Lecture Quantitative Finance Spring Term 2015

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Introduction

Introduction

- Previous chapter: introduction of discrete time models in one period.
- Today: introduction of multiperiod discrete-time models
  - Redefine all concepts of the one-period model in this multiperiod setup.
  - Focus on the binomial model.
  - Derive the Black-Scholes' formula in a discrete time setting.

3.1: The multiperiod model

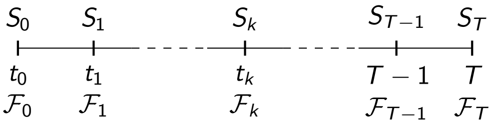
Model setup

- Trading periods: $t \in \{0, \ldots, T\}$ for some $0 \leq T < \infty$.
- Market consisting of $d+1$ assets:
  - ∗ asset 0 is consider as a riskless bond,
  - ∗ assets $1, \ldots d$ are risky assets.
- The price at time t of the riskless bond is given by $S_t^0 = (1+r)^t$, where $r > -1$ denotes the risk-free interest rate.
- The price of asset i at time t is modeled by a non-negative random variable S_t^i, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume $\bar{S}_t = \left(S_t^0, S_t\right) = \left(S_t^0, S_t^1, \ldots, S_t^d\right)$ to be measurable with respect to some σ-field $\mathcal{F}_t \subseteq \mathcal{F}$ for every $t \in \{0, \ldots, T\}$.
- Assume $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_T \subseteq \mathcal{F}$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$.
  $\Rightarrow (\mathcal{F}_t)_{t=0,\ldots,T}$ form a **filtration** on the given probability space.

Model setup

Interpretation

- $\mathcal{F}_t$ represents all the information available up to time t.
  $\Rightarrow$ It is natural to assume $\mathcal{F}_s \subseteq \mathcal{F}_t$ for any $s \leq t$, since there is no loss of information over time.
- S_t^i $\mathcal{F}_t$-measurable means that the price at time t of the i^{th} asset is based only on the past of the market and not on its future behaviour.



Adaptedness and Predictability

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_t)_{t=0,\dots,T}$ be a filtration on it. Then:

- A stochastic process $Z = (Z_t)_{t=0,\dots,T}$ is called adapted (with respect to the filtration) if Z_t is $\mathcal{F}_t$-measurable for every $t = 0, \dots, T$.
- A stochastic process $Y = (Y_t)_{t=1,\dots,T}$ is called predictable (with respect to the filtration) if Y_t is $\mathcal{F}_{t-1}$-measurable for every $t = 1, \dots, T$.

$\Rightarrow$ In our model, the price process $\bar{S} = \left(\bar{S}_t\right)_{t=0,\dots,T}$ is adapted to the filtration $(\mathcal{F}_t)_{t=0,\dots,T}$.

Trading strategy: Definition

- An $\mathbb{R}^{d+1}$-valued process $\bar{\xi} = (\xi^0, \xi) = (\xi^0, \xi^1, \dots, \xi^d)$ is called a **trading strategy** if it is predictable with respect to the filtration $(\mathcal{F}_t)_{t=0,\dots,T}$.

  In other words, ξ_t^i is $\mathcal{F}_{t-1}$-measurable for every $t \in \{1, \dots, T\}$, $i \in \{0, \dots, d\}$.

Interpretation

- ξ_t^i represents the number of shares of asset i held during the t^{th} trading period between times $t-1$ and t.
- $\xi_t^i S_{t-1}^i$ denotes the amount invested in the i^{th} asset at time $t-1$, while $\xi_t^i S_t^i$ is the resulting value at time t.
- Predictability of the strategy represents the fact that any investment must be allocated at the beginning of each trading period, without anticipating future prices.

Self-financing strategy

- A trading strategy $\bar{\xi} \in \mathbb{R}^{d+1}$ is called **self-financing** if:

$$\bar{\xi}_t \cdot \bar{S}_t = \bar{\xi}_{t+1} \cdot \bar{S}_t \quad \text{for every } t = 1, \ldots, T-1.$$

This means that for every $t \in \{1, \ldots, T-1\}$:

$$\sum_{i=0}^{d} \xi_t^i \cdot S_t^i = \sum_{i=0}^{d} \xi_{t+1}^i \cdot S_t^i.$$

- Therefore, an equivalent condition for self-financing is:

$$\sum_{i=0}^{d} \left(\xi_{t+1}^i - \xi_t^i\right) \cdot S_t^i = 0.$$

$\Rightarrow$ The portfolio of a self-financing strategy is rearranged in such a way that its **present value is preserved**.
$\Rightarrow$ Any change in the portfolio value is due to **price fluctuations** of the assets and not to some external factors.

