Lecture Quantitative Finance Spring Term 2014

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Put-call parity relationship

- There is a delightful simple argument that defines a relationship between the value (premium) $\pi_t(C)$ of a European call option and the value (premium) $\pi_t(P)$ of a European put option with the same strike price K and expiry date T.
- Consider the following two portfolios:
  - A: one call option (with strike K and maturity T) plus $K \cdot e^{-r(T-t)}$ cash (invested in a bank); and
  - B: one put option (with strike K and maturity T) plus one unit of the underlying asset.

Put-call parity relationship

- At the expiry date T, we have that
  - the portfolio A is worth $\max(S_T - K, 0) + K = \max(S_T, K)$; and
  - the portfolio B is worth $\max(K - S_T, 0) + S_T = \max(K, S_T)$.
- According to the principle of no-arbitrage (previous lecture: positions with **same** payoffs have the **same** price) the two portfolios have the same value, i.e.,

$$\pi_t(C) + K \cdot e^{-r(T-t)} = \pi_t(P) + S_t$$

Put-call parity relationship

- Put-call parity relationship:

$$\pi_t(C) - \pi_t(P) = S_t - K \cdot e^{-r(T-t)}.$$

- Put-call parity relationship with a dividend rate of r^*

$$\pi_t(C) - \pi_t(P) = S_t \cdot e^{-r^*(T-t)} - K \cdot e^{-r(T-t)}.$$

- **Example:** A 2-year European call option has a market price of 50 CHF and a strike of 140 CHF. The underlying stock price is 100 CHF and the stock pays a dividend yield of 2% (annualized). The 2-year (annualized) interest rate is 5%. What is the market price of a 2-year European put struck at 140 CHF?

  Solution. Using the put-call parity relationship, we have

$$\begin{aligned}
\pi_t(P) &= \pi_t(C) - S_t \cdot e^{-r^*(T-t)} + K \cdot e^{-r(T-t)} \\
&= 50 - 100 \cdot e^{-2\% \times (2-0)} + 140 \cdot e^{-5\% \times (2-0)} \\
&= 80.60.
\end{aligned}$$

Put-call parity: Exercise

Exercise 1: The price of a non-dividend paying stock is 20 CHF. A 6-month European call option with strike 18 CHF sells for 4 CHF. A European put option on the same stock, with the same strike price and the same maturity, sells for 1.47 CHF. The continuously compounded risk-free interest rate is 6% per annum. Are these three securities (the stock and the two options) consistently priced?

1. No, there is an arbitrage opportunity worth 2.00.
2. No, there is an arbitrage opportunity worth 2.53.
3. No, there is an arbitrage opportunity worth 14.
4. Yes.

Upper and lower bounds on option prices

- The price of a call increases with the price of the underlying; this increase is convex (non-linear).
- The price of a put, on the other hand, increases when the price of the underlying drops; this increase is (also) convex.
- The price of an option can be decomposed into
  - its intrinsic value, which is the option's value/payoff when it would be exercised today, i.e., $\max(S_t - K, 0)$ for a call and $\max(K - S_t, 0)$ for a put; and
  - its time value, which is the remainder, reflecting the possibility that the option will create further gains in the future.

Upper and lower bounds on option prices

- Consider for example a 1-year call with strike $K = 100$ CHF. The current price is $S = 120$ CHF and the (risk-free) interest rate is 5%. The asset pays no dividends. Say the call premium is 26.17 CHF: this can be decomposed into an intrinsic value of $120 - 100 = 20$ CHF and time value of 6.17 CHF. The time value increases with the volatility of the underlying asset.

Upper and lower bounds on option prices

- For simplicity, assume there are no dividend payments.
- The current value of a call must be less than, or equal to, the asset price:
$$C_t \leq \pi_t(C) \leq S_t.$$
(In the limit, an option with zero exercise price is equivalent to holding the stock.)
- Stronger: the value of an European call must be greater than, or equal to, the price of the asset minus the present value of the strike price:
$$\pi_t(C) \geq S_t - K \cdot e^{-r(T-t)}.$$
- As an example, take our call option from the previous slide: The lower bound is:
$S_t - K \cdot e^{-r(T-t)} = 120 - 100 \cdot e^{-5\% \times (1-0)} = 24.88$ CHF, which is a sharper bound than $S - K = 20$ CHF.

Upper and lower bounds on option prices

- For put options: $\pi_t(P) \leq K$. (The upper bound is attained when price of the underlying falls to zero.)
- Using put-call parity we get: $\pi_t(P) \geq K \cdot e^{-r(T-t)} - S_t$.
- Consider an American call on a non-dividend paying stock:
  - by exercising early, the holder gets exactly $S_t - K$; and
  - we know that $\pi_t(C) \geq S_t - K \cdot e^{-r(T-t)}$, which is greater than or equal to $S_t - K$.
- Hence an American call on a non-dividend paying stock should never be exercised early.
- In our example (from the previous two slides) we found a lower bound of 24.88 CHF for a European call; if we would exercise an American call (with the same strike) we would only get $120 - 100 = 20$ CHF; because this is less than the minimum value of the European call, the American call should not be exercised.

