Lecture Quantitative Finance Spring Term 2015

Guest Lecture:
Dr. Robert Huitema

Lecture 4: March 12, 2015

Introduction

- Previous chapters: basic tools for quantitative analysis in a discrete time model setting
- Today: introduction to derivatives
  - financial contracts traded in private over-the-counter (OTC) markets or on organised exchanges
  - derive their value from some underlying index, typically the price of an asset
- the contract must specify a principal (notional amount) which is defined in terms of currency, shares, or some other unit
- movements in the value of the derivative depend on the notional and the underlying price or index
- derivatives are contracts (private agreements) between two parties, thus the sum of gains and losses on derivatives contracts must be zero

Overview

- derivatives markets can be classified

  by the type of trading:

  - OTC
  - exchange traded instruments

  or by the underlying instruments:

  - interest rate contracts (forwards, swaps, options)
  - foreign exchange contracts (forwards, swaps, options)
  - equity linked contracts
  - commodity contracts

- derivatives can be broadly classified into

  - linear instruments (ex. forward contracts, futures, and swaps) $\longrightarrow$ today
  - non-linear instruments (options)

- these markets have grown exponentially!

Forward contracts: Definition

- Today: general characteristics as well as the pricing of linear derivatives
- Note: pricing is only the first step towards risk management; the second step consists of combining the valuation formula with the distribution of underlying risk factors to derive the distribution of contract values (later: market risk)
- most common transactions are spot transactions: for physical delivery as soon as practical (in two business days or in a week)
- historically: grain farmers went to a centralised location to meet buyers for their product
- as markets developed: farmers realised that it would be beneficial to trade for delivery at some future date and this allowed them to hedge out price fluctuations for the sale of their anticipated production

- Forwards are private agreements:
  - to exchange a given asset against cash (another asset)
  - at a fixed point in the future
- The terms of the contract are:
  - quantity (number of units or shares)
  - date
  - price at which the exchange will be done
- a position that implies buying the asset is said to be **long**; a position to sell is said to be **short**
- any gain to one party must be a loss to the other
- these instruments represent contractual obligations: the exchange must occur whatever happens to the intervening price (unless default occurs); unlike the options (later!) there is **no choice** in taking delivery or not
- to avoid possibility of losses, the farmer could enter a forward sale of grain for CHF; he locks a price now for delivery in future (terminology: the farmer is hedged against movements in the price)

Forward contracts: Notation

$$
\begin{aligned}
t &= \text{current time} \\
T &= \text{time of delivery} \\
\tau = T - t &= \text{time to maturity} \\
S_t &= \text{current spot price of the asset} \\
F_t(T) &= \text{current forward price of the asset for delivery at } T \\
&\ \text{also written } F_t \text{ or } F \text{ to avoid clutter} \\
V_t &= \text{current value of contract} \\
r &= \text{current risk-free rate for delivery at } T \\
n &= \text{quantity, number of units in contract}
\end{aligned}
$$

- The face amount (principal value) of the contract: the amount nF to pay at maturity (like a bond); also called the notional amount
- interest rates are continuously compounded; the present value of a CHF paid at expiration is

$$PV(1\text{ CHF}) = \frac{1}{e^{r(T-t)}} = e^{-r(T-t)}$$

- say that the initial forward price is $F_t = 100$ CHF
- a speculator agrees to buy $n = 500$ units for F_t at time T
- at expiration:
  - the speculator pays $nF = 50'000$ CHF in cash and receives 500 units of the underlying
  - the speculator could then sell the underlying at the prevailing spot price S_T for a profit $n(S_T - F)$
    for example if the spot price is at $S_T = 120$ CHF the profit is $500 \times (120 - 100) = 10'000$ CHF; this is also the mark-to-market value of the contract at expiration

- the value of the forward contract at expiration, for one unit of the underlying asset is:

$$V_T = S_T - F$$

- the value is derived from the purchase and physical delivery of the underlying asset: there is a payment of cash in exchange for the actual asset
- another mode: cash settlement; this involves simply measuring the market value of the asset upon maturity S_T and agreeing for the long to receive $nV_T = n(S_T - F)$ (can be positive or not, involving a profit or a loss)
- the payoff patterns on long and short position in a forward contract are linear in the underlying spot price; the two figures are symmetrical around the horizontal axis
- for a given spot price, the sum of the profit or loss for the long and the short is zero