Discounted Price Process

- To compare prices at different trading times, we have to consider discounted values.
- Using the riskless asset as a **numeraire**, we define the **discounted price process** X as follows:

$$X_t^0 = \frac{S_t^0}{S_t^0} \equiv 1;$$

$$X_t^i = \frac{S_t^i}{S_t^0} = \frac{S_t^i}{(1+r)^t} \text{ for } t \in \{1, \ldots, T\}, \, i \in \{1, \ldots d\}.$$

Value and Gain process: Definition

- The (discounted) **value process** $V = (V_t)_{t \in \{0,\dots,T\}}$ of a trading strategy $\bar{\xi}$ is given by:

$$V_0 = \bar{\xi}_1 \cdot \bar{X}_0 \quad \text{and} \quad V_t = \bar{\xi}_t \cdot \bar{X}_t \text{ for } t = 1, \dots, T.$$

- The **gain process** $G = (G_t)_{t \in \{0,\dots,T\}}$ of $\bar{\xi}$ is then defined as:

$$G_0 = 0 \quad \text{and} \quad G_t = \sum_{k=1}^{t} \xi_k \left(X_k - X_{k-1} \right) \text{ for } t = 1, \dots, T.$$

$\Rightarrow$ V_t represents the portfolio value at the end of the t^{th} trading period, G_t represents the net gains accumulated through following the stategy $\bar{\xi}$ up to time t.

Self-financing: Characterisation

Lemma

Let $\bar{\xi}$ be a trading strategy. Then, the following are equivalent:

1. *$\bar{\xi}$ is self-financing.*
2. *$\bar{\xi}_t \cdot \bar{X}_t = \bar{\xi}_{t+1} \cdot \bar{X}_t$ for any $t = 1, \ldots, T-1$.*
3. *The value process associated to $\bar{\xi}$ can be written as*

$$V_t = V_0 + G_t = V_0 + \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1})$$

for any $t = 0, \ldots, T$.

Arbitrage Opportunity: Definition

- A self financing trading strategy $\bar{\xi} = (\xi^0, \xi)$ is called an **arbitrage opportunity** if the corresponding value process V satisfies

$$V_0 \leq 0, \quad V_T \geq 0 \quad \mathbb{P}\text{-a.s.} \quad \text{and} \quad \mathbb{P}[V_T > 0] > 0.$$

- The market model is **arbitrage-free** if no such arbitrage opportunity exists.
- The market is arbitrage-free if and only if there are no arbitrage opportunities for each single trading period.

Martingale: Definition

- Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, \mathbb{P})$ be a filtered probability space. Then, a stochastic process $M = (M_t)_{t=0,\dots,T}$ is called a **martingale** if:
  - M is adapted with respect to $(\mathcal{F}_t)_{t=0,\dots,T}$;
  - $\mathbb{E}_{\mathbb{P}}[|M_t|] < \infty$ for every $t \in \{0, \dots, T\}$;
  - $\mathbb{E}_{\mathbb{P}}[M_t \mid \mathcal{F}_s] = M_s$, for all $0 \leq s \leq t \leq T$.
- The **best prediction** is exactly the current value $\Rightarrow$ **fair game**.
- A stochastic process M is a martingale under the underlying probability measure $\mathbb{P}$.
  $\Rightarrow$ To highlight the measure we are working with, we say that M is a $\mathbb{P}$-martingale or a martingale under the measure $\mathbb{P}$.

Martingale measure: Definition

- A probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ is called a **martingale measure** if the discounted price process $(X_t)_{t=0,\cdots,T}$ is a $\mathbb{P}^*$-martingale, i.e.

$$\begin{aligned} \mathbb{E}_{\mathbb{P}^*}[X_t] &< \infty \\ \mathbb{E}_{\mathbb{P}^*}\left[X_t^i \mid \mathcal{F}_s\right] &= X_s^i \text{ for } 0 \leq s \leq t \leq T,\ i = 1, \ldots, d. \end{aligned}$$

- Recall that two probability measures $\mathbb{P}$ and $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ are equivalent if for any set $A \subseteq \mathcal{F}$ we have:

$$\mathbb{P}[A] = 0 \text{ if and only if } \mathbb{P}^*[A] = 0.$$

In this case, we write $\mathbb{P} \approx \mathbb{P}^*$.

