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An overview of Jordan measure

Bachelor Thesis

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Chapter 1

Introduction

The goal of this work is to present the Jordan measure and give an overview of its main properties. In particular, we want to explain how the Jordan measure generalizes the concept of volume: defined first only for boxes, then for *elementary sets* and lastly for the so-called *measurable sets*. The latter, turns out, define a quite large family, containing for example any triangle, any ball and actually any bounded set that has the graph of a continuous function as boundary (a proof of that will be provided in chapter 4). The first results we introduce regard additivity, monotonicity and translation invariance of the Jordan measure J, which are proven initially only for the family of elementary sets and then generalized to the collection of measurable sets. The same procedure is used also when showing that, up to normalization, the previous properties uniquely determine J. This, somehow surprisingly, observation is followed by the *discretization formula*, which can be seen as an additional motivation for the claim that J extends the concept of volume naturally.

In the second part, that is chapters 3 and 4, we exploit the properties of the maps $\overline{J}$ and $\underline{J}$, used for the definition of the Jordan measure, to obtain measurability criteria. For that purpose, we need to introduce the formalism used in [2], and in particular the idea of splitting $\mathbb{R}^d$ into cubes of side length $1/n$ and counting how many of them are contained in some fixed set A. To motivate this procedure we consider the example of the unit disc D in two dimensions, where one can explicitly give an infinite sequence of elementary sets converging, as set-theoretic limit, to D.

The main theorem we obtain (using the approach proposed in [2]) tells us that a set A is measurable if and only if its boundary has measure zero. We can use this result, which is crucial for proving measurability of many sets, to analyze in detail two subsets of $\mathbb{R}^2$: the unit ball and the triangle. For the latter we also provide an explicit formula, where, interestingly, the derivation we provide mimic the one usually proposed in Euclidean geometry.

In the last part of this work, we give a few examples of non-measurable sets, which indicate some limitations of the Jordan measure.

Chapter 2

Elementary sets

2.1 Definition and first properties

We'll start with some definitions, which are needed for the construction of the so-called *elementary sets*.

Definition 2.1 *An interval I is a subset of $\mathbb{R}$ of the form $[a, b] = \{x \in \mathbb{R} \,|a \leq x \leq b\}$, $[a, b) = \{x \in \mathbb{R} \,|a \leq x < b\}$, $(a, b] = \{x \in \mathbb{R} \,|a < x \leq b\}$, or $(a, b) = \{x \in \mathbb{R} \,|a < x < b\}$, where $a \leq b$. In that case, we call a and b the endpoints of I.*

Note that we allow $a = b$, which means in particular that the empty set $\emptyset = (a, a)$ and isolated points $\{a\} = [a, a]$, $a \in \mathbb{R}$, are also considered intervals. These two special cases of intervals are exactly the ones having zero *length*, which we define as follows:

Definition 2.2 *The length of an interval $I \subset \mathbb{R}$ with endpoints $a \leq b$ is defined by the quantity $|I| := b - a$.*

Remark 2.3 *To denote the cardinality of a set, we will always use $\#\{\cdot\}$. In particular, $|\,.\,|$ it's to be interpreted only in the sense of the previous definition and not as the cardinality mapping.*

We may notice that, as we would expect, the length of an interval is depending only on the endpoints, in particular $[a, b]$, $(a, b]$, $[a, b)$ and (a, b) have all the same length.

Using Cartesian products of intervals, we now can define boxes.

Definition 2.4 *A box B is a subset of $\mathbb{R}^d$ of the form $I_1 \times I_2 \times \cdots \times I_d$, where $I_1, ..., I_d$ are intervals*[1]*. Moreover, if each I_i has endpoints a_i and b_i we define the volume of B as $|B| := \prod_{i=1}^{d}(b_i - a_i)$.*

[1] Whenever we write $\mathbb{R}^d$, it's implicit that we're considering only the cases $d \geq 1$.

Note that $|B| = |I_1| \cdots |I_d|$ and that a 1-dimensional box is an interval with volume equal to its length.

We now move to the main object of this section, i.e. the class of sets that can be constructed using (finite) unions of boxes.

Definition 2.5 *$E \subset \mathbb{R}^d$ is an elementary set if it can be written as a finite union of boxes. We denote with $\mathcal{E}_d$ the class of all elementary sets in $\mathbb{R}^d$.*

When we try to combine elementary sets by taking (finitely many) unions and intersections, we notice that the resulting sets are still elementary. This fact is easy to visualize geometrically when $d = 2$ and can be proven for any dimension d, as we shall see in the next lemma. The idea of the proof is really to think as if d was equal to 2 and to use that boxes behave similarly in any dimension.

Lemma 2.6 (Boolean closure of $\mathcal{E}_d$) *Let $E, F \subset \mathbb{R}^d$ elementary sets, then $E \cup F$, $E \cap F$, $E \setminus F$ and $E \triangle F$ are also elementary, where $E \setminus F := \{x \in E \,|x \notin F\}$ and $E \triangle F := (E \setminus F) \cup (F \setminus E)$. Moreover, also the translation of E by $x \in \mathbb{R}^d$ $E + x := \{y + x \,|x \in E\}$ is an elementary set.*

Proof Let $E = B_1 \cup \cdots \cup B_n$ and $F = C_1 \cup \cdots \cup C_m$, both in $\mathcal{E}_d$. First, we want to prove that $E \cup F \in \mathcal{E}_d$. But $E \cup F = B_1 \cup \cdots \cup B_n \cup C_1 \cup \cdots \cup C_m$, which is a finite union of boxes, and so we directly get the aimed result.

We now consider $E \cap F$. Notice that

$$E \cap F = (B_1 \cup \cdots \cup B_n) \cap (C_1 \cup \cdots \cup C_m) = \bigcup_{i=1}^{n} B_i \cap (C_1 \cup \cdots \cup C_m)$$
$$= \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} B_i \cap C_j.$$

Therefore, it suffices to show that for any fixed $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$ the set $B_i \cap C_j$ is a box. Fix i and j and let's first assume that B_i and C_j are both closed, i.e. that $B_i \cap C_j$ can be written, for some $\{a_k, b_k, c_k, d_k\}_{1 \le k \le d}$, as

$$\left(\prod_{k=1}^{d} [a_k, b_k] \right) \cap \left(\prod_{k=1}^{d} [c_k, d_k] \right).$$

The latter is either the empty set or of the form

$$\prod_{k=1}^{d} [\max\{a_k, c_k\}, \min\{b_k, d_k\}], \tag{2.1}$$

and thus in particular a box. Consider now the general case, where we do not necessarily have closedness of the boxes. Then the intersection of B_i and C_j

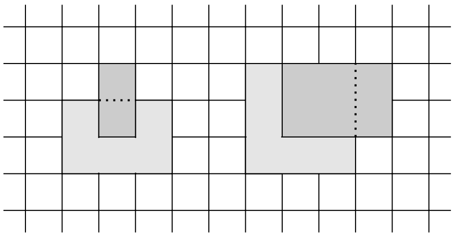

Figure 2.1: Possible shapes for $E \setminus F$.

can also be written as in 2.1, with the only difference that the corresponding intervals may be open/half-open/half-closed or empty. In particular, $B_i \cap C_j$ is a box.

We now want to show that $E \setminus F$ is elementary. There is a main difference to the previous part, which consists in the fact that the difference of two boxes is general not a box (while the intersection of two boxes is a box itself). Indeed, even in two dimensions, this set could be for example "U" or "L" shaped (see figure 2.1). A possible solution to this issue is to divide $E \cup F$ in a union of small enough sub-boxes. As first, we notice that, exactly like we did previously, we can reduce the problem to the case where E and F are boxes. Indeed

$$E \setminus F = \left(\bigcup_{i=1}^{n} B_i \right) \setminus \bigcap_{j=1}^{m} C_j = \left(\bigcup_{i=1}^{n} B_i \right) \cap \left(\bigcap_{j=1}^{m} C_j \right)^C = \bigcup_{i=1}^{n} \left(B_i \cap \bigcup_{j=1}^{m} C_j^C \right)$$
$$= \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \left(B_i \cap C_j^C \right),$$

and therefore we only need to prove that, for fixed i and j, $B_i \setminus C_j$ is elementary, since then the (above proven) closure of $\mathcal{E}_d$ under finite unions directly gives the result.

To this purpose we take $B_i \cup C_j$ and divide it in small disjoint boxes $D_1, ..., D_N$ which are either fully contained in $B_i \setminus C_j$ or fully contained in C_j (figure 2.1 could be helpful to convince yourself that this is always possible). But then $B_i \setminus C_j$ is a finite union of boxes, and thus elementary.

The claim that $E \triangle F := (E \setminus F) \cup (F \setminus E)$ is elementary follows directly from what was shown above, as it's a union of elementary sets.

Lastly, for $x = (x_1, ..., x_d)$, we notice that $E + x = (B_1 + x) \cup \cdots \cup (B_d + x)$. Moreover, for each $1 \leq i \leq d$, $B_i + x$ (where B_i has endpoints $\{a_k, b_k\}_{1 \leq k \leq b_k}$) is a product of intervals (with endpoints $\{a_k + x_k, b_k + x_k\}_{1 \leq k \leq b_k}$) and in particular a box. This concludes the proof. $\square$

There is a second result related to elementary sets which we would like to prove. This observation will allow us to define properly the measure of an elementary set.

Lemma 2.7 *Any elementary $E \subset \mathbb{R}^d$ can be written as a disjoint union of boxes.*

Proof The idea is to divide E is small enough boxes, similarly to what we did in the proof of the previous lemma. Following the approach used by Tao in lemma 1.1.2 of [6], we start with the case where the dimension d is equal to 1.

Let $E = I_1 \cup \cdots \cup I_n$, where each I_i has endpoints a_{2i-1} and a_{2i}. Write then the $2k$ endpoints $a_1, ..., a_{2k}$ in ascending order, that is $a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(2k)}$. Consider now a family of intervals $\{J_j\}_{j \in \{1,...,2k-1\}}$, where each J_j has endpoints $a_{(j)}$ and $a_{(j+1)}$. Then every I_i can be written as a disjoint union of the $J_1, ..., J_{2k-1}$, where for each j we only need to decide correctly which boundaries of J_j must be open and which closed. We can thus get a disjoint family of intervals whose union is exactly E.

Consider now the general case $d \geq 1$.

Let $E = B_1 \cup \cdots \cup B_n$, where $B_i = I_{i,1} \times \cdots \times I_{i,d}$, $i \in \{1, ..., k\}$. Now for each $j \in \{1, ..., d\}$ consider the family $I_{1,j}, ..., I_{n,j}$ and apply to it what we've just proven for the case $d = 1$. We thus get a family of disjoint intervals $J_{1,j}, ..., J_{\tilde{n},j}$ such that each $I_{i,j}$ can be written as a union of them. But then, by taking Cartesian product, we can express each B_k as a disjoint union of boxes of the form $J_{i_1,1} \times \cdots \times J_{i_d,d}$, $i_1, ..., i_d \geq 1$. Since the union of those disjoint boxes is exactly E, we conclude the proof. □

2.2 Measure of an elementary set

Now we want to give each elementary set a measure, and the idea is simply to extend the concept of "volume of boxes" to elementary sets.

Theorem 2.8 *Assume that $E \in \mathcal{E}_d$ can be written as disjoint union $B_1 \cup \cdots \cup B_n$. Let $m(E) = |B_1| + \cdots + |B_n|$, then $m : \mathcal{E}_d \to \mathbb{R}_+$ is a well defined map*[2]*, in the sense that is independent of the choice of the partition $B_1, ..., B_n$.*

Proof Let $B_1, ..., B_n$ and $C_1, ...C_m$ be boxes in $\mathbb{R}^d$ such that $E = B_1 \sqcup \cdots \sqcup B_n = C_1 \sqcup \cdots \sqcup C_m$ [3]. For $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$ we define $A_{i,j} := B_i \cap C_j$,

[2]The set $\mathbb{R}_+$ stands here for the non-negative real numbers.