Valuing forward contracts

- two important questions arise
  - how is the current forward price F_t determined?
  - what is the current value V_t of a forward contract?
- Assumptions:
  - the underlying asset pays no income (later we will generalise it)
  - no transaction costs
  - ability to lend and borrow at the same risk-free rate
- forward contracts are established so that their initial value is zero
- achieved by setting F_t appropriately by a no-arbitrage relationship between the cash and forward markets
- arbitrage is a zero-risk, zero-net investment strategy that still generates profit
- no-arbitrage is a situation where positions with the same payoffs have the same price

Valuing forward contracts

- Strategies:
  - buy one share / unit of the underlying asset at the spot price S_t and hold it to time T
  - enter a forward contract to buy one share / unit of the same underlying asset at the forward price F_t;
    in order to have sufficient funds at maturity to pay F_t, we invest the present value of F_t in an interest bearing account; the present value is $F_t \cdot e^{-r(T-t)}$;
    the forward price F_t is set so that the initial cost of the forward contract V_t is zero
- the two portfolios are economically equivalent (because they will be identical at maturity); hence their up-front cost must be the same; to avoid arbitrage we must have:

$$S_t = F_t \cdot e^{-r(T-t)}$$

Valuing forward contracts

- define the fair forward price F_t such that the initial value of the contract is zero
- the term multiplying F_t is the present value factor for maturity $T - t$ or $PV(1\,\text{CHF})$.
- assuming $S_t = 100$, $r = 5\%$, $\tau = T - t = 1$ we have

$$F_t = S_t \cdot e^{r(T-t)} = 100 \cdot \exp(0.05 \times 1) = 105.13$$

- no down payment to enter the forward contract, unlike for the cash position: forward price must be higher then the spot price to reflect the time value of the money
- abstracting from transaction costs, any deviation creates an arbitrage opportunity

Valuing forward contracts: Ex. 1 – arbitrage

- assume that $F = 110$
- we determined the fair value $S_t \cdot e^{r(T-t)} = 105.13$ based on the cash price
- apply the principle: buy low at 105.13 and sell high at 110
- we can lock in sure profit by
    - buying now the asset spot at 100 (e.g. borrowing 100 to buy the asset now)
    - selling now the asset forward at 110
- at expiration we will owe the principal plus interest, or 105.13, but receive 110 for a profit of 4.87
- this would be a blatant arbitrage opportunity or *money machine*

Valuing forward contracts: Ex. 2 – arbitrage

- assume that $F = 102$
- we determined the fair value $S_t \cdot e^{r(T-t)} = 105.13$
- apply the principle: buy low at 102 and sell high at 105.13
- we can lock in sure profit by
    - short selling the asset spot at 100
    - buying the asset forward at 102
- from the short sale, we invest the cash, which will grow to 105.13
- at expiration we will have to deliver the stock, but this will be acquire through the forward purchase: we pay 102 for this and are left with a profit of 3.13
- this transaction involves the short-sale of the asset;
- when purchasing, we pay 100 and receive one share of the asset
- when short-selling, we borrow one share of the asset and promise to give it back at a future date; in the meantime we sell it at 100

Valuing an off-market forward contract

- problem of evaluating an outstanding forward contract with a locked-in delivery price of K
- in general will have a non-zero value because K differs from the prevailing forward rate; such a contract is said to be off-market
- strategies
  - buy one share / unit of the underlying asset at the spot price and hold it until time T
  - enter a forward contract to buy one share / unit of the same underlying asset at the price K;
    in order to have sufficient funds at maturity to pay K, we invest the present value of K in an interest-bearing account;
    this present value is also $Ke^{-r(T-t)}$;
    in addition we have to pay the market value of the forward contract, or V_t

Valuing an off-market forward contract

- the up-front cost of the two portfolios must be identical, hence

$$S_t = V_t + K \cdot e^{-r(T-t)}$$

- consequently

$$V_t = S_t - K \cdot e^{-r(T-t)}$$

  which defines the market value of an outstanding long position
- a short position would have the reverse sign
- for risk management: one needs to consider the distribution of the underlying risk factors, S_t and r.

