Lecture Quantitative Finance Spring Term 2015

Prof. Dr. Erich Walter Farkas

Lecture 1: February 19, 2015

Administrative information

- Room: HAH E 11 at UZH
- Thursday, 12.15 - 13.45: **no break!**
- First lecture: Thursday, February 19, 2015
- No lecture: Thursday, April 09, 2015: Easter Holidays
- No lecture: Thursday, May 14, 2015: Ascension Day
- Last lecture: Thursday, May 28, 2015
- Exam date: Thursday, June 4, 2015, 12.00 - 14.00
- Exam location: Room: TBD at UZH
- Exam details: closed books
- Material: see OLAT

Target audience of this course

- UZH – MA: Pflichtmodule BF
- UZH ETH – Master of Science in Quantitative Finance (elective area: MF)
- anybody interested in an introduction to quantitative finance

Goals of this course

At the end of this course you

- will be able to understand and apply the fundamental concepts of quantitative finance;
- will have learned the (fundamental) aspects of valuing financial instruments (bonds, forwards, options, etc.) and the role of asset price sensitivities;
- will have the ability to comprehend and manage (market) risk and to use quantitative techniques to model these risks.

Selected literature

- J. Hull,
  Options, Futures and Other Derivatives,
  Prentice Hall Series in Finance, 2008
- P. Wilmott,
  Paul Wilmott on Quantitative Finance,
  John Wiley & Sons, 2006
- P. Jorion,
  Financial Risk Manager Handbook,
  Wiley Finance, 2007
- J. Cvitanic and F. Zapatero,
  Introduction to the Econ. and Math. of Financial Markets,
  The MIT Press, 2004
- H. Föllmer and A. Schied, *Stochastic finance: An introduction in discrete time*, De Gruyter Berlin, 2002

Selected literature

- T. Bjork, *Arbitrage Theory in Continuous Time*, Oxford Univ Press, 2004
- M. Musiela, M. Rutkowski, *Martingale Methods In Financial Modelling*, Springer Verlag, 2005
- D. Lamberton, B. Lapeyre, *Introduction to Stochastic Calculus Applied to Finance*, Chapman & Hall, 2007
- P. Wilmott, *Derivatives: The Theory and Practice of Financial Engineering*, John Wiley & Son, 1998
- T. Mikosch, *Elementary Stochastic Calculus with Finance in View*, World Scientific, 2000
- I. Karatzas and S. Shreve, *Brownian motion and stochastic calculus*, Springer-Verlag, New York, 1991

Lecturers

Prof. Dr. Erich Walter Farkas

- Dipl. Math., MSc. Math: University of Bucharest
- Dr. rer. nat.: Friedrich-Schiller-University of Jena
- Habilitation: Ludwig-Maximilians-University of Munich
- Since 1. Oct. 2003 at UZH & ETH:
  PD (reader) and wissenschaftlicher Abteilungsleiter (Director)
  in charge for the UZH ETH – Quantitative Finance Master
  first joint degree of UZH and ETH
- Since 1. Feb. 2009:
  Associate Professor for Quantitative Finance at UZH
  Program Director MSc Quantitative Finance (joint degree UZH ETH)
- Associate Faculty, Department of Mathematics, ETH Zürich
- Faculty member of the Swiss Finance Institute

PhD students

- Giada Bordogna
- Fulvia Fringuellotti
- Kevin Meyer

Guest lecturers

- Dr. Pedro Fonseca, Former Head Risk Analytics & Reporting, SIX Management AG
- Marek Krynski, Executive Director at UBS
- Robert Huitema, Associate Director at UBS

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- Appointment via E-mail is kindly requested

Further lectures in Mathematical / Quantitative Finance

- Direct continuation of this lecture
  - Fall 2015:
    Mathematical Foundations of Finance
    ETH: W. Farkas, M. Schweizer
  - Spring 2016:
    Continuous Time Quantitative Finance
    UZH: M. Chesney
- Related lectures
  - Fall 2015: Financial Engineering
  - Spring 2015 and Spring 2016: Asset Management; Quantitative Risk Management

