Modeling Dependencies

Guest Lecture

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• This lecture aims at providing a practical overview on the importance of dependencies and how to model them.

• Exercises in Excel (without VBA) will be used to explain the associated techniques. Solutions will be made available.

If possible, the students should bring a Laptop with Excel.

A very short list of software used for calculations:

Excel based: VBA and @Risk (commercial)
Open source: R and Octave
Commercial: Matlab and Mathematica
Library-based: C#, C++, Java, etc.
Welcome

1. Introduction to Dependencies
2. Methods for Modeling Dependencies
3. Model Selection and Calibration
4. Trusted Data

Closing Remarks
First, a Notation Issue

- “Correlation” is usually meant as a measure of “dependency”
- “Dependency” is more general
- Indeed, it can happen that dependency exists, but the usual correlation measures are not able to capture it – for instance:

```
Correlation Measures:
Pearson = zero
Spearman = zero
Kendall = zero
```
1. Introduction to Dependencies

- Dependencies Between What?
- Why Model Dependencies?
- How to Measure Dependencies?
Dependencies Between What?

The short answer (in banking and insurance industries):

“Between risks”

With the usual suspects:

• Credit Risks (counterparty)
• Operational Risks
• Investment Risks (asset)
• Underwriting Risks (liability)
### Dependencies Between What?

The not-so-short answer:

<table>
<thead>
<tr>
<th>Context</th>
<th>Range</th>
<th>Time Dependency</th>
<th>Observation Period</th>
<th>Time Separation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Event Frequencies</td>
<td>Within a Risk</td>
<td>Static (Snapshot)</td>
<td>During 1 Second</td>
<td>At coincident times</td>
</tr>
<tr>
<td>Severities</td>
<td>Across Risks e.g. underwriting, investment, credit, operational</td>
<td>Across Time</td>
<td>During 5 Minutes</td>
<td>With Time-Lag of 1 Second</td>
</tr>
<tr>
<td>Stock Returns</td>
<td></td>
<td></td>
<td>During 1 Day</td>
<td>With Time-lag of 1 Year</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

The main focus during this lecture
Why Model Dependencies?
Example #1 – Mergers

Merged Companies:
New fat-tail risks due to new strong dependencies
Why Model Dependencies?

Dependencies create riskier worlds

Positive dependency typically generates fatter-tails, leading to:

- Less diversification effect
- Higher frequency of “rare” events
- Increased Value-at-Risk (VaR)
- Increased Expected-Shortfall* (ES)

* Also known as “Tail-Value-at-Risk” or “Conditional-Value-at-Risk”.
Why Model Dependencies?
Example #2 – Less Diversification

If losses from various business lines are dependent, then the *diversification effect is smaller*.

10 Aggregated Risks

- **No correlation**: 99% ES=48
- **30% Spearman correlation**: 99% ES=75

[Graph showing probability density functions (pdf) for different correlation levels]
Why Model Dependencies?
Example #3 – Higher Frequency of Rare Events

If rare events are correlated, then the **probability of joint events is higher**.

![Joint Defaults (100 counterparties)](chart)

(Individual default prob. = 4%)

- **No correlation**
- **8% correlation** between defaults

number of joint defaults
Why Model Dependencies?

- Regulation:
  - Basel III and FINMA “Swiss finish” (for Banks)
  - Swiss Solvency Test and Solvency II (for Insurers)
- Reserving / Risk Adjusted Capital
- Pricing
- Capital Allocation
- Business Planning
- Portfolio and Risk Management
- ...

To Improve Strategy
(Profitability, Survival, ...)

Internal models are important
Why Model Dependencies?

• **Negative dependencies** usually have the opposite effect of positive dependencies, but are less frequent.

• Example: mean-reversal behaviour of stock prices, i.e. relation between the price “before” and “now”.

• (A positive cash-flow means a negative cash-flow for someone else, but that does not count as negative dependency for neither of them.)
Exercise #1

Random variables can be generated using the fact that the cumulative distribution function $F(x)$ follows a Uniform(0,1) distribution, i.e. $F(x) \sim \text{Uniform}(0,1)$.

