LECTURE 11. 03. 2020

$A \subseteq \mathbb{R}^n$ bdd Jordan meas
$\Rightarrow$ A is $\mathcal{L}^n$-meas. and $\mu(A) = \mathcal{L}^n(A)$

EX $E = \mathbb{Q} \cap [0,1]$

- $\underline{\mu}(E) = 0 = \sup\{ \sum_{k=1}^{m} vol(I_k) \quad \bigcup_{k=1}^{m} I_k \subseteq E \}$
- $\overline{\mu}(E) = 1 \qquad E \subseteq \bigcup_{J=1}^{m} I_J$

CLAIM $\sum_{J=1}^{m} vol(I_J) \geq 1$ ⊛

$\Rightarrow \overline{\mu}(E) \geq 1$. Actually $\overline{\mu}(E) = 1$ because $[0,1]$ covers E.

Suppose by contradiction $\exists$
$\{ I_J, J = 1 .. m\}$ $E \subseteq \bigcup_{J=1}^{m} I_J$

$$\sum_{J=1}^{m} vol(I_J) < 1$$

$\Rightarrow$ the lower Jordan measure

$[0,1] \setminus \bigcup_{j=1}^{m} I_j$ is positive

$\Rightarrow$ it contains $\vee$ $\underset{\text{interval}}{\tilde{I}} \subseteq [0,1] \setminus \bigcup_{j=1}^{m} I_j$

Since $\tilde{I} \cap \mathbb{Q} \neq \emptyset \Rightarrow$

$\bigcup_{j} I_j$ cannot cover E $\lightning$

$\underline{\mu}(E) = 0$ & $\bar{\mu}(E) = 1$

But $\mathbb{Q} \cap [0,1]$ is $\mathcal{L}^1$-measu.

$$\mathbb{Q} \cap [0,1] = \bigcup_{q \in \mathbb{Q} \cap [0,1]} \{q_n\}$$

$\{q_n\}$ are $\mathcal{L}^1$-meas and

its $\boxed{\mathcal{L}^1 \{q_n\} = 0}$ //

Def 1.3.13

A Borel meas. in $\mathbb{R}^n$ μ is Borel regular if $\forall A \subseteq \mathbb{R}^n$ $\exists$ $\underset{\underset{A}{\cup\!\!\!\mid}}{B} \subseteq \mathbb{R}^n$ Borel : $\mu(B) = \mu(A)$.

Cor $\mathcal{L}^n$ is Borel regular

Proof

$A \subseteq \mathbb{R}^n$

- $\mathcal{L}^n(A) = +\infty \Rightarrow B = \mathbb{R}^n$

$\mathcal{L}^n(\mathbb{R}^n) = \mathcal{L}^n(A) = +\infty$

- $\mathcal{L}^n(A) < +\infty$

$$\mathcal{L}^n(A) = \inf\{\mathcal{L}^n(G) \quad \underset{\text{open}}{G \supseteq A}\}$$

$\forall k \geq 0 \quad \exists \; G_k$ open :

$$\mathcal{L}^n(G_k) \leq \mathcal{L}^n(A) + \frac{1}{2^k}$$

$$\mathcal{L}^n(G_1) \leq \mathcal{L}^n(A) + 1 < +\infty$$

- We can assume w.l.o.g that

$$G_{k+1} \subseteq G_k$$

$$\left[\ \tilde{G}_1 = G_1, \quad \tilde{G}_2 = G_1 \cap G_2, \quad \tilde{G}_3 = \tilde{G}_2 \cap G_3 \ \ldots \right]$$

$$\mathcal{L}^n(\tilde{G}_k) \leq \mathcal{L}^n(G_k) \leq \mathcal{L}^n(A) + \frac{1}{2^k}$$

$$G = \bigcap_{k=1}^{\infty} G_k$$

$$\mathcal{L}^n(G_1) < +\infty$$

$$\Rightarrow \mathcal{L}^n(G) = \lim_{n \to +\infty} \mathcal{L}^n(G_k).$$

$$\mathcal{L}^n(G_k) \leq \mathcal{L}^n(A) + \frac{1}{2^k}$$

Let $k \to +\infty$

$$\mathcal{L}^n(G) \leq \mathcal{L}^n(A) + 0$$

$$\mathcal{L}^n(A) \leq \mathcal{L}^n(G)$$

$\uparrow$

$$A \subseteq G_k \quad \forall k \Rightarrow A \subset \bigcap_{k=1}^{\infty} G_k$$

$$\bigcap_{k=1}^{\infty} G_k = G$$

$$\Rightarrow \mathcal{L}^n(A) = \mathcal{L}^n(G)$$

and G is a Borel set because it is the intersection of open sets!