FTAP: Dynamic version

- We denote by $\mathcal{P}$ the set of all martingale measures, which are equivalent to $\mathbb{P}$, i.e.

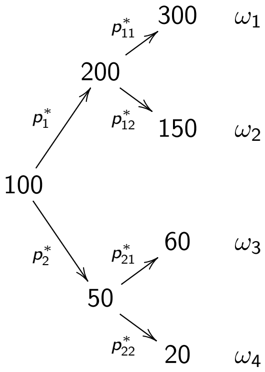
$$\mathcal{P} = \{\mathbb{P}^* \ : \ \mathbb{P}^* \text{ is a martingale measure and } \mathbb{P}^* \approx \mathbb{P}\}.$$

Theorem (Dynamic FTAP)

Consider a multiperiod market model. Then, the market is free of arbitrage if and only if the set $\mathcal{P}$ of all equivalent martingale measures is not empty.

Example (Two-period Model)

- Consider a two-period model (i.e. $T = 2$) consisting of a riskless and a risky asset with dynamics:



where $r = 0$ is assumed for simplicity.

Example (continue)

Goal Find the martingale measure $\mathbb{P}^*$, so that there is no arbitrage in both trading periods.

$\Rightarrow$ We have to solve the following systems of equations:

$$\begin{cases} 1 &= p_1^* + p_2^* \\ 100 &= 200p_1^* + 50p_2^* \end{cases} \Leftrightarrow p_1^* = \frac{1}{3},\ p_2^* = \frac{2}{3}$$

$$\begin{cases} 1 &= p_{11}^* + p_{12}^* \\ 200 &= 300p_{11}^* + 150p_{12}^* \end{cases} \Leftrightarrow p_{11}^* = \frac{1}{3},\ p_{12}^* = \frac{2}{3}$$

$$\begin{cases} 1 &= p_{21}^* + p_{22}^* \\ 50 &= 60p_{21}^* + 20p_{22}^* \end{cases} \Leftrightarrow p_{21}^* = \frac{3}{4},\ p_{22}^* = \frac{1}{4}$$

Example (continue)

- The martingale measure is given by:

$$\begin{aligned}
\mathbb{P}^*[\{\omega_1\}] &= p_1^* \cdot p_{11}^* = \tfrac{1}{9};\\
\mathbb{P}^*[\{\omega_2\}] &= p_1^* \cdot p_{12}^* = \tfrac{2}{9};\\
\mathbb{P}^*[\{\omega_3\}] &= p_2^* \cdot p_{21}^* = \tfrac{1}{2};\\
\mathbb{P}^*[\{\omega_4\}] &= p_2^* \cdot p_{22}^* = \tfrac{1}{6}.
\end{aligned}$$

Call and Put options

- An **European call option** on the i^{th} asset **with maturity T** and strike price $K > 0$ gives its owner the right to buy asset i at time T for the fixed price K. Its payoff is given by:

$$C^{call,i} = \left(S_T^i - K\right)^+ = \begin{cases} S_T^i - K, & \text{if } S_T^i - K \geq 0; \\ 0, & \text{if } S_T^i - K < 0. \end{cases}$$

- An **European put option** on the i^{th} asset **with maturity T** and strike price $K > 0$ gives its owner the right to sell asset i at time T for the fixed price K. Its payoff is given by:

$$C^{put,i} = \left(K - S_T^i\right)^+ = \begin{cases} 0, & \text{if } S_T^i - K \geq 0; \\ K - S_T^i, & \text{if } S_T^i - K < 0. \end{cases}$$

- **Put-Call-Parity:**

$$C^{call,i} - C^{put,i} = S_T^i - K.$$

Asian option

- The outcome of an **Asian option** on the i^{th} underlying asset depends on the average price

$$S^i_{av} = \frac{1}{|M|} \sum_{t \in M} S^i_t$$

for $M \subset \{0, 1, \cdots, T\}$ a subset of predetermined time periods.

