cases}$$

Joint distribution functions

- a function $H : [-\infty, \infty] \times [-\infty, \infty] \to [0, 1]$ is a joint distribution function if
  - H is 2-increasing
  - $H(x, -\infty) = 0 = H(-\infty, y)$ and $H(\infty, \infty) = 1$
- **Remark:** note that H is grounded and has margins

$$F(x) = H(x, \infty) \quad \text{and} \quad G(y) = H(\infty, y)$$

which are distribution functions

Joint distribution functions: Examples

- $$H(x,y) = \begin{cases} \frac{(x+1)(e^y-1)}{x+2e^y-1} & , \quad (x,y) \in [-1,1] \times [0,\infty) \\ 1-e^{-y} & , \quad (x,y) \in (1,\infty) \times [0,\infty] \\ 0, & , \quad \text{elsewhere.} \end{cases}$$
- margins:

$$F(x) = U_{-1,1}(x) = \begin{cases} 0 & , \quad x \in [-\infty,-1) \\ \frac{x+1}{2} & , \quad x \in [-1,1] \\ 1 & , \quad x \in (1,\infty] \end{cases}$$

$$G(y) = \begin{cases} 0 & , \quad y \in [-\infty,0) \\ 1-e^{-y} & , \quad y \in [0,\infty] \end{cases}$$

Sklar's theorem

1. let H be a joint distribution function with margins F and G
   - then there exists a copula C such that for all $x, y \in [-\infty, \infty]$

$$H(x, y) = C(F(x), G(y))$$

   - if F and G are continuous, then C is unique; otherwise, C is uniquely determined on $\mathrm{Ran}F \times \mathrm{Ran}G$
2. conversely, if C is a copula and F and G are distribution functions, then the function H defined as a above is a joint distribution function with margins F and G

Sklar's theorem: Example

- consider

$$H(x,y)=\begin{cases}\frac{(x+1)(e^y-1)}{x+2e^y-1} & , \quad (x,y)\in[-1,1]\times[0,\infty)\\ 1-e^{-y} & , \quad (x,y)\in(1,\infty)\times[0,\infty]\\ 0, & , \quad \text{elsewhere.}\end{cases}$$

- then the associated copula is

$$C(u,v)=\frac{uv}{u+v-uv}$$

- check it in the class!

Quasi-inverse of a distribution function

- let F be a distribution function
- a *quasi-inverse* of F is any function $F^{(-1)} : [0,1] \to \overline{\mathbb{R}}$ such that
  1. if $t \in \operatorname{Ran} F$ then $F^{(-1)}(t)$ is any number in $[-\infty, \infty]$ such that $F(x) = t$, i.e. for all $t \in \operatorname{Ran} F$
     $$F(F^{(-1)}(t)) = t;$$
  2. if $t \notin \operatorname{Ran} F$, then
     $$F^{(-1)}(t) = \inf\{x \ : \ F(x) \geq t\} = \sup\{x \ : \ F(x) \leq t\}.$$
- if F is strictly increasing, then it has a single quasi-inverse, which is of course the ordinary inverse, for which we use the customary notation F^{-1}

Quasi-inverse of a distribution function: Example

- the quasi-inverses of ε_a, the unit step at a are the functions given by:

$$\varepsilon_a^{(-1)}(t) = \begin{cases} a_0 & , \quad t = 0 \\ a & , \quad t \in (0,1) \\ a_1, & , \quad t = 1 \end{cases}$$

where a_0 and a_1 are any numbers in $[\infty, \infty]$ such that $a_0 < a \leq a_1$

Sklar's theorem using quasi-inverse of distribution functions

- let H be a joint distribution function with margins F and G
  - let the copula C given by Sklar's theorem, i.e. such that for all $x, y \in [-\infty, \infty]$
  $$H(x, y) = C(F(x), G(y))$$
  - let $F^{(-1)}$ and $G^{(-1)}$ be quasi-inverses of F and G respectively
- then for any $u, v \in [0, 1]$
$$C(u, v) = H(F^{(-1)}(u), G^{(-1)}(v))$$

Gumbel's bivariate exponential distribution

- let $\theta \in [0,1]$ and let H_θ be the joint distribution function given by

$$H_\theta(x,y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y+\theta xy)} & , \quad x \geq 0, y \geq 0 \\ 0 & , \quad \text{otherwise} \end{cases}$$