Generate the variable $u$ from Uniform(0,1) and plug it in the inverse of the cumulative function. The result $x = F^{-1}(u)$ has the desired distribution.

**Goals:**

1) Generate 100 realizations of a Gaussian variable $X \sim \text{Gaussian}(0,1)$.

2) Generate 100 realizations of a Poisson variable $Y \sim \text{Poisson}(3)$.
   
   **Hint:** Use a table and the *vlookup* formula to find the inverse $F^{-1}(u)$.

3) Estimate their expected value and standard deviation based on the realizations (the error goes down with larger number of realizations).
How to Measure Dependencies?

E.g. with correlation measures, such as:

\[ \rho(X_1, X_2) = \frac{\text{Cov}(X_1, X_2)}{\sqrt{\text{Var}(X_1)\text{Var}(X_2)}} \]

Eq. 1

- **Pearson:** \( \rho(X_1, X_2) = \frac{\text{Cov}(X_1, X_2)}{\sqrt{\text{Var}(X_1)\text{Var}(X_2)}} \)

  (or "linear")

- **Spearman:** \( \rho_S(X_1, X_2) = \rho(F_1(X_1), F_2(X_2)) \)

- **Kendall:** \( \tau(X_1, X_2) = E[\text{sign}((X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2))] \)

  where \( \tilde{X}_1 \) and \( \tilde{X}_2 \) have the same joint distribution, but are independent of \( X_1 \) and \( X_2 \).

- **Parametric**
- **Only captures linear dependency**

- **Not sensitive to outliers**
- **Good for heavy-tailed distributions**
- **Better than linear correlation**

Where \( \text{Cov}(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) \)
Exercise #2

In the **1-Factor Gaussian Model** all variables (e.g. stocks) share a common-factor, the “**market**” $M$:

$$X_1 = \sqrt{1 - \rho} Y_1 + \sqrt{\rho} M$$

$$X_2 = \sqrt{1 - \rho} Y_2 + \sqrt{\rho} M$$

etc.

All variables have distribution $\text{Gaussian}(0, 1)$*. The market $M$ and the idiosyncratic components $Y_i$ are independent.

**Goals:**

1) Generate 1000 realizations of $X_1$ and $X_2$ with $\rho = 30\%$.

2) Estimate the various correlations: a) Pearson (linear); b) Spearman; c) Kendall, and compare with the theory:

   - Pearson: $\rho$
   - Spearman: $\frac{6}{\pi} \arcsin(\rho/2)$
   - Kendall: $\frac{2}{\pi} \arcsin(\rho)$

* From $\mu + \sigma X_i$ one can get other parameters.
2. Methods for Modeling Dependencies

- Explicit vs. Implicit Models
- Common-Factors
- Copulas
- Tail Dependency
Modeling Dependencies

To model independency there is only one choice.
To model dependency there are infinitely many choices.
Explicit vs. Implicit Models

Example #4 – Natural Catastrophes

(Re)Insurer

Property in France

Windstorm in France

Fire in France

Property in Germany

Windstorm in Germany

Fire in Germany

Not accounting for dependencies can lead to insolvency.

Note: Diagram inspired from work of Roland Bürgi, Michel M. Dacorogna & Roger Iles (2008).
Explicit Models
Example #4 – Common-Factors

(Re)Insurer

Property in France

Property in Germany

Windstorm in France

Fire in France

Windstorm in Germany

Fire in Germany

Common Windstorm (stochastic variable)

Note: Diagram inspired from work of Roland Bürgi, Michel M. Dacorogna & Roger Iles (2008).
Implicit Models
Example #4 – Copulas

Note: Diagram inspired from work of Roland Bürgi, Michel M. Dacorogna & Roger Iles (2008).
Explicit vs. Implicit Models

Explicit Models
- e.g. Common Factors/Shocks,
  Causal, ...
  Regression,
  Frailty, ...