- Therefore, the **average price call option** on the i^{th} asset with strike price K corresponds to the payoff:

$$C^{\text{call}}_{av} = \left(S^i_{av} - K\right)^+ .$$

- The **average price put option** on the i^{th} asset is described by:

$$C^{\text{put}}_{av} = \left(K - S^i_{av}\right)^+ .$$

Barrier option

- A **barrier option** is a contingent claim whose payoff depends on whether the price of the underlying asset reaches a certain level before maturity or not. Usually of two types: knock-out or knock-in options.
- A **knock-out option** has zero payoff once the price of the underlying asset S^i reaches the barrier $B \in \mathbb{R}$. For example, an up-and-out call option is described by:

$$C^{\text{call}}_{u\&o} = \begin{cases} \left(S^i_T - K\right)^+, & \text{if } \max_{0 \leq t \leq T} S^i_t < B; \\ 0, & \text{otherwise.} \end{cases}$$

- A **knock-in option** pays off only if the barrier B is reached. For example, a down-and-in put option with strike price K is given by:

$$C^{\text{put}}_{d\&i} = \begin{cases} \left(K - S^i_T\right)^+, & \text{if } \min_{0 \leq t \leq T} S^i_t \leq B; \\ 0, & \text{otherwise.} \end{cases}$$

Attainable payoffs: Definition

- A contingent claim C with maturity T is said to be **attainable** (or replicable) if there exists a self-financing strategy $\bar{\xi}$, whose terminal portfolio value coincides with C, i.e.

$$C = \bar{\xi}_T \cdot \bar{S}_T \quad \mathbb{P}\text{-a.s.}$$

  In this case, the trading strategy $\bar{\xi}$ is called **replicating strategy** for C.

- C is attainable if and only if its corresponding discounted claim $H = C/(1+r)^T$ can be written as

$$H = \bar{\xi}_T \cdot \bar{X}_T = V_T = V_0 + \sum_{t=1}^{T} \xi_t \cdot (X_t - X_{t-1}),$$

  for a self-financing trading strategy $\bar{\xi}$ with value process V.

  In this case, we say that the discounted claim H is attainable with the replicating strategy $\bar{\xi}$.

Value process: Characterisation

Theorem

Let H be a discounted, attainable contingent claim. Then, H is integrable with respect to any equivalent martingale measure, i.e.

$$\mathbb{E}_{\mathbb{P}^*}[H] < \infty \quad \text{for all } \mathbb{P}^* \in \mathcal{P}.$$

Moreover, for every $\mathbb{P}^* \in \mathcal{P}$, the value process associated with the replicating strategy of H can be written as

$$V_t = \mathbb{E}_{\mathbb{P}^*}[H \mid \mathcal{F}_t] \quad \mathbb{P}\text{-a.s.} \quad \text{for } t = 0, \dots, T.$$

In particular, V is a non-negative $\mathbb{P}^*$-martingale for every equivalent martingale measure $\mathbb{P}^* \in \mathcal{P}$.

Arbitrage-free prices

Goal Price a discounted contingent claim H without introducing arbitrage in the market.

- If H is attainable, then the discounted initial investment needed for replicating H

$$\bar{\xi}_1 \cdot \bar{X}_0 = V_0 = \mathbb{E}_{\mathbb{P}^*}[H] \quad \text{for } \mathbb{P}^* \in \mathcal{P}$$

  can be interpreted as the unique discounted arbitrage-free price for H.

- If the claim H is not attainable, how should we proceed?

Arbitrage-free prices

- Let H be a discounted contingent claim. Then, a real number $\pi^H \geq 0$ is an **arbitrage-free price** for H if there exists an adapted stochastic process X^{d+1} such that:

$$\begin{aligned} X_0^{d+1} &= \pi^H, \\ X_t^{d+1} &\geq 0 \quad \text{for } t = 1, \ldots, T-1, \\ X_T^{d+1} &= H; \end{aligned}$$

  and such that the enlarged market model with price process $(X^0, \ldots, X^d, X^{d+1})$ is arbitrage-free.

Arbitrage-free prices

- A priori, the claim H could have more than one arbitrage-free price.
- Denote by $\Pi(H)$ the set of all arbitrage-free prices for H:

$$\Pi(H) = \left\{\pi^H \in \mathbb{R} : \pi^H \text{ is an arbitrage-free price for H}\right\}.$$

- Its lower and upper bounds are then defined by:

$$\Pi^{\downarrow}(H) = \inf \Pi(H) \quad \text{and} \quad \Pi^{\uparrow}(H) = \sup \Pi(H).$$

Theorem

Let H be a discounted contingent claim. Then, the set $\Pi(H)$ is non-empty and given by:

$$\Pi(H) = \{\mathbb{E}_{\mathbb{P}^*}[H] : \mathbb{P}^* \in \mathcal{P} \textit{ such that } \mathbb{E}_{\mathbb{P}^*}[H] < \infty\}.$$

In addition, the lower and upper bound of $\Pi(H)$ can be written as:

$$\Pi^{\downarrow}(H) = \inf_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}_{\mathbb{P}^*}[H] \quad \textit{and} \quad \Pi^{\uparrow}(H) = \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}_{\mathbb{P}^*}[H].$$

