

The Banach-Tarski Theorem*

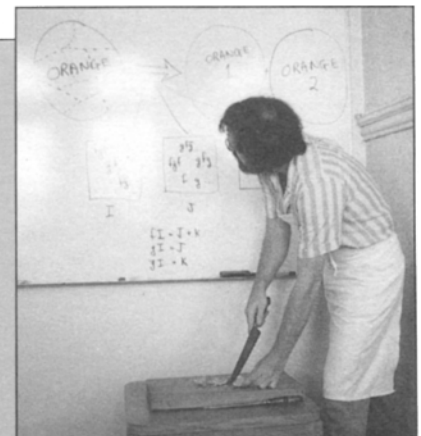
Robert M. French

It is theoretically possible, believe it or not, to cut an orange into a finite number of pieces that can then be reassembled to produce two oranges, each having exactly the same size and volume as the first one. That's right: with sufficient diligence and dexterity, from any three-dimensional solid we can produce two new objects exactly the same as the first one!

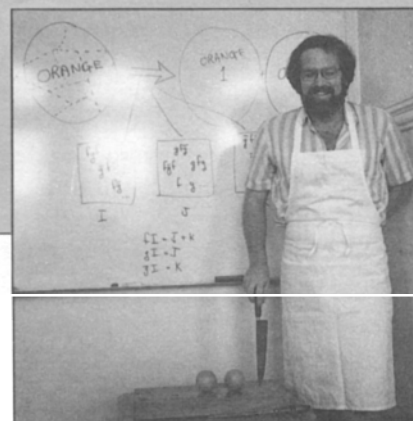
Mathematicians, upon first hearing of this result (otherwise known as the Banach-Tarski Theorem), are generally somewhat blasé; they know that funny

counterintuitive things crop up all the time whenever infinity is involved. Most mathematicians encounter the result for the first time in graduate school and file it away in their strange results category (along with space-filling curves, Cantor functions, and non-measurable sets). But in spite of the relative simplicity of the proof, discovered by Stefan Banach and Alfred Tarski in 1924 and hinging on the Axiom of Choice, many mathematicians go no further than the lay scientist who comes across the result.

The author uses the Banach-Tarski Theorem to cut up an orange and reassemble it into two oranges.



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The mathematics of infinity is almost always counterintuitive and has been so ever since its inception at the end of the nineteenth century when Georg Cantor proved the completely astounding result that infinity came in different sizes. This result initially so upset the mathematical community that Henri Poincaré once maligned it as a disease from which mathematics would have to recover.¹

The purpose of this article is neither to explain the subtleties of infinity nor to give a rigorous proof of the Banach-Tarski Theorem. Instead, a few simple notions about infinity will be explained that will serve as the basis for the subsequent explanation of the main ideas of the proof of this wonderful theorem.

Let's start with a bit of elementary geometry. Two subsets of the plane are said to be *congruent* when one can be made to coincide precisely with the other using only translations and rotations in the plane. The essence of congruence is that the distances between the points of the first set remain unchanged after it has been moved to coincide with the second set. Congruence, however, is not to be confused with one-to-one correspondence. The set of even numbers $\{2, 4, 6, \dots\}$, for example, is not congruent to the set of natural numbers $\{1, 2, 3, \dots\}$ because there is no way to *overlay* one set on the other even though the two sets can be put into one-to-one correspondence. Nothing prevents an infinite set from being congruent to a proper subset of itself, however. Consider, for example, the two infinite sets $\{1, 2, 3, \dots\}$ and $\{5, 6, 7, \dots\}$. Congruence is demonstrated by shifting all of the elements of the first set four units to the right.

Equivalence by Finite Decomposition

Let's return to our orange. Before we actually begin converting it into two oranges, we need the notion of "equivalence by finite decomposition." In spite of its complex name, the idea is simple. Basically, we divide an object X into a finite number of disjoint parts and then rearrange them into a new object Y . (Note that "rearranging" a given set means that the set, in its initial position, is congruent to the set in its final position.) Under these circumstances, we say that X is equivalent by finite decomposition to Y . This type of equivalence is transitive. In other words, if a set X is equivalent to Y , which in turn is equivalent to Z , then X and Z are also equivalent by finite decomposition.