- Intuitive
- Potentially accurate
- Give insight into business
- But can lead to a false sense of accuracy

Implicit Models
- e.g. Copulas,
  Lévy-Copulas,
  Pareto-Copulas, ...

- Many types of dependencies
- Explicit tail dependency
- But calibration is complicated,
  and causality might not be known
Implicit Models
Example #5 – Common-Factors

Same stochastic variable in two or more risks
(e.g. default of a counterparty)

Risks faced by a certain company:
Risk 1: money deposited in Bank A
Risk 2: outstanding receivables from client X, which happens to be the same Bank A
Common-Shock: default of Bank A
Implicit Models
Example #6 – Common-Factors

Adding common stochastic variables to other variables
(e.g. Merton Model, Poisson Shocks), or to their parameters

Risks in a portfolio:
Portfolio: composed by stocks with returns modeled as $X_i \sim \text{Gaussian}$.

1-Factor Gaussian Model: all stocks share a common factor: the market.

\[
X_1 = \sqrt{1 - \rho} Y_1 + \sqrt{\rho} M \\
X_2 = \sqrt{1 - \rho} Y_2 + \sqrt{\rho} M \\
\text{etc.}
\]

where the idiosyncratic components $Y_i \sim \text{Gaussian}$ are independent.
Exercise #3
Merton Model

• The Merton model is commonly used for credit risk (e.g. Basel III)
• It uses the 1-factor Gaussian model (equivalent to the Gaussian copula) to model the assets $A_i$ of the counterparties ($i = 1, 2, ...$):

$$A_i = \sqrt{1 - \rho} Y_i + \sqrt{\rho} M$$

where $Y_i$ are idiosyncratic factors and $M$ the market (see exercise #2).
• Default occurs when the assets $A_i$ drops below a certain value, which is set by the default probability $p_i$:

$$B_i = \begin{cases} 1 & \text{if } A_i < \Phi^{-1}(p_i) \sim \text{Bernoulli}(p_i) \\ 0 & \text{otherwise} \end{cases}$$

Goal: Simulate 3 counterparties with assets correlated with $\rho = 30\%$ and default probabilities $p_i = 5\%$. Estimate numerically the (linear) correlation between defaults.

Hint: Results from exercise #2 can be re-used here.

Note: a similar model was used to produce the chart on page 11.
Exercise #4
Poisson Common-Shocks

Risk 1: Windstorms in France with frequency $X_F \sim \text{Poisson}(5)$, i.e. average of 5 windstorms per year.

Risk 2: Windstorms in Germany with frequency $X_G \sim \text{Poisson}(6)$.

Common-Shock: Pan-European windstorm with frequency $X_C \sim \text{Poisson}(2)$, modeled as:

$$X_F = X_1 + X_C$$
$$X_G = X_2 + X_C$$

where $X_1 \sim \text{Poisson}(3)$ and $X_2 \sim \text{Poisson}(4)$.

Goal: Use a Monte-Carlo simulation to estimate the Pearson correlation coefficient. Compare against the theoretical result:

$$\rho = \frac{\lambda_C}{\sqrt{\lambda_F \lambda_G}}$$

between 0 and $\min(\lambda_F, \lambda_G)$

Where $\lambda_F$, $\lambda_G$ and $\lambda_C$ are the expected values of $X_F$, $X_G$ and $X_C$.

Or "Default" instead of "Windstorm" and "Portfolio" instead of "Country"

Try to derive this using equations (1-2) in pg. 15
Copulas

• The principle behind copulas is fairly simple. In the bivariate case, consider two random variables $X_1$ and $X_2$. A copula $C$ is a parameterization of the joint cumulative distribution function:

$$F_{1,2}(X_1, X_2) = P(X_1 \leq x_1; X_2 \leq x_2) = C(F_1(X_1), F_2(X_2))$$

• No dependency: $C(u_1, u_2) = u_1 u_2$ like $P(A \wedge B) = P(A)P(B)$

• FGM*: $C(u_1, u_2) = u_1 u_2 (1 + \theta (1 - u_1)(1 - u_2))$ $|\theta| \leq 1$ and $\tau = \frac{2}{9} \theta$