[3]We will use many times the notation $X \sqcup Y$ for the union of two disjoint sets. Similarly, we write $X_1 \sqcup \cdots \sqcup X_n$ for a union of n pairwise disjoint sets.

which is a box as intersection of boxes. Observe then that

$$B_i = E \cap B_i = \left(\bigcup_{j=1}^{m} C_j \right) \cap B_i = \bigcup_{j=1}^{m} (C_j \cap B_i) = \bigcup_{j=1}^{m} A_{i,j} \,,$$
$$C_j = E \cap C_j = \left(\bigcup_{i=1}^{n} B_i \right) \cap C_j = \bigcup_{i=1}^{n} (B_i \cap C_j) = \bigcup_{i=1}^{n} A_{i,j}.$$

On the other hand, for any boxes $D, D_1, ..., D_k$ such that $D = D_1 \sqcup \cdots \sqcup D_k$ we have $|D| = |D_1| + \cdots + |D_k|$ [4]. This shows in particular, since the $\{A_{i,j}\}_{i,j}$ are disjoint, $|\bigcup_i A_{i,j}| = \sum_i |A_{i,j}|$ and $|\bigcup_j A_{i,j}| = \sum_j |A_{i,j}|$. Therefore, we obtain

$$\sum_{i=1}^{n} |B_i| = \sum_{i=1}^{n} \sum_{j=1}^{m} |A_{i,j}| = \sum_{j=1}^{m} \sum_{i=1}^{n} |A_{i,j}| = \sum_{j=1}^{m} |C_j|. \qquad \square$$

Now that we have a definition for the measure m, it's natural to ask ourselves what are the properties of such a mapping. In particular, we would like that, for boxes, m is consistent with the previously defined volume. At the same time, since we know that an object doesn't change its volume when translated, m should also satisfy translation invariance. These (and many other) properties are indeed true and summarized in the following theorem.

Theorem 2.9 *Let $E, F \in \mathcal{E}_d \subset \mathbb{R}^d$. The map $m : \mathcal{E}_d \to \mathbb{R}_+$ satisfies the following properties:*

(1) $m(\emptyset) = 0$.

(2) $m(B) = |B|$ for any box B.

(3) (Additivity) $m(E \sqcup F) = m(E) + m(F)$.

(4) (Monotonicity) If $E \subset F$ then $m(E) \leq m(F)$.

(5) (Subadditivity) $m(E \cup F) \leq m(E) + m(F)$.

(6) (Translation invariance) For any $x \in \mathbb{R}^d$, $m(E + x) = m(E)$.

(7) (Normalisation) $m([0,1)^d) = m([0,1]^d) = 1$.

Proof We start with (1), which is just a consequence of the empty set being a box with volume zero. For (2) we notice that B can be written as a disjoint union of boxes $B_1, ..., B_n$ by simply choosing $n = 1$ and $B_1 = B$, and so $m(B) = |B|$ by independence of the choice of the partition (theorem 2.8). Note that this also proves normalisation.

[4] This can be proved by using a subdivision, as we did for example in lemma 2.6.

Let's now prove the finite additivity. Write $E = B_1 \sqcup \cdots \sqcup B_n$ and $F = C_1 \sqcup \cdots \sqcup C_m$ as union of boxes. Then $E \sqcup F = B_1 \sqcup \cdots \sqcup B_n \sqcup C_1 \sqcup \cdots \sqcup C_m$ and so by the independence of the choice of the partition

$$m(E \cup F) = \sum_{i=1}^{n} |B_i| + \sum_{j=1}^{m} |C_j| = m(E) + m(F).$$

Monotonicity and subadditivity are direct consequences of additivity, since

$$m(F) = m((F \setminus E) \sqcup (F \cap E)) = m(F \setminus E) + m(F \cap E) \geq m(F \cap E) = m(E)$$

whenever $E \subset F$, and therefore

$$m(E \cup F) = m(F \sqcup (E \setminus F)) = m(F) + m(E \setminus F) \leq m(F) + m(E).$$

For the translation invariance we observe first that if E is a disjoint union of boxes, say $B_1 \sqcup \cdots \sqcup B_n$, then same holds for $E + x = (B_1 + x) \sqcup \cdots \sqcup (B_n + x)$. Moreover, for any box $B = \prod_{k=1}^{d}(a_k, b_k)$ we have

$$|B + (x_1, ..., x_d)| = \prod_{k=1}^{d}(b_k + x_k - (a_k + x_k)) = \prod_{k=1}^{d}(b_k - a_k) = |B|.$$

By combining both observations, we obtain $m(E + x) = \sum_{i=1}^{n} |B_i + x| = \sum_{i=1}^{n} |B_i| = m(E)$. □

The properties proved in the previous theorem are not very surprising and seem quite natural. In fact, the proof was straightforward. There is an other interesting feature: those properties uniquely define the measure m (actually a few of them are enough, see following theorem).

Theorem 2.10 *Let $\tilde{m} : \mathcal{E}_d \to \mathbb{R}_+$ be a map satisfying finite additivity, translation invariance and normalisation (as defined in theorem 2.9). Then $\tilde{m} = m$.*

Proof We follow the steps proposed in exercise 1.5 of [7]: we first prove that m and $\tilde{m}$ are equal for sets of the form $[p, q]^d,\ p, q \in \mathbb{Q}$, and then we extend this result to any box B by using density of $\mathbb{Q}^d$ in $\mathbb{R}^d$.

We fix $n \in \mathbb{N}$ and we observe that

$$[0, 1) = \bigsqcup_{j=1}^{n} \left[\frac{j-1}{n}, \frac{j}{n}\right), \tag{2.2}$$

$$[0, 1)^d = \bigsqcup_{j_1, ..., j_d \in \{1, ..., n\}} \left[\frac{j_1 - 1}{n}, \frac{j_1}{n}\right) \times \cdots \times \left[\frac{j_d - 1}{n}, \frac{j_d}{n}\right). \tag{2.3}$$

Moreover, by translation invariance we get, for all $j_1, ..., j_d \in \{1, ..., n\}$,

$$\tilde{m}\left(\left[0, \frac{1}{n}\right)^d\right) = \tilde{m}\left(\left[\frac{j_1 - 1}{n}, \frac{j_1}{n}\right) \times \cdots \times \left[\frac{j_d - 1}{n}, \frac{j_d}{n}\right)\right).$$

Since the disjoint union in 2.3 contains exactly n^d terms we have, by additivity of $\tilde{m}$,

$$\tilde{m}\left([0,1)^d\right) = n^d \cdot \tilde{m}\left(\left[0,\frac{1}{n}\right)^d\right). \tag{2.4}$$

Similarly, for any $k \in \mathbb{N}$

$$\tilde{m}\left(\left[0,\frac{k}{n}\right)\right) = k^d \cdot \tilde{m}\left(\left[0,\frac{1}{n}\right)^d\right) = \frac{k^d}{n^d} \cdot \tilde{m}\left([0,1)^d\right) = \frac{k^d}{n^d},$$

where we used 2.4 and normalisation of $\tilde{m}$. We conclude that for any $q \in \mathbb{Q}$ we have $\tilde{m}\left([0,q]^d\right) = q^d$.

At this point, for any $a_j, b_j \in \mathbb{Q}$ we can use translation invariance, with $x = (-a_1, ..., -a_d)$, to get that

$$\tilde{m}\left([a_1,b_1) \times \cdots \times [a_d,b_d)\right) = \prod_{j=1}^{d}(b_j - a_j).$$

Consider now a box $B = I_1 \times \cdots \times I_d$, where each interval I_j has endpoints $a_j < b_j$. We want to approximate this box by two other boxes having rational endpoints. Let thus $\varepsilon > 0$ arbitrary. First, by density of $\mathbb{Q} \subset \mathbb{R}$ we can find, for each $j \in \{1, ..., d\}$, points $a_j^-, a_j^+, b_j^-, b_j^+ \in \mathbb{R}$ such that

$$a_j^- < a_j < a_j^+, \quad b_j^- < b_j < b_j^+, \quad |a_j^\pm - a_j| < \varepsilon, \quad |b_j^\pm - b_j| < \varepsilon.$$

We then define $I_j^\pm := [a_j^\mp, b_j^\pm)$ and the two boxes $B^\pm := I_1^\pm \times \cdots \times I_d^\pm$. Now, since $B^- \subset B \subset B^+$, we get by monotonicity $\tilde{m}(B^-) \le \tilde{m}(B) \le \tilde{m}(B^+)$. Let's try to compute more explicitly what this inequality means:

$$\tilde{m}(B^+) = \prod_{i=1}^{d}(b_j^+ - a_j^-) \le \prod_{i=1}^{d}(b_j - a_j + 2\varepsilon)$$
$$\tilde{m}(B^-) = \prod_{i=1}^{d}(b_j^- - a_j^+) \ge \prod_{i=1}^{d}(b_j - a_j - 2\varepsilon)$$

which implies, for $M = \max_j \underbrace{(b_j - a_j)}_{:=m_j}$,

$$|\tilde{m}(B^\pm) - \prod_{i=1}^{d}(b_j - a_j)| \le 2\varepsilon \cdot M^{d-1} \cdot d. \tag{2.5}$$

For the last inequality we used the following general fact:

Claim 2.11 *Let* $B = \prod_{i=1}^{d}(a_i, b_i)$ *and* $B_\varepsilon = \prod_{i=1}^{d}(a_i + \varepsilon, b_i - \varepsilon)$ *for some* $\varepsilon > 0$. *Then* $m(B \setminus B_\varepsilon) \leq 2\varepsilon \cdot d \cdot M^{d-1}$, *where* $M := \max_{1 \leq i \leq d}(b_i - a_i)$. *Moreover,* $|D_n| \xrightarrow{n\to\infty} |B|$ *for any sequence of boxes* $\{D_n\}_{n\in\mathbb{N}}$ *satisfying that for all* $\varepsilon > 0$ *there exists a* $N \in \mathbb{N}$ *such that* $n \geq N$ *implies* $B_\varepsilon \subset D_n \subset B$.

Proof We first define, for $1 \leq i \leq d$, the sets

$$A_i := \bigcup_{i=1}^{d}(a_1, b_1) \times \cdots \times (a_{i-1}, b_{i-1}) \times [b_i - \varepsilon, b_i) \times (a_{i+1}, b_{i+1}) \times \cdots \times (a_d, b_d)$$

and

$$\tilde{A}_i := \bigcup_{i=1}^{d}(a_1, b_1) \times \cdots \times (a_{i-1}, b_{i-1}) \times (a_i, a_i + \varepsilon] \times (a_{i+1}, b_{i+1}) \times \cdots \times (a_d, b_d).$$

But since $B \setminus B_\varepsilon$ is contained in $\bigcup_{i=1}^{d}(A_i \cup \tilde{A}_i)$, we get that

$$m(B \setminus B_\varepsilon) \leq \sum_{i=1}^{d} 2\left(\varepsilon \cdot \prod_{j\neq i}(b_j - a_j)\right) \leq \sum_{i=1}^{d} 2\varepsilon \cdot M^{d-1} = d \cdot 2\varepsilon \cdot M^{d-1}.$$

For the second part of the claim we notice that, by additivity of m,

$$m(B) - m(B_\varepsilon) = m(B \setminus B_\varepsilon) \xrightarrow{\varepsilon\to 0} 0.$$

But for n large enough we have $B_\varepsilon \subset D_n \subset B$ and therefore, by monotonicity of m,

$$m(B) \geq \limsup_{n\to\infty} m(D_n) \geq \liminf_{n\to\infty} m(D_n) \geq m(B_\varepsilon).$$

By taking $\varepsilon \to 0$ we obtain the desired result. □

A clear consequence of 2.5 is that $m(B)$ and $\tilde{m}(B)$ are equal for any box B. The last step needed, in order to conclude the proof of the theorem, is to generalize this equality to any elementary set. But this can be done by simply using the definition of m and additivity of $\tilde{m}$. Indeed, for any partition of E into boxes, $m(E) = m(B_1) + \cdots + m(B_n) = \tilde{m}(B_1) + \cdots + \tilde{m}(B_n) = \tilde{m}(E)$. □

Remark 2.12 *Claim 2.11 is a very useful result, which we are going to use also in the next chapters. Note that, by reversing the roles of* B *and* B_ε, *one can show that the second part of the claim is true also if we have the condition* $B \subset D_n \subset B_{-\varepsilon}$ *instead of* $B_\varepsilon \subset D_n \subset B$.

