Lecture Quantitative Finance Spring Term 2015

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2.1: The one-period model

Model setup

- Two trading periods: today and tomorrow denoted by $t_0 = 0$ and $t_1 = 1$.
- Market consisting of $d + 1$ assets:
  - ∗ asset 0 is consider as a riskless bond,
  - ∗ assets $1, \dots d$ are risky assets.
- Present prices are given by $\bar{\pi} = \left(\pi^0, \dots, \pi^d\right)$ where $\pi^i \geq 0$ for every $i = 0, \dots, d$.
- Future prices are modeled by a vector of non-negative random variables $\bar{S} = \left(S^0, S\right) = \left(S^0, S^1, \dots, S^d\right)$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

  $\Rightarrow$ We interpret $S^i(\omega)$ as the price of asset i at time 1 if the market scenario $\omega \in \Omega$ occurs.
- For the riskless bond, we assume that $\pi^0 = 1$ and $S^0(\omega) = 1 + r$ for every $\omega \in \Omega$, where $r > -1$ denotes the risk-free interest rate.

Portfolio: Definition

- A **portfolio** is a vector $\bar{\xi} \in \mathbb{R}^{d+1}$ with $\bar{\xi} = \left(\xi^0, \ldots, \xi^d\right)$.
  $\Rightarrow$ The value ξ^i represents the number of shares of asset i held in the portolio.
- The **portfolio price** at time $t_0 = 0$ and $t_1 = 1$, respectively, is given by:

$$
\begin{aligned}
\bar{\pi} \cdot \bar{\xi} &= \sum_{i=0}^{d} \pi^i \xi^i \\
\bar{\xi} \cdot \bar{S}(\omega) &= \sum_{i=0}^{d} S^i(\omega) \xi^i .
\end{aligned}
$$

- Note that the map $\omega \in \Omega \mapsto \bar{\xi} \cdot \bar{S}(\omega)$ defines a random variable.

Example

Example

- $\Omega = \{\omega_1, \omega_2\}$, where ω_1 represents the event of a price's increase, while ω_2 denotes a price's decrease.
- $p \in (0, 1)$ is the probability of a price's increase
- $\bar{\xi} = (-2, 1, 2, 0, \ldots, 0) \in \mathbb{R}^{d+1}$ is a portfolio consisting of a short position in the bank account, and long positions in the first and second asset.

Arbitrage opportunities

- An **arbitrage opportunity** is the possibility of making money out of nothing, without being exposed to any kind of loss risk.
- Supposing that $\Omega = \{\omega_1, \ldots, \omega_n\}$ for some finite $n \in \mathbb{N}$, a portfolio $\bar{\xi} \in \mathbb{R}^{d+1}$ is an arbitrage opportunity if, and only if, it solves the following system, with at least one strict inequality:

$$\begin{cases} \xi^0 1 + \xi^1 \pi^1 + \cdots + \xi^d \pi^d & \leq & 0; \\ \xi^0(1+r) + \xi^1 S^1(\omega_1) + \cdots + \xi^d S^d(\omega_1) & \geq & 0; \\ \vdots & & \\ \xi^0(1+r) + \xi^1 S^1(\omega_n) + \cdots + \xi^d S^d(\omega_n) & \geq & 0; \end{cases}$$

where $r > -1$ is the risk-free interest rate.

- Aim: characterize market model which do not allow for the existence of arbitrage opportunities.

Risk-neutral measure

- A probability measure $\mathbb{P}^*$ is called a **risk-neutral measure** or **martingale measure** if for any asset $i = 0, \dots, d$ we have:

$$\pi^i = \mathbb{E}_{\mathbb{P}^*}\left[\frac{S^i}{1+r}\right].$$

$\Rightarrow$ The expected value of the discounted price of asset i coincides with its initial price.

- Recall that for a random variable $X : \Omega \to \mathbb{R}$, the expected value of X under the probability measure $\mathbb{P}^*$ is given by

$$\mathbb{E}_{\mathbb{P}^*}[X] = \sum_{\omega\in\Omega} X(\omega)\cdot\mathbb{P}^*[\{\omega\}], \text{ for } \Omega \text{ countable;}$$

$$\mathbb{E}_{\mathbb{P}^*}[X] = \int_\Omega X d\mathbb{P}^*, \text{ for general } \Omega.$$