Now, let's consider our first little "paradox": the set of positive integers, \mathbf{N} , is equivalent by finite decomposition to the integers with a one-element "hole" in them—for example, the set of integers with one of its members, say 5, removed. There are various demonstrations of this fact but the one given below will turn

out to be the most instructive for what follows. First, create two subsets of \mathbf{N} : the set B consisting of all multiples of 5 (i.e., $\{5, 10, 15, \dots\}$) and its complement A containing all non-multiples of 5 (i.e., $\{1, 2, 3, 4, 6, 7, \dots\}$). By definition, these two sets are disjoint, and their union is equal to all of \mathbf{N} . We now are in a position to introduce the key technique of this proof and all of the others in this article, including the Banach-Tarski Theorem itself. We will call this technique "shifting toward infinity." We shift B toward infinity by 5 units, thus producing a new set B' equal to $\{10, 15, 20, \dots\}$ which is, by definition, congruent to B . We now have, on the one hand, a disjoint union of sets $(A \cup B)$, which is equal to the positive integers, and a second disjoint union of sets $(A \cup B')$, which is equal to the positive integers with the element 5 removed. But, as we have said, B and B' are obviously congruent, as is, even more obviously, A with itself. We can therefore conclude that the set of integers and the set of integers with 5 removed are equivalent by finite decomposition.

The next proof is slightly more complicated but is based on the same principle of shifting toward infinity that was used to show that \mathbf{N} and $\mathbf{N} \setminus \{5\}$ were equivalent by finite decomposition. This time we will consider a circle and a circle with a one-point "hole" in it. The claim is that these two sets are equivalent by finite decomposition. Here is an outline of how the proof goes.

Let C be a circle with radius 1 unit, and let 0 be some point (in fact, the one we are going to "remove") on the circumference of C . From point 0 , we move counterclockwise along the circumference a distance of exactly one unit, the radius of the circle. Call the point at which we stopped 1, and then continue walking. Exactly one unit later, stop and mark the point where you stopped by 2, etc. Call B the set of all points $\{0, 1, 2, 3, \dots\}$.

Just as in the previous demonstration, A will designate the points (of the circle this time) that are not in B . Now imagine that the set B is the channel selector dial of a television. Turn the dial one click to the left. This dial-turning superimposes the set $\{0, 1, 2, \dots\}$ on the set $\{1, 2, 3, \dots\}$. The latter set we call B' ; and obviously B and B' are congruent. Since the circle C is equal to $A \cup B$ and the circle without the point 0 is equal to $A \cup B'$, we conclude that the circle and the circle with a one-point hole are equivalent to each other by finite decomposition.

Next, we wish to demonstrate that a closed one-by-one square can be decomposed and reassembled to form a closed isosceles triangle whose altitude is equal to one of the sides of the original square.

The first thing we might try is to cut the square along one of its diagonals, thus obtaining two right triangles (see Figure 1). The desired isosceles triangle would be produced by reassembling the two right tri-

¹ Martin Gardner, *Mathematical Carnival*, New York: Vintage Books (1965), Ch. 3, p. 27.

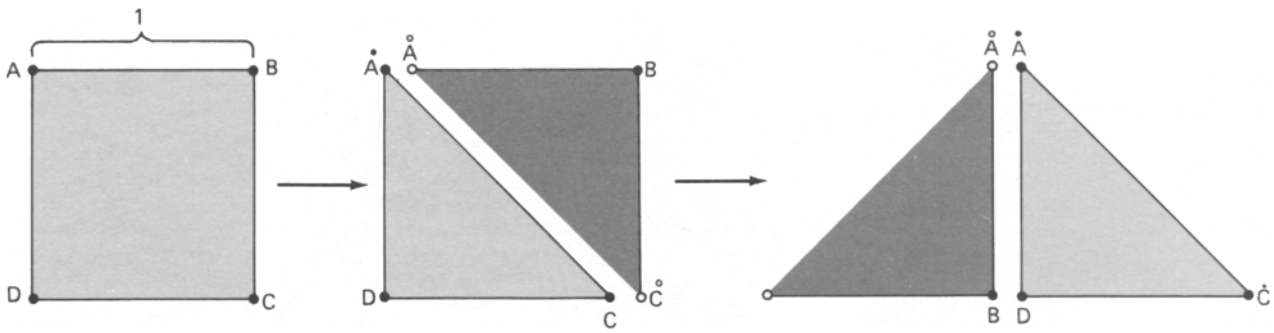


Figure 1. How to transform a square into an isosceles triangle.

angles in such a way that two of their legs would coincide and their hypotenuses would meet at a point. This method, however, does not work. When we cut the square, we do not produce two complete right triangles. The diagonal of the square can only be used to constitute *one* of the hypotenuses, not both. Furthermore, the definition of equivalence by finite decomposition requires that the constituent parts must be *dis-joint* (this, at least, does correspond to our real-world notion of what is meant by the separate parts of something). When we cut an object in half we do not allow some points to belong to both halves. Figure 1 shows a way of cutting the square that satisfies this condition.