• Clayton: $C(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}$, $\theta \geq 0$ and $\tau = \frac{\theta}{\theta + 2}$

• Gumbel: $C(u_1, u_2) = \exp\left[-\left((-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right)^{1/\theta}\right]$ $\theta \geq 1$ and $\tau = 1 - \frac{1}{\theta}$

• Gaussian: $C(u_1, u_2) = \Phi_{\rho}(\Phi^{-1}(u_1), \Phi^{-1}(u_2))$ $|\rho| < 1$ and $\tau = \frac{2}{\pi} \text{asin } \rho$

• Student’s t: $C(u_1, u_2) = t_{\nu,\rho}(t_{\nu}^{-1}(u_1), t_{\nu}^{-1}(u_2))$

* Fairly-Gumbel-Morgenstern. Only adequate to model small dependencies.
Copulas
A Short Graphical Overview

Source: SAS
Copulas
Algorithm to generate bivariate copulas*

One uses the fact that derivatives of the copula yield conditional distribution functions – for instance:

\[ P(X_2 \leq x_2 | X_1 = x_1) = \frac{\partial}{\partial u_1} C(u_1, u_2) \]

where \( u_1 = F_1(X_1) \), \( u_2 = F_2(X_2) \). The outcome follows a Uniform(0,1) and is independent of \( U_1 \).

**First, generate dependent uniform variables \( u_1 \) and \( u_2 \):**
1. Draw independent \( \nu_1 \) and \( \nu_2 \) from Uniform(0,1).
2. Set \( u_1 = \nu_1 \)
3. Set \( \nu_2 = \frac{\partial C}{\partial u_1} \) and solve for \( u_2 \), i.e. \( u_2 = \left( \frac{\partial C}{\partial u_1} \right)^{-1} \), where the right end side contains \( \nu_1 \) and \( \nu_2 \) - see next page for a few examples.

**Second, generate the marginal variables \( x_1 \) and \( x_2 \):**
4. Simply use \( x_1 = F_1^{-1}(u_1) \) and \( x_2 = F_2^{-1}(u_2) \)

* A similar algorithm applies to the multivariate case as well.
Explicit expression for $u_2$:

- **FGM:**
  
  $u_2 = \frac{2v_2}{(\sqrt{B} + A)}$
  
  where $A = 1 - \theta (2v_1 - 1)$
  
  and $B = [1 - \theta (2v_1 - 1)]^2 + 4\theta v_2 (2v_1 - 1)$

- **Gumbel:**
  
  $\frac{\partial c}{\partial u_1}$ is not invertible...

- **Clayton:**
  
  $\frac{\partial c}{\partial u_1} = \left[1 + \left(\frac{u_1}{u_2}\right)^\delta - u_1^\delta\right]^{-\frac{1}{\delta} - 1}$, so that $v_2 = \frac{\partial c}{\partial u_1}$ yields
  
  $u_2 = \left[v_1^{-\theta} \left(v_2^{-\theta/(\theta+1)} - 1\right) + 1\right]^{-1/\theta}$
Exercise #5

The losses from windstorms in France and Germany, $S_1$ and $S_2$ respectively, follow a Pareto distribution with scale parameter 3 and shape parameter $\alpha = 4$, i.e.

$$F_i(s_i) = 1 - \left(\frac{3}{3+s_i}\right)^4$$

with $s_i > 0$ and $i = 1, 2$.

**Goal:** Introduce a Kendall correlation of 50% between $S_1$ and $S_2$ by using the Clayton copula with the tail dependency on the upper side, i.e. use the formulas from pages 29-30 and then use

$$\tilde{u}_i = 1 - u_i$$

to plug in

$$F_i^{-1}(\tilde{u}_i) = 3(1 - \tilde{u}_i)^{-1/4} - 3$$
Exercise #6

The **Gumbel copula** can be generated using the algorithm*:

1. Generate independent $v_1, v_2 \sim \text{Uniform}(0, 1)$.
2. Find $w$ such that $w(1 - \ln(w)/\theta) = v_2$, where $0 \leq w \leq 1$.
3. Set $u_1 = \exp[v_1^{1/\theta} \ln(w)]$ and $u_2 = \exp[(1 - v_1)^{1/\theta} \ln(w)]$.