Risk-neutral measure: example

Example

- Consider a market with one riskless asset and one risky asset (i.e. $d = 1$).
- Suppose $\Omega = \{\omega_1, \omega_2, \omega_3\}$.
- Then, the risk-neutral prices of these assets are given by the following equations:

$$
\begin{aligned}
1 &= \frac{1+r}{1+r} \cdot \mathbb{P}^*[\{\omega_1\}] + \frac{1+r}{1+r} \cdot \mathbb{P}^*[\{\omega_2\}] + \frac{1+r}{1+r} \cdot \mathbb{P}^*[\{\omega_3\}]; \\
\pi^1 &= \frac{S^1(\omega_1)}{1+r} \cdot \mathbb{P}^*[\{\omega_1\}] + \frac{S^1(\omega_2)}{1+r} \cdot \mathbb{P}^*[\{\omega_2\}] + \frac{S^1(\omega_3)}{1+r} \cdot \mathbb{P}^*[\{\omega_3\}].
\end{aligned}
$$

Equivalent measures

- Two probability measures $\mathbb{P}$ and $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ are said to be equivalent if for any set $A \subseteq \mathcal{F}$ we have:

$$\mathbb{P}[A] = 0 \text{ if, and only if, } \mathbb{P}^*[A] = 0.$$

In this case, we write $\mathbb{P} \approx \mathbb{P}^*$.

- Denote by $\mathcal{P}$ the set of all martingale measures, which are equivalent to $\mathbb{P}$:

$$\mathcal{P} = \{\mathbb{P}^* \,:\, \mathbb{P}^* \text{ is a risk-neutral measure and } \mathbb{P}^* \approx \mathbb{P}\}$$

Fundamental Theorem of Asset Pricing

Theorem (The Fundamental Theorem of Asset Pricing)

A market model is arbitrage-free if, and only if, there exists a risk-neutral measure $\mathbb{P}^*$ *on* $(\Omega, \mathcal{F})$*, which is equivalent to* $\mathbb{P}$*.*

Note: The Fundamental Theorem of Asset Pricing is equivalent to $\mathcal{P} \neq \emptyset$.

Free lunch

- A portfolio $\bar{\xi} \in \mathbb{R}^{d+1}$ is called a **free lunch** if $\bar{\xi} \cdot \bar{\pi} < 0$ and $\bar{\xi} \cdot \bar{S} \geq 0$ $\mathbb{P}$-a.s.
- If no such strategy exists, we say that the market model satisfies the No-Free-Lunch condition (NFL)

Theorem

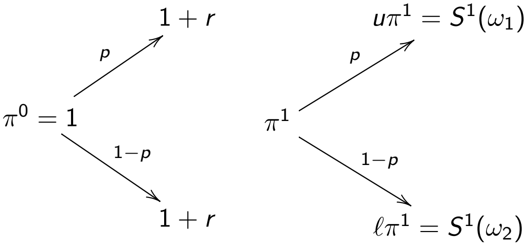
Consider a market model with $n \in \mathbb{N}$ possible future states described by $\Omega = \{\omega_1, \ldots, \omega_n\}$ with $\mathbb{P}[\{\omega_i\}] > 0$, $\forall i = 1, \ldots, n$. Then, the following are equivalent:

1. *The market model satisfies the (NFL) condition.*
2. *There exists a risk-neutral measure $\mathbb{P}^*$.*

Note: instead of the FTAP, here nothing is said about the equivalence of the risk-neutral measure $\mathbb{P}^*$ with the underlying measure $\mathbb{P}$.

Example (Binomial Model)

- Consider a one-period model with a risk-free asset with interest rate $r > 0$ and a risky asset with initial price $S_0 > 0$.
- $\Omega = \{\omega_1, \omega_2\}$, where ω_1 and ω_2 represent the event of an up and down price move, respectively.
- Take $\ell, u > 0$ with $\ell < u$ and let $p \in (0,1)$ be the probability of an up-move, i.e. $0 < p = \mathbb{P}[\{\omega_1\}] < 1$.



Binomial Model: Example 1

Example (continue)

Claim 1 No free lunch if, and only if, $\ell \leq 1 + r \leq u$.