Chapter 1: Bond fundamentals

Bonds: Definition and Examples

- Bonds are financial claims which entitle the holder to receive a stream of periodic payments, known as *coupons*, as well as a final payment, known as the *principal* (or *face value*).
- In practice, depending on the nature of the issues, one distinguishes between different types of bonds; important examples include:
  - **Government or Treasury Bonds**: issued by governments, primarily to finance the shortfall between public revenues and expenditures and to pay off earlier debts;
  - **Municipal Bonds**: issued by municipalities, e.g., cities and towns, to raise the capital needed for various infrastructure works such as roads, bridges, sewer systems, etc.;
  - **Mortgage Bonds**: issued by special agencies who use the proceeds to purchase real estate loans extended by commercial banks;
  - **Corporate Bonds**: issued by large corporations to finance the purchase of property, plant and equipment.
- Among all assets, the simplest (most basic) to study are fixed-coupon bonds as their cash-flows are predetermined.
- The valuation of bonds requires a good understanding of concepts such as *compound interest*, *discounting*, *present value* and *yield*.
- For **hedging** and **risk management** of bond portfolios (risk) sensitivities such as *duration* and *convexity* are important.

Zero-Coupon Bonds

- A *zero-coupon* bond promises no coupon payments, only the repayment of the principal at maturity.
- Consider an investor who wants a zero-coupon bond, which
  - pays 100 CHF
  - in 10 years, and
  - has no default risk.
- Since the payment occurs at a future data – in our case after 10 years – the value of this investment is surely less than an up-front payment of 100 CHF.
- To *value* this payment one needs two ingredients:
  - the prevailing *interest rate*, or *yield*, per period
  - and the *tenor*, denoted T, which gives the number of periods until maturity expressed in years.

- The **present value** (PV) of a zero-coupon bond can be computed as:

$$PV = \frac{C_T}{(1+y)^T},$$

  where C_T is the principal (or face value) and y is the discount rate.
- For instance, a payment of $C_T = 100$ CHF in 10 years discounted at 6% is (only) worth 55.84 CHF.

Note:

- The (market) value of zero-coupon bonds decreases with longer maturities;
- keeping T fixed, the value of the zero-coupon bond decreases as the yield increases.

- Analogously to the notion of *present value*, we can define the notion of *future value* (FV) for an initial investment of amount PV:

$$FV = PV \times (1 + y)^T$$

- For example, an investment now worth $PV = 100$ CHF growing at 6% per year will have a future value of 179.08 CHF in 10 years.

- The internal rate of return of a bond, or annual growth rate, is called the *yield*, or *yield-to-maturity* (YTM).
- Yields are usually easier to deal with than CHF values.
- Rates of return are directly comparable across assets (when expressed in percentage terms and on an annual basis).
- The yield y of a bond is the solution to the (non-linear) equation:

$$P = P(y),$$

where "P" is the (market) price of the bond and $P(\cdot)$ is the price of the bond as a function of the yield y; in case of a zero-coupon bond

$$P(y) := \frac{C_T}{(1+y)^T}.$$

- The yield of bonds with the same characteristics but with different maturities can differ strongly; i.e., the yield (usually) depends upon the maturity of the bond.
- The *yield curve* is the set of yields as a function of maturity.
- Under "normal" circumstances, the yield curve is upward sloping; i.e., the longer you lock in your money, the higher your return.

Important: state the method used for compounding:

- **annual compounding** (usually the norm):

$$PV = \frac{C_T}{(1+y)^T}.$$

- **semi-annual compounding** (e.g. used in the U.S. Treasury bond market): interest rate y_s is derived from:

$$PV = \frac{C_T}{(1+y_s/2)^{2T}}$$

  where $2T$ is the number of periods.