- the marginal distribution functions are exponentials, with quasi-inverses given for $u, v \in [0,1]$ by

$$F^{(-1)}(u) = -\log(1-u) \quad \text{and} \quad G^{(-1)}(v) = -\log(1-v)$$

- hence the corresponding copula is given by

$$C_\theta(u,v) = u + v - 1 + (1-u)(1-v)\, e^{-\theta \log(1-u) \log(1-v)}$$

Copulas as joint distribution with uniform margins

- let C be a copula and define $H_C : [-\infty, \infty] \times [-\infty, \infty] \to [0,1]$

$$H_C(x,y) = \begin{cases} 0 & , \quad x < 0 \text{ and } y < 0, \\ C(x,y) & , \quad x, y \in [0,1] \\ x & , \quad y > 1, x \in [0,1] \\ y & , \quad x > 1, y \in [0,1] \\ 1 & , \quad x > 1 \text{ and } y > 1 \end{cases}$$

- then H_C is a distribution function both of whose margins are readily seen to be U_{01}
- **Interpretation:** copulas are restrictions to $[0,1] \times [0,1]$ of joint distributions whose margins are U_{01}

Copulas and random variables

- consider X and Y two random variables with distribution functions F and G respectively, i.e.

$$F(x) = P[X \leq x] \quad G(y) = P[Y \leq y],$$

and joint distribution function H, i.e.

$$H(x, y) = P[X \leq x, Y \leq y]$$

- then the copula C given by Sklar's Theorem is called the copula of X and Y and is denoted C_{XY}
- **Theorem:**
  - let X and Y two random variables with continuous distribution functions
  - then X and Y are independent if, and only if, $C_{XY} = \Pi$

Copulas and dependence structure

- Sklars theorem shows how a unique copula C describes in a sense the dependence structure of the multivariate distribution function of a random vector $X = (X_1, X_2)$
- This motivates a further

  Definition:

  The copula of (X_1, X_2) is the distribution function C of $(F_1(X_1), F_2(X_2))$

Theorem:

- let X and Y two random variables with continuous distribution functions
- then C_{XY} is invariant under strictly increasing transformations of X and Y,
  i.e. if α and β are strictly increasing on Ran$F$ and Ran$G$ then $C_{\alpha(X)\beta(Y)} = C_{XY}$.

Construction of bivariate distributions

- if we have a collection of copulas, then, as a consequence of Sklar's theorem we automatically have a bivariate or multivariate distributions with whatever marginal distributions we desire
- by the invariance of the copula under strictly increasing transformations of the random variables if follows that the nonparametric nature of the dependence between two random variables is expressed by a copula
- $\longrightarrow$ we need to have a variety of copulas at our disposal!

Archimedean copulas: pseudo-inverses

- **Definition:**
  - let $\varphi : [0,1] \to [0,\infty]$ be a continuous strictly decreasing function such that $\varphi(1) = 0$
  - the pseudo-inverse of φ is the function

$$\varphi^{[-1]} : [0,\infty] \to [0,1]$$

given by: $$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & , \quad 0 \leq t \leq \varphi(0), \\ 0 & , \quad \varphi(0) \leq t \leq \infty. \end{cases}$$

- note that $\varphi^{[-1]}$ is continuous and non-increasing on $[0,\infty]$ and strictly decreasing on $[0,\varphi(0)]$.
- note that if $\varphi(0) = \infty$ then $\varphi^{[-1]} = \varphi^{-1}$

Archimedean copulas: definition

- **Theorem:**
  - let $\varphi : [0,1] \to [0,\infty]$ be a continuous strictly decreasing function such that $\varphi(1) = 0$
  - $\varphi^{[-1]}$ be the pseudo-inverse of φ
  - let $C : [0,1] \times [0,1] \to [0,1]$ given by:

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$$

  - then C is a copula if, and only if, φ is convex
- those copulas are called Archimedean and the function φ is called the generator of the copula
- if $\varphi(0) = \infty$ we say φ is a strict generator and C is a strict Archimedean copula

Archimedean copulas: Example 1

- let $\varphi : [0,1] \to [0,\infty]$, $\varphi(t) = -\log t$
- then $\varphi(0) = \infty$ and φ is strict
- thus $\varphi^{[-1]}(t) = \varphi^{-1}(t) = \exp(-t)$ and the generated copula is

$$C(u,v) = \exp(\log u + \log v) = uv = \Pi(u,v)$$

- consequently Π is a strict Archimedean copula

Archimedean copulas: Example 2

- let $\varphi : [0,1] \to [0,\infty]$, $\varphi(t) = 1 - t$
- then $\varphi^{[-1]}(t) = 1 - t$ and 0 for $t > 1$; i.e. $\varphi^{[-1]}(t) = \max(1-t, 0)$
- consequently the generated copula is

$$C(u,v) = \max(u+v-1, 0) = W(u,v).$$

- this means W is also an Archimedean copula
- note that the copula $M(u,v) = \min(u,v)$ is not Archimedean