Proof

- Absence of free lunches is equivalent to the existence of a risk-neutral measure $\mathbb{P}^*$ satisfying:

$$\begin{pmatrix} \frac{S^0(\omega_1)}{1+r} & \frac{S^0(\omega_2)}{1+r} \\ \frac{S^1(\omega_1)}{1+r} & \frac{S^1(\omega_2)}{1+r} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{P}^*[\{\omega_1\}] \\ \mathbb{P}^*[\{\omega_2\}] \end{pmatrix} = \begin{pmatrix} 1 \\ \pi^1 \end{pmatrix}.$$

- This is equivalent to:

$$\begin{cases} \mathbb{P}^*[\{\omega_1\}] + \mathbb{P}^*[\{\omega_2\}] = 1; \\ \frac{u\pi^1}{1+r} \cdot \mathbb{P}^*[\{\omega_1\}] + \frac{\ell\pi^1}{1+r} \cdot \mathbb{P}^*[\{\omega_2\}] = \pi^1. \end{cases}$$

Binomial Model: Example 1

Example (continue)

- Solution:

$$\mathbb{P}^*\left[\{\omega_1\}\right] = \frac{(1+r)-\ell}{u-\ell}, \; \mathbb{P}^*\left[\{\omega_2\}\right] = \frac{u-(1+r)}{u-\ell}.$$

- Since these probabilities have to be ≥ 0, we get:

$$\ell \leq 1+r \leq u.$$

■

Binomial Model: Example 1

Example (continue)

Claim 2 No Arbitrage if, and only if, $\ell < 1 + r < u$.

Proof

- Again, we have to solve the sistem of equations

$$\begin{pmatrix} \frac{S^0(\omega_1)}{1+r} & \frac{S^0(\omega_2)}{1+r} \\ \frac{S^1(\omega_1)}{1+r} & \frac{S^1(\omega_2)}{1+r} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{P}^*[\{\omega_1\}] \\ \mathbb{P}^*[\{\omega_2\}] \end{pmatrix} = \begin{pmatrix} 1 \\ \pi^1 \end{pmatrix}.$$

- The model is arbitrage-free if, and only if, $\mathbb{P}^*$ is equivalent to $\mathbb{P}$. Since $\mathbb{P}[\{\omega_1\}], \mathbb{P}[\{\omega_2\}] > 0$ we get:

$$\mathbb{P}^*[\{\omega_1\}] = \frac{(1+r) - \ell}{u - \ell} > 0,\ \mathbb{P}^*[\{\omega_2\}] = \frac{u - (1+r)}{u - \ell} > 0.$$

- This implies $\ell < 1 + r < u$. ■

Attainable Payoffs

- To compare the prices π and of S, we have to convert them to a common standard $\Rightarrow$ Use the riskless asset as a **numeraire** and consider the present value $S/(1+r)$ of the future price S.
- $\mathcal{V} = \{\bar{\xi} \cdot \bar{S} \mid \text{ for } \bar{\xi} \in \mathbb{R}^{d+1}\}$ is the set of all **attainable payoffs**.
- For $V \in \mathcal{V}$, the price of V at time t_0 is given by

$$\pi(V) = \bar{\xi} \cdot \bar{\pi}.$$

Lemma

Suppose we are in an arbitrage-free model. Let $V \in \mathcal{V}$ be an attainable payoff generated by two different strategies $\bar{\xi}, \bar{\eta} \in \mathbb{R}^{d+1}$. Then, we have: $\bar{\xi} \cdot \bar{\pi} = \bar{\eta} \cdot \bar{\pi}$.

Derivatives: Call and Put options

- An **European call option** on the i^{th} asset with strike price $K > 0$ gives its owner the right to buy asset i at time t_1 for the fixed price K. The payoff of the call option is given by:

$$C^{call,i} = (S^i - K)^+ = \begin{cases} S^i - K, & \text{if } S^i - K \geq 0; \\ 0, & \text{if } S^i - K < 0. \end{cases}$$

- An **European put option** on the i^{th} asset with strike price $K > 0$ gives its owner the right to sell asset i at time t_1 for the fixed price K. The payoff of the put option is given by:

$$C^{put,i} = (K - S^i)^+ = \begin{cases} 0, & \text{if } S^i - K \geq 0; \\ K - S^i, & \text{if } S^i - K < 0. \end{cases}$$

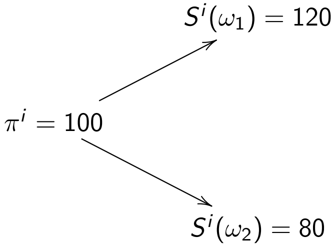
- The payoffs of call and put options are related by the following equation, known as the **Put-Call-Parity:**

$$C^{call,i} - C^{put,i} = S^i - K$$

Call and Put options: Example

Example

- Suppose the prices of asset i are as follows:



- In addition, assume that $K = 110$.