Attainable Claims: Characterisation

Theorem

Consider an arbitrage-free primary market model and a discounted contingent claim H such that $H \geq 0$. Then, we have:

1. *If H is attainable, then $\Pi(H)$ consists of the unique element V_0, where V denotes the value process corresponding to the replicating strategy of H.*
2. *If H is not attainable, then $\Pi^{\downarrow}(H) < \Pi^{\uparrow}(H)$ and the set of arbitrage-free price is an open interval of the form $\Pi(H) = (\Pi^{\downarrow}(H), \Pi^{\uparrow}(H))$.*

Complete Markets

- A multiperiod arbitrage-free market model is called **complete** if every contingent claim is attainable.
- From the previous theorem, it follows that in a complete market model every contingent claim has a unique arbitrage-free price.

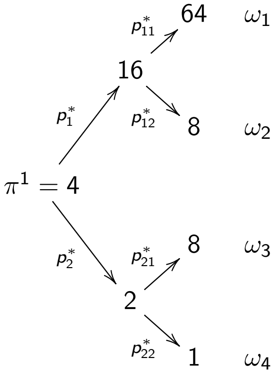
Theorem

An arbitrage-free market model is complete if and only if there exists just one equivalent martingale measure, i.e. $|\mathcal{P}| = 1$.

Two-period Model: Example 2

Example

- Consider a two-period model consisting of a riskless and risky asset with undiscounted dynamics:



where $u = 4$, $\ell = \frac{1}{2}$, $\pi^1 = 4$ and $r = 1$.

Two-period Model: Example 2

Example (continue)

- First, we want to exclude arbitrage from the market by finding an equivalent martingale measure $\mathbb{P}^*$.
- Therefore, we have to solve the following systems of equations:

$$\begin{cases} 1 &= p_1^* + p_2^* \\ 4 &= 8p_1^* + 1p_2^* \end{cases} \Leftrightarrow p_1^* = \frac{3}{7},\ p_2^* = \frac{4}{7}$$

$$\begin{cases} 1 &= p_{11}^* + p_{12}^* \\ 8 &= 16p_{11}^* + 2p_{12}^* \end{cases} \Leftrightarrow p_{11}^* = \frac{3}{7},\ p_{12}^* = \frac{4}{7}$$

$$\begin{cases} 1 &= p_{21}^* + p_{22}^* \\ 1 &= 2p_{21}^* + \frac{1}{4}p_{22}^* \end{cases} \Leftrightarrow p_{21}^* = \frac{3}{7},\ p_{22}^* = \frac{4}{7}$$

Example (continue)

- **Note:** since $r = 1$, we have used discounted price values to find the equivalent martingale measure.
- The equivalent martingale measure $\mathbb{P}^*$ is given by:

$$\begin{aligned}
\mathbb{P}^*[\{\omega_1\}] &= p_1^* \cdot p_{11}^* &= \frac{9}{49}, \\
\mathbb{P}^*[\{\omega_2\}] &= p_1^* \cdot p_{12}^* &= \frac{12}{49}, \\
\mathbb{P}^*[\{\omega_3\}] &= p_2^* \cdot p_{21}^* &= \frac{12}{49}, \\
\mathbb{P}^*[\{\omega_4\}] &= p_2^* \cdot p_{22}^* &= \frac{16}{49}.
\end{aligned}$$

$\Rightarrow$ There exists a unique equivalent martingale measure $\mathbb{P}^*$. Hence, the market model is arbitrage-free and complete.

Example (continue)

- Consider a call option with payoff $C = \left(S_2^1 - 4\right)^+$.
- Since the market is complete, the discounted claim $H = \frac{C}{(1+r)^2}$ is attainable and so its arbitrage-free price equals:

$$
\begin{aligned}
\pi^H &= \mathbb{E}_{\mathbb{P}^*}[H] \\
&= \sum_{\omega \in \Omega} H(\omega) \cdot \mathbb{P}^*[\{\omega\}] \\
&= H(\omega_1) \cdot \mathbb{P}^*[\{\omega_1\}] + \cdots + H(\omega_4) \cdot \mathbb{P}^*[\{\omega_4\}] \\
&= \frac{60}{4} \cdot \frac{9}{49} + \frac{4}{4} \cdot \frac{12}{49} + \frac{4}{4} \cdot \frac{12}{49} + 0 \\
&= \frac{159}{49}.
\end{aligned}
$$

3.2: The binomial model

Model setup

- Consider a market model with one risky asset in which trading is executed at time $t \in \{0, 1, \dots, T\}$.
- Look at asset 0 as a riskless asset with price at time t given by $S_t^0 = (1+r)^t$, where $r > -1$ denotes the risk-free interest rate.
- The price of the risky asset at time t is given by a non-negative random variable S_t^1, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which we will explicitly define later.