Upper and lower bounds on option values

- The only reason to exercise an American call early is to capture a dividend payment; intuitively, a high-income payment makes holding the asset more attractive than holding the option.
- Thus American options on income-paying assets may be exercised early.

Upper and lower bounds on option values: Example

What is a lower bound on the price of a European call option with a strike of 80 CHF and 1 year till maturity? The price of the underlying asset is 90 CHF, and the 1-year interest rate is 5% per annum. Assume that interests are compounded continuously.

1. 14.61;
2. 13.90;
3. 10.00; or
4. 5.90.

The second answer is correct. The call lower bound (when there are no income payments) is

$$S_t - K \cdot e^{-r(T-t)} = 90 - 80 \cdot e^{-0.05 \times 1} = 90 - 76.10 = 13.90.$$

Exercise

Exercise 2 The best way to close out a long American call option position early (option written on a stock that pays no dividends) would be to:

1. exercise the call;
2. sell the call;
3. deliver the call; or
4. do none of the above.

Exercise

Exercise 3 Which of the following statements about options on futures is *true*?

1. an American call is equal in value to a European call;
2. an American put is equal in value to a European put;
3. put-call parity holds for both American and European options; or
4. none of the above statements is true.

Pricing by replication

- To derive the price of an option contract one (typically) constructs a portfolio that perfectly replicates the payoff of the option.
- To avoid arbitrage opportunities from occurring, the price of this replicating portfolio must then be equal to the price of the option.

Pricing by replication

- Consider a call option on a stock whose price evolves according to a binomial tree model:
  - the stock's initial price of $S_0 = 100$ CHF can only move up or down to either $S_{1,up} = 150$ CHF or $S_{1,down} = 50$ CHF;
  - the option is a call with strike $K = 100$ CHF;
  - so the payoff of the option is either $c_{1,up} = 50$ CHF or $c_{1,down} = 0$ CHF;
  - furthermore, assume an interest rate of $r = 25\%$.

- We replicate the payoff of the option by combining the stock with a risk-free asset in a portfolio.
- This is feasible, because there are 2 states of the world and 2 instruments: the stock and the risk-free asset.
- To not introduce an arbitrage opportunity, we must have that the value of the option is equal to that of the portfolio.
- The portfolio consists of n shares and a risk-free investment B (a negative value implies borrowing).
- We set $c_{1,up} = nS_{1,up} + B$ or $50 = n \cdot 150 + B$; and
- $c_{1,down} = nS_{1,down} + B$ or $0 = n \cdot 50 + B$.
- Solving (the 2×2 system) gives $n = 0.5$ and $B = -25$.
- At time $t = 0$ the value of the loan is $B_0 = -25/(1 + 25\%) = -20$.
- At time $t = 0$ the value of the portfolio is $nS_0 + B_0 = 0.5 \times 100 - 20 = 30$.
- Hence the current value of the option must be $\pi_0(C) = 30$.

Risk-neutral pricing

- Note that we did not use the actual probability of an up/down move to compute the price of the option; so what probability is (implicitly) used then? Define the probability of an up-move as p.
- Now, write the value of the stock as the discounted expected payoff (w.r.t. the probability p) assuming investor's are risk-neutral, i.e., write

$$S_0 = \frac{1}{1+r}\left[p \times S_{1,up} + (1-p) \times S_{1,down}\right],$$

where the term in brackets is the expectation of tomorrow's stock price.
- Filling in the numbers and solving for p gives

$$100 = \frac{1}{1.25}\left[p \times 150 + (1-p) \times 50\right],$$

we find a probability of $p = 0.75$.

- We now value the option in the same fashion

$$\pi_0(C) = \frac{1}{1+r}\left[p \times c_1 + (1-p) \times c_2\right],$$

- and this gives $\pi_0(C) = \frac{1}{1.25}\left[0.75 \times 50 + (1 - 0.75) \times 0\right] = 30.$
- We find that the price of an option is equal to the discounted risk-neutral expectation of its payoff; hence, we call p a **risk-neutral probability**.

Outlook:

- The Black-Scholes model is based on these ideas and provides an elegant closed-form solution for the price of European call and put options.

Plain vanilla vs. exotic options

- The European/American call and put options considered so far are so-called **plain vanilla option contracts**.
- Many other types of option contracts exist.
- **Binary options** (also called digital options) pay a fixed amount, say, Q, when the price of the underlying asset ends up above the strike price, i.e., their payoff is

$$c_T = Q \times 1_{\{S_T - K \geq 0\}}$$

- These options have a sharp discontinuity around the strike price: just below K their value is zero; just above, their value is equal to Q.
- Due to this discontinuity the payoff of these options is very difficult to replicate.