**Goal:** Take $X_1, X_2 \sim \text{Gamma}(3, 1)$ and correlate them to 70% (Pearson correlation) using:

- **a)** A Gumbel copula (**Hint:** use a vlookup table to solve for $w$ above);
- **b)** And a Gaussian copula.

For each, estimate:

- **c)** The tail probability $P(X_2 > u \mid X_1 > u)$
  at $u = F^{-1}(99\%)$.
- **d)** The $t\text{VaR}$ of the sum, at 99% confidence level.

**Result:** while $t\text{VaRs}$ are similar, the tail prob. are not.

* See general Algorithm 6.1. in Embrechts et al (2001) and pages therein for its derivation.
Copulas
Algorithm for Gaussian and Student’s t- copulas

Given a correlation matrix \( \Sigma \):

**Gaussian copula:**
1. Perform a Cholesky-decomposition \( \Sigma = A^T A \).
2. Generate independent \( \tilde{X}_1, ..., \tilde{X}_d \sim \text{Gaussian}(0,1) \).
3. Compute \( (X_1, ..., X_d) = A \tilde{X} \).
4. Finally, compute \( U_i = \Phi(X_i) \).

**Student’s t-copula:**
   Do steps 1 to 3 above.
4. Generate \( \xi = \sum_{i=1}^{v} Y_i^2 \), where \( Y_i \sim \text{Gaussian}(0,1) \) are independent.
5. Finally, compute \( U_i = t_v(X_i / \sqrt{\xi / v}) \), where \( t_v \) is the cumulative distribution function of the t-distribution.
Exercise #7

Goal: Simulate the bivariate $u_1, u_2$ from the Student’s t-copula with $\nu = 3$ and $\rho = 70\%$.

Compare the estimate of Kendall’s tau with the theoretical value.
Tail Dependency

- Tail dependency can be easily modeled by an appropriate copula.


- **FGM** copula: None
- **Clayton** copula: On one side
- **Gumbel** copula: On one side
- **Gaussian** copula: None
- **Student’s t**-copula: On both sides (non-zero even if $\rho = 0$)
Tail Dependency
Bivariate Case

• Upper- and Lower-Tail Dependency Coefficients:

\[
\lambda_L = \lim_{u \searrow 0} P(X_2 \leq F_2^{-1}(u) \mid X_1 \leq F_1^{-1}(u)) = \lim_{u \searrow 0} \frac{C(u, u)}{u} \\
\lambda_U = \lim_{u \nearrow 1} P(X_2 > F_2^{-1}(u) \mid X_1 > F_1^{-1}(u)) = \lim_{u \nearrow 1} \frac{1 - 2u + C(u, u)}{1 - u}
\]

• FGM copula: \( \lambda_L = 0 \) and \( \lambda_U = 0 \).
• Clayton copula: \( \lambda_L = 2^{-1/\theta} \) and \( \lambda_U = 0 \).
• Gumbel copula: \( \lambda_L = 0 \) and \( \lambda_U = 2 - 2^{1/\theta} \).
• Gaussian copula: \( \lambda_L = 0 \) and \( \lambda_U = 0 \).
• Student’s t-copula: \( \lambda_L = \lambda_U = 2t_{v+1} \left( -\sqrt{\frac{(v+1)(1-\rho)}{1+\rho}} \right) \). Non-zero even if \( \rho = 0 \).
• Model Selection
• Model Calibration
Model Selection

- A model should be a good balance between accurateness and complexity.

Can the model be simplified?
- Can business and management understand the model?
- Was the model selection properly documented?
Model Selection

• Modeling with **common-factors** provides with a relatively natural selection of the model and a path to sequential improvements.