  In addition, suppose that $S_0^1 = \pi^1 > 0$.
- Suppose that the **return** R_t of the t^{th} trading period can only take the values

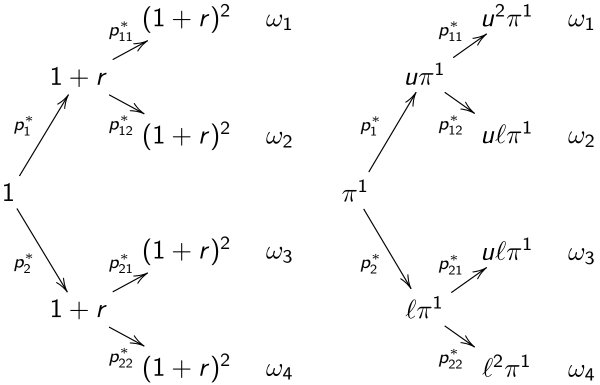
$$R_t = \frac{S_t^1 - S_{t-1}^1}{S_{t-1}^1} \in \{\ell - 1, u - 1\},$$

where $\ell, u \in \mathbb{R}$ are such that $0 < \ell < u$.

$\Rightarrow$ The stock price moves from S_{t-1}^1 to either the higher value $S_t^1 = S_{t-1}^1 u$ or the lower value $S_t^1 = S_{t-1}^1 \ell$.

Model setup

- In the first two trading times, we have the following dynamics:



- In general:

$$S_t^1(\omega) = \pi^1 \cdot u^{j_t(\omega)} \cdot \ell^{t-j_t(\omega)}$$

where $j_t(\omega)$ is the number of up-moves in a total of t moves, when the event ω occurs.

Model setup: probability space

- Let $\Omega = \{-1,1\}^T = \{\omega = (y_1,\ldots,y_T) : y_i \in \{-1,1\}\}$ be the sample space.
- Denote by $Y_t(\omega) = y_t$ for $\omega = (y_1,\ldots,y_T)$ the projection on the t^{th} coordinate of ω, so that:

$$R_t(\omega) = \ell\,\frac{1-Y_t(\omega)}{2} + u\,\frac{1+Y_t(\omega)}{2} - 1 = \begin{cases} u-1, & \text{if } Y_t(\omega) = 1, \\ \ell - 1, & \text{if } Y_t(\omega) = -1. \end{cases}$$

- The price process of the risky asset can be written as

$$S_t = \pi^1 \cdot \prod_{k=1}^{t} (1+R_k).$$

- The discounted price process is of the form:

$$X_t = \frac{S_t^1}{S_t^0} = \pi^1 \cdot \prod_{k=1}^{t} \frac{1+R_k}{1+r}.$$

Model setup: probability space

- For a filtration, we take $\mathcal{F}_t = \sigma(S_1, \dots, S_t) = \sigma(X_0, \dots, X_t)$ for any $t = 0, \dots, T$.

  $\Rightarrow \mathcal{F}_0$ is the trivial sigma field, $\mathcal{F} := \mathcal{F}_T$ coincides with the power set of Ω and the random variables Y_t, R_t are $\mathcal{F}_t$ measurable for any trading time.
- Fix any probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ with $\mathbb{P}[\{\omega\}] > 0$ for all $\omega \in \Omega$.
- The above model is called **T-period binomial model** or **CRR model**.

Completeness and Absence of Arbitrage

Theorem

The binomial model is arbitrage free if, and only if, $\ell < 1 + r < u$*. In this case, the market is complete and so there exists a unique equivalent martingale measure* $\mathbb{P}^*$*.*
In addition, the random variables $R_1, \ldots, R_T$ *are independent under* $\mathbb{P}^*$ *with joint distribution*

$$\mathbb{P}^*[R_t = u - 1] = \quad p^* = \frac{(1+r) - \ell}{u - \ell},$$

$$\mathbb{P}^*[R_t = \ell - 1] = 1 - p^* = \frac{u - (1+r)}{u - \ell}.$$

Arbitrage-free prices

- Since the binomial model is arbitrage-free and complete, any contingent claim C is attainable.
- Thus, we can extend the model by defining the arbitrage-free discounted price process S^2 for C as follows:
  - $S_0^2 = \pi^C = \mathbb{E}_{\mathbb{P}^*}\left[\frac{C}{(1+r)^T}\right]$, since $\mathbb{P}^*$ is unique;
  - $S_t^2 = (1+r)^t \cdot \mathbb{E}_{\mathbb{P}^*}\left[\frac{C}{(1+r)^T} \mid \mathcal{F}_t\right]$, for $t = 1, \ldots, T$.