Example (continue)

- Then, the payoff of a call option is:

$$C^{call,i}(\omega) = \begin{cases} S^i(\omega_1) - K = 10, & \text{if } \omega = \omega_1; \\ 0, & \text{if } \omega = \omega_2. \end{cases}$$

- Instead, for the put option we get:

$$C^{put,i}(\omega) = \begin{cases} 0, & \text{if } \omega = \omega_1; \\ K - S^i(\omega_2) = 30, & \text{if } \omega = \omega_2. \end{cases}$$

- In particular, note that the Put-Call-Parity is satisfied.

Basket option and Straddle

- An option on the value $V = \bar{\xi} \cdot \bar{S}$ of a portfolio $\bar{\xi} \in \mathbb{R}^{d+1}$ is called a **basket** or **index option**.

  $\Rightarrow$ For example, the payoff of a basket put option would be of the form $(K - V)^+$.

- A collection of an "at-the-money" call and put option with strike K on a portfolio $V = \bar{\xi} \cdot \bar{S}$ is called a **straddle** and its payoff is of the form:

$$C^{straddle} = (V - K)^+ + (K - V)^+ .$$

Arbitrage-free prices: Definition

- In general, it is not clear how to price a contingent claim C.
- To this end, suppose that we trade C at time t_0 for a price π^C. This is equivalent to say that we introduce a new asset in the market model with prices:

$$\pi^{d+1} := \pi^C \text{ and } S^{d+1} := C.$$

- We refer to this new market with $d+2$ assets (i.e. the original ones plus the contingent claim) as the **extended market model**.
- A real number $\pi^c \geq 0$ is called an **arbitrage-free price** for the contingent claim C, if the extended market model is arbitrage-free.

Arbitrage-free prices

- Denote by $\Pi(C)$ the set of all arbitrage-free prices for a contingent claim C.

Theorem

Suppose the initial market model is arbitrage-free and let C be a contingent claim. Then, the set of arbitrage-free prices for C is non-empty and of the form:

$$\Pi(C) = \left\{ \mathbb{E}_{\mathbb{P}^*}\left[\frac{C}{1+r}\right] \; : \; \mathbb{P}^* \in \mathcal{P}, \; \mathbb{E}_{\mathbb{P}^*}[C] < \infty \right\}.$$

Arbitrage-free prices

- By the previous theorem, we know that $\Pi(C)$ is not empty, but how many elements are in there?
- Define the lower and upper bound for this set, respectively, as:

$$\pi^{\downarrow}(C) = \inf \Pi(C) = \inf_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}_{\mathbb{P}^*}\left[\frac{C}{1+r}\right].$$

$$\pi^{\uparrow}(C) = \sup \Pi(C) = \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}_{\mathbb{P}^*}\left[\frac{C}{1+r}\right].$$

Attainable Claims: Definition and Characterisation

- A contingent claim C is called **attainable** if there exists a portfolio $\bar{\xi} \in \mathbb{R}^{d+1}$ such that $C = \bar{\xi} \cdot \bar{S}$ $\mathbb{P}$-a.s.
- The vector $\bar{\xi}$ is then called a **replicating portfolio** for C.

Theorem

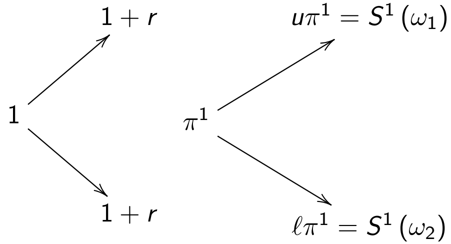
Consider an arbitrage-free market model and let C be an arbitrary contingent claim. Then:

1. *The claim C is attainable if and only if $\Pi(C)$ consists of exactly one element, i.e. $\Pi(C) = \{\bar{\xi} \cdot \bar{\pi}\}$ where $\bar{\xi}$ is a replicating portfolio.*
2. *If C is not attainable, then $\pi^{\downarrow}(C) < \pi^{\uparrow}(C)$ and $\Pi(C) = \left(\pi^{\downarrow}(C), \pi^{\uparrow}(C)\right)$.*

Binomial model: Example 2

Example

- Consider again the one-period binomial model with the following dynamics:



- Recall that the market is arbitrage-free if, and only if, $\ell < 1 + r < u$.

Example (continue)

- The **unique** risk-neutral probability measure $\mathbb{P}^*$, equivalent to $\mathbb{P}$, was given by:

$$\mathbb{P}^*\left[\{\omega_1\}\right] = \frac{(1+r)-\ell}{u-\ell}, \; \mathbb{P}^*\left[\{\omega_2\}\right] = \frac{u-(1+r)}{u-\ell}.$$

- Let $C^{call} = \left(S^1 - K\right)^+$ be the payoff of a call option on the risky asset with strike price $K > 0$.
- We want to find the exact value of π^C.