$\boxed{\text{N}}$

Question : $\Sigma = \mathcal{B}(\mathbb{R}^n)$?

1.4 VITALI SET

AXIOM OF CHOICE

Suppose E is a set and $\{E_\alpha\}_{\alpha \in A}$

$E_\alpha \subseteq \cancel{A}$ (A is not assumed countable)

$E_\alpha \neq \emptyset$

Then there is $\alpha \xmapsto{f} x_\alpha \quad x_\alpha \in E_\alpha \quad \forall \alpha$

$f(\alpha)$ is called a choice function.

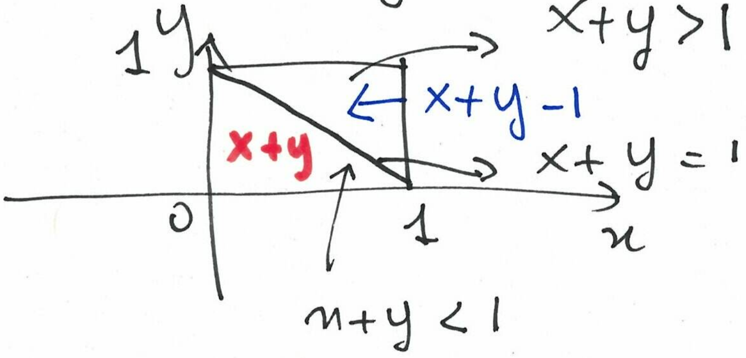
$E = \{ \{1, 7, 9\}, \{100, 8, 1000\}, \{9, 30, 20\} \}.$

$\{1, 8, 9\}.$

$\mathbb{X} = [0,1)$

For $x, y \in \mathbb{X}$ we define the so-called sum of x and y modulo 1

$$x \oplus y = \begin{cases} x+y & x+y<1 \\ x+y-1 & x+y \geq 1 \end{cases}$$



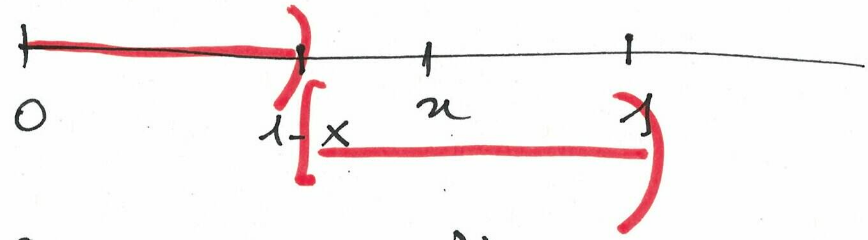
Remark

$E \subseteq \mathbb{X}$ $\mathcal{L}^1$-meas., $x \in \mathbb{X}$ $\Rightarrow$ $E \oplus x \subseteq \mathbb{X}$ $\mathcal{L}^1$-meas and $\mathcal{L}^1(E \oplus x) = \mathcal{L}^1(E)$

$$E \oplus x = \boxed{E_1 \cup E_2}$$

$$E_1 = (E \cap [0, 1-x)) \oplus x$$

$$E_2 = (E \cap [1-x, 1)) \oplus x$$



$$E_1 = \big(E \cap [0, 1-x)\big) \oplus x = \big(E \cap [0,1-x)\big) + x$$

$$\forall y \in [0, 1-x) \Rightarrow 0 \leq y \leq 1-x$$

$$\Rightarrow y + x \leq 1$$

$$E_2 = \big(E \cap [1-x, 1)\big) \oplus x = \big(E \cap [1-x,1)\big) + x - 1$$

$$\downarrow$$

$$\boxed{\text{EX}}$$

<u>See</u> <u>Serie 4</u>: $\mathcal{L}^1$ is translation invariant: $E \subseteq \mathbb{R}$ $\mathcal{L}^1$ meas, $x \in \mathbb{R} \Rightarrow x + E$ is $\mathcal{L}^1$-meas and $\mathcal{L}^1(x+E) = \mathcal{L}^1(E)$. (Thm 3.1.12)