Unfortunately, there are two candidates for the altitude of the isosceles triangle and no points at all along one of its sides. Why not try to remove one of the extra altitudes and “paste” it along the edge of the triangle that is without points? It turns out that this trick almost works, but, as you can see in Figure 2, it falls slightly short of the mark. Even after pasting in this altitude of length 1, a “hole” remains. We still need a segment of length $\sqrt{2} - 1$. We will use the technique of shifting toward infinity to excise from the original square a line segment of the required length and then will show that this “theft” is of absolutely no deleterious mathematical consequence. Then, our minds at ease, we will finish our construction by plugging the hole in the side of the triangle with the purloined line segment.

How do we filch the required line segment from the original square? Basically, we show that the square and the square minus the desired line segment are equivalent by finite decomposition. The precise specifications of this line segment require it to have a length of $\sqrt{2} - 1$ with one end including its endpoint and the other end not. The excision technique is virtually identical to the one that allowed us to show a circle and a circle missing a point were equivalent by finite decomposition. Now, instead of removing a point from a circle, we will remove a line segment from a disk. We therefore begin by inscribing the circle C of

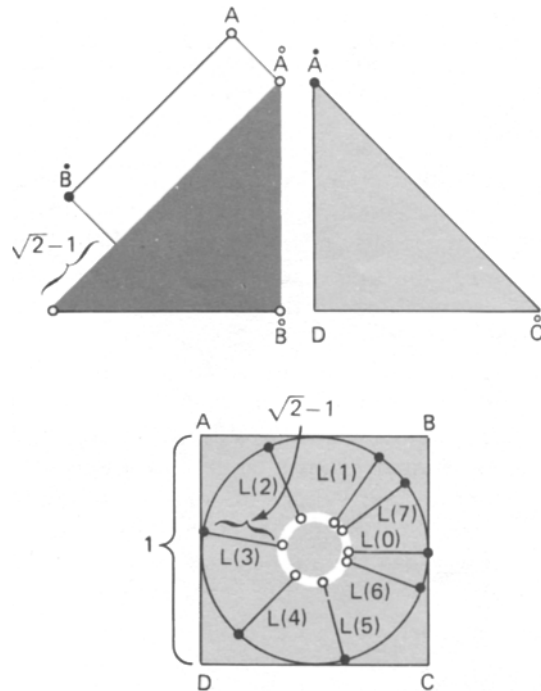


Figure 2. The unit square is equivalent to the square minus a line segment of length $\sqrt{2} - 1$.

the preceding demonstration in the square. We will be considering the closed disk D whose boundary is C . To each of the points $0, 1, 2, \dots$ attach a segment of length $\sqrt{2} - 1$ (see Figure 2). Call these segments $L(0), L(1), \dots$. The remainder of the proof is exactly the same as before except that C , the circle, is now replaced by D , the disk, and the point 0 by the line segment $L(0)$. We have therefore shown that the disk and the disk with a line segment missing are equivalent by finite decomposition. Further, because our theft of the line segment didn't affect any part of the