2.3 The discretization formula

To compare the size of two finite sets the most natural tool is cardinality, which is however not very useful for comparing intervals, since in that case the only possible cardinalities are $0,1$ and ∞. To solve this problem, one can for example introduce the idea of volume, as we did in section 2.1. But is there a direct connection between the two concepts (cardinality and volume)? The answer is actually yes, and what is very surprising is the simplicity of the formula that allows this connection. Indeed, we have the following result.

Lemma 2.13 *Let $I \subset \mathbb{R}$ be an interval, then*

$$|I| = \lim_{n\to\infty} D_n(I) := \lim_{n\to\infty} \frac{1}{n}\#\left\{ I \cap \frac{\mathbb{Z}}{n} \right\},$$

where for $n \in \mathbb{N}$ we define $\frac{\mathbb{Z}}{n} = \{\frac{k}{n} | \, k \in \mathbb{Z}\}$.

Proof Let first take a look at a very simple case: $I = [0,1]$. Then we have that $\frac{1}{n}\#\{\frac{0}{n}, ..., \frac{n}{n}\} = \frac{n}{n+1}$ converges to 1, i.e. to the length of $[0,1]$. Consider now a general I with endpoints $a < b$ (the case $I = \{a\}$ is trivial). Then there exist $k, m \in \mathbb{Z}$ such that $\frac{k}{n} \notin I$ but $\frac{k+1}{n} \in I$, and $\frac{m}{n} \in I$ but $\frac{m+1}{n} \notin I$. In particular

$$|(b-a) - \frac{m-(k+1)}{n}| \le \frac{2}{n},$$

but also $D_n(I) = \frac{1}{n}\#\{\frac{k+1}{n}, ..., \frac{m}{n}\} = \frac{1}{n}\cdot(m-k)$. Therefore,

$$|D_n(I) - |I|| \le \left| \frac{m-(k+1)}{n} - |I| \right| + \left| D_n(I) - \frac{m-(k+1)}{n} \right| \le \frac{2+1}{n} \xrightarrow{n\to\infty} 0.$$

And we conclude the proof. □

This result can actually be extended to boxes and even elementary sets, as the next proposition shows.

Proposition 2.14 *Let $B \subset \mathbb{R}^d$ be a box and $E \subset \mathbb{R}^d$ an elementary set. Then*

$$\begin{aligned} |B| &= \lim_{n\to\infty} D_n(B) := \lim_{n\to\infty} \frac{1}{n^d}\#\left\{ B \cap \frac{\mathbb{Z}^d}{n} \right\}, \\ m(E) &= \lim_{n\to\infty} D_n(E) := \lim_{n\to\infty} \frac{1}{n^d}\#\left\{ E \cap \frac{\mathbb{Z}^d}{n} \right\}. \end{aligned} \tag{2.6}$$

Proof Let $B = I_1 \times \cdots \times I_d$, then $\#\{B \cap \frac{\mathbb{Z}^d}{n}\} = \prod_{i=1}^d \#\{I_i \cap \frac{\mathbb{Z}}{n}\}$ and therefore

$$D_n(B) = \prod_{i=1}^d \frac{\#\{I_i \cap \frac{\mathbb{Z}}{n}\}}{n} = \prod_{i=1}^d D_n(I_i) \xrightarrow{n\to\infty} \prod_{i=1}^d |I_i| = |B|$$

Take now E elementary, then we know that $m(E) = |B_1| + \cdots + |B_k|$ for some disjoint boxes $B_1, ..., B_k$ (where, as already said before, the choice of the boxes is arbitrary). Then for any i, j we have, by disjointness, $\#\{B_i\} + \#\{B_j\} = \#\{B_i \cup B_j\}$, and therefore

$$|B_1| + \cdots + |B_k| = \sum_{i=1}^{k} \lim_{n\to\infty} \frac{1}{n^d} \#\{B_i \cap \frac{\mathbb{Z}^d}{n}\} = \lim_{n\to\infty} \frac{1}{n^d} \sum_{i=1}^{k} \#\{B_i \cap \frac{\mathbb{Z}^d}{n}\}$$
$$= \lim_{n\to\infty} \frac{1}{n^d} \#\{E \cap \frac{\mathbb{Z}^d}{n}\}.$$

Note that we can interchange limit and sum as the latter has only finitely many terms. □

Remark 2.15 *Here we would like to point out that this result also provide an alternative proof that $m(E)$ is independent of the choice of the partition $B_1, ..., B_k$. Indeed, one sees that the right-hand side of 2.6 depends directly on E. In particular, one could use the discretization formula to provide an alternative definition of $m : \mathcal{E}_d \to \mathbb{R}^d$.*

Chapter 3

Definition and first properties of the Jordan measure

3.1 Area of the two-dimensional unit ball

In this first section, we want to see a possible way of approximating the area of the two-dimensional unit ball using elementary sets.

To this purpose we define, for $k \in \mathbb{N}$, the set

$$\mathcal{Q}_k := \left\{ q \in \mathbb{R} \,\middle|\, q = \frac{i}{2^k} \text{ for some } i \in \mathbb{Z} \right\}.$$

We then define the family of all cubes having vertices in $\mathcal{Q}_k$ and side length $\frac{1}{2^k}$, that is

$$\mathcal{D}_k := \left\{ \left(p, q + \frac{1}{2^k} \right) \times \left(p, q + \frac{1}{2^k} \right) \,\middle|\, p, q \in \mathcal{Q}_k \right\}.$$

It's clear that for every $k \in \mathbb{N}$ and each $x \in \mathcal{Q}_k + (\frac{1}{2^{k+1}}, \frac{1}{2^{k+1}})$ there exists exactly one cube D_k in $\mathcal{D}_k$ having center in x. We are going to denote this unique cube as $D_k(x)$ (see figure 3.1).

We want to approximate the area of the closed unit ball[1] $\overline{B} \subset \mathbb{R}^2$ from the inside, by using the previously constructed cubes. We can start with $D_1(0)$, which is of course not a very precise approximation. Let's thus use cubes of side length $\frac{1}{2}$. Then the best we can do is to take

$$A_1 = D_2\left(\frac{1}{4}, \frac{1}{4}\right) \cup D_2\left(-\frac{1}{4}, \frac{1}{4}\right) \cup D_2\left(\frac{1}{4}, -\frac{1}{4}\right) \cup D_2\left(-\frac{1}{4}, -\frac{1}{4}\right),$$

[1] From now on we will use for open and closed unit balls the notations $B := \{x \in \mathbb{R}^d |\ \|x\| < 1\}$ and $\overline{B} := \{x \in \mathbb{R}^d |\ \|x\| \le 1\}$. Similarly for $y \in \mathbb{R}^d$ and $r > 0$ we write $B_r(y) := \{x \in \mathbb{R}^d |\ \|x - y\| < r\}$ and $\overline{B_r(y)} := \{x \in \mathbb{R}^d |\ \|x - y\| \le r\}$, as well as $B_r := B_r(0)$ and $\overline{B_r} := \overline{B_r(0)}$.

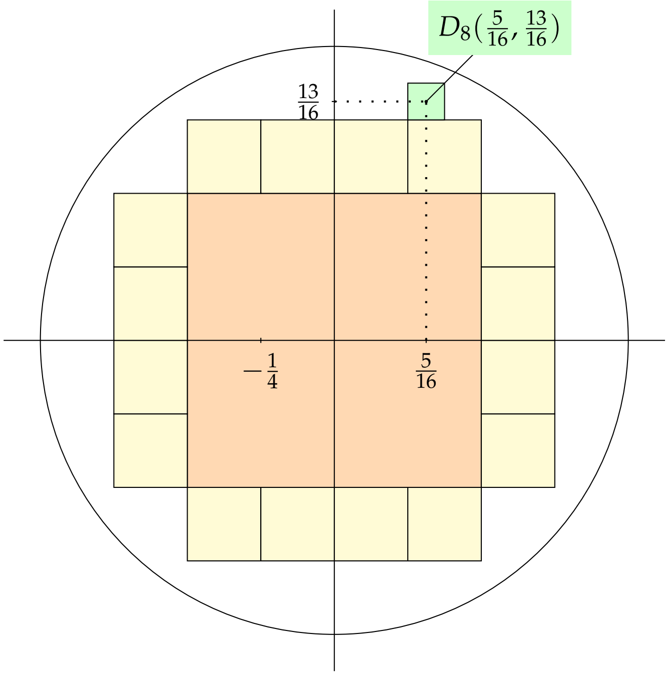

Figure 3.1: In orange the set A_1, in yellow the set $A_2 \setminus A_1$, and in green an example for a cube $D_k(x)$.

which is still equal to $D_1(0)$. Things start to improve when we move to cubes with side length $\frac{1}{4}$, since now we get $A_2 = A_1 \cup (A_2 \setminus A_1)$, where the part we're adding to the previous approximation, $A_2 \setminus A_1$, has area $4 \cdot \frac{1}{4} = 1$ (see figure 3.1). One continue this procedure inductively, getting for each $k \geq 1$ an elementary set A_k (containing A_{k-1}) which is a disjoint union of cubes in $\mathcal{D}_k$.

Remark 3.1 *Formally, we could define A_k as the union of all sets D_k that lies in the family $\{D_k \in \mathcal{D}_k | D_k \subset \overline{B}\}$.*

The $\{A_k\}_{k\geq 1}$ are elementary, but there is still an issue that we would like to solve: these sets are not closed and in particular any point that lies on the boundary of the cubes composing A_k is itself not in A_k. We thus define $C_k := \overline{A_k}$, which is still an elementary set. We now claim that we can quantify explicitly the accuracy of this approximation.

Claim 3.2 *Let $k \geq 1$ and $r_k = 1 - \frac{\sqrt{2}}{2^k}$. Consider the closed balls $\overline{B_{r_k}} := \{x \in \mathbb{R}^2 | \; \|x\| \leq r_k\}$. Then*

$$\overline{B_{r_k}} \subset C_k \subset \overline{B} \subset \frac{1}{r_k} C_k.$$

Proof We start with the inclusion $\overline{B_{r_k}} \subset C_k$. Pick any point $x = (x_1, x_2) \in \overline{B_{r_k}}$, and let $a_i = \max\{a \in \mathbb{Z} | \frac{a}{2^k} \le x_i\}$, $i \in \{1, 2\}$. Notice that any cube in $\mathcal{D}_k$ has a diagonal of length $\sqrt{2}/2^k$, i.e. exactly $1 - r_k$. In particular we deduce that $D := (\frac{a_1}{2^k}, \frac{a_1+1}{2^k}) \times (\frac{a_2}{2^k}, \frac{a_2+1}{2^k})$ is a cube fully contained in B, and hence one of the cubes composing A_k. But since x, which is contained in $\overline{D}$, was arbitrary, we get that $\overline{B_{r_k}} \subset C_k$. On the other hand, the inclusion we have just proved also implies $\overline{B} = \frac{1}{r_k}\overline{B_{r_k}} \subset \frac{1}{r_k}\overline{C_k} = \frac{1}{r_k}C_k$, and we conclude. □

Remark 3.3 *Note that, as k goes to infinity, r_k converges to 1. In particular, we expect to see $|m(C_k) - m(\tilde{C}_k)| \xrightarrow{k \to \infty} 0$, for $\tilde{C}_k = \frac{1}{r_k}C_k$. We will see a formal proof of that in section 3.3.*

3.2 The Jordan measure

The example in the previous section can be seen as a motivation for the following definitions, which describe a natural way of defining the Jordan measure on $\mathbb{R}^d$.