Archimedean copulas: Example 3

- let $\theta \in (0,1]$ and $\varphi_\theta : [0,1] \to [0,\infty]$, $\varphi_\theta(t) = \log(1 - \theta \log t)$
- then $\varphi_\theta(0) = \infty$, φ_θ is strict, and

$$\varphi_\theta^{[-1]}(t) = \varphi_\theta^{-1}(t) = \exp[(1 - e^t)/\theta]$$

- consequently the generated copula is

$$C_\theta(u,v) = uv \exp(-\theta \log u \log v)$$

Archimedean copulas: further examples

- two-parameter family of Archimedean copulas
- $\alpha > 0$, $\beta \geq 1$

$$C_{\alpha,\beta}(x,y) = \left\{ [(x^{-\alpha} - 1)^{\beta} + (y^{-\alpha} - 1)^{\beta}]^{1/\beta} + 1 \right\}^{-1/\alpha}$$

Looking forward

The next slides are optional material!!!

Generalities on multivariate copulas: Outline

- Definition
- Sklar's theorem in n-dimensions
- Multivariate Archimedean copulas, Gumbel copula, Clayton copula
- Implicit copulas

Multivariate copulas: definition

A function $C : [0,1] \times \cdots \times [0,1] \to [0,1]$ is a copula if

1. C is grounded,
2. for every $i = 1, ..., n$ and any $u_i \in [0,1]$
$$C(1, ..., 1, u_i, 1, ..., 1) = u_i$$
3. C is n-increasing (i.e. for all $(x_1, ...x_n), (y_1, ..., y_n) \in [0,1]^n$ with $x_j \leq y_i$ we have
$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\ldots+i_n} C(u_{1i_1}, ..., u_{ni_n}) \geq 0$$
where $u_{j1} = x_j$ and $u_{j2} = y_j$ for all $j = 1, ..., n$.)

Multivariate copulas: further properties

- n-dimensional copulas are Lipschitz

$$\mid C(v_1, ..., v_n) - C(u_1, ..., u_n) \mid \leq \sum_{k=1}^{n} \mid v_k - u_k \mid .$$

- **Definition:** n-dimensional distribution functions are functions $H : [-\infty, \infty] \times \cdots \times [-\infty, \infty] \to \mathbb{R}$ such that
  - H is n-increasing
  - $H(x_1, ..., x_n) = 0$ for all such that $x_k = -\infty$ for at least one k and $H(\infty, ..., \infty) = 1$
- thus H is grounded and the one-dimensional margins are distribution functions: $F_1, ..., F_n$

Sklar's theorem in n-dimensions

- let H be an n-dimensional distribution function with margins $F_1, ..., F_n$
  - then there exists an n-copula C such that for all $x_1, ..., x_n \in [-\infty, \infty]$

$$H(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)).$$

  - if $F_1, ..., F_n$ are continuous, then C is unique; otherwise, C is uniquely determined on $\mathrm{Ran}F_1 \times ... \times \mathrm{Ran}F_n$
- conversely,
  - if C is an n-copula and $F_1, ..., F_n$ are distribution functions
  - then the function H defined as a above is an n-dimensional distribution function with margins $F_1, ..., F_n$

Copulas and dependence structures

- Sklars theorem shows how a unique copula C describes in a sense the dependence structure of the multivariate distribution function of a random vector $X = (X_1, ..., X_n)$
- this motivates the further
  Definition:

  the copula of $(X_1, ..., X_n)$ is the distribution function C of $(F_1(X_1), ..., F_n(X_n))$!

Multivariate Archimedean copulas

- **Theorem:**
  - let $\varphi : [0,1] \to [0,\infty]$ be a continuous strictly decreasing function such that $\varphi(1) = 0$ and $\varphi(0) = \infty$
  - let φ^{-1} be the inverse of φ
  - let $C : [0,1] \times ... \times [0,1] \to [0,1]$ given by:

$$C(u_1, ..., u_n) = \varphi^{-1}(\varphi(u_1) + ... + \varphi(u_n)).$$

  - then C is a copula if, and only if, φ^{-1} is completely monotone on $[0,\infty)$, i.e. has derivatives of all orders that alternate in sign.
- those copulas are called Archimedean and the function φ is called the generator of the copula.