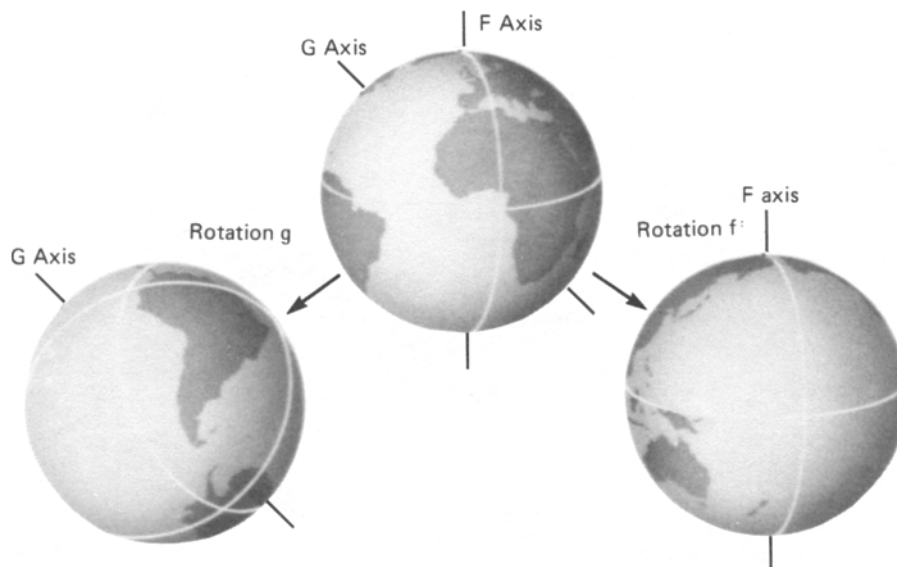


Figure 3. The transformations of the sphere are composed of rotations about the F and G axes. The angle between the axis is chosen so that fg is not equal to gf .

square outside of the disk, we can safely assert that the square and the square without the missing line segment are also equivalent by finite decomposition. Finally, we insert $L(0)$ in the hole along the side of the triangle and obtain the desired result; the closed square is equivalent by finite decomposition to the closed isosceles triangle.

Until now we have only performed our shift-toward-infinity vanishing act on sets whose size was insignificantly small compared to the sets that contained them: a point taken from a circle and a line segment excised from a disk. While this may be mildly interesting, it's hardly spectacular. Let's now take a look and see how these techniques can also be applied to much larger sets, such as the entire volume of a solid ball.

Hausdorff's Paradox

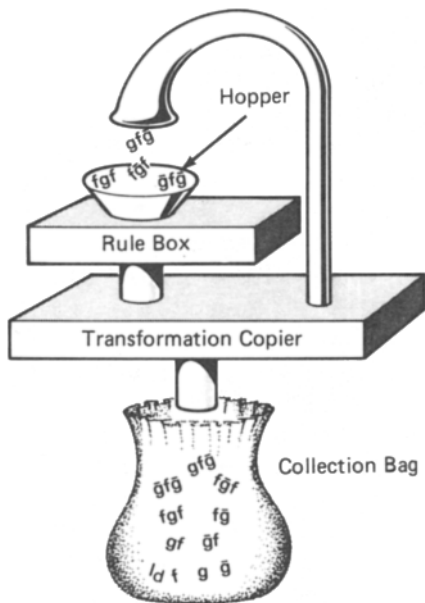
We now have the tools necessary to produce two oranges from one. The heart of the proof of the Banach-Tarski Theorem is based on a result of Felix Hausdorff. Hausdorff's result concerns only the skin of the orange (a "skin" of thickness zero). The Hausdorff paradox, as it is called, shows that it is possible to divide this skin, once an insignificantly small (more precisely, countably infinite) set of points has been removed, into three disjoint sets of points A , B , and C such that A , B , C , and $B \cup C$ are all congruent to one another. Now, that is positively weird! The mutual congruence of these three sets means that A is congruent to the disjoint union of *two copies of itself*. This is referred to as a paradoxical decomposition of A . Essentially, by carefully reassembling these sets, A , B , and C , we obtain a set of two complete orange skins each of which is equivalent by finite decomposition to the original orange skin. In other words, one sphere can be cut up

and reassembled into two spheres identical to the first.

Can we do the same thing to solid balls? The answer is yes; intuitively, we must imagine applying the Hausdorff technique to hollow balls whose skins get progressively thicker and thicker. Finally, we apply this construction to a "hollow" ball whose inside consists only of a single point to produce two equivalent balls each missing its center point. We have thus done the construction for the closest thing possible to a solid ball—namely, a solid ball without its center. Having gone that far, it is a relatively easy matter to show that the solid ball and the solid ball without its center are equivalent by finite decomposition. This completes the proof of the Banach-Tarski Theorem: a ball is equivalent by finite decomposition to two copies of itself.

Now let's take a look at the proof of Hausdorff's paradox, the mainstay of the Banach-Tarski Theorem. Recall that Hausdorff's construction is only concerned with the surface of the ball (i.e., the sphere) and not the ball.