- **continuous compounding** (used ubiquitously in the quantitative finance literature) interest rate y_c is derived from:

$$PV = \frac{C_T}{\exp(y_c T)}.$$

Example: Consider our example of the zero-coupon bond, which pays 100 CHF in 10 years, once again. Recall that the PV of the bond is equal to 55.8395 CHF. Now, we can compute the 3 yields as follows:

- annual compounding:

$$PV = \frac{C_T}{(1+y)^{10}} \Rightarrow y = 6\%$$

- semi-annual compounding:

$$PV = \frac{C_T}{(1+y_s/2)^{20}} \Rightarrow (1+y_s/2)^2 = 1+y \Rightarrow y_s = 5.91\%$$

- continuously compounding:

$$PV = \frac{C_T}{\exp(y_c T)} \Rightarrow \exp(y_c) = 1+y \Rightarrow y_c = 5.83\%$$

Note: increasing the (compounding) frequency results in a lower equivalent yield.

Exercise L01.1: Assume a semi-annual compounded rate of 8% (per annum). What is the equivalent annual compounded rate?

1. 9.20%
2. 8.16%
3. 7.45%
4. 8.00%.

Exercise L01.2: Assume a continuously compounded rate of 10% (per annum). What is the equivalent semi-annual compounded rate?

1. 10.25%
2. 9.88%
3. 9.76%
4. 10.52%.

- While zero coupon bonds are a very useful (theoretical) concept, the bonds usually issued and traded are coupon bearing bonds.

Note:

- A zero coupon bond is a special case of a coupon bond (with zero coupon);
- and a coupon bond can be seen as a portfolio of zero coupon bonds.

Price-yield relationship

Consider now the price (or present value) of a coupon bond with a general pattern of fixed cash-flows. We define the price-yield relationship as follows:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}.$$

Here we have adopted the following notations:

- C_t: the cash-flow (coupon or principal) in period t;
- t: the number of periods (e.g. half-years) to each payment;
- T: the number of periods to final maturity;
- y: the discounting yield.

- As indicated earlier, the typical cash-flow pattern for bonds traded in reality consists of regular coupon payments plus repayment of the principal (or face value) at the expiration.
- Specifically, if we denote c the coupon rate and F the face value, then the bond will generate the following stream of cash flows:

$$\begin{aligned} C_t &= cF && \text{prior to expiration} \\ C_T &= cF + F && \text{at expiration.} \end{aligned}$$

- Using this particular cash-flow pattern, we can arrive (with the use of the geometric series formula) arrive at a more compact formula for the price of a coupon bond:

$$\begin{aligned} P &= \frac{cF}{1+y} + \frac{cF}{(1+y)^2} + \cdots + \frac{cF}{(1+y)^{T-1}} + \frac{cF+F}{(1+y)^T} \\ &= cF \cdot \frac{\frac{1}{1+y} - \frac{1}{(1+y)^{T+1}}}{1 - \frac{1}{1+y}} + \frac{F}{(1+y)^T} \\ &= \frac{cF}{y} \cdot \left(1 - \frac{1}{(1+y)^T}\right) + \frac{F}{(1+y)^T}. \end{aligned}$$

Remark. If the coupon rate matches the yield ($c = y$) (using the same compounding frequency) then the price of the bond equals its face value; such a bond is said to be priced *at par*.

Example: Consider a bond that pays 100 CHF in 10 years and has a 6% annual coupon.

a.) What is the market value of the bond if the yield is 6%?

b.) What is the market value of the bond if the yield falls to 5%?

Solution: The cash flows are $C_1 = 6$, $C_2 = 6, \ldots, C_{10} = 106$. Discounting at 6% gives PVs of 5.66, 5.34,..., 59.19, which sum up to 100 CHF; so the bond is selling at par. Alternatively, discounting at 5% leads to a price of 107.72 CHF.

Exercise L01.3: Consider a 1-year fixed-rate bond currently priced at 102.9 CHF and paying a 8% coupon (semi-annually). What is the yield of the bond?

1. 8%
2. 7%
3. 6%
4. 5%.