Definition 3.4 *Let $A \subset \mathbb{R}^d$ be a bounded set.*

- *The Jordan inner measure $\underline{J}(A)$ of A is defined as*

$$\underline{J}(A) = \sup\{m(E) |\ E \in \mathcal{E}_d, E \subset A\}.$$

- *The Jordan outer measure $\overline{J}(A)$ of A is defined as*

$$\overline{J}(A) = \inf\{m(F) |\ F \in \mathcal{E}_d, A \subset F\}.$$

Definition 3.5 *We say that a bounded set $A \subset \mathbb{R}^d$ is Jordan measurable if $\underline{J}(A) = \overline{J}(A)$, and it that case we call $J(A) = \underline{J}(A) = \overline{J}(A)$ the Jordan measure of A. Let $\mathcal{J}_d$ be the family of all Jordan measurable sets in $\mathbb{R}^d$, then J is the map*

$$\begin{aligned} J : \mathcal{J}_d &\to \mathbb{R}_+ \\ A &\mapsto J(A). \end{aligned}$$

Before saying anything about J we would like to observe a few properties of $\underline{J}$ and $\overline{J}$.

Lemma 3.6 *$\underline{J}$ and $\overline{J}$ are monotone. Moreover, $\overline{J}$ is subadditive.*

Proof Consider $A \subset B \subset \mathbb{R}^d$. Any elementary set E contained in A satisfies $E \subset B$ and therefore, by taking the supremum as in the definition of $\underline{J}$,

$$\sup\{m(E) |\ E \in \mathcal{E}_d, E \subset A\} \le \sup\{m(E) |\ E \in \mathcal{E}_d, E \subset B\}.$$

Similarly, any F containing B satisfies $A \subset F$ and so, by taking the infimum as in the definition of $\overline{J}$,

$$\inf\{m(F)|\ F \in \mathcal{E}_d, A \subset F\} \leq \inf\{m(E)|\ F \in \mathcal{E}_d, B \subset F\}.$$

This proves monotonicity of the inner and outer Jordan measures. Now, for the second claim, consider $C, D \subset \mathbb{R}^d$ arbitrary and elementary sets G, H with $C \subset G$ and $D \subset H$. By subadditivity of m

$$\overline{J}(C \cup D) \leq m(G \cup H) \leq m(G) + m(H),$$

and by taking the infimum as in the definition of $\overline{J}(C)$ and $\overline{J}(D)$ we get $\overline{J}(C \cup D) \leq \overline{J}(C) + \overline{J}(D)$. □

It's difficult to imagine, directly from definition, how $\mathcal{J}_d$ looks like, and in particular it's not clear which sets exactly are Jordan measurable. A useful tool in that sense is the following theorem, which can be interpreted as "measurable sets are almost elementary". We would like also to point out that we can see, directly from definition 3.4, that elementary sets are Jordan measurable (since both the supremum and infimum in the definition are taken at the set A itself).

Theorem 3.7 (Characterization of Jordan measurability) *Let $A \subset \mathbb{R}^d$ be a bounded set. Then the following are equivalent:*

(1) A is Jordan measurable.

(2) For every $\varepsilon > 0$, there exist elementary sets E and F such that $E \subset A \subset F$ and $m(F \setminus E) \leq \varepsilon$.

(3) For every $\varepsilon > 0$, there exists an elementary set $E \subset A$ such that $\overline{J}(A \setminus E) \leq \varepsilon$.

(4) For every $\varepsilon > 0$, there exists an elementary set E such that $\overline{J}(E \triangle A) \leq \varepsilon$.

Proof We are going to prove $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ [2]. Let $\varepsilon > 0$.

(1) ⇒ (2): By definition of $\underline{J}(A)$ there exists $E \subset A$ elementary such that $m(E) > \underline{J}(A) - \frac{\varepsilon}{2}$, and similarly $F \supset A$ elementary such that $m(F) < \overline{J}(A) + \frac{\varepsilon}{2}$. Then, since A Jordan measurable,

$$m(F) < \overline{J}(A) + \frac{\varepsilon}{2} = \underline{J}(A) + \frac{\varepsilon}{2} < m(E) + 2 \cdot \frac{\varepsilon}{2}.$$

Moreover, $E \subset F$ and so by theorem 2.9 we have $m(F) = m(E) + m(F \setminus E)$. Combining both observations, we get $m(F \setminus E) < \varepsilon$.

(2) ⇒ (3): Let E, F as in (2). Since $A \setminus E \subset F \setminus E$ we get, by definition of $\overline{J}$, $\overline{J}(A \setminus E) \leq m(F \setminus E) < \varepsilon$.

[2]For the first three implications we follow the proof proposed for exercise 2.2 in [7].

<u>**(3) ⇒ (4):**</u> This implication is direct since for E as in (3) holds $A \triangle E = (A \setminus E) \cup \emptyset$.

<u>**(4) ⇒ (1):**</u> For E as in (4) we have $\overline{J}(A \setminus E) + \overline{J}(E \setminus A) < \varepsilon$, and so in particular $\overline{J}(A \setminus E) < \varepsilon$ and $\overline{J}(E \setminus A) < \varepsilon$. By definition of $\overline{J}$ we can find elementary F and G such that $(A \setminus E) \subset F$, $(E \setminus A) \subset G$ and

$$m(F) \leq \overline{J}(A \setminus E) + \varepsilon \leq 2\varepsilon,$$
$$m(G) \leq \overline{J}(E \setminus A) + \varepsilon \leq 2\varepsilon.$$

But then the elementary sets $H := E \setminus (F \cup G)$ and $K := F \cup G \cup H$ satisfy $H \subset A \subset K$ and

$$m(K) - m(H) = m(F \cup G) \leq m(F) + m(G) \leq 4\varepsilon,$$

which, by taking the limit $\varepsilon \to 0$, concludes the proof. □

Now we would like to analyze more in detail Jordan measurable sets. With the following few results we are going to show that $\mathcal{J}_d$ satisfies properties which are similar to those of $\mathcal{E}_d$, and that in particular we can find statements that are analogous to lemma 2.6 and theorems 2.9, 2.10.

Lemma 3.8 (Boolean closure of $\mathcal{J}_d$) *Assume $A, B \in \mathcal{J}_d$, then $A \cap B$, $A \cup B$, $A \setminus B$ and $A \triangle B$ are in $\mathcal{J}_d$ too.*

Proof All the claims are direct consequences of lemma 2.6, and to prove them we're going to use the representations of measurability (3) and (4) of theorem 3.7. Fix $\varepsilon > 0$ arbitrary. Let E, F elementary such that $E \subset A$, $F \subset B$ and $\overline{J}(A \setminus E) < \varepsilon$, $\overline{J}(B \setminus F) < \varepsilon$. Then by subadditivity of $\overline{J}$:

$$\begin{aligned}
\overline{J}((A \cup B) \setminus (E \cup F)) &\leq \overline{J}(A \setminus (E \cup F)) + \overline{J}(B \setminus (E \cup F)) \\
&\leq \overline{J}(A \setminus E) + \overline{J}(B \setminus F) < 2\varepsilon, \\
\overline{J}((A \cap B) \setminus (E \cap F)) &\leq \overline{J}((A \cap B) \setminus E) + \overline{J}((A \cap B) \setminus F) \\
&\leq \overline{J}(A \setminus E) + \overline{J}(B \setminus F) < 2\varepsilon.
\end{aligned}$$

Which, by (3) in theorem 3.7, shows that $A \cup B$ and $A \cap B$ are in $\mathcal{J}_d$.

Moreover, regarding $A \setminus B$, we have

$$\begin{aligned}
(A \setminus B) \triangle (E \setminus F) &= ((A \setminus B) \setminus (E \setminus F)) \cup ((E \setminus F) \setminus (A \setminus B)) \\
&\subset ((A \setminus B) \setminus (E \setminus B)) \cup ((E \setminus F) \setminus (E \setminus B)) \\
&\subset (A \setminus E) \cup (B \setminus F).
\end{aligned}$$

Thus by subadditivity and monotonicity $\overline{J}((A \setminus B) \triangle (E \setminus F)) < 2\varepsilon$, and we conclude by (4) in theorem 3.7. The case $A \triangle B$ follows exactly as in lemma 2.6. □

Theorem 3.9 *Let $A, B \subset \mathbb{R}^d$ be Jordan measurable sets. Then:*

(1) *(Additivity) If A, B disjoint, then $J(A \cup B) = J(A) + J(B)$.*

(2) *(Monotonicity) If $A \subset B$, then $J(A) \leq J(B)$.*

(3) *(Subadditivity) $J(A \cup B) \leq J(A) + J(B)$.*

(4) *(Translation invariance) For any $x \in \mathbb{R}^d$, $A + x \in \mathcal{J}_d$ and $J(A + x) = J(A)$.*

Proof As in theorem 2.9, the key part is to show additivity, since then monotonicity and subadditivity follow directly:

$$J(B) \overset{(1)}{=} J(B \setminus A) + J(B \cap A) \geq 0 + J(A), \text{ if } A \subset B,$$
$$J(A \cup B) \overset{(1)}{=} J(B \setminus A) + J(A) \overset{(2)}{\leq} J(B) + J(A).$$

Let thus $A, B \subset \mathcal{J}_d$ be disjoint and $\varepsilon > 0$. Then we can find elementary sets E_-, E_+, F_-, F_+ such that $E_- \subset A \subset E_+$, $F_- \subset B \subset F_+$ and

$$J(A) - \varepsilon \leq m(E_-) \leq J(A) \leq m(E_+) \leq J(A) + \varepsilon,$$
$$J(B) - \varepsilon \leq m(F_-) \leq J(B) \leq m(F_+) \leq J(B) + \varepsilon.$$

Then, since $E_- \sqcup F_- \subset A \sqcup B \subset E_+ \sqcup F_+$,

$$J(A \cup B) \geq m(E_- \sqcup F_-) \overset{2.9}{=} m(E_-) + m(F_-) \geq J(A) + J(B) - 2\varepsilon,$$
$$J(A \cup B) \leq m(E_+ \sqcup F_+) \overset{2.9}{\leq} m(E_+) + m(F_+) \leq J(A) + J(B) + 2\varepsilon.$$

By taking the limit $\varepsilon \to 0$ we obtain the desired additivity.

We are left with translation invariance, which is a direct consequence of the translation invariance of m. Indeed, for $\varepsilon > 0$ and $E_\pm$ as above,

$$\overline{J}(A + x) - \varepsilon \leq m(E_+ + x) - \varepsilon \overset{2.9}{=} m(E_+) - \varepsilon \leq J(A) \leq$$
$$\leq m(E_-) + \varepsilon \overset{2.9}{=} m(E_- + x) + \varepsilon \leq \underline{J}(A + x) + \varepsilon.$$

Taking the limit $\varepsilon \to 0$ gives both measurability and the desired equality. $\square$

At this point it may seem natural to claim that not only m (see theorem 2.10), but also J are unique when we require additivity, non-negativity, translation invariance and normalisation. This assertion is indeed true, and in section 4.2 we will see how it can be proved.