Multivariate Archimedean copulas: Examples

- Gumbel copula: $\theta \geq 1$, $\varphi_\theta(t) = (-\log t)^\theta$

$$C_\theta^{Gu}(u_1, \ldots, u_n) = \exp\left(-\left[(-\log u_1)^\theta + \ldots(-\log u_n)^\theta\right]^{1/\theta}\right)$$

$\theta = 1$ gives independence, $\theta \to \infty$ gives comonotonicity

- Clayton copula: $\theta > 0$, $\varphi_\theta(t) = t^{-\theta} - 1$.

$$C_\theta^{Cl}(u_1, \ldots, u_n) = \left(u_1^{-\theta} + \ldots u_n^{-\theta} - d + 1\right)^{-1/\theta}$$

$\theta \to 0$ gives independence, $\theta \to \infty$ gives comonotonicity

Multivariate Archimedean copulas: some remarks

- Pro: multivariate Archimedean copulas can be generated fairly simple
- Con: all the k-margins of an n-Archimedean copula are identical
- Con: there are only one or two parameters and this limits the nature of the dependence structure in these families

Parametric copulas

there are essentially two possibilities:

1. copulas implicit in well-known parametric distributions
   - Sklar's theorem states that we can always find a copula in a parametric distribution function
   - let H be the distribution function and let $F_1, \dots, F_n$ its continuous margins
   - then the implied copula is

$$C(u_1, \dots, u_n) = H(F_1^{(-1)}(u_1), \dots, F_n^{(-1)}(u_n))$$

   - such a copula may not have a simple closed form

Parametric copulas

- closed form parametric copula families generated by some explicit construction that is known to yield copulas
- the best example is the Archimedean copula family
- these generally have limited numbers of parameters

Implicit copulas: Example

- Gaussian Copula:

$$C_P^{Ga}(u_1, ..., u_n) = \mathbf{N}_P\left(N^{-1}(u_1), ..., N^{-1}(u_n)\right)$$

where N denotes the standard univariate distribution function

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt,$$

$\mathbf{N}_P$ denotes the joint distribution function of $\mathbf{X} \sim N_n(0, P)$ and P is a correlation matrix

Key facts for simulation: Probability and Quantile Transform

- **Proposition 1:** (Probability transform)
  - let X be a random variable with continuous distribution function F
  - then $F(X) \sim U_{01}$ (standard uniform)
  - $u \in (0, 1)$

$$P(F(X) \leq u) = P(X \leq F^{-1}(u)) = F(F^{-1}(u)) = u,$$

- **Proposition 2:** (Quantile transform)
  - let U be uniform and F the distribution function of any rv X
  - then $F^{-1}(U)$ has the same distribution with X so that $P(F^{-1}(U) \leq x) = F(x)$

Key facts for simulation: Probability and Quantile Transform

- These facts are the key to all statistical simulation and essential in dealing with copulas
- Simulating Gaussian copula
  - Simulate $\mathbf{X} \sim N_n(\mathbf{0}, P)$
  - Set $U = (\Phi(X_1), ..., \Phi(X_n))$ (probability transformation)

Meta-distributions and their simulation

- by the converse of Sklars Theorem we know that if C is a copula and F_1,..., F_d are univariate distribution functions, then $F(x) = C(F_1(x_1), ..., F_n(x_n))$ is a multivariate distribution functions with margins $F_1, ..., F_n$
- we refer to F as a meta-distribution with the dependence structure represented by C
- for example, if C is a Gaussian copula we get a meta-Gaussian distribution and if C is a t copula we get a meta-t distribution
- if we can sample from the copula C, then it is easy to sample from F: we generate a vector $(U_1, ..., U_n)$ with distribution function C and then return $(F_1^{(-1)}(U_1), ..., F_n^{(-1)}(U_n))$

Some additional comments

- correlation is defined only when the variances of the two random variables are finite
- $\longrightarrow$ not ideal when we work with heavy tailed distributions
- example: actuaries who model losses in different business lines with infinite variance distributions may not describe the dependence of their risk using correlation!
- correlation of two risks does not depend only on their copula
- correlation is linked to the marginal distributions of the risks!

Some additional comments

- often very difficult (in particular in higher dimensions and in situations where we are dealing with heterogeneous risk factors) to find a good multivariate model that describes both marginal behavior and dependence structure effectively
- the copula approach to multivariate models allows us to consider marginal modeling and dependence modeling issues

Some conclusions

- copulas help in the understanding of dependence at a deeper level
- they show us potential pitfalls of approaches to dependence that focus only on correlation
- they allow us to define alternative dependence structures
- they express dependence on a quantile scale ($\longrightarrow$ QRM!)
- they facilitate a bottom-up approach to multivariate model building
- they are easily simulated and thus lend themselves to Monte Carlo risk studies

  $\longrightarrow$ useful in risk management where we often have a much better idea about the marginal behavior of individual risk factors than we do about their dependence structure

References

Books:

1.
   - *An Introduction to Copulas*
   - Author:
     Roger Nelsen
   - Springer 2007
2.
   - *Quantitative Risk Management: Concepts, Techniques and Tools*
   - Authors:
     Paul Embrechts, Rüdiger Frey, Alexander McNeil