Black-Scholes' Formula for Binomial Model

Theorem (Black-Scholes' formula for Binomial Model)

Suppose we are in an arbitrage-free, complete binomial model. Then, the price of an undiscounted call option C on the underlying risky asset with maturity T and strike price $K > 0$ is given by:

$$\pi^C = \frac{1}{(1+r)^T} \sum_{i=0}^{T} \binom{T}{i} (p^*)^i (1-p^*)^{T-i} \left(\pi^1 u^i \ell^{T-i} - K\right)^+ ,$$

$$S_t^2 = \frac{1}{(1+r)^{T-t}} \sum_{i=0}^{T-t} \binom{T-t}{i} (p^*)^i (1-p^*)^{T-t-i} \left(S_t^1 u^i \ell^{T-i} - K\right)^+ .$$

3.3: Exercises

Exercise 1: Question

Question

Consider a two period ($T = 2$) binomial model with one risky asset ($d = 1$) and assume $u = 4$, $\ell = 1/2$, $r = 1$, $\pi^1 = S_0^1 = 4$.

Compute the price of an European put option on the risky asset with maturity T:

$$C = \left(K - S_2^1\right)^+,$$

for a strike price $K = 4$.

Exercise 1: Solution

Solution

Since $\ell < 1 + r < u$, the binomial model is arbitrage-free and complete.

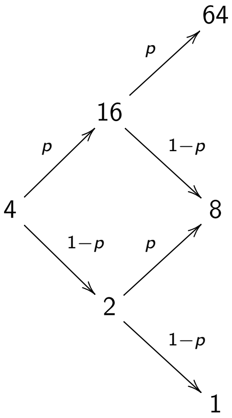
$\Rightarrow$ There exists a unique martingale measure $\mathbb{P}^*$ such that

$$\pi^C = \mathbb{E}_{\mathbb{P}^*}\left[\frac{C}{(1+r)^T}\right].$$

As we have a recombining binomial tree, we need to determine the probability of an upward movement p and the probability of a downward movement $1 - p$.

Exercise 1: Solution

In particular here:



Exercise 1: Solution

We know that in the binomial model the probability of an up/down move are given by:

$$\begin{cases} p^* = \frac{(1+r)-\ell}{u-\ell} = \frac{3}{7}; \\ 1 - p^* = \frac{u-(1+r)}{u-\ell} = \frac{4}{7}. \end{cases}$$

Therefore:

$$\mathbb{P}^*[\{\omega_1\}] = \frac{3}{7} \cdot \frac{3}{7} = \frac{9}{49};$$

$$\mathbb{P}^*[\{\omega_2\}] = \frac{3}{7} \cdot \frac{4}{7} = \frac{12}{49};$$

$$\mathbb{P}^*[\{\omega_3\}] = \frac{4}{7} \cdot \frac{3}{7} = \frac{12}{49};$$

$$\mathbb{P}^*[\{\omega_4\}] = \frac{4}{7} \cdot \frac{4}{7} = \frac{16}{49}.$$

Exercise 1: Solution

Hence, the price of the put option is given by:

$$\begin{aligned}\pi^C &= \mathbb{E}_{\mathbb{P}^*}\left[\frac{C}{(1+r)^T}\right] \\ &= \sum_{\omega_i \in \Omega} \frac{C(\omega_i)}{(1+r)^T} \cdot \mathbb{P}^*[\{\omega_i\}] \\ &= \frac{1}{4} \cdot \left(0+0+0+3 \cdot \frac{16}{49}\right) = \frac{12}{49}.\end{aligned}$$

Exercise 2: Question

Question

Consider again a two period ($T = 2$) binomial model with one risky asset ($d = 1$) and assume $u = 4$, $\ell = 1/2$, $r = 1$ and $\pi^1 = S_0^1 = 4$.