Given a sphere S , we will select two axes of this sphere, F and G . The angle formed by these two axes at the center of the sphere is to be 45° . We will designate by f a clockwise rotation of the sphere by 180° about the F axis and by g a clockwise rotation of the sphere by 120° about the G axis. We call f and g *transformations of the sphere* (see Figure 3). We will use combinations of these transformations to describe different sequences of rotations of the sphere. For example, the composite transformation g^2f specifies the operation consisting of turning the sphere 180° about F , followed by two rotations of 120° about G . To avoid an unnecessary proliferation of exponents, we will write \bar{g} to designate g^2 . (Remember that \bar{g} represents not only a clockwise rotation of 240° but also a rotation of 120° in the opposite direction.) From now on we will call f , g , and \bar{g} *elementary transformations*. From a given position



Rule 1: $1d \rightarrow f, g, \bar{g}$
 Rule 2: $f... \rightarrow gf..., \bar{g}f...$
 Rule 3: $g... \rightarrow fg...$
 $\bar{g}... \rightarrow \bar{g}f...$

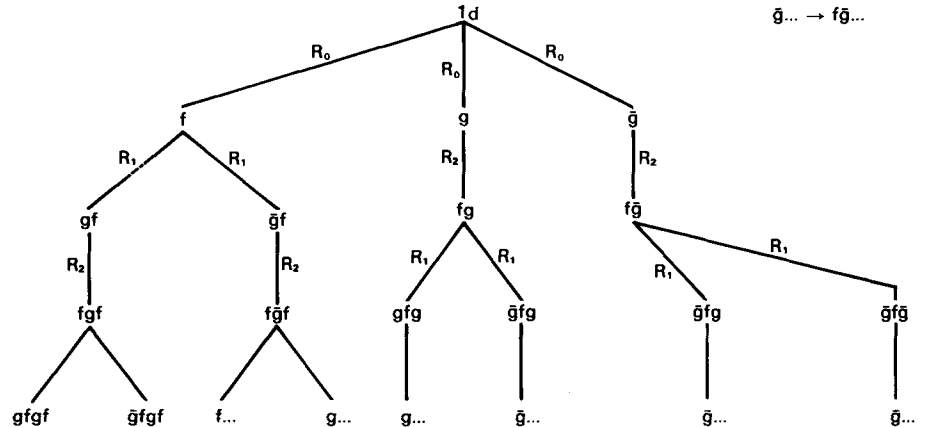


Figure 4. The simple machine used to produce the set of all possible transformations.

of the sphere, if we apply f twice in a row, the sphere will be returned to its initial position. We write $f^2 = 1$, where 1 is the identity transformation—in other words, the transformation that does not change the position of the sphere. Similarly, since g represents a rotation of 120° , $g^3 = 1$.

These two observations allow us to reduce complex transformations to a simpler form. For example, $g^5 f^3 = (g^3)(g^2)(f^2)(f) = 1 \cdot (g^2) \cdot 1 \cdot f = g^2 f$. On the other hand, there is no way to simplify the composite transformation $gfgfg\bar{g}$ because the position of the two axes F and G was carefully selected in such a way that fg does not equal gf . In other words, starting with a given position of the sphere, when we perform the transformations in the order g , then f , the sphere will be in a different position than had we done first f , then g .

We are only interested in transformations reduced to their lowest form, and we will use an iterative machine (see Figure 4) to produce the set Q of all of these transformations. To start the machine running, we put the identity transformation 1 into the hopper. The machine executes the following three rules: (1) when 1 is the only transformation in the hopper, it produces the three elementary transformations f , g , and \bar{g} ; (2) when a transformation whose leftmost element is f goes through the rule box, two new transformations are produced—the first by adding an additional rotation by g to the transformation in the box, and the second by adding an additional rotation by \bar{g} (for example; if $f\bar{g}f$ goes into the rule box, $gf\bar{g}f$ and $\bar{g}f\bar{g}f$ will come out); (3) when a transformation whose leftmost element is g or \bar{g} goes through the rule box, a new transformation is produced by adding an additional rotation by f to it (for example, if gfg goes into the rule box, $fgfg$ will come out).

The transformations produced in the rule box then

drop into a transformation copier, which produces a copy of each transformation and sends it back up to the hopper; the original transformation then drops into the large collection bag below the machine. This is how Q , the set of transformations $1, f, g, \bar{g}, fg, gf, \bar{g}f, \dots$, is produced. The angle between the axes F and G was chosen to ensure that each of the elements of Q represents a *unique* position of the sphere with respect to its initial position.