Valuing an off-market forward contract: Example

- assume we still hold the previous forward contract with $F_t = 105.13$
- after one month the spot price moves to $S_t = 110$
- the fixed rate is $K = 105.13$ throughout the life of the contract
- the interest has not changed at $r = 5\%$
- maturity is now shorter by one month: $\tau = T - t = 11/12$
- the new value of the contract is

$$\begin{aligned} V_t &= S_t - K \cdot e^{-r(T-t)} \\ &= 110 - 105.13 \cdot \exp(-0.05 \times 11/12) \\ &= 110 - 100.42 = 9.58 \end{aligned}$$

- the contract is more valuable than before, because the spot price has moved up

Valuing forward contracts with income payments

- Previously considered a situation where the asset produces no income payment
- In practice the asset may be:
    - a stock that pays a regular dividend
    - a bond that pays a regular coupon
    - a stock index that pays a dividend stream approximated by a continuous yield
    - a foreign currency that pays a foreign-currency denominated interest rate
- whichever income is paid on the asset we may distinguish:
    - discrete payments (fixed amounts at regular points in time)
    - continuous payments (accrued in proportion to the time the asset is held)
- a storage cost is equivalent to a negative dividend

Example (fwd. contr. with income payments):

- consider a stock priced at 100 that pays a dividend of $D = 1$ in three months
- the present value of this payment, discounted over three months is
$$D \cdot e^{-r(T-t)} = 1 \cdot \exp(-0.05 \times 3/12) = 0.99$$
- we only need to put up $S_t - PV(D) = 100 - 0.99 = 99.01$ to get one share in one year
- put differently: we buy 0.9901 fractional shares now and borrow against the (sure) dividend payment of 1 to buy an additional 0.0099 fractional share for a total of one share
- for a discrete dividend the pricing formula is extended to:
$$F_t \cdot e^{-r(T-t)} = S_t - PV(D)$$
where $PV(D)$ is the present value of the dividend / coupon payment
- with storage costs we need to *add* the present value of storage costs $PV(C)$ to the right side of the last equation

Valuing forward contracts with income payments

The approach is similar for an asset that pays a continuous income, defined per unit time, instead of discrete amounts

- denote $r_t^*(T) =$ foreign risk-free rate for delivery at T
- holding a foreign currency should be done through an interest-bearing account paying interest that accrues with time
- over the horizon $\tau = T - t$, we can afford to invest less up-front, $S_t \cdot e^{-r^*(T-t)}$ in order to receive one unit at maturity
- then

$$F_t \cdot e^{-r(T-\tau)} = S_t \cdot e^{-r^*(T-t)}$$

- hence the forward price should be

$$F_t = S_t \cdot e^{-r^*(T-t)} / e^{-r(T-t)}$$

Example (fwd. contr. with income payments)

Consider an eight-month forward contract on a stock with a price of CHF 98 per share. The delivery date is eight months hence. The firm is expected to pay a CHF 1.80 per share dividend in four months. Risk-less zero-coupon interest rates (continuously compounded) are 4% for six months, and 4.5% for eight months.
Compute the theoretical forward price!

Solution. We need first to compute the PV of the dividend payment, which is

$$PV(D) = 1.8 \exp(-0.04 \times 4/12) = 1.776.$$

We know

$$F_t \cdot e^{-r(T-t)} = S_t - PV(D)$$

which implies

$$F = (98 - 1.776) \cdot \exp(0.045 \times 8/12) = 99.15.$$

Example (fwd. contr. with storage costs)

Assume the storage cost for gold is $ 5.00 per ounce, with payment made at the end of the year. Spot gold is $ 290 per ounce and the risk-free rate is 5%.
Compute the price of a 1-year forward contract on gold!