Determine a hedging strategy $\overline{\xi} = (\overline{\xi}_1, \overline{\xi}_2)$, where $\overline{\xi}_1 = (\xi_1^0, \xi_1^1)$, $\overline{\xi}_2 = (\xi_2^0, \xi_2^1)$, for the European put option on the risky asset with strike price $K = 4$:

$$C = (K - S_2^1)^+.$$

Exercise 2: Solution

Solution

From the previous exercise, we know that the market is arbitrage-free, complete and the unique equivalent martingale measure $\mathbb{P}^*$ is determined by:

$$p^* = \frac{(1+r) - \ell}{u - \ell} = \frac{3}{7}.$$

A hedging strategy $\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2)$ is a portfolio such that:

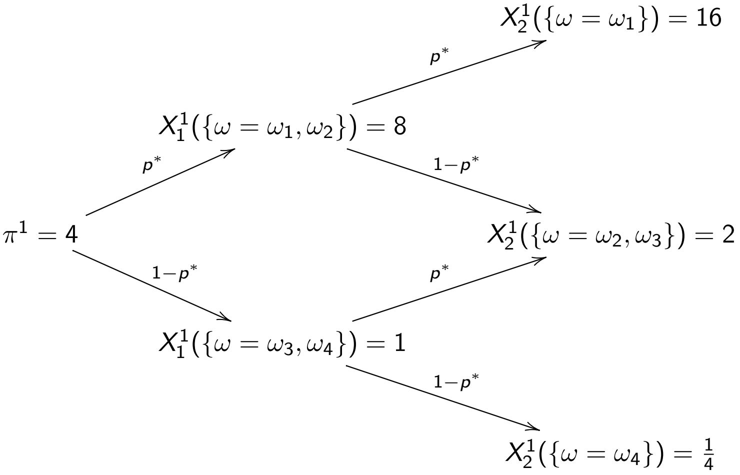
- the value process V_2 of the portfolio equals the payoff of the discounted European put option

$$\frac{C}{(1+r)^2} = \frac{\left(K - S_2^1\right)^+}{(1+r)^2};$$

- $V_1 = \mathbb{E}_{\mathbb{P}^*}[\frac{C}{(1+r)^2} \mid S_1^1]$.

Exercise 2: Solution

The dynamic of the discounted price process X is given by:



Exercise 2: Solution

For time $t = 1$ we have to solve the following equations:

$$\begin{cases} X_1^0 \cdot \xi_1^0 + X_1^1(\{\omega = \omega_1, \omega_2\}) \cdot \xi_1^1 = 0, \\ X_1^0 \cdot \xi_1^0 + X_1^1(\{\omega = \omega_3, \omega_4\}) \cdot \xi_1^1 = \frac{3}{4} \cdot (1 - p^*); \end{cases}$$

putting in the numbers

$$\begin{cases} \xi_1^0 + 8 \cdot \xi_1^1 = 0, \\ \xi_1^0 + \xi_1^1 = \frac{12}{28}. \end{cases}$$

we get the solution:

$$\begin{cases} \xi_1^0 = \frac{24}{49}, \\ \xi_1^1 = -\frac{3}{49}; \end{cases}$$

Exercise 2: Solution

For $t = 2$ we have to consider the upward and downward paths separately.

- For the up-case, it follows:

$$\begin{cases} X_2^0 \cdot \xi_2^0 + X_2^1(\{\omega = \omega_1\}) \cdot \xi_2^1 = 0, \\ X_2^0 \cdot \xi_2^0 + X_2^1(\{\omega = \omega_2\}) \cdot \xi_2^1 = 0; \end{cases}$$

and so:

$$\begin{cases} \xi_2^0 + 16 \cdot \xi_2^1 = 0, \\ \xi_2^0 + 2 \cdot \xi_2^1 = 0. \end{cases}$$

Hence:

$$\begin{cases} \xi_2^0 = 0, \\ \xi_2^1 = 0; \end{cases}$$

Exercise 2: Solution

- For the down-case we have:

$$\begin{cases} X_2^0 \cdot \xi_2^0 + X_2^1(\{\omega = \omega_3\}) \cdot \xi_2^1 = 0, \\ X_2^0 \cdot \xi_2^0 + X_2^1(\{\omega = \omega_4\}) \cdot \xi_2^1 = \frac{3}{4}; \end{cases}$$

so it follows:

$$\begin{cases} \xi_2^0 + 2 \cdot \xi_2^1 = 0, \\ \xi_2^0 + \frac{1}{4} \cdot \xi_2^1 = \frac{3}{4}. \end{cases}$$

Thus:

$$\begin{cases} \xi_2^0 = \frac{6}{7}, \\ \xi_2^1 = -\frac{3}{7}. \end{cases}$$