- Another special case of a general coupon bond is the so-called *perpetual bond*, or *consol*.
- These are bonds with regular coupon payments of $C_t = cF$ and with infinite maturity.
- The price of a consol is given by:

$$P = \frac{c}{y} F.$$

Derivation:

$$
\begin{aligned}
P &= \frac{cF}{1+y} + \frac{cF}{(1+y)^2} + \frac{cF}{(1+y)^3} + \cdots \\
&= cF\left[\frac{1}{1+y} + \frac{1}{(1+y)^2} + \frac{1}{(1+y)^3} + \cdots\right] \\
&= cF\,\frac{1}{1+y}\left[1 + \frac{1}{(1+y)} + \frac{1}{(1+y)^2} + \cdots\right] \\
&= cF\,\frac{1}{1+y}\left[\frac{1}{1-(1/(1+y))}\right] \\
&= cF\,\frac{1}{1+y}\,\frac{1+y}{y} \\
&= \frac{c}{y}F.
\end{aligned}
$$

- We will now address the question: what happens to the price of the bond when the yield changes from its initial value say, y_0, to a new value $y_1 = y_0 + \Delta y$, where Δy is assumed to be 'small'.
- Assessing the effect of changes in risk factors (in our case, the yield) on the price of assets is of key importance for **hedging** and **risk management**.
- We start from the price-yield relationship $P = P(y)$. We now have an initial value of the bond $P_0 = P(y_0)$, and a new value of the bond $P_1 = P(y_1)$.
- For a 'small' yield change Δy, we can approximate P_1 from a Taylor expansion,

$$P_1 = P_0 + P'(y_0)\Delta y + \frac{1}{2}P''(y_0)(\Delta y)^2 + \dots .$$

- This is an infinite expansion with increasing powers of Δy; only the first two terms (linear and quadratic) are usually used by finance practitioners.

- The first- and second-order derivative of the bond price w.r.t. yield are very important, so they have been given special names.
- The negative of the first-order derivative is the *dollar duration* (DD):

$$DD = -P'(y) = -\frac{dP}{dy} = D^* \times P,$$

  where D^* is the *modified duration*.
- Another duration measure is the so-called *Macaulay duration* (D), which is defined as:

$$D = \frac{1}{P}\left(\sum_{t=1}^{T} \frac{t \times cF}{(1+y)^t} + \frac{T \times F}{(1+y)^T}\right).$$

- Often risk is measured as the *dollar value of a basis point* (DVBP) (also known as DV01):

$$DV01 = (D^* \times P) \times BP,$$

  where BP stands for *basis point* $(= 0.01\%)$.

- The second-order derivative is the dollar convexity (DC):

$$DC = P''(y) = \frac{d^2P}{dy^2} = \kappa \times P,$$

where κ is called the convexity.

- For fixed-coupon bonds, the cash-flow pattern is known and we have an explicit price-yield function; therefore one can compute analytically the first- and second-order derivatives.

Example: Recall that a zero-coupon bond has only payment at maturity equal to the face value $C_T = F$:

$$P(y) = \frac{F}{(1+y)^T}.$$

Then, we have $D = T$ and

$$\frac{dP}{dy} = (-T) \times \frac{F}{(1+y)^{T+1}} = -\frac{T}{1+y} \times P,$$

so the modified duration is $D^* = T/(1+y)$. Additionally, we have

$$\frac{d^2P}{dy^2} = -(T+1) \times (-T) \times \frac{F}{(1+y)^{T+2}} = \frac{(T+1)T}{(1+y)^2} \times P,$$

so that the convexity is $\kappa = \frac{(T+1)T}{(1+y)^2}$.

Remarks:

- note the difference between the modified duration $D^* = T/(1+y)$ and the Macaulay duration $(D = T)$;
- duration is measured in periods, like T;
- considering annual compounding, duration is measured in years, whereas with semi-annual compounding duration is in half-years and has to be divided by two for conversion to years;
- dimension of convexity is expressed in periods squared;
- considering semi-annual compounding, convexity is measured in half-years squared and has to be divided by four for conversion to years squared.