A Full Iterative Machine

This four-part machine with its hopper, rule box, transformation copier, and collection bag forms the basis of the more powerful iterative machine that we need for the Banach-Tarski Theorem. The full iterative machine not only must be capable of producing all of the transformations in Q , but also must be able to *sort* them into three disjoint subsets, I, J , and K whose union is equal to all of Q and that have the following properties:

$$fI = J \cup K; gI = J; \bar{g}I = K.$$

What do these equalities mean? Consider the first one, $fI = J \cup K$, which means that if you apply f (a clockwise rotation of 180°) to all of the transformations in I , you obtain exactly the set of transformations $J \cup K$. In other words, I is congruent to $J \cup K$. Similarly, $gI = J$ means that upon applying g (a clockwise rotation of 120°) to all of the transformations in I , you obtain exactly all of the transformations in J . Thus, I is congruent to J . Similarly, we find that I is congruent to K .

Figure 5 shows the full-blown iterative machine that will create these three sets of transformations I, J , and K . The major conceptual difference with the basic machine (Figure 4) is the addition of three transformation

sorters. The role of these sorters is simple: based solely on the leftmost elementary transformation of any transformation entering the sorter, they determine the tube down which it will be sent. The machine operates sequentially. First, it processes all of the transformations in its I hopper, then everything in its J hopper, and finally everything in its K hopper before returning to the I hopper. For this reason, we can talk of cycles of the machine. Were I to be asked to put my finger on the key technique of the proof of the Banach-Tarski Theorem, I wouldn't hesitate to single out this clever way of generating the three disjoint subsets, I , J , and K of the set Q of all transformations of the sphere. Figure 6 indicates several stages of production of this machine.

It should be clear, at least empirically, that we now have the desired relationships between the various subsets of Q , namely: $fI = J \cup K$; $gI = J$; $\bar{g}I = K$.

Are some of the pieces starting to fall together? The Hausdorff paradox states that we can divide the sphere (minus a countable set) into three disjoint subsets of *points* A , B , and C such that A , B , C , and $B \cup C$ are pairwise congruent. We produced with our iterative machine three disjoint subsets of *transformations* of the sphere I , J , and K such that I , J , K , and $J \cup K$ are pairwise congruent. If you think this is too much of a coincidence to be an accident, you are right. We are indeed closing in on the result.

Two Spheres from One

Let's return to our sphere. No matter how many times you rotate it in any imaginable way about a fixed center, when you are finally done, you can always find exactly one axis that would have allowed you to go from the initial position of the sphere to its final position in just one rotation. This is what we do for all of the transformations in Q . For each transformation, regardless of its length, we determine the axis of rotation that would have allowed us to go from the initial position directly to the final position of the sphere. This axis cuts the sphere at two points that we call, not surprisingly, *poles*. We then collect in a set D both poles associated with each transformation in Q . This set D represents the points on the sphere that, for at least one transformation of Q , *do not move*. (It turns out that D , being a countable set, is infinitesimally small compared to the entire sphere.) All of the other points on the sphere move for every transformation in Q . This set of points, which we will call D^* or, alternately, $S \setminus D$ (where S is the sphere), is the one that interests us and is virtually the entire sphere anyway.

How should we go about defining the three other sets of points A , B , and C whose disjoint union will be equal to D^* ? To each point p in D^* , apply all of the transformations in Q , collecting the resulting points in

a set called $Q(p) = \{p, f(p), g(p), \bar{g}(p), fg(p), \dots\}$. It is easy to show that for any two distinct points p and p' , the sets $Q(p)$ and $Q(p')$ are either identical or disjoint. From each of the sets created in this way, pick a point. Collect all of these points together in a set M . (The possibility of creating this set M implies our tacit acceptance of the Axiom of Choice. Before devoting time to a discussion of this axiom, let's finish the proof.) A moment's reflection will convince you that the set D^* is equal to the set obtained by applying all of the transformations of Q to the points in M .

The last little step in the proof consists of dividing D^* into three disjoint subsets A , B , and C such that A , B , C , and $B \cup C$ are pairwise congruent. With the means now at our disposal, this will be easy. Recall the three subsets of transformations I , J , and K that we constructed so carefully. Define A as the set of points resulting from the application of all of the transformations of I to the set M . Similarly, B and C will be produced by the application of all of the transformations of J and K , respectively, to M . This construction gives the desired disjoint decomposition of D^* into A , B , and C . Because fI is equal to $J \cup K$, however, $f(A)$ is obviously equal to $B \cup C$. Since f is simply a rotation by 180° , we can conclude that A and $B \cup C$ are congruent. By similar reasoning, clearly $gI = J$ implies that A and B are congruent, and $\bar{g}I = K$ implies that A and C are congruent. The transitivity of congruence allows us to conclude that A , B , C , and $B \cup C$ are all pairwise congruent.