Example (continue)

- By the previous theorem, we get:

$$\begin{aligned}\pi^C &= \mathbb{E}_{\mathbb{P}^*}\left[\frac{C^{call}}{1+r}\right] \\ &= \frac{C^{call}(\{\omega_1\})}{1+r}\mathbb{P}^*[\{\omega_1\}] + \frac{C^{call}(\{\omega_2\})}{1+r}\mathbb{P}^*[\{\omega_2\}] \\ &= \frac{\left(u\pi^1 - K\right)^+}{1+r}\,\frac{1+r-\ell}{u-\ell} + \frac{\left(\ell\pi^1 - K\right)^+}{1+r}\,\frac{u-(1+r)}{u-\ell}.\end{aligned}$$

$\Rightarrow$ A call option is an attainable payoff. Later, we will derive its replicating strategy.

Complete Markets

- Consider an arbitrage-free model. We say that the market is **complete**, if every contingent claim is attainable.

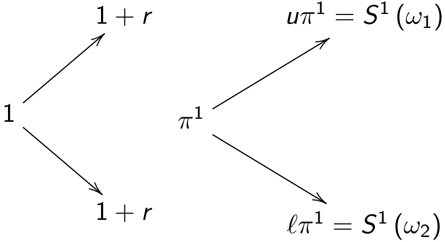
Theorem

The binomial model is complete if, and only if, $\ell < 1 + r < u$*. In this case, every claim is attainable.*

Binomial model: Example 3

Example

- Consider again the one-period binomial model with the following dynamics:



- Recall that the market is arbitrage-free **and** complete if, and only if, $\ell < 1 + r < u$.

Example (continue)

- Let C^{call} be a call option on the underlying risky asset with strike price $K > 0$.
- To replicate C^{call}, we need to determine a replicating strategy $\bar{\xi} = (\xi^0, \xi^1)$ satisfying:

$$\left(S^1(\omega) - K\right)^+ = \xi^0 \cdot (1+r) + \xi^1 \cdot S^1(\omega)$$

for every $\omega \in \Omega$.

- This translates into the following system of two equations:

$$\begin{cases} \left(S^1 - K\right)^+ (\omega_1) = \xi^0 \cdot (1+r) + \xi^1 \cdot S^1(\omega_1); \\ \left(S^1 - K\right)^+ (\omega_2) = \xi^0 \cdot (1+r) + \xi^1 \cdot S^1(\omega_2). \end{cases}$$

Example (continue)

- Hence, the replicating strategy of a call option is of the form:

$$\xi^0 = \frac{u\left(\ell\pi^1 - K\right)^+ - \ell\left(u\pi^1 - K\right)^+}{(1+r)(u-\ell)},$$

$$\xi^1 = \frac{\left(u\pi^1 - K\right)^+ - \left(\ell\pi^1 - K\right)^+}{\pi^1 (u-\ell)}.$$

- **Note:** $\xi^0 \leq 0$.
  $\Rightarrow$ To replicate a call option, we have to borrow money at the risk-free rate.

Binomial model: Example 3

Example (continue)

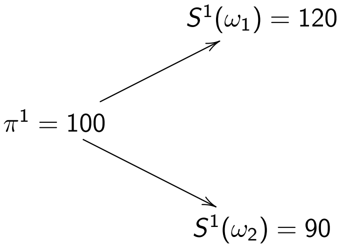
- The unique arbitrage-free price of C^{call} is then given by:

$$\begin{aligned}\pi\left(C^{call}\right) &= \bar{\xi}\cdot\bar{\pi} \\ &= \frac{\left(u\pi^1-K\right)^+\cdot(1+r-\ell)}{(1+r)(u-\ell)}+\frac{\left(\ell\pi^1-K\right)^+\cdot(u-1-r)}{(1+r)(u-\ell)}.\end{aligned}$$

- **Note:** This formula coincides with the one of Example 2.