3.3 The d-dimensional unit ball

Let's go back to the unit ball example seen in section 3.1. Recall that for $\overline{B} = \{x \in \mathbb{R}^2 | \ \|x\| \leq 1\}$ we were able to find sequences of elementary sets $\{C_k\}_{k\in\mathbb{N}}, \{\tilde{C}_k\}_{k\in\mathbb{N}}$ such that for all k we have $C_k \subset \overline{B} \subset \tilde{C}_k$ and $\tilde{C}_k = \frac{1}{r_k} C_k$. In order to show that $\overline{B}$ is Jordan measurable it would be enough to prove

$$m\left(\frac{1}{r_k} C_k\right) = \frac{1}{r_k^2} m(C_k),$$

since then we can use that $r_k = 1 - \frac{\sqrt{2}}{2^k} \xrightarrow{k\to\infty} 1$ to conclude $\underline{J}(\overline{B}) = \overline{J}(\overline{B})$ (by definition). As one may expect, we can actually prove a more general result, telling us that homothetic transformations $\psi_{\lambda,a}(x) = \lambda x + a$, $a \in \mathbb{R}^d, \lambda > 0$, preserve measurability and stretch the Jordan measure by a factor λ^d, that is

$$J(\psi_{\lambda,a}(A)) = \lambda^d J(A). \tag{3.1}$$

The key observation is that the family of elementary sets contained in $A \subset \mathbb{R}^d$

$$\{E \subset A \mid E \text{ elementary }\}$$

is in bijection with the family of elementary sets contained in $\lambda A + a$

$$\{F \subset \lambda A + a \mid F \text{ elementary }\}$$

through the map $E \mapsto \lambda E + a$. By taking infimum and supremum we get that, assuming A to be measurable, the upper and lower Jordan measure of $\lambda A + a$ coincide. Moreover, $m(\lambda E) = \lambda^d m(E)$ for every elementary $E = I_1 \times \cdots \times I_d$, since the endpoints of each I_i are stretched by a factor λ, and so, because of the translation invariance of m, we conclude the desired property 3.1.

We now want to generalize the derivation we saw in section 3.1 to the more general case of the d-dimensional unit ball. We proceed in the exact same way as before, as we shall see in the following.

Claim 3.10 $\overline{J}(\overline{B}) = \underline{J}(\overline{B})$.

Proof Recall the notation $B_r(y) = \{x \in \mathbb{R}^d | \ \|x - y\| < r\}$ for the open ball of radius r with center $x \in \mathbb{R}^d$. Let $\mathcal{Q}_k := \{q \in \mathbb{R} | \ q = \frac{i}{2^k} \text{ for some } i \in \mathbb{Z}\}$ as defined before, and

$$\mathcal{D}_k := \left\{ D_k = \prod_{j=1}^d I_j \ \middle| \ I_j = \left(p_j, p_j + \frac{1}{2^k}\right),\ p_j \in \mathcal{Q}_k \right\},$$
$$A_k := \{D_k \in \mathcal{D}_k \mid D_k \subset B\}.$$

With this setup we now take $k \in \mathbb{N}$ large enough so that $\delta_k := \sqrt{n}/2^k < 1$ and fix the radius $r_k := 1 - \delta_k = 1 - \sqrt{n}/2^k$. We then claim that for each

$x = (x_1, ..., x_n) \in \overline{B_{r_k}}$ the box $D := \prod(x_i - \frac{1}{2^k}, x_i + \frac{1}{2^k})$ is fully contained in B. But this is indeed a direct consequence of D having diagonal $diag(D) = \max_{x,y \in D} \|x - y\| = \delta_k$. This allows us to find a $D_k \in \mathcal{D}_k$ such that $x \in \overline{D_k}$: for each $1 \leq i \leq d$ let $a_i = \max\{a \in \mathbb{Z} | \ \frac{a}{2^k} \leq x_i\}$, then

$$\left(\frac{a_i}{2^k}, \frac{a_i + 1}{2^k}\right) \subset \left(x_i - \frac{1}{2^k}, x_i + \frac{1}{2^k}\right),$$

and thus $D_k := \prod_i \left(\frac{a_i}{2^k}, \frac{a_i+1}{2^k}\right) \subset B$. Therefore, since x was arbitrary, $\overline{B_{r_k}} \subset C_k := \overline{A_k}$ and we get the exact same result as in 3.2:

$$\overline{B_{r_k}} \subset C_k \subset \overline{B} \subset \frac{1}{r_k} C_k.$$

The latter can be used to get

$$\underline{J}(\overline{B}) \geq \sup_{k \in \mathbb{N}} m(C_k) = \lim_{k \to \infty} m(C_k) = \lim_{k \to \infty} \frac{1}{r_k^d} m(C_k) \overset{3.1}{=} \lim_{k \to \infty} m(\frac{1}{r_k} C_k) \geq \overline{J}(\overline{B}).$$

But since by definition $\underline{J}(\overline{B}) \leq \overline{J}(\overline{B})$, we conclude $\underline{J}(\overline{B}) = \overline{J}(\overline{B})$. □

Chapter 4

Jordan measurability

4.1 Characterization of the Jordan measure using cubes

The above example of the $d-$dimensional unit ball may suggest that, when checking measurability of a set, we do not really need to work with the whole range of elementary sets: it might be enough to look only at finite disjoint unions of cubes with fixed side length. The purpose of this section is to show how this can be done and what are the consequences of this procedure.

The construction we're proposing in the next two sections is mainly based on the approach used by Laczkovich et al. in chapter 3 of [2]. However, some of the definitions (and most of the proofs) are modified in a way that allows the reader to connect them better to the content of the previous chapters.

We'll start with a generalization of the sets $\mathcal{D}_k$ seen in section 3.3.

Definition 4.1 *For $n \in \mathbb{N}$ we let $\mathcal{K}_n$ be the family of closed cubes in $\mathbb{R}^d$ having side length $\frac{1}{n}$ and vertices in $\frac{\mathbb{Z}}{n}$.*

The first result we present, which follows the observations in section 3.1 of [2], regards the partition of $\mathcal{K}_n$ into three groups.

Claim 4.2 *Let $A \subset R^d$ be a bounded set. Then each $K \in \mathcal{K}_n$ is in exactly one of the following families:*

- *K is an interior cube, that is $K \subset \mathring{A}$.*
- *K is an exterior cube, that is $K \cap \overline{A} = \emptyset$.*
- *K is a boundary cube, that is $K \cap \partial A \neq \emptyset$.*

In particular, we say that K is an interior or boundary cube of A if $K \cap \overline{A} \neq \emptyset$.

Proof We would like to show that K being a boundary cube is the same as being neither an interior nor an exterior cube. The fact that K being a boundary cube implies that it cannot be neither an interior nor an exterior cube it's clear. For the other implication, assume by contradiction that exists $K \subset \mathring{A} \cup \mathring{A}^c$ such that K is neither an interior nor an exterior cube[1]. Then $\exists x, y \in K$ such that $x \in \mathring{A}, y \in \mathring{A}^c$. But by convexity of K we get that the segment from x to y is contained in K and therefore also the point $z = x + t_0(y - x)$, where $t_0 = \sup\{t \geq 0 | \; z = x + t(y - x) \in A\}$. But any open ball around z intersects both $\mathring{A}$ and $\mathring{A}^c$ and so $z \in \partial A$, which contradicts the condition $K \subset \mathring{A} \cup \mathring{A}^c$. □

Now, for any bounded $A \subset \mathbb{R}^d$ we define $\underline{J_n}(A)$ as $\frac{1}{n^d}$ times the number of interior cubes $K \in \mathcal{K}_n$ of A, and similarly we define $\overline{J_n}(A)$ as $\frac{1}{n^d}$ times the number of cubes $K \in \mathcal{K}_n$ which are interior or boundary cubes, that is

$$\underline{J_n}(A) = \frac{1}{n^d} \sum_{\substack{K \in \mathcal{K}_n \\ K \subset \mathring{A}}} 1, \quad \overline{J_n}(A) = \frac{1}{n^d} \sum_{\substack{K \in \mathcal{K}_n \\ K \cap \overline{A} \neq \emptyset}} 1.$$

Remark 4.3 *Let $A_{(n)} := \{K \in \mathcal{K}_n | \; K \subset \mathring{A}\}$ be the elementary set given by the union of all interior cubes of A, then*

$$|\underline{J_n}(A) - J(A_{(n)})| = |\sum_{\substack{K \in \mathcal{K}_n \\ K \subset \mathring{A}}} |K| - J(\bigcup_{\substack{K \in \mathcal{K}_n \\ K \subset \mathring{A}}} K)| \leq const(d) \cdot J(\bigcup_{\substack{K \in \mathcal{K}_n \\ K \subset \mathring{A}}} \partial K)$$
$$\leq const(d) \cdot \sum_{\substack{K \in \mathcal{K}_n \\ K \subset \mathring{A}}} J(\partial K) = 0,$$

since, for each K, ∂K is a union of boxes with zero volume. Similarly, for $A^{(n)} := \{K \in \mathcal{K}_n | \; K \cap \overline{A} \neq \emptyset\}$ we have $\overline{J_n}(A) = J(A^{(n)})$.

Note moreover that $A \subset B \subset \mathbb{R}^d$ implies $A^{(n)} \subset B^{(n)}$, and therefore we can already see that both $\overline{J_n}$ and $\underline{J_n}$ are monotone.

The constructed mapping $\underline{J_n}$ and $\overline{J_n}$ are exactly the formalization of the procedure used in section 3.3 to check measurability of the unit ball[2]. With the following results we want to show that this approach works actually for any bounded set, in the sense that the obtained sequences converge, as n goes to infinity, to the inner and outer Jordan measures.

Lemma 4.4 *For any closed box B we have that*

$$\lim_{n \to \infty} \underline{J_n}(B) = \lim_{n \to \infty} \overline{J_n}(B) = |B|.$$

[1]Here we write $\mathring{A}^c$ for the set of points in the interior of $A^c = \mathbb{R}^d \setminus A$.

[2]Note that actually in section 3.3 we were considering the sequences $\{\underline{J_{2^n}}\}_n$ and $\{\overline{J_{2^n}}\}_n$, however, since we are only interested in the limit as n goes to infinity, this is the same as considering the sequences $\{\underline{J_n}\}_n$ and $\{\overline{J_n}\}_n$.

Proof The proof, whose first part is taken from lemma 3.3 in [2], has some analogies with the proof of the discretization formula 2.13, and indeed the two results are strictly related[3]. Let $B = \prod_{i=1}^{d}[a_i, b_i]$ and $B^{(n)}, B_{(n)}$ as defined in remark 4.3, and for each $1 \le i \le d$ let $p_i, q_i \in \mathbb{Z}$ such that $\frac{p_i - 1}{n} < a_i \le \frac{p_i}{n}$ and $\frac{q_i - 1}{n} \le b_i < \frac{q_i}{n}$. We notice that a cube $K = \prod_{i=1}^{d}[\frac{k_i}{n}, \frac{k_i+1}{n}]$ is an interior or boundary cube if and only if $p_i \le k_i + 1 \le q_i$ for all $1 \le i \le d$. In particular, $B^{(n)}$ and $B_{(n)}$ are boxes of the form $B_{(n)} = \prod_{i=1}^{d}(a_i + \varepsilon_i, b_i - \varepsilon_i)$, $B^{(n)} = \prod_{i=1}^{d}(a_i - \delta_i, b_i + \delta_i)$ for some $\varepsilon_1, ..., \varepsilon_d, \delta_1, ..., \delta_d \le \frac{1}{n}$. We can therefore apply claim 2.11 with $\{D_n\}_n = \{B_{(n)}\}_n$ and remark 2.12 with $\{D_n\}_n = \{B^{(n)}\}_n$, giving us $\lim_{n\to\infty} J(B_{(n)}) = \lim_{n\to\infty} J(B^{(n)}) = |B|$. We conclude by remark 4.3. □

Before proving the aimed convergence of $\underline{J_n}$ and $\overline{J_n}$ we need find a way to describe the Jordan measure using only closed boxes, which can be done as follows.