Now we are ready to use our trick of shifting to infinity. Recall the image of the television channel selector knob. Imagine instead that we now have a spherical knob with two axes of rotation. Suppose that the transformation f represents a click of the button by 180° about its first axis; to make the set A coincide with the set $B \cup C$, we need only turn the knob by one click about this axis. In a similar fashion, one 120° click about the second axis (i.e., the transformation g) brings the set A directly onto B , while two clicks make A coincide with C .

Finally, we are in possession of the result that will take us directly to the decomposition we need to finish the Banach-Tarski construction. Remember that our goal is to cut the ball into a finite number of pieces that will be reassembled into two balls of the same size and volume as the first one. As we have already said, our starting point will be the Hausdorff paradox. The idea is as follows: Given that the surface of the ball can be cut into four disjoint sets A , B , C , and D such that A , B , C , and $B \cup C$ are *all* mutually congruent, we can use the set $B \cup C$ as a "cutting template" to produce the pairs of sets that will eventually be reassembled into two separate spheres. Lay this template on top of A and cut out two sets A_1 and A_2 that are congruent to B and C , respectively. Since both B and C are each congruent to A , the decomposition of A into A_1 and A_2 is

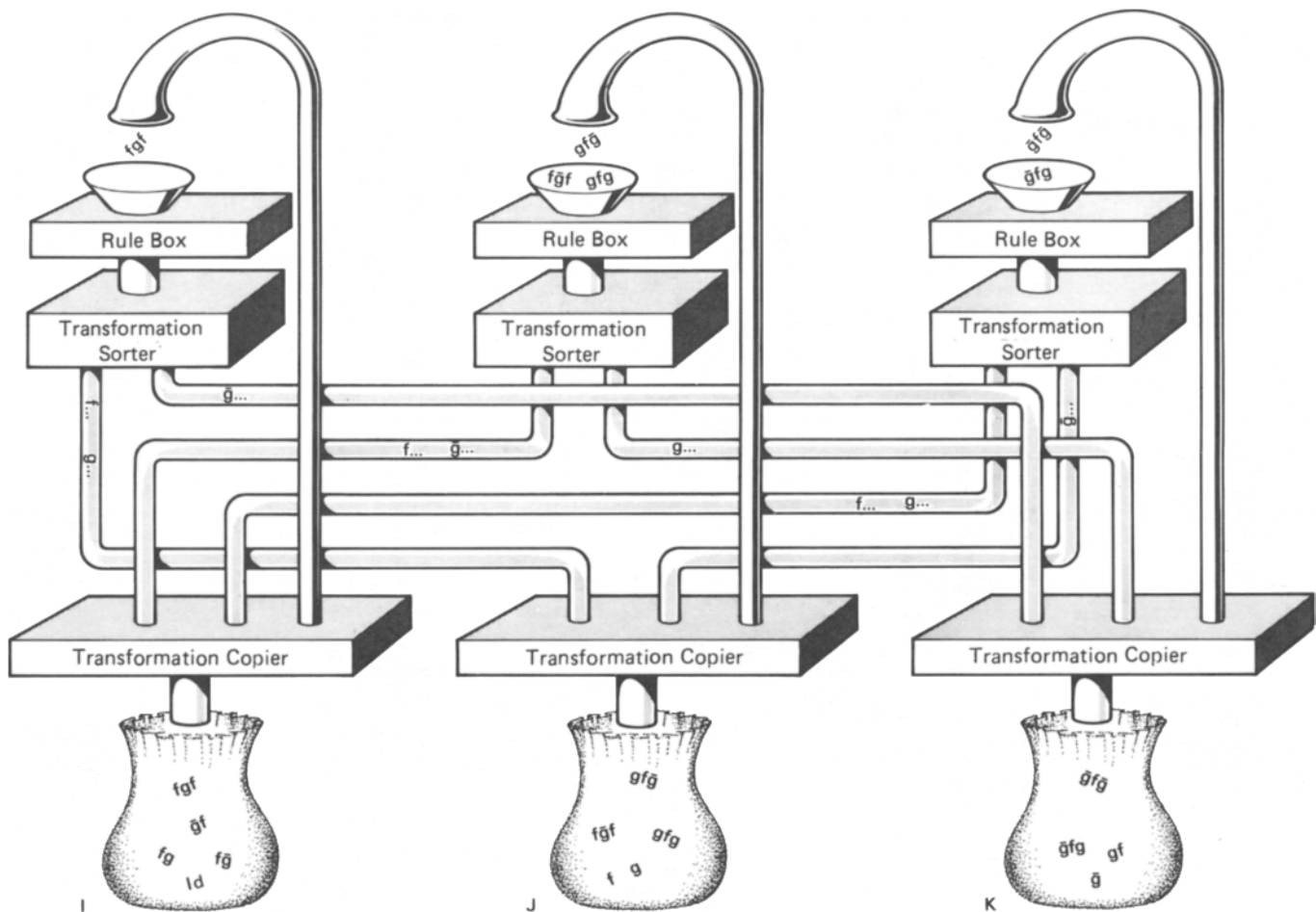


Figure 5. The full-blown machine used to generate and sort all of the transformations.

	Cycle 0	Cycle 1	Cycle 2	Cycle 3
Contents of Bag I	1	1	1, fg, gf, fg	1, fg, gf, fg, fgf
Contents of Bag J	empty	f, g	f, g	f, g, fgf, gfg, gfg
Contents of Bag K	empty	g	g, gf	g, gf, gfg, gfg

Figure 6. The results of the first iterations of the transformation-producing and sorting machine.

paradoxical. We then decompose B and C in a similar fashion into B_1 and B_2 , and C_1 and C_2 . In other words, we can decompose S into disjoint subsets as follows:

$$\begin{aligned}
 S &= A \cup B \cup C \cup D \\
 &= (A_1 \cup A_2) \cup (B_1 \cup B_2) \cup (C_1 \cup C_2) \cup D \\
 &= (A_1 \cup B_1 \cup C_1 \cup D) \cup (A_2 \cup B_2 \cup C_2).
 \end{aligned}$$

From $(A_1 \cup B_1 \cup C_1 \cup D)$ we make one sphere S_1 , which is equivalent by finite decomposition to the

original sphere S (since A is congruent to A_1 , B is congruent to B_1 , etc.). Only one tiny detail remains to be shown. Can we construct a second sphere from $(A_2 \cup B_2 \cup C_2)$?