Summary: Using the duration-convexity terminology developed so far, we can rewrite the Taylor expansion for the change in the price of a bond, as follows:

$$\Delta P = -\left(D^* \times P_0\right)(\Delta y) + \frac{1}{2}\left(\kappa \times P_0\right)(\Delta y)^2 + \cdots,$$

where

- duration measures the first-order (linear) effect of changes in yield,
- convexity measures the second-order (quadratic) term,

and recall that $P_0 = P(y_0)$.

Example: Consider a zero-coupon bond with $T = 10$ years to maturity and a yield of $y_0 = 6\%$ (semi-annually). The (initial) price of this bond is $P_0 = 55.368$ CHF as obtained from:

$$P_0 = \frac{100}{(1 + 6\%/2)^{2 \cdot 10}} = 55.368.$$

We now compute the various sensitivities of this bond:

- Macaulay duration $D = T = 10$ years;
- Modified duration is given by $\frac{dP_0}{d(y/2)} = -D^* \times P_0$:

$$D^* = \frac{2 \cdot 10}{1 + 6\%/2} = 19.42 \quad \text{half-years},$$

  or $D^* = 9.71$ years;
- Dollar duration $DD = D^* \times P_0 = 9.71 \times 55.37 = 537.55$;
- Dollar value of a basis point is $DVBP = DD \times 0.0001 = 0.0538$;
- Convexity is

$$\frac{21 \times 20}{(1 + 6\%/2)^2} = 395.89 \quad \text{half-years squared},$$

  or $\kappa = 98.97$ years squared.

Finally, we can now turn to the problem of estimating the change in the value of the bond if the yield goes from $y_0 = 6\%$ to, say, $y_1 = 7\%$, i.e., $\Delta y = 1\%$:

$$\begin{aligned}
\Delta P &\sim -(D^* \times P_0)(\Delta y) + \frac{1}{2}(\kappa \times P_0)(\Delta y)^2 \\
&= -(9.71 \times 55.37) \cdot 1\% + \frac{1}{2}(98.97 \times 55.37) \cdot (1\%)^2 \\
&= -5.101.
\end{aligned}$$

Note that the exact value for the yield $y_1 = 7\%$ is 50.257 CHF. Thus:

- using only the first term in the expansion, the predicted price is $55.368 - 5.375 = 49.992$ CHF, and
- the linear approximation has a pricing error of -0.53% (not bad given the large change in the yield).
- Using the first two terms in the expansion, the predicted price is $55.368 - 5.101 = 50.266$ CHF,
- thus adding the second term reduces the approximation error to 0.02%.

Exercise L01.4: What is the price impact of a 10-BP increase in yield on a 10-year zero-coupon bond whose price, duration and convexity are $P = 100$ CHF, $D = 7$ and $\kappa = 50$, respectively.

1. -0.705
2. -0.700
3. -0.698
4. -0.690.

Having done the numerical calculations, it is now helpful to have a graphical representation of the duration-convexity approximation. The graph (on the next slide) compares the following three curves:

1. the actual, exact price-yield relationship:

$$P = P(y);$$

2. the duration based estimate (first-order approximation):

$$P = P_0 - D^* P_0 \cdot \Delta y;$$

3. the duration and convexity estimate (second-order approximation):

$$P = P_0 - D^* \times P_0 \cdot \Delta y + \frac{1}{2}\kappa \times P_0 \times (\Delta y)^2.$$