$$\mathcal{L}^1(E_1) = \mathcal{L}^1\big(E \cap [0, 1-x)\big)$$

$$\mathcal{L}^2(E_2) = \mathcal{L}^1\big(E \cap [1-x, 1)\big)$$

$\Rightarrow$ $E \oplus x$ is $\underline{\mathcal{L}^1\text{-meas}}$

$$\mathcal{L}^1(E \oplus x) = \mathcal{L}^1(E_1) + \mathcal{L}^1(E_2)$$
$$= \mathcal{L}^1(\underbrace{E \cap [0, 1-x)}) + \mathcal{L}^1(\underbrace{E \cap [1-x, 1)})$$
$$= \mathcal{L}^1(E)$$

- In $X = [0,1)$: $x, y \in X$ $x \sim y$ iff $\boxed{x - y \in \mathbb{Q}}$

$\{ [x], \ x \in [0,1) \}$

By AC there exists $P \subseteq [0,1)$
$\boxed{P}$ consists of exactly one representative point from each equivalent class.

- Let $\mathbb{Q} \cap [0,1) = \{ r_J \}_{J \in \mathbb{N}}$, $r_0 = 0$

$\forall J \in \mathbb{N}$:
$$\boxed{P_J = P \oplus r_J}$$

- CLAIM 1 (P_J) are mutually disjoint.

$$P_{J_1} \cap P_{J_2} = \phi \text{ if } J_1 \neq J_2 .$$

Proof of claim 1

$$x \in P_{J_1} \cap P_{J_2} \quad (J_1 \neq J_2)$$

$$\Rightarrow \quad x = P_{J_1} \oplus r_{J_1} = P_{J_2} \oplus r_{J_2}$$

$$P_{J_1}, P_{J_2} \in P \qquad r_{J_1}, r_{J_2} \in \mathbb{Q}$$

$$P_{J_1} - P_{J_2} \in \mathbb{Q} \qquad P_{J_1} \sim P_{J_2}$$

$$\Rightarrow [P_{J_1}] = [P_{J_2}] \Rightarrow \boxed{P_{J_1} = P_{J_2}}$$

$$\Rightarrow r_{J_1} = r_{J_2} \Rightarrow P_{J_1} = P_{J_2}$$

$\square$

- $$\bigcup_{J=0}^{\infty} P_J = [0,1)$$

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$X = [0,1)$

$x, y \in X \quad x \sim y \iff x - y \in \mathbb{Q}$

$\{[x],\ x \in X\}$

AC $\Rightarrow \exists\ P \subseteq [0,1)$ such that

CLAIM $\quad P$ is NOT $\mathcal{L}^1$-measurable

$\mathbb{Q} \cap [0,1) = \{r_J\}. \quad J \in \mathbb{N} \quad (r_0 = 0)$

$$P_J = P \oplus r_J$$

$$x \oplus y = \begin{cases} x + y & \text{if } x + y < 1 \\ x + y - 1 & \text{if } x + 1 \geq 1 \end{cases}$$

1) $P_{J_1} \cap P_{J_2} = \emptyset$ if $J1 \neq J2$

2) $[0,1) \overset{?}{=} \bigcup_{J=1}^{\infty} P_J$

"$\subseteq$"

$x \in [0,1) \quad [x] \Rightarrow \exists!\ p \in P \quad p \sim x$

- $x = p \Rightarrow x \in P = P_0$
- $x > p \Rightarrow x = p + \underset{\mathbb{Q}}{\underset{\cap}{r_J}} = p \oplus r_J$

$\Rightarrow \underline{x \in P_J}$

$\cdot\ x < p \Rightarrow -1 < x - p < 0$

$\Rightarrow 0 < x - p + 1 < 1$

$x - p + 1 \in [0, 1)$

$x - p + 1 = r_k \in [0, 1)$

$x = p + r_k - 1 = p \oplus r_k$

$\Rightarrow x \in P_k$

$[0, 1) \subseteq \bigcup_{J=0}^{\infty} P_J$

$\square$

<u>CLAIM</u> <u>P is NOT $\mathcal{L}^1$-measurable</u>

<u>Proof of CLAIM</u>

If P were $\mathcal{L}^1$-meas. $\Rightarrow P_J$ is $\mathcal{L}^1$ meas. $\forall J$ $\quad P_J = P \oplus r_J$