Barrier options

- **Barrier options** are options whose payoff depends on whether the price of the underlying asset S hits a certain level (barrier) during the option's lifetime.
- A knock-out option loses its value when S hits a certain level H.
- A knock-in option comes into existence when S hits a certain level H.

Barrier options

- An example of a knock-out option is the down-and-out call: this option loses its value when S hits H during the lifetime of the option; in this case the knock-out price H must be lower than the initial price S_0.
- The option that "appears" at H is the down-and-in call; with identical parameters, the down-and-out call and the down-and-in call are complementary: when one appears, the other disappears.
- Similarly for up-and-out call and up-and-in call.

Barrier options

- Barrier options can be attractive because they are cheaper than equivalent European options (which of course reflects the fact that they are less likely to be exercised than other options).
- Barrier options are difficult to hedge/replicate due to the fact that the barrier introduces a discontinuity.

Barrier options: Example

Of the following options, which one does not benefit from an increase in the stock price when the current stock price is 100 CHF and the barrier has not yet been crossed.

1. A down-and-out call with out barrier at 90 CHF and strike at 110 CHF; or
2. a down-and-in call with in barrier at 90 CHF and strike at 110 CHF.

Solution.

- A down-and-out call where the barrier has not been touched is still alive and hence benefits from an increase in S so the first statement is incorrect.
- A down-and-in call only comes alive when the barrier is touched – so an increase in S brings it further away from the barrier. So the second statement is correct.

Barrier options: Example

Of the following options, which one does not benefit from an increase in the stock price when the current stock price is 100 CHF and the barrier has not yet been crossed.

1. An up-and-in put with barrier at 110 CHF and strike at 100 CHF; or
2. an up-and-in call with barrier at 110 CHF and strike at 100 CHF.

Solution.

- An up-and-in put would benefit from an increase in S as this brings it closer to the barrier of 110 CHF so the first statement is incorrect.
- Finally, an up-and-in call would also benefit if S gets closer to the barrier. So the second statement is also incorrect.

Asian options

- **Asian options** (or average rate options) generate payoffs that depend on the average value of the underlying stock price during the lifetime of the option, instead of just the final value.
- Define this average value as $S_{ave}(t, T)$; the final payoff for an Asian call is
$$c_T = \max(S_{ave}(t, T) - K, 0).$$
- Because an average is value less volatile than a final value Asian options are cheaper than European options.

Look-back options

- **Look-back options** have payoffs that depend on the extreme values of S over the option's lifetime.
- Define S_{max} as the maximum and S_{min} as the minimum value of S over a certain period of time.
- A fixed-strike look-back call option pays $\max(S_{max} - K, 0)$.
- A floating-strike look-back call option pays $\max(S_{max} - S_{min}, 0)$.
- These options are more expensive than European options.

Which of the following options is path-dependent?

1. An Asian Option;
2. a binary option;
3. an American option; or
4. a European call option.

Solution. The payoff of an Asian option depends on the average value of S and therefore is path-dependent.

Don't forget: important formulae

- Payoff of a long call and put option:

$$C_T = \max(S_T - K, 0) \quad \text{and} \quad P_T = \max(K - S_T, 0).$$

- Put-call parity relationship:

$$\pi_t(C) - \pi_t(P) = S_t \cdot e^{-r^*(T-t)} - K \cdot e^{-r(T-t)}$$

- Bounds on the value of a call (no dividends):

$$S_t - K \cdot e^{-r(T-t)} \leq \pi_t(C) \leq S_t.$$

- Bounds on the value of a put (no dividends)

$$K \cdot e^{-r(T-t)} - S_t \leq \pi_t(P) \leq K.$$

Answers to the Exercises

Exercise 1 Put-call parity applies to these European options. With no dividend, the relationship is

$$\pi_t(C) - \pi_t(P) = S_t - K \cdot e^{-r(T-t)}.$$

The first term is

$$\pi_t(C) - \pi_t(P) = 4 - 1.47 = 2.53.$$

The second term is:
$S_t - K \cdot e^{-r(T-t)} = 20 - 18 \cdot \exp[-6\%(6/12)] = 2.53$. Because the two values are equal there is no arbitrage opportunity. Thus, number 4 is correct.

Exercise 2 The correct answer is the second one. When there is no dividend, there is never any reason to exercise an American call early. Instead, the option should be sold to another party.

Answers to the Exercises

Exercise 3 The last answer is correct. Futures have an implied income stream equal to the risk-free rate. As a result an American call may be exercised early. Similarly, the American put maybe exercised early. Also, the put-call parity only works when there is no possibility of early exercise, or with European options.

Remember

- Chapter 3:
  - options are financial contracts traded in private over-the-counter (OTC) markets or on organised exchanges;
  - their value is derived from the price of the underlying asset;
- put-call parity
- upper and lower bounds on option values
- fundamentals of option valuation
  - pricing by replication
  - risk-neutral pricing
- target: Black-Scholes PDE and formulas