Solution. Assuming continuous compounding, the present value factor is

$$PV = \exp(-0.05) = 0.951.$$

Here the storage cost C is equivalent to a negative dividend and must be evaluated as of now; this gives

$$PV(C) = 5 \times 0.951 = 4.756.$$

Generalising equation

$$F_t \cdot e^{-r(T-t)} = S_t - PV(D)$$

we get

$$F = (S + PV(C))/PV(1\$) = (290 + 4.756)/0.951 = 309.87$$

Exercise 1

If the observed two-year forward price of one unit of the US$ is 0.850 units of the CHF, while the fair price is 0.710, what is your strategy to make an arbitrage profit

- borrow US$, buy CHF, and enter a short forward contract on CHF
- borrow US$, buy CHF, and enter a short forward contract on US$
- borrow CHF, buy US$, and enter a short forward contract on CHF
- borrow CHF, buy US$, and enter a short forward contract on US$

Futures contracts: Definition

- are standardised, negotiable, and exchange traded contracts to buy or sell an underlying asset
- they differ from forward contracts as follows
  - in contrast to forwards, which are OTC contracts tailored to customers' needs, futures are traded on organized exchanges (either with a physical location or electronic)
  - standardisation: futures are offered with a limited choice of expiration dates;
    they trade in fixed contract sizes;
    $\longrightarrow$ for most future contracts a good liquidity;
    but: less precisely suited to the need of some hedgers
  - clearinghouse: after each transaction is confirmed, the clearing house basically interposes itself between the buyer and the seller, ensuring the performance of the contract; thus, unlike forward contracts, counter-parties do not have to worry about the credit risk of the other side of the trade

Futures contracts: further difference to forwards

- marking to market: as the clearinghouse has to deal with the credit risk of two original counterparties, it has to monitor credit risk closely;
  achieved by daily marking-to-market and this involves settlement of the gains and losses on the contract every day.
  This will avoid the accumulation of large losses over time, potentially leading to an expensive default
- margins: although daily settlement accounts for past losses, it does not provide a buffer against future losses.
  This is the goal of **margins**, which represent upfront posting of collateral that can be seized should the other party default; if the equity in the account falls below the maintenance margin, the customer is required to provide additional funds to cover the initial margin;
  the level of margin depend on the instrument and the type of position – in general less volatile instruments or hedged positions require lower margins
- futures trading is centralized on an exchange: volume is the number of contracts traded during the day; open interest represents the outstanding number of contracts at the close of the day

Futures contracts: Example 1

- Consider a futures contract of $1'000$ units of an asset worth \$100
- a long futures position is economically equivalent to holding \$100'000 worth of the asset directly
- to enter the futures position, a speculator has to post for example only \$5'000; this amount is placed in an equity account with the broker
- the next day, the futures price moves down by \$3, leading to a loss of \$3'000 for the speculator
- the profit or loss is added to the equity account bringing it down to $\$5'000 - \$3'000 = \$2'000$
- the speculator would then receive a margin call from the broker, asking to have an additional \$3'000 of capital posted to the account
- if she or he fails to meet the margin call, the broker has the right to liquidate the position

Futures contracts: Example 2

Which one of the following statements is *incorrect* regarding the margining of exchange-traded futures contracts?

1. day trades and spread transactions require lower margin levels
2. if an investor fails to deposit variation margin in a timely manner, the position may be liquidated by the carrying broker
3. initial margin is the amount of money that must be deposited when a futures contract is opened
4. a margin call will be issued only if the investor's margin account balance becomes negative

Solution. All the answers are correct, except the last one. If the margin account balance falls below the maintenance margin (not zero), a margin call will be issued.

Swap contracts: Definition

- are OTC agreements to exchange a *series* of cash flows according to pre-specified terms
- the underlying asset can be an interest rate, an exchange rate, an equity, a commodity price, or any other index
- typically, swaps are established for longer periods than forwards and futures
- Example: a 10-year *currency swap* could involve an agreement to exchange every year 5 million dollars against 3 million pounds over the next ten years, in addition to a principal amount of 100 million dollars against 50 million pounds at expiration.
- the principal is also called notional principal