Example (continue)

- Next: how can we use options to modify the risk of a position?
- Assume that the risky asset can be bought at time t_0 for the price $\pi^i = 100$ and at time t_1 for the prices:



Example (continue)

- If we invest in the risky asset, the corresponding returns $R(S^1) = \frac{S^1 - \pi^1}{\pi^1}$ are given by:

$$\begin{cases} R(S^1)(\omega_1) = \frac{S^1(\omega_1) - \pi^1}{\pi^1} = 20\%; \\ R(S^1)(\omega_2) = \frac{S^1(\omega_2) - \pi^1}{\pi^1} = -10\%. \end{cases}$$

- On the other hand, let $C^{call} = \left(S^1 - 100\right)^+$ be a call option on the risky asset with strike price $K = 100$, and suppose in addition that $r = 0$.
- According to the above formula, its price is then given by:

$$\pi^C = \pi(C^{call}) = \frac{20}{3} = 6.67$$

Example (continue)

- Therefore, the return $R(C) = \frac{(S^1 - K)^+ - \pi^C}{\pi^C}$ of an initial investment of π^C is equal to:

$$
\begin{aligned}
R(C)(\omega_1) &= \frac{(120-100)^+ - \frac{20}{3}}{\frac{20}{3}} = 200\%, \\
R(C)(\omega_2) &= \frac{0 - \frac{20}{3}}{\frac{20}{3}} = -100\%.
\end{aligned}
$$

$\Rightarrow$ There is a dramatic increase of profit opportunities and losses. This is called the **leverage effect** of options.

Binomial model: Example 3

Example (continue)

- To reduce the risk of holding the asset, we can hold the portfolio $\tilde{C} = \left(K - S^1\right)^+ + S^1$, which consists of a put option and the asset itself.
- Clearly, this "portfolio insurance" will involve an additional cost.
- Indeed, from the Put-Call Parity, we can derive the price of the put option:

$$\pi(C^{put}) = \pi(C^{call}) - \pi^1 + \frac{K}{1+r} = \frac{20}{3};$$

thus in order to hold S^1 and a put on S^1 we need to pay the amount $100 + 20/3$ at time $t_0 = 0$.

Binomial model: Example 3

Example (continue)

- The return at $t_1 = 1$ is then given by:

$$R(\tilde{C})(\omega_1) = 12.5\% \quad R(\tilde{C})(\omega_2) = -6.25\%.$$

- Hence, by holding this portfolio insurance we have reduce the risk of a loss, althought the possibility of a big gain has decreased as well.

2.2: Exercises

Exercise 1: Question

Question

Consider a one-period market model with $d + 1$ assets.

1. Show that if there exists a free lunch (i.e. a strategy $\bar{\xi} \in \mathbb{R}^{d+1}$ such that $\bar{\xi} \cdot \bar{\pi} < 0$ and $\bar{\xi} \cdot \bar{S} \geq 0$ $\mathbb{P}$-a.s.), then there exists an arbitrage opportunity.
2. Give an example to show that the reverse is not true.

Exercise 1: Solution

Solution

1. This part is optional.
   Suppose there exists a free-lunch, i.e. there exists $\overline{\xi} \in \mathbb{R}^{d+1}$ such that $\overline{\xi} \cdot \overline{\pi} < 0$ and $\overline{\xi} \cdot \overline{S} \geq 0$ $\mathbb{P}$-a.s..

   We want to show that this implies the existence of an arbitrage opportunity, i.e. there exists $\overline{\eta} \in \mathbb{R}^{d+1}$ such that:

   - $\overline{\eta} \cdot \overline{\pi} \leq 0$,
   - $\overline{\eta} \cdot \overline{S} \geq 0$ $\mathbb{P}$-a.s.,
   - $\mathbb{P}\left[\overline{\eta} \cdot \overline{S} > 0\right] > 0$.

   By definition, a free-lunch implies a negative initial portfolio, i.e.

$$0 > \overline{\xi} \cdot \overline{\pi} = \xi^0 \cdot 1 + \xi \cdot \pi =: -\delta;$$

   so that $\delta > 0$.

Exercise 1: Solution

Now, we define $\eta^0 := \xi^0 + \delta$ and $\eta^i := \xi^i$ for $i = 1 \ldots d$. Then, for the new strategy η we get:

$$\begin{aligned} \overline{\eta} \cdot \overline{\pi} &= (\xi^0 + \delta) \cdot 1 + \eta \cdot \pi \\ &= 0. \end{aligned}$$

Moreover:

$$\begin{aligned} \overline{\eta} \cdot \overline{S} &= (\xi^0 + \delta) \cdot (1 + r) + \eta \cdot S \\ &= \eta^0 (1 + r) + \delta (1 + r) + \xi \cdot S \\ &= \overline{\xi} \cdot \overline{S} + \delta \cdot (1 + r). \end{aligned}$$

By definition of free-lunch, we know that $\mathbb{P}\left[\overline{\xi} \cdot \overline{S} \geq 0\right] = 1$. But since $\delta \cdot (1 + r) > 0$, it follows that

$$\mathbb{P}\left[\overline{\eta} \cdot \overline{S} > 0\right] = \mathbb{P}\left[\overline{\xi} \cdot \overline{S} + \delta \cdot (1 + r) > 0\right] = 1.$$

This proves that $\overline{\eta}$ is an arbitrage opportunity.