$\mathcal{L}^1(P_J) = \mathcal{L}^1(P)$

$[0, 1) = \bigcup_{J=0}^{\infty} P_J$

$1 = \mathcal{L}^1([0, 1)) = \sum_{J=0}^{\infty} \mathcal{L}^1(P_J) = \sum_{J=0}^{\infty} \mathcal{L}^1(P)$

If $\mathcal{L}^1(P) = 0 \quad \Downarrow \quad 0 \neq 1$

If $\mathcal{L}^1(P) > 0 \quad \Downarrow \quad \infty \neq 1$

$\Rightarrow$ P is NOT $\mathcal{L}^1$-measurable

$\Rightarrow \exists\, B \subseteq \mathbb{R}$ such that

$$\mathcal{L}^1(B) < \mathcal{L}^1(B \cap P) + \mathcal{L}^1(B \setminus P)$$

$B \cap P$, $B \setminus P$.

- $\underline{\mathcal{L}^1(P) > 0}$ otherwise

if $\mathcal{L}^1(P) = 0 \Rightarrow P$ is $\mathcal{L}^1$-meas.

- $E \subseteq P \Rightarrow E$ is $\mathcal{L}^1$-meas and $\mathcal{L}^1(E) = 0$

<u>Proof</u> Set $E_i = E \oplus r_i \qquad \mathbb{Q} \cap [0,1) = \{r_i\}_{i \in \mathbb{N}}$

E_i is $\mathcal{L}^1$-meas. and

$\mathcal{L}^1(E_i) = \mathcal{L}^1(E)$

$F = \bigcup_{i=0}^{\infty} E_i \subseteq [0,1)$, it is $\mathcal{L}^1$-meas.

$1 = \mathcal{L}^1([0,1)) \geq \mathcal{L}^1(F) = \sum_{i=0}^{\infty} \mathcal{L}^1(E_i)$

$= \sum_{i=0}^{\infty} \mathcal{L}^1(E)$

$\Leftrightarrow \boxed{\mathcal{L}^1(E) = 0}$

Ex 1. 4. 1

$\forall A \subseteq \mathbb{R}$, $\mathcal{L}^1(A) > 0$

$\exists B \subseteq A$, B is NOT $\mathcal{L}^1$- meas.

Proof $A \subseteq [0,1)$

$B = A \cap P$, $B_i = A \cap P_i$

CLAIM B is NOT $\mathcal{L}^1$ meas

Proof of CLAIM If B is $\mathcal{L}^1$ meas

then $\mathcal{L}^1(B_i) = \mathcal{L}^1(B) = 0$

$(B_i \subseteq P_i \quad \forall i = 0, 1, \ldots) = \sum_{i=0}^{\infty} \mathcal{L}^1(B_i) = 0$

$$A = \bigcup_{i=0}^{\infty} B_i \Rightarrow \mathcal{L}^1(A) = \sum_{i=0}^{\infty} \mathcal{L}^1(B_i) = 0$$

$$\downarrow \qquad\qquad \vee \qquad\qquad \Downarrow$$

$$\qquad\qquad 0$$

$$[0,1) = \bigcup_{i=0}^{\infty} P_i \Rightarrow A = [0,1) \cap A$$

$$= \bigcup_{i=0}^{\infty} \underbrace{(P_i \cap A)}_{B_i} \qquad \blacksquare$$

1.5 CANTOR TRIADIC SET

If $A = \{a_m\}_{m\in\mathbb{N}} \subseteq \mathbb{R}$ $\mathcal{L}^1(A) = 0$

$X = [0,1]$

For any fixed integer $\boxed{b \geq 2}$, any $x \in [0,1]$ one can expand x in base b:

$$x = \sum_{i=1}^{\infty} \frac{d_i(x)}{b^i} \qquad d_i(x) \in \{0, \dots, b-1\}$$