Swap contracts

- Example: a 5-year *interest rate swap* in which one party pays 8% of the principal amount of 100 million dollars in exchange for receiving an interest payment indexed to a floating interest rate; in this case, since both payments are the same amount in the same currency, there is no need to exchange principal at maturity
- swaps can be viewed as a portfolio of forward contracts and they can be priced using valuation formulas for forwards: our currency swap contract from above can be viewed as a combination of 10 forward contracts with various face values, maturity dates, and rates of exchange

Don't forget: important formulae

- forward price, no income on the asset

$$S_t = F_t \cdot e^{-r(T-t)}$$

- forward price, income on the asset
  - discrete dividend:

$$S_t - PV(D) = F_t \cdot e^{-r(T-t)}$$

  - continuous dividend:

$$S_t \cdot e^{-r^*(T-t)} = F_t \cdot e^{-r(T-t)}$$

Answer to the **Exercise 1**

The correct answer is the fourth one. Because the forward price at 0.850 is greater than the fair price at 0.71 we can create a profit opportunity by buying US$ at the cheap rate and selling at the expensive one. So, we borrow CHF, exchange CHF into US$, invest in US$, and at the same time sell forward.

Introduction to Derivatives (cont'd)

- Previous chapter: introduction to derivatives
  - financial contracts traded in private over-the-counter (OTC) markets or on organised exchanges
  - derive their value from some underlying index, typically the price of an asset
- the contract must specify a principal (notional amount) which is defined in terms of currency, shares, or some other unit
- movements in the value of the derivative depend on the notional and the underlying price or index
- derivatives can be broadly classified into
  - linear instruments (ex. forward contracts, futures, and swaps)
  - non-linear instruments (options)

Introduction to Derivatives (cont'd)

- Previous chapter: introduction to derivatives
  - financial contracts traded in private over-the-counter (OTC) markets or on organised exchanges
  - derive their value from some underlying index, typically the price of an asset
- the contract must specify a principal (notional amount) which is defined in terms of currency, shares, or some other unit
- movements in the value of the derivative depend on the notional and the underlying price or index
- derivatives can be broadly classified into
  - linear instruments (ex. forward contracts, futures, and swaps)
  - non-linear instruments (options) $\longrightarrow$ now!

3.2: Options (fundamentals)

- Recall the terminology: asset is any financial object whose value is known at present but is liable to change in the future
- typical examples are: shares in a company, commodities (e.g. gold, oil, electricity) currencies (e.g. the value of 100$ in euros)

Definition

Options are instruments / contracts that give the holder

- the right to buy or to sell an asset
- at a specified price (delivery price, exercise price, strike price), usually denoted K
- until a specified expiration date.

- options to *buy*: **call** options; options to *sell*: **put** options
- they give a right to the purchaser, but not an obligation; will be exercised only if they generate profit
- contrast to forwards: they involve an obligation (to buy or to sell) and can generate profits or losses
- the seller of the option is said to write the option (the writer)

European and American Options

Depending on the time of exercise:

- European options: can be exercised only at maturity
- American options can be exercised at any time, before or at maturity
- because American options include the right to exercise at maturity, they must be at least as valuable as European options

Example: European call option

Today Professor Smart (the writer) writes an European call option that gives you (the holder):

- the right to buy 100 shares of IBM
- for $1'000$ \$
- in three months from now on

After those three months have elapsed, you would then take one of these two actions

- if the actual value of 100 IBM shares turns out to be more than $1'000$ \$ you would exercise your right to buy the shares from Professor Smart (why: because you could immediately sell them for a profit)
- if the actual value of 100 IBM shares turns out to be less than $1'000$ \$ you would not exercise your right to buy the shares from Professor Smart - the deal would not be worthwhile

Example: European call option

- Summary:
    - You: are not obliged to purchase the shares $\longrightarrow$ you do not lose money (in the first case you gain money; in the second case you neither gain nor lose)
    - Professor Smart: will not gain any money and may lose
- To compensate this imbalance, when the option is agreed (today) you would be expected to pay Professor Smart an amount of money known as the **value (price, premium) of the option**.
- Key question(s): how much should the holder pay for the privilege of holding an option?
  What is a **fair option value**?
- To answer these questions one has to:
    - develop a math. model for the behaviour of the asset price
    - precise interpretation of *fairness*
    - do some analysis