Rest assured, we haven't come this far for the answer to be no! First, notice that $A_2 \cup B_2 \cup C_2$ can almost be reassembled to make a second sphere S_2 , identical to the original one; all that is missing is the set D whose size, as we have already pointed out is "insignificant" compared to that of S . The demonstration

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that a sphere and a sphere with the set D removed are equivalent by finite decomposition is essentially the same as the proof that a circle and the same circle with a point missing are equivalent by finite decomposition. Thus $S_2 \setminus D$ and S_2 are equivalent by finite decomposition, and that concludes the proof. We have shown that the sphere S , when properly dissected, can be decomposed and then reassembled into two spheres S_1 and S_2 , each of which is equivalent by finite decomposition to S !

Applications

So we have now shown that one basketball, if it is cut up carefully enough, can spawn two. So much the better for the sports world, but what about the banking community? Can a bank note, even of the smallest denomination, produce two of its kind? Unfortunately not. The mathematician A. Lindenbaum proved that no bounded set in the plane can have a paradoxical decomposition,² and a bank note, sad to say, is a bounded set in the plane.

We have already described the "thickening" technique by which we transform the spheres into balls. To produce these two copies by means of the Banach-Tarski Theorem, we need the Axiom of Choice. What could be more intuitively obvious than this axiom, which claims that it is possible to start with any collection of non-empty sets and create a new set by selecting one element from each of the sets in the collection.

The validity of the Axiom of Choice, like that of Euclid's Fifth Postulate some two hundred years before, was a hotly debated subject within the mathematical community this century. The question was finally resolved around the beginning of the 1960s. The fate of this axiom resembled that of the Fifth Postulate. It turned out that the Axiom of Choice, like the Fifth Postulate, is neither true nor false, but independent of the other axioms of the system. If we accept it as true—and what could be more natural?—we are mathematically obliged to accept the strange result of Banach and Tarski that derives from it.

So much for theory. Now let's move on to some amusing practical applications. All you need is a sharp knife, a small loaf of bread, a few fish, and a large audience. Then if you go about carefully doing the cuts and reassemblies indicated in this article, who knows where it all might lead.

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² A. Lindenbaum, *Fund. Math.* (1926), 8, p. 218.