• When modeling with **copulas**, the large number of choices can actually be a disadvantage.

A possible procedure for selecting an appropriate copula:

– Start by choosing/fit the marginal distributions.

– Then choosing a copula which will bring the desired/expected dependency structure, e.g. based on tail dependency. There are also various statistical tests: Akaike information criterion, Pseudo-likelihood ratio tests, Bayes factor.
Model Calibration

• One has to pay close attention to the fact that dependencies:
  – Are subject to **measure uncertainties**
  – Usually **change over time** (e.g. increasing in times of stress)
  – Might result from **spurious relationships**

• Visual inspection is important

• Calibration of **common-factors** parameters is usually based on exposure/volume data, market data and expert estimates.

• Calibration of **copula** parameters typically uses:
  – Method of moments. For instance, estimating Kendall’s tau
  – Maximum likelihood estimation
Model Calibration
Dependency Structure

- It is difficult to calibrate a correlation matrix:
  - It usually has many entries
    E.g. given \( N \) risks or risks elements, there are \( N(N - 1)/2 \) pair-wise correlations (e.g. 435 entries for 30 risks).
  - It must be symmetric and positive-semidefinite, i.e. cannot have negative eigenvalues.
  - The entries will have a lot of random error
• Trusted Data
• Overview of Systemorph
Why is trusted data important?

- Good decisions and good modelling need comprehensive and reliable data
- Growing data sources, competition and reporting requirements – e.g.:
  - SST
  - Solvency I / II
  - ORSA
  - BCBS 239
  - Basel III
  - NBA / NBO
  - MiFID I / II
  - Dodd-Frank
  - EMIR I / II
  - FATCA
  - IFRS4
  - etc.

Potential game changers:
- Swiss Solvency Test (2006)
- Own Risk and Solvency Assessment (2015)
- Effective risk data aggregation and risk reporting (2016)
- Third Basel accord (2019), plus Swiss finish
- US regulation on OTCs (2010 & ongoing)
- European regulation on OTCs (2012 & ongoing)
- Foreign Account Tax Compliance Act (2014 & ongoing)
Systemorph at a Glance

**Background**
- Founded 2011, headquartered in Zurich

**Customers**
- Global financial services firms, focus on (re)insurance and banks

**Mission**
- Revolutionize software solutions for financial institutions
- Streamline and simplify risk reporting, analysis and actuarial functions

**Team**
- Risk and capital management, modeling, enterprise systems, information management
- All hold advanced degrees in computer science, physics or mathematics
Do you trust your data?

- How are they changing?
- Who owns them?
- Are they accurate?
- How do we share them?
- How do we use them for modelling?
C⁴ – Data Intelligence Re-imagined ...

Systemorph C⁴ – Smart Data Layer

Import / Export of data & files from and to DB’s, Apps, Excel files ...  

Development of data apps in Systemorph

Serving of data to Business apps

• Link data assets  
• Build powerful apps  
• Decentralize ownership  
• Collaborate  
• Integrate analytics  
• Track data history and changes  
• Manage data quality  
• Report to stakeholders

Trusted data!

Calculation servers – Matlab, R, Java, C# ...

Self-Service Reporting – LinqPad, Excel, Word, mobile apps
• Closing Remarks
• Literature
Closing Remarks
Principles for effective risk aggregation and risk reporting

• Following the 2007 financial crisis, the Bank for International Settlements has issued Basel III, and BCBS 239 (Principles for effective risk data aggregation and risk reporting):

• “One of the most significant lessons learned from the [2007] global financial crisis [was that] many banks lacked the ability to aggregate risk exposures and identify concentrations quickly and accurately at the bank group level, across business lines and between legal entities.

Some banks were unable to manage their risks properly because of weak risk data aggregation capabilities and risk reporting practices. This had severe consequences to the banks themselves and to the stability of the financial system as a whole.”

• **Systemically important** banks will comply first. Others will follow.
**General:**


**Copulas:**

Thank you very much for your attention.

If you have questions, comments or suggestions, please contact:
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