Why do we study options

- have become extremely popular (in many cases more money is invested in them than in the underlying assets)
- they are extremely attractive to investors, both for speculation and for hedging
- there exists a systematic way to determine how much they are worth (hence they can be bought and sold with some confidence)
- a further attraction is: combining different types of options an investor can take a position that reaps benefits form various types of asset behaviour

Payoff profiles

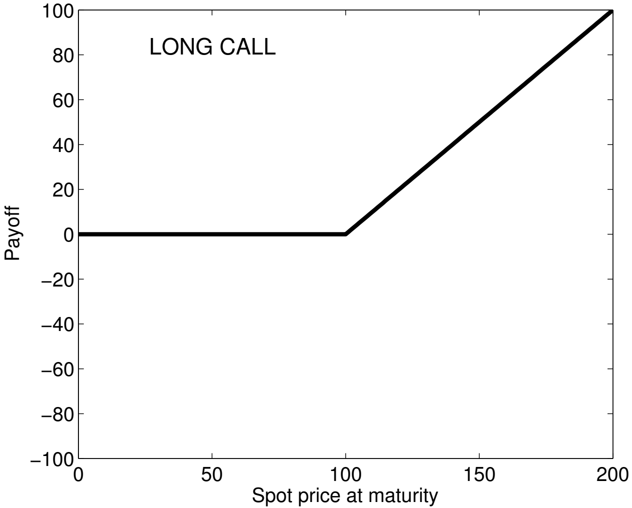
- Useful: visualise *payoff diagrams*; these are non-linear!
- the payoff profile of a long position in a call option at expiration:

$$C_T = \max(S_T - K, 0)$$

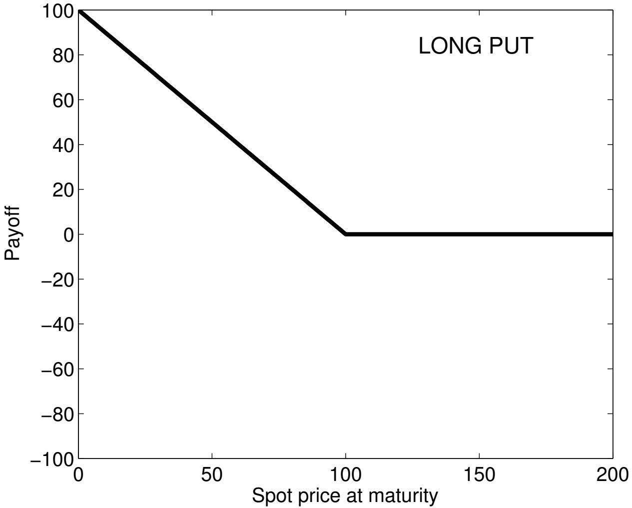
- the payoff profile of a long position in a put option at expiration:

$$P_T = \max(K - S_T, 0)$$

Call Payoff


LONG CALL
Payoff
Spot price at maturity
100
80
60
40
20
0
–20
–40
–60
–80
–100
0
50
100
150
200

Put Payoff


LONG PUT
Payoff
100
80
60
40
20
0
−20
−40
−60
−80
−100
0
50
100
150
200
Spot price at maturity

At-the-money, in-the-money; out-of-the-money

- if the current asset price S_t is close to the strike price: option is said to be *at-the-money*
- if the current asset price S_t is such that the option could be exercised now at a profit: option is said to be *in-the-money*
- in the remaining situation: the option is said to be *out-of-the-money*

- long options positions have limited downside risk (the loss of the premium)
- short call options have unlimited downside risk because there is no upper limit on S
- worst loss on short positions occurs if S goes to zero