Exercise 1: Solution

2. This part is not optional.
Consider a binomial model with 2 assets ($d = 1$) and $\Omega = \{\omega_1, \omega_2\}$ such that $\mathbb{P}[\{\omega_1\}] = \mathbb{P}[\{\omega_2\}] = \frac{1}{2}$.
Assume $\pi^1 = 2$, $S^1(\omega_1) = 3$, $S^1(\omega_2) = 2$ and $r = 0$.

- The necessary and sufficient condition to have no free-lunch is: $\ell \leq 1 + r \leq u$.
  $\Rightarrow$ Here, we have $\frac{2}{2} \leq 1 \leq \frac{3}{2}$, therefore there is no free-lunch.
- On the other hand, the necessary and sufficient condition to have no arbitrage opportunities is $\ell < 1 + r < u$.
  $\Rightarrow$ Since this is not satisfied, there exists an arbitrage opportunity in our market model.

Exercise 2: Question

Question

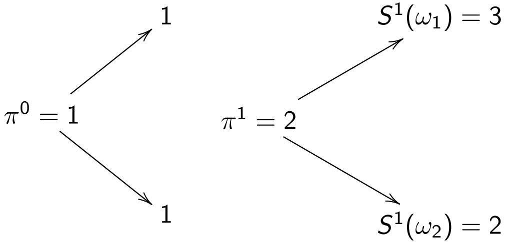
Consider a one-period model with 2 assets (i.e. $d = 1$) and $\Omega = \{\omega_1, \omega_2\}$ such that $\mathbb{P}[\{\omega_1\}] = \mathbb{P}[\{\omega_2\}] = \frac{1}{2}$. Assume in addition that $\pi^1 = 2$, $S^1(\omega_1) = 3$ and $S^1(\omega_2) = 2$.

1. Show that if $r = 0$, then there are no free-lunches in the model.
2. Determine a risk-neutral measure $\mathbb{P}^*$ for this market. Observe that in general $\mathbb{P} \neq \mathbb{P}^*$.

Exercise 2: Solution

Solution

1. Suppose $r = 0$. Then we have $\pi^0 = S^0 = 1$, so that the price process follows the dynamics:



By definition, a free-lunch is a strategy $\overline{\xi} \in \mathbb{R}^2$ such that $\overline{\xi} \cdot \overline{\pi} < 0$ and $\overline{\xi} \cdot \overline{S} \geq 0$, $\mathbb{P}$-a.s.

Exercise 2: Solution

To show its existence, the following system of linear inequalities has to be solvable:

$$\begin{cases} 1 \cdot \xi^0 + 2 \cdot \xi^1 < 0; \\ 1 \cdot \xi^0 + 2 \cdot \xi^1 \geq 0; \\ 1 \cdot \xi^0 + 3 \cdot \xi^1 \geq 0. \end{cases}$$

But this system has no solution, therefore this model satisfies the (NFL) condition.

Exercise 2: Solution

2. To determine a risk-neutral measure $\mathbb{P}^*$ we have to solve the following system of equations:

$$\begin{pmatrix} \frac{S^0(\omega_1)}{1+r} & \frac{S^0(\omega_2)}{1+r} \\ \frac{S^1(\omega_1)}{1+r} & \frac{S^1(\omega_2)}{1+r} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{P}^*[\{\omega_1\}] \\ \mathbb{P}^*[\{\omega_2\}] \end{pmatrix} = \begin{pmatrix} \pi^0 \\ \pi^1 \end{pmatrix}.$$

Substituting the numbers given in the problem, this is equivalent to:

$$\begin{cases} \mathbb{P}^*[\{\omega_1\}] + \mathbb{P}^*[\{\omega_2\}] = 1; \\ 2\mathbb{P}^*[\{\omega_2\}] + 3\mathbb{P}^*[\{\omega_1\}] = 2. \end{cases}$$

We conclude that:

$$\begin{cases} \mathbb{P}^*[\{\omega_2\}] = 1, \\ \mathbb{P}^*[\{\omega_1\}] = 0 \end{cases}.$$

In particular, observe that $\mathbb{P} \neq \mathbb{P}^*$.