Lemma 4.5 *For any bounded $A \in \mathbb{R}^d$ we have*

$$\overline{J}(A) = \inf\left\{ \sum_{i=1}^{N} |K_i| \;\middle|\; A \subset \bigcup_{i=1}^{N} K_i,\ K_1, ..., K_N \text{ closed boxes} \right\}, \tag{4.1}$$

$$\underline{J}(A) = \inf\left\{ \sum_{i=1}^{N} |K_i| \;\middle|\; \bigcup_{i=1}^{N} K_i \subset A,\ K_1, ..., K_N \text{ closed boxes s.t.} \mathring{K}_1, ..., \mathring{K}_N \text{ disjoint} \right\}.$$

Proof We start with the first equality. Let $\overline{\mu}(A)$ be the right-hand side and note that, by definition 3.4 and lemma 2.7,

$$\overline{J}(A) = \inf\left\{ \sum_{i=1}^{N} |B_i| \;\middle|\; A \subset \bigcup_{i=1}^{N} B_i,\ B_1, ..., B_N \text{ disjoint boxes} \right\}.$$

Let now $B_1, ..., B_N$ be disjoint boxes whose union contains A. Then in particular $A \subset \cup_i \overline{B_i}$ and choosing $K_i = B_i$, $1 \le i \le N$, gives $\sum_i |B_i| = \sum_i |\overline{B_i}| = \sum_i |K_i| \ge \overline{\mu}(A)$. Since the $B_1, ..., B_N$ were arbitrary, we get $\overline{J}(A) \ge \overline{\mu}(A)$. On the other hand, for any family of closed boxes $K_1, ..., K_N$ with $A \subset \cup_i K_i =: E$ we can write E as disjoint union of boxes $B_1, ..., B_M$ (by lemma 2.7). Therefore $\sum_i |B_i| = m(E) \le \sum_i |K_i|$ and since the $K_1, ..., K_N$ were arbitrary we get $\overline{J}(A) \le \overline{\mu}(A)$. Thus, the first of the two equalities is proven.

For the second one, denote the right-hand side with $\underline{\mu}(A)$ and recall the definition of the inner Jordan measure

$$\underline{J}(A) = \sup\left\{ \sum_{i=1}^{N} |B_i| \;\middle|\; \bigcup_{i=1}^{N} B_i \subset A,\ B_1, ..., B_N \text{ disjoint boxes} \right\}.$$

[3] Specifically, cubes in $\mathcal{K}_n$ have vertices in $\frac{\mathbb{Z}}{n}$, which is exactly the set used for stating the discretization formula.

Let $K_1, ..., K_N$ be closed boxes contained in A with pairwise disjoint interiors. Then also the disjoint boxes $\{B_i\}_{1\le i\le N} := \{\mathring{K}_i\}_{1\le i\le N}$ are contained in A, where $\sum_i |B_i| = \sum_i |K_i|$. Therefore $\sum_i |K_i| \le \underline{J}(A)$ and, since the $K_1, ...K_N$ were arbitrary, $\underline{\mu}(A) \le \underline{J}(A)$. Finally, let $B_1, ..., B_N$ be disjoint boxes contained in A and $E := \sqcup_i B_i$. For any $\varepsilon > 0$ we can find closed boxes $K_i \subset B_i$ such that $|K_i| \ge |B_i| - \frac{\varepsilon}{N}$. In particular, the sets $\mathring{K}_1, ..., \mathring{K}_N$ are disjoint and satisfy $\cup_i K_i \subset A$, as well as $\sum_i |K_i| \ge m(E) - \varepsilon$. By taking the limit $\varepsilon \to 0$ we obtain $m(E) \le \underline{\mu}(A)$ and by arbitrariness of the $B_1, ..., B_N$ we conclude $\underline{J}(A) \le \underline{\mu}(A)$ □

Theorem 4.6 *For any bounded set A*

$$\lim_{n\to\infty} \overline{J_n}(A) = \overline{J}(A), \tag{4.2}$$

$$\lim_{n\to\infty} \underline{J_n}(A) = \underline{J}(A). \tag{4.3}$$

Proof For the proof we are going to follow the structure proposed for theorem 3.4 in [2]. We start with equation 4.2. Since, for all $n \in \mathbb{N}$, $\overline{J_n}(A) = J(A^{(n)})$ and $A^{(n)}$ is an elementary set containing A, we get $\overline{J_n}(A) \ge \overline{J}(A)$ and so $\liminf_{n\to\infty} \overline{J_n}(A) \ge \overline{J}(A)$. We now want to find an analogous inequality for $\limsup_{n\to\infty} \overline{J_n}(A)$. Let for that purpose $K_1, ..., K_N$ be closed boxes with $A \subset E := K_1 \cup \cdots \cup K_N$. Since $\overline{J_n}$ is monotone (remark 4.3) and subadditive[4],

$$\overline{J_n}(A) \le \overline{J_n}(\bigcup_{i=1}^N K_i) \le \sum_{i=1}^N \overline{J_n}(K_i),$$

but we also have, by lemma 4.4, $\overline{J_n}(K_i) \xrightarrow{n\to\infty} |K_i|$ for each $1 \le i \le N$. Therefore, by combining both observations, we obtain $\limsup_{n\to\infty} \overline{J_n}(A) \le \sum_{i=1}^N |K_i|$. But, since the cubes $K_1, ..., K_N$ were arbitrary, we can take the infimum as in 4.1, and we get $\limsup_{n\to\infty} \overline{J_n}(A) \le \overline{J}(A)$, which concludes the proof of 4.2.

Now we move to 4.3. First notice that if $\mathring{A}$ is empty then $\underline{J}(A) = 0 = \underline{J_n}(A)$. Assume therefore that A has non-empty interior. Then, by remark 4.3,

$$\limsup_{n\to\infty} \underline{J_n}(A) = \limsup_{n\to\infty} J(A_{(n)}) \le \underline{J}(A).$$

Consider now any family of closed boxes $K_1, ..., K_N$ contained in A with pairwise disjoint interiors. Notice first that any cube which is in the interior of $\mathring{K}_i$ for some $1 \le i \le N$ is also in the interior of $\bigcup_{i=1}^N \mathring{K}_i$, and so, since the $K_1, ..., K_N$ are disjoint, we get

$$\underline{J_n}(\bigcup_{i=1}^N K_i) \ge \sum_{i=1}^N \underline{J_n}(K_i).$$

[4]For all $A, B \subset \mathbb{R}^d$ we have that any cube K intersecting $\overline{A \cup B}$ is intersecting also $\overline{A}$ or $\overline{B}$ (or both), and so in particular $\overline{J_n}$ must be subadditive.

But then, by monotonicity of $\underline{J_n}$ and lemma 4.4,

$$\liminf_{n\to\infty} \underline{J_n}(A) \geq \liminf_{n\to\infty} \underline{J_n}(\bigcup_{i=1}^{N} K_i) \geq \liminf_{n\to\infty} \sum_{i=1}^{N} \underline{J_n}(K_i) = \sum_{i=1}^{N} |K_i|,$$

and since the $K_1, ..., K_N$ were arbitrary we obtain, by taking the supremum as in lemma 4.5, $\liminf_{n\to\infty} \underline{J_n}(A) \geq \underline{J}(A)$. This concludes the proof of 4.3. □

4.2 Measurability criteria

With the above theorem, one can now prove a very important criterion for measurability. We are going to use the proof presented for theorem 3.7 in [2], which is surprisingly short.

Theorem 4.7 *For every bounded set A*

$$\overline{J}(A) = \overline{J}(\overline{A}) = \underline{J}(A) + \overline{J}(\partial A).$$

In particular, A is measurable if and only if $\overline{J}(\partial A) = 0$.

Proof Let $n \in \mathbb{N}$. The number of cubes intersecting $\overline{A}$ is the same as the number of interior or boundary cubes of A, that is, the number of interior cubes plus the number of cubes intersecting ∂A. But since $\overline{\partial A} = \partial A$ we obtain

$$\overline{J_n}(A) = \overline{J_n}(\overline{A}) = \underline{J_n}(A) + \overline{J_n}(\partial A).$$

By taking the limit $n \to \infty$ and using theorem 4.6, we get the aimed equalities. □

Remark 4.8 *If $A \subset \mathbb{R}^d$ is contained in a measurable set C satisfying $\overline{J}(C) = 0$, then by monotonicity $\overline{J}(A) = 0 = J(A)$. We call such a set A a null set. In particular, any null set is measurable and any set whose boundary is a null set is measurable.*

A first evidence that this theorem is quite important is for example the fact that, with this new measurability criterion, we can now prove uniqueness of the Jordan measure $J : \mathcal{J}_d \to \mathbb{R}_+$.

Theorem 4.9 *Let $\mu : \mathcal{J}_d \to \mathbb{R}_+$ be a map satisfying additivity, translation invariance and the normalisation $\mu([0,1)^d) = 1$. Then $\mu = J$.*

Proof The structure of the proof is similar to the one of theorem 3.18 in [2]. Let μ be a map satisfying the above properties. Then we claim that μ is necessarily monotone. Indeed, if A and B are measurable with $A \subset B$, we get by additivity

$$\mu(B) = \mu((B \setminus A) \sqcup A) = \mu(B \setminus A) + \mu(A) \geq \mu(A).$$

Let K, K' be cubes of the form $\prod_{i=1}^{d}[\frac{a_i}{n}, \frac{a_i+1}{n})$ for some $a_1, ..., a_d \in \mathbb{Z}$, and let $\mathcal{G}_n$ be the family of all such cubes that are contained in $[0,1)^d$. By translation invariance we get $\mu(K) = \mu(K')$, and so by additivity

$$1 = \mu\left([0,1)^d\right) = \mu\left(\bigcup_{K \in \mathcal{G}_n} K\right) = n^d \cdot \mu(K),$$

which implies $\mu(K) = \frac{1}{n^d}$.

Claim 4.10 *For K as above we have $\mu(K) = \mu(\overline{K})$.*

Proof Notice first that by monotonicity, additivity and translation invariance $\mu(\overline{K}) - \mu(K) \leq \mu(\partial K) \leq \mu(\partial[0,1]^d)$. For the case $d = 1$ we have

$$\{0,1\} \subset \left[0, \frac{1}{N}\right) \cup \left[1, 1 + \frac{1}{N}\right) \text{ for all } N \in \mathbb{N},$$

and so $\mu(\partial[0,1]) \leq 2 \cdot \frac{2}{N} \xrightarrow{N \to \infty} 0$. While for the case $d \leq 2$, we notice that we can cover (similarly to the case $d = 1$) the boundary of $[0,1]^d$ with $C(d) \cdot N$ many cubes of side length $\frac{1}{N}$, where $C(d)$ is a fixed constant depending only on the dimension d. Therefore, we get

$$\mu(\partial[0,1]^d) \leq \frac{C(d) \cdot N}{N^d} \xrightarrow{N \to \infty} 0. \qquad \square$$

We can now use the claim to get that μ is equal to J on $\mathcal{K}_n$ and also that

$$\mu(A_{(n)}) = J(A_{(n)}), \quad \mu(A^{(n)}) = J(A^{(n)}).$$

Hence for all n we have

$$\underline{J_n}(A) = J(A_{(n)}) = \mu(A_{(n)}) \leq \mu(A)$$

and

$$\overline{J_n}(A) = J(A^{(n)}) = \mu(A^{(n)}) \geq \mu(A).$$

But $\underline{J_n}(A)$ and $\overline{J_n}(A)$ converge, respectively, to $\overline{J}(A)$ and $\underline{J}(A)$, which are both equal to $J(A)$ by measurability of A. Therefore, we conclude $J(A) \leq \mu(A) \leq J(A)$. $\square$

Before moving to the next section, we would like to present one more criterion for measurability. Recall that theorem 4.7 was proven using the formalism constructed in section 4.1, that is by considering interior and boundary cubes. One may notice that this is somehow similar to the approach used in section 2.3 for the proof of the discretization formula (equation 2.6), where we were counting the points of the form $\frac{a}{n}$, $a \in \mathbb{Z}$, contained in an elementary set. The next proposition shows that the discretization formula holds actually for any measurable set.