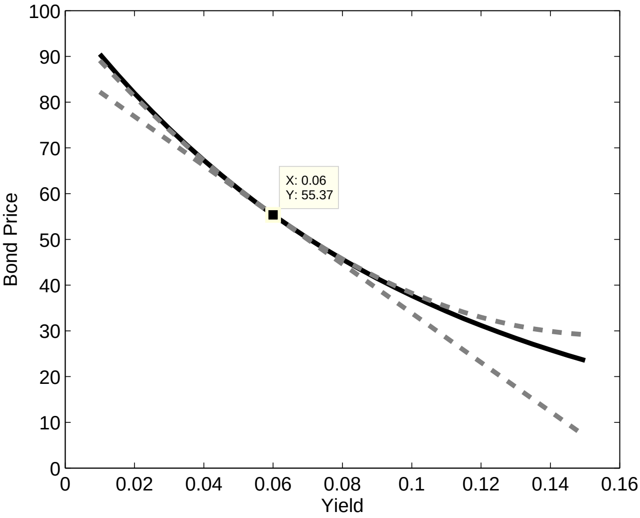

Figure : The price-yield relationship and the duration-convexity based approximation. *Solid black*: The exact price-yield relationship, *Dashed gray*: Linear and Second order approximations.

Conclusions:

- for small movements in the yield, the duration-based linear approximation provides a reasonable fit to the exact price; including the convexity term, increases the range of yields over which the approximation remains reasonable;
- Dollar duration measures the (negative) slope of the tangent to the price-yield curve at the starting point y_0;
- when the yield rises, the price drops but less than predicted by the tangent; if the yield falls, the price increases faster than the duration model. In other words, the quadratic term is always beneficial.

Notes:

- In economic terms, duration is the average time to wait for each payment weighted by their present values.
- For the standard bonds considered so far, we have been able to compute duration and convexity analytically. However, in practice there exist bonds with more complicated features (such as mortgage-backed securities with an embedded prepayment option), for which it is not possible to compute duration and convexity in closed form.
- Instead, we need to resort to numerical method, in particular, approximating the bond price sensitivities with finite differences.

- Choose a change in the yield, Δy, and reprice the bond under an up-move scenario $P_+ = P(y_0 + \Delta y)$ and a down-move scenario $P_- = P(y_0 - \Delta y)$.
- Then approximate the first-order derivative with a centered finite difference. From

$$D^* = -\frac{1}{P}\frac{dP}{dy}$$

effective duration is estimated as:

$$D^* \approx -\frac{1}{P_0} \times \frac{P_+ - P_-}{2\Delta y} = \frac{1}{P_0} \times \frac{P(y_0 - \Delta y) - P(y_0 + \Delta y)}{2\Delta y}.$$

- Similarly, from

$$\kappa = \frac{1}{P}\frac{d^2P}{dy^2}$$

effective convexity is estimated as:

$$\kappa \approx \frac{1}{P_0} \times \left[\frac{P(y_0 - \Delta y) - P_0}{\Delta y} - \frac{P_0 - P_0(y_0 + \Delta y)}{\Delta y}\right] \times \frac{1}{\Delta y}.$$

Answers

Exercise L01.1: b) This is derived from $(1 + y_s/2)^2 = (1 + y)$ or, equivalently, $(1 + 0.08/2)^2 = (1 + y)$, which gives $y = 8.16\%$. compounding.

Exercise L01.2: a) This is derived from $(1 + y_s/2)^2 = \exp(y_c)$ or, equivalently, $(1 + y_s/2)^2 = 1.1056$, which gives $y_s = 10.25\%$.

Exercise L01.3: d) We need to find y such that $4/(1 + y_s/2) + 104/(1 + y_s/2)^2 = 102.9$. Solving, we find $y_s = 5\%$.

Exercise L01.4: c) The initial price is $P_0 = 100$. The yield increase is 10-BP, which means $\Delta y = 10 \cdot 0.0001 = 0.001$. The price impact is

$$\begin{aligned}\Delta P &= -\left(D^* \times P_0\right)(\Delta y) + \frac{1}{2}\left(\kappa \times P_0\right)(\Delta y)^2 \\ &= -7 \cdot 100 \cdot (0.001) + \frac{1}{2} \cdot 50 \cdot 100 \cdot (0.001)^2 \\ &= -0.6975.\end{aligned}$$