Exercise 3: Question

Question

Consider a one-period model with 2 assets. Let $\Omega = \{\omega_1, \omega_2\}$ and consider a probability measure $\mathbb{P}$ with $\mathbb{P}[\{\omega_1\}], \mathbb{P}[\{\omega_2\}] > 0$.

Assume $r = 0$, $\pi^0 = 1$, $\pi^1 = 100$, $S^1(\omega_1) = 120$, $S^1(\omega_2) = 80$ and let $80 < K < 120$.

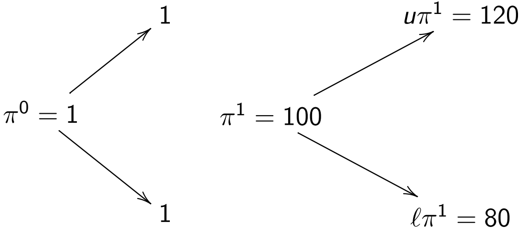
Consider a call and put option on the risky asset, with payoffs $C^{call} = \left(S^1 - K\right)^+$ and $C^{put} = \left(K - S^1\right)^+$, respectively.

For which values of K do we have $\pi\left(C^{call}\right) \leq \pi\left(C^{put}\right)$?

Exercise 3: Solution

Solution

Consider a one-period model with 2 assets ($d = 1$). We assume $r = 0$, $\pi^0 = 1$, $\pi^1 = 100$, $S^1(\omega_1) = 120$, $S^1(\omega_2) = 80$ and $80 < K < 120$.



Let $C^{call} = (S^1 - K)^+$ and $C^{put} = (K - S^1)^+$ be the payoffs of a call and put option on S^1, respectively.

Exercise 3: Solution

Step 1 Derive an equivalent risk-neutral measure and see if it is unique.

$\Rightarrow$ We have to solve the following equations:

$$\begin{pmatrix} \frac{S^0(\omega_1)}{1+r} & \frac{S^0(\omega_2)}{1+r} \\ \frac{S^1(\omega_1)}{1+r} & \frac{S^1(\omega_2)}{1+r} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{P}^*[\{\omega_1\}] \\ \mathbb{P}^*[\{\omega_2\}] \end{pmatrix} = \begin{pmatrix} \pi^0 \\ \pi^1 \end{pmatrix}.$$

Substituting the given numbers, we get:

$$\begin{cases} \mathbb{P}^*[\{\omega_1\}] + \mathbb{P}^*[\{\omega_2\}] = 1 \\ 80 \cdot \mathbb{P}^*[\{\omega_1\}] + 120 \cdot \mathbb{P}^*[\{\omega_2\}] = 100 \end{cases}$$

Therefore we get $\mathbb{P}^*[\{\omega_1\}] = \mathbb{P}^*[\{\omega_2\}] = \frac{1}{2}$, which means that $\mathbb{P}^*$ is unique.

Exercise 3: Solution

Step 2 Compute π^C.

$\Rightarrow$ Uniqueness of the equivalent risk-neutral measure implies that the price of a contingent claim C is given by

$$\pi^C = \mathbb{E}_{\mathbb{P}^*}\left[\frac{C}{1+r}\right].$$

Thus, for the call option:

$$\begin{aligned}
\pi(C^{call}) &= \frac{\left(S^1(\omega_1)-K\right)^+}{1+r}\cdot\mathbb{P}^*[\{\omega_1\}] + \frac{\left(S^1(\omega_2)-K\right)^+}{1+r}\cdot\mathbb{P}^*[\{\omega_2\}] \\
&= (120-K)\cdot\frac{1}{2} + 0\cdot\frac{1}{2} \\
&= 60-\frac{K}{2}.
\end{aligned}$$

Exercise 3: Solution

On the other hand, for the put option:

$$\begin{aligned}\pi(C^{put}) &= \frac{\left(K - S^1(\omega_1)\right)^+}{1+r} \cdot \mathbb{P}^*\left[\{\omega_1\}\right] + \frac{K - \left(S^1(\omega_2)\right)^+}{1+r} \cdot \mathbb{P}^*\left[\{\omega_2\}\right] \\ &= 0 \cdot \frac{1}{2} + (K - 80) \cdot \frac{1}{2} \\ &= \frac{K}{2} - 40.\end{aligned}$$

Hence, $\pi(C^{call}) \leq \pi(C^{put})$ is equivalent to $60 - \frac{K}{2} \leq \frac{K}{2} - 40$, i.e. $K \geq 100$. But $K < 120$ by assumption, so we have:

$$100 \leq K < 120.$$