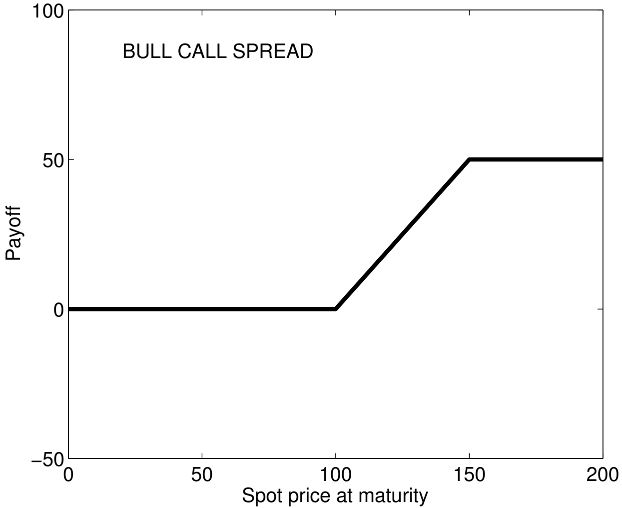
Example: bull spread

- hold a call option with exercise price K_1
- write a call option for the same asset, and same expiry date, with exercise price K_2 with $K_2 > K_1$
- at the expiry date, the payoff of the first option is $\max(S_T - K_1, 0)$
- at the expiry date, the payoff of the second option is $-\max(S_T - K_2, 0)$
- hence the overall payoff at expiration is

$$\max(S_T - K_1, 0) - \max(S_T - K_2, 0)$$

- the holder benefits when the asset price finishes above K_1 but gets no extra benefit if it is above K_2

Bull Spread Payoff



How are options traded

- options can be traded on a number of official exchanges
- the first of these: Chicago Board of Options Exchange (CBOE) started in 1973
- most exchanges operate through the use of *market makers*: individuals who are obliged to buy or to sell options whenever asked to do so
  - on request the market maker will quote a price for the option
  - more precisely he will quote two prices: the bid and the ask
  - the bid is the price at which the market maker will buy the option from you
  - the ask is the price at which the market maker will sell it to you
  - the bid is lower than the ask; the difference is called the *bid-ask spread*
  - typically, market makers aim to make their profits from the bid-ask spread and do not wish to speculate on the market

How are options traded

- options are also traded directly between large financial institutions - so called *over-the-counter* or OTC deals
- they often have non-standard features that are tailored to the particular needs of the parties involved
- there are many introductory texts that explain how stock markets operate; in particular Chapter 6 of J. Hull is also a good source of basic practical information about option pricing trading including
  - what range of expiry dates and exercise prices are typically offered
  - how dividends and stock splits are dealt with
  - how money and products actually change hands
- the Financial Times newspaper tabulates the prices of some options that may be traded on the London International Financial Futures & Options Exchange; for example the issue from Friday 19. September 2003 included the information

How are options traded

		Calls			Puts		
Option		*Oct*	*Nov*	*Dec*	*Oct*	*Nov*	*Dec*
RBS	1600	67.0	92.5	109.5	29.0	49.0	62.5
(1634.0)	1700	19.5	43.5	59.0	82.0	100.0	112.5

- the number 1634.0 is the closing price of Royal Bank of Scotland's shares from the previous day
- the numbers 1600 and 1700 are two exercise prices, in pence
- the numbers 67.0, 92.5, 109.5 are the prices of the call options with exercise price 1600 and expiry dates in Oct., Nov., and Dec. respectively (more precisely for 18:00 on the third Wednesday of the month)
- similarly 19.5, 43.5, 59.0 are the prices of call options with exercise price 1700 for those expiry dates
- similarly one interprets the other numbers for put options
- the numbers quoted lie somewhere between the bid and the ask

- usually the term portfolio is used to describe a combination of: assets, options, and cash (invested in a bank)
- it is possible to hold negative amounts of each
    - a negative amount of cash has the obvious interpretation that cash has been borrowed
    - owning a negative amount of an asset or of an option: possible through *short-selling*: selling an item that is not owned with the intention of buying it back at a later date
- in practice to short-sell an item one must first borrow it from somebody who owns it and give it back later
- we will assume that this is always possible, at no cost, and that the short seller is free to choose when to buy back and return the item
- if we short sell an asset at time t_1 and buy it back at time t_2 then we have: gained an amount S_{t_1} at time t_1 from the short sale and paid out an amount S_{t_2} at time t_2 from the buy back; the overall profit /loss at time $t = t_2$ is therefore $e^{r(t_2-t_1)}S(t_1) - S(t_2)$