Proposition 4.11 *Let $A \subset \mathbb{R}^d$ be a Jordan measurable set, then*

$$J(A) = \lim_{n\to\infty} D_n(A) := \lim_{n\to\infty} \frac{1}{n^d} \# \left\{ A \cap \frac{\mathbb{Z}^d}{n} \right\}.$$

Proof The proof is actually a direct consequence of the discretization formula for elementary sets (proposition 2.14). In particular, we would like to point out that one can prove the claim without using any of the results seen in the previous section.

Consider elementary E, F such that $E \subset A \subset F$. Then

$$\left\{ E \cap \frac{\mathbb{Z}^d}{n} \right\} \subset \left\{ A \cap \frac{\mathbb{Z}^d}{n} \right\} \subset \left\{ F \cap \frac{\mathbb{Z}^d}{n} \right\}$$

and so, for all $n \in \mathbb{N}$, $D_n(E) \leq D_n(A) \leq D_n(F)$. Note moreover that the discretization formula for elementary sets gives us

$$m(E) = \lim_{n\to\infty} D_n(E) \leq \lim_{n\to\infty} D_n(F) = m(F).$$

At this point we take the supremum over all elementary $E \subset A$ and the infimum over all elementary $F \supset A$, and we conclude $J(A) \leq \lim_{n\to\infty} D_n(A) \leq J(A)$. □

4.3 Examples of measurable sets

Now that we have all the necessary tools, we can check the measurability of many different sets.

Lemma 4.12 *Products $A \times A' \subset \mathbb{R}^{d+e}$ of Jordan measurable sets $A \subset \mathbb{R}^d$ and $A' \subset \mathbb{R}^e$ are Jordan measurable with measure $J(A \times A') = J(A) \cdot J(A')$.*

Proof We first prove that products $E \times F$ of elementary sets $E \subset \mathbb{R}^d, F \subset \mathbb{R}^e$ are elementary with $m(E \times F) = m(E) \cdot m(F)$.

Indeed, for $E = \sqcup_{i=1}^n B_i, F = \sqcup_{j=1}^m C_j$ we have that $E \times F = \sqcup_{i,j} B_i \times C_j$ is elementary. But since $|B_i \times C_j| = |B_i| \cdot |C_j|$, we get

$$\begin{aligned} m(E \times F) &= \sum_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} |B_i \times C_j| = \sum_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} |B_i| \cdot |C_j| = \left(\sum_{i=1}^n |B_i| \right) \left(\sum_{j=1}^m |C_j| \right) \\ &= m(E) \cdot m(F). \end{aligned}$$

Now let's prove the result for general measurable sets. Let $\varepsilon > 0$ arbitrary. Consider E, F, E', F' elementary such that $E \subset A \subset F$, $E' \subset A' \subset F'$ and

$$m(F) - \varepsilon \leq J(A) \leq m(E) + \varepsilon, \quad m(F') - \varepsilon \leq J(A') \leq m(E') + \varepsilon.$$

Then, by the previous observations, we get elementary sets $E \times E'$, $F \times F'$ such that $E \times E' \subset A \times A' \subset F \times F'$ and

$$\begin{aligned}(J(A) - \varepsilon)(J(A') - \varepsilon) &\leq m(E) \cdot m(E') = m(E \times E') \leq \underline{J}(A \times A') \\ &\leq \overline{J}(A \times A') \leq m(F \times F') = m(F) \cdot m(F') \\ &\leq (J(A) + \varepsilon)(J(A') + \varepsilon).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we conclude. □

The proof of the next result is based on the one proposed by Laczkovich et al. for theorem 3.13 in [2].

Proposition 4.13 *Let $H \subset \mathbb{R}^d$ be a compact set and $f : H \to \mathbb{R}$ a continuous function. Then $graph(f) := \{(x, f(x)) | x \in H\} \subset \mathbb{R}^{d+1}$ has measure zero.*

Proof The idea of the proof is to use uniform continuity of f (f is a continuous function on a compact set) to approximate its graph with finitely many small cubes. Let thus $\varepsilon > 0$ and choose $\delta > 0$ such that for all $x, y \in H$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$ (possible by uniform continuity).

Since H is bounded we can find a cube K containing H. Let now n large enough such that $\frac{\sqrt{d}}{n} < \delta$, and consider the family $\{K_i\}_{i \in I}$ of cubes in $\mathcal{K}_n$ which are intersecting K (the union of those cubes gives then $K^{(n)}$). The family of closed sets $\{H_i\}_{i \in I} := \{K_i \cap H\}_{i \in I}$ satisfies then $H \subset \cup_{i \in I} H_i$ and $diag(H_i) \leq diag(K_i) = \frac{\sqrt{d}}{n} < \delta$, $i \in I$. In particular, since H_i compact, the restriction of f to H_i attains both its maximum M_i and its minimum m_i. But $diag(H_i) < \delta$ implies that for all $i \in I$ we have the bound $M_i - m_i < \varepsilon$. Finally, since

$$graph(f) \subset \bigcup_{i \in I} K_i \times [m_i, M_i],$$

we obtain

$$J(graph(f)) \leq J\left(\bigcup_{i \in I} K_i \times [m_i, M_i]\right),$$

and so by lemma 4.12 and subadditivity

$$\begin{aligned}J(graph(f)) &\leq \sum_{i \in I} J(K_i) \cdot (M_i - m_i) \leq \sum_{i \in I} J(K_i) \cdot \varepsilon = \sum_{i \in I} J(\mathring{K}_i) \cdot \varepsilon = J(\bigcup_{i \in I} \mathring{K}_i) \cdot \varepsilon \\ &\leq J(K) \cdot \varepsilon.\end{aligned}$$

But $\varepsilon > 0$ was arbitrary, and we're done. □

Corollary 4.14 *The area under the graph of a continuous, real-valued, non-negative function on a compact set $H \subset \mathbb{R}^d$, i.e. the set $A(f) := \{(x, y) |\ x \in H,\ 0 \leq y \leq f(x)\}$, is measurable.*

Proof By theorem 4.7 we only need to check that the boundary of $A(f)$ is a null set. But $\partial A(f)$ is contained in $\partial B \cup graph(f)$, where B is the closed box $H \times [0, max_{x\in H} f(x)]$. Therefore, by proposition 4.13 we get

$$\overline{J}(\partial A(f)) \leq \overline{J}(\partial B) + \overline{J}(graph(A)) = 0. \qquad \square$$

Remark 4.15 *The previous corollary can be generalized to continuous real-valued functions on compact sets. Indeed, if f is of this form, we can define the non-negative functions $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$, which satisfy $f = f_+ - f_-$ and all the assumptions of corollary 4.7. But then*

$$\begin{aligned}\partial A(f) :=\ & \partial\{(x,y) |\ x \in H,\ 0 \leq y \leq f(x) \text{ or } f(x) \leq y \leq 0\} \\ & = \partial A(f_+) \cup \partial A(-f_-)\end{aligned}$$

is a null set[5]*, and thus $A(f)$ is measurable.*

Corollary 4.16 *Any open or closed ball is measurable.*

Proof Since the open and the closed ball have the same boundary, it's enough to check that $\partial B_r(x) = \{y \in \mathbb{R}^d |\ \|y - x\| = r\}$ is a null set. Moreover, since the homothetic transformation $\psi_{r,x}(y) = ry + x$ preserves measurability, it's enough to consider the case $r = 1, x = 0$. Define the functions

$$f : \{y = (y_1, ..., y_{d-1}) \in \mathbb{R}^{d-1} |\ \|y\| \leq 1\} \to \mathbb{R},\ f(y) := \sqrt{1 - \|y\|^2}$$

and $g := -f$, both continuous by continuity of the norm. Then the boundary of $B_1(0)$ is contained in the union of $graph(f)$ and $graph(g)$, which are both null sets by the previous proposition. Therefore $\partial B_1(0)$ is a null set itself. $\square$

Remark 4.17 *Note that we can actually compute the Jordan measure of any ball $B_r(x) \subset \mathbb{R}^d$ up to a constant C_d depending only on the dimension. Indeed,*

$$J(B_r(x)) = J(\psi_{r,x}(B_1(0))) = r^d \cdot J(B_1(0)) = r^d \cdot C_d,$$

for $C_d := J(B_1(0))$.

Proposition 4.13 is a key result, which can be used for many objects in $\mathbb{R}^2$ or $\mathbb{R}^3$. As a motivating example, we consider triangles in two dimensions. Let T be a triangle in $\mathbb{R}^2$ with vertices $a, b, c \in \mathbb{R}^2$. The boundary of T is the union of three line segments, and for each of them there are two options:

- the segment is the graph of an affine transformation $L : [0, 1] \subset \mathbb{R} \to \mathbb{R}$, and thus in particular the graph of a continuous function,
- the segment is a vertical line, i.e. of the form $\{(x_0, y_0 + t)\}_{0\leq t\leq t_0}$, and therefore contained in the boundary of some box B.

[5] Here we use that, by symmetry, $\partial A(-f_-)$ and $\partial A(f_-)$ have the same Jordan measure.

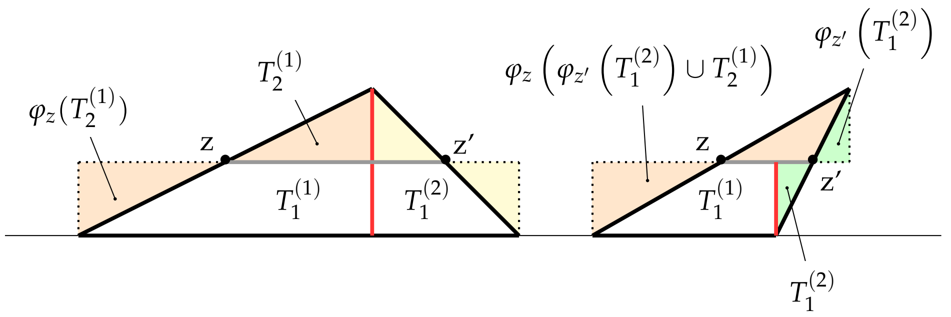

Figure 4.1: Representation of the point reflections applied to two different triangles. For the triangle on the right we have $T_2^{(2)} = \emptyset$.

In both cases the line segment has measure zero and so the same holds for ∂T, which implies that T is measurable.

We now want to show how one can explicitly compute the Jordan measure of such a triangle. For that, we first need to prove invariance of the Jordan measure under point reflections.