$$x = 0.d_1 d_2 \dots$$

This expansion is NOT UNIQUE

$\boxed{b=10}$

$$0.1 = 0.0\bar{9} = 0.0999\ldots$$

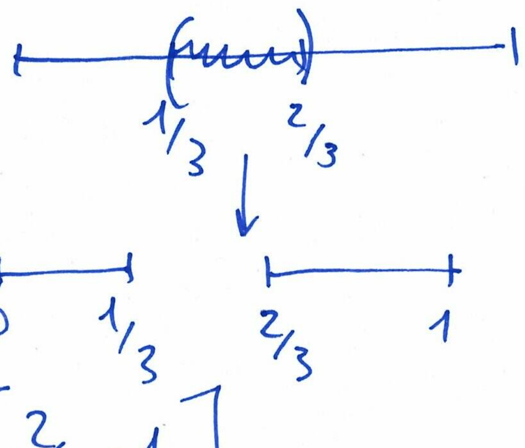
$$\frac{1}{10} = \sum_{j=2}^{\infty} \frac{9}{10^j}$$

CANTOR SET $\mathcal{C}$

Construction of a sequence of sets in $[0,1]$

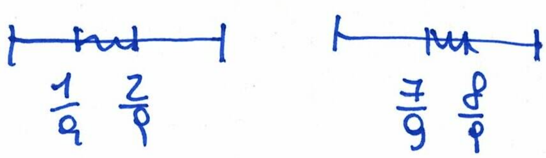
- $\mathcal{C}_0 = [0,1]$

- $\mathcal{C}_1$



$$\mathcal{C}_1 = [0, 1/3] \cup [2/3, 1]$$

$\mathcal{C}_2$



$\mathcal{C}_2 = [0, 1/9] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

⋮

$\mathcal{C}_n$ is a set ~~sets~~ which consists of 2^n disjoint closed INTERVALS of length 3^{-n}

$\Downarrow$

$$\mathcal{L}^1(\mathcal{C}_n) = \underbrace{2^n}_{\text{\# intervals}} \cdot \underbrace{\frac{1}{3^n}}_{\text{length of intervals}}$$

$$\lim_{n \to \infty} \mathcal{L}^1(\mathcal{C}_n) = 0$$

$$\mathcal{C} = \bigcap_{n=0}^{\infty} \mathcal{C}_n$$

<u>$\mathcal{C}$ is closed</u> $\qquad C_{n+1} \subseteq C_n$

$$\mathcal{L}^1(\mathcal{C}_0) = 1 < +\infty$$

$$\boxed{\mathcal{L}^1(\mathcal{C}) = \lim_{n \to +\infty} \mathcal{L}^1(\mathcal{C}_n) = 0}$$

- The set $\mathcal{C}$ has the cardinality of CONTINUUM

Recall $\forall x \in [0,1]$ can be represented with a ternary expansion $\overset{(b=2)}{\Rightarrow}$

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} = 0.a_1 a_2 a_3 \ldots$$

$a_i \in \{0, 1, 2\}$

Remark

$$\frac{1}{3} = 0.1 = 0.0222\ldots = 0.0\overline{2}$$

$$\left(\underline{Ex} \quad \sum_{j=2}^{\infty} \frac{2}{3^j} = 1 \right)$$

$$\frac{2}{3} = 0.2 = 0.1\overline{2}$$

Check

$$C_1 = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{3^k},\ a_1 \neq 1 \right\}$$

$$C_2 = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{3^k},\ a_2 \neq 1 \right\}$$

$$\vdots$$

$$C_n = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{3^k}\quad a_n \neq 1 \right\}$$

$$\Rightarrow \mathcal{C} = \underbrace{\left\{ \sum_{k=1}^{\infty} \frac{a_k}{3^k},\quad a_k \neq 1 \quad \forall k \geq 1 \right\}}.$$

$$\underbrace{(a_1, a_2, a_3, \dots)}_{\{a_n\}_{n \in \mathbb{N}}} \in \underbrace{\{0, 2\}}^{\mathbb{N}^*} \xrightarrow{f} \underbrace{x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}}$$

this has the card. of continuum

$\Rightarrow f$ is a bijection

$\Rightarrow \mathcal{C}$ has the card. of continuum as well

1.6 The Lebesgue-Stieltjes MEASURE

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing ($x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$), continuous from the left.

$$\forall x_0 \in \mathbb{R} : \quad F(x_0) = \