Lemma 4.18 *For $z \in \mathbb{R}^d$, let $\varphi_z(x) = 2z - x$ be the reflection through the point z. Then for any $A \subset \mathbb{R}^d$ measurable we have that $\varphi_z(A)$ is measurable with $J(\varphi_z(A)) = J(A)$.*

Proof Consider $c \in \mathbb{R}$ and any interval $I \subset \mathbb{R}$ with endpoints $a \leq b$. Then $\varphi_c(I)$ is an interval with endpoints $2c - b$ and $2c - a$, that is an interval with length $2c - a - (2c - b) = b - a$. Similarly, for $z = (z_1, ..., z_d) \in \mathbb{R}^d$ and any box $B \subset \mathbb{R}^d$ having endpoints $a_i \leq b_i$, $1 \leq i \leq d$, we have that $\varphi_z(B)$ is a box satisfying

$$|\varphi_z(B)| = \prod_{i=1}^{d}(2z_i - a_i - (2z_i - b_i)) = \prod_{i=1}^{d}(b_i - a_i) = |B|.$$

Therefore, for any elementary set $E = B_1 \sqcup \cdots \sqcup B_n$ we have that $\varphi_z(E) = \varphi_z(B_1) \sqcup \cdots \sqcup \varphi_z(B_n)$ and $m(\varphi_z(E)) = |B_1| + \cdots + |B_n| = m(E)$. We conclude by observing that for any elementary set E contained (resp. containing) A, $\varphi_z(E)$ is an elementary set contained (resp. containing) $\varphi_z(A)$, and so, since φ_z is invariant on $\mathcal{E}_d$, $\underline{J}(A) = \underline{J}(\varphi_z(A))$ (resp. $\overline{J}(A) = \overline{J}(\varphi_z(A))$). □

The next step consists in considering the special case of a triangle T where both vertices a and b are contained in the x-axis $\mathbb{R} \times \{0\}$.

Using the height $h := \max\{|y| \mid (x, y) \in \partial T\}$ we can now split T in two parts, by defining

$$T_1 := \{(x, y) \in T \mid |y| \leq \frac{h}{2}\},\ T_2 := \{(x, y) \in T \mid |y| > \frac{h}{2}\}.$$

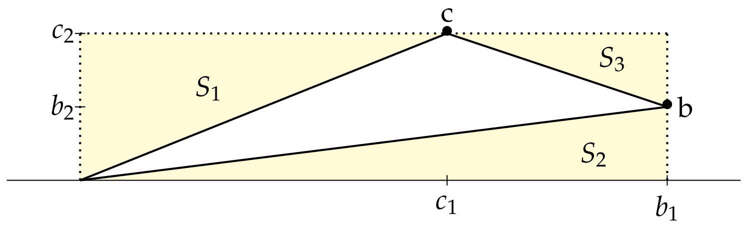

Figure 4.2: The three right triangles S_1, S_2 and S_3.

Now we split both T_1 and T_2 into two pieces $T_1^{(1)}, T_1^{(2)}$ and $T_2^{(1)}, T_2^{(2)}$ (possibly empty) so that there exist $z, z' \in \mathbb{R}^2$ and point reflections $\varphi_z, \varphi_{z'}$ for which

$$B_1 := T_1^{(1)} \cup \varphi_z \left(T_2^{(1)}\right) \cup T_1^{(2)} \cup \varphi_{z'} \left(T_2^{(2)}\right)$$

(see figure 4.1, left part) or

$$B_2 := T_1^{(1)} \cup \varphi_z \left(\varphi_{z'} \left(T_1^{(2)}\right) \cup T_2^{(1)}\right) \cup T_2^{(2)}$$

(see figure 4.1, right part) is a box[6].

By applying the previous lemma, we thus obtain that, for some $i \in \{1, 2\}$, $J(T) = J(B_i) = |\overline{B_i}|$, which is equal to $\frac{1}{2} \cdot h \cdot |b - a|$.

Remark 4.19 *The procedure we used to obtain the formula for the measure of such a triangle is exactly the one that's used in Euclidean geometry to motivate the formula "half the base times the height". Therefore, if we define the area of a two-dimensional triangle as it's Jordan measure, we see that this intuitive idea of "splitting the triangle into pieces" corresponds to an actual formal proof.*

The last step now consists in the generalization of the previous formula to any arbitrary triangle T. By translation invariance it's enough to consider the case where $a = (a_1, a_2) = (0, 0)$. For simplicity, we consider the case where $c_2 \geq b_2$ and $b_1 \geq c_1$ [7]. Then $J(T)$ is equal to the Jordan measure of the box $[0, b_1] \times [0, c_2]$ minus the Jordan measure of three right triangles S_1, S_2, S_3 (see figure 4.2), for which we can use the above formula. Hence,

$$\begin{aligned} J(T) &= J([0, b_1] \times [0, c_2]) - (J(S_1) + J(S_2) + J(S_3)) \\ &= \left| c_2 b_1 - \left(\frac{c_2 c_1}{2} + \frac{b_2 b_1}{2} + \frac{(b_1 - c_1)(c_2 - b_2)}{2} \right) \right| \\ &= \left| c_2 b_1 - \frac{1}{2} b_1 c_2 - \frac{1}{2} c_1 b_2 \right| = \frac{1}{2} |c_1 b_2 - c_2 b_1|. \end{aligned}$$

[6] Actually we should say that $\overline{B_i}$ is a box, since the boundary of B_i is in part contained in B_i and in part not contained, and so B_i is not a box in the sense of definition 2.4. However we still have $J(B_i) = |\overline{B_i}|$.

[7] In the general case one gets something of the form $J(T) = |B| - J(S_1) - J(S_2) - J(S_3) - |B'|$, for some boxes B and B'.

To summarize, we've shown that for any triangle T with vertices a, b, c we have

$$J(T) = \frac{1}{2} \left| (c_1 - a_1)(b_2 - a_2) - (c_2 - a_2)(b_1 - a_1) \right|.$$

Remark 4.20 *The above result can be used when considering convex polygons: by use of fan triangulation, that is splitting the n-polygon into $n-2$ different triangles, one gets a procedure for computing the Jordan measure of any convex polygon. For more details about triangulations we refer to [1].*

4.4 Examples of non-measurable sets

Now that we have seen a few examples of measurable sets, the question that arises is how one can find non-measurable sets. For this purpose, the most known example is probably $Q := \mathbb{Q}^d \cap [0,1]^d$. The reason for Q being non-measurable is strictly related to the definition of an elementary set, which states that any elementary E must be a *finite* union of boxes. In particular, we cannot use the fact that Q can be written as a countable union of isolated points $q \in \mathbb{Q}^d$ to conclude $J(Q) = 0$. In the following we show non-measurability of Q in three different ways.

- The first approach consists in applying directly the definitions of $\underline{J}$ and $\overline{J}$ to Q. Let for that purpose B be a box contained in Q, then we have two options: either $\mathring{B} \neq \emptyset$ or B is a box with measure zero. The first case is actually not possible since otherwise $\mathring{B}$ contains a point $r \in [0,1]^d \setminus \mathbb{Q}^d$, which gives a contradiction. Therefore, we conclude that any elementary $E = B_1 \sqcup \cdots \sqcup B_n$ contained in Q satisfies $J(E) = J(B_1) + \cdots + J(B_n) = 0 + \cdots + 0 = 0$, and we get $\underline{J}(Q) = 0$. On the other hand, if E is an elementary set containing Q, then $[0,1]^d = \overline{Q} \subset \overline{E}$. But since E is measurable we have $J(\overline{E}) = J(E)$, and therefore $J(E) \geq 1$. Taking the infimum over all elementary E containing Q gives $\overline{J}(Q) \geq 1$. Actually, since $E = [0,1]^d$ is elementary we get $\overline{J}(Q) = 1$, which proves, together with $\underline{J}(Q) = 0$, non-measurability of Q.
- A second, way shorter, approach consists in observing that the boundary of Q is exactly $[0,1]^d$. In fact this implies $\overline{J}(\partial Q) = 1$, which proves, by failure of the boundary condition in theorem 4.7, non-measurability of Q.
- Lastly, we want to use the discretization formula to show that Q cannot be measurable. Assume by contradiction that Q was measurable, then by proposition 4.11 we have

$$J(Q) = \lim_{n\to\infty} \frac{\#\{Q \cap \frac{\mathbb{Z}^d}{n}\}}{n^d} = \lim_{n\to\infty} \frac{(n+1)^d}{n^d} = 1.$$

On the other hand, for the translated set $Q' := Q + (\sqrt{2}, ..., \sqrt{2})$ we have, since $Q' \cap \mathbb{Q}^d = \emptyset$,

$$J(Q') = \lim_{n\to\infty} \frac{\#\{Q' \cap \frac{\mathbb{Z}^d}{n}\}}{n^d} = 0.$$

Therefore, combining both observations, we get a contradiction to translation invariance of J, which implies that Q must be a non-measurable set.

A direct consequence of what we've just observed is that, for any $0 \leq a < b$, one can find a non-measurable set $C \subset \mathbb{R}^d$ with inner measure a and outer measure b. In fact, we can consider the set $C' = \mathbb{Q}^d \cap [0, (b-a)^{\frac{1}{d}}]^d$, which satisfies $\underline{J}(C') = 0$ and, by repeating the argument used for Q, also

$$\overline{J}(C') = \overline{J}\left([0, (b-a)^{\frac{1}{d}}]^d\right) = b - a.$$

Therefore, by defining $C = C' \sqcup [b, b + a^{\frac{1}{d}}]^d$ we obtain a set with

$$\underline{J}(C) = \underline{J}\left([b, b + a^{\frac{1}{d}}]^d\right) = a \quad \text{and} \quad \overline{J}(C) = (b-a) + a = b.$$

At this point, it's natural to ask ourselves if the previous result is true also when we impose additional conditions on the non-measurable set we're considering. In particular, we could for example try to see if, for all $\varepsilon > 0$, there exists an open set $U \subset \mathbb{R}$ with inner Jordan measure smaller than ε and outer Jordan measure 1. This is actually possible and is a consequence of the following claim, which provides an open set A having upper measure $b \geq 1$ and therefore also, by picking $U := \frac{1}{b}A$, the aimed set U.

Claim 4.21 *For all $\varepsilon > 0$ there exists an open subset A of $\mathbb{R}$ with $\underline{J}(A) \leq \varepsilon$ and $\overline{J}(A) \geq 1$.*

Proof Let $\{q_n\}_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q} \cap [0,1]$ and, for all $n \geq 1$, define the interval $A_n := (q_n - \frac{1}{2^{n+1}}, q_n + \frac{1}{2^{n+1}})$. Then choose $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} \frac{1}{2^n} < \varepsilon$ and define $A = \bigcup_{n\geq N} A_n$.

First we look at $\underline{J}(A)$. Assume that E is a closed elementary set contained in A. Then, since E is compact and $\{A_n\}_{n\geq N}$ an open cover of E, there must exist a $M \in \mathbb{N}$ such that $E \subset \bigcup_{n=N}^{M} A_n$. In particular,

$$J(E) \leq J\left(\bigcup_{n=N}^{M} A_n\right) \leq \sum_{n=N}^{M} |A_n| \leq \sum_{n=N}^{\infty} \frac{1}{2^n} < \varepsilon.$$

Since in general the (arbitrary) elementary set E does not need to be closed, we need to approximate it with a closed elementary set $\tilde{E} \subset E \subset A$. By the above argument, we then have that $J(\tilde{E}) < \varepsilon$. However, $\tilde{E}$ can be chosen so that we at least have the upper bound $J(E) \leq \varepsilon$. Hence, for any elementary $E \subset A$ we get $J(E) \leq \varepsilon$, which implies that $\underline{J}(A) \leq \varepsilon$. Now, for $\overline{J}(A)$, consider some E elementary containing A. Then in particular, by density of the $\{q_n\}_{n \geq N}$, $[0,1] \subset \overline{A} \subset \overline{E}$. Therefore $J(E) = J(\overline{E}) \geq 1$ and since E was arbitrary, we get $\overline{J}(A) \geq 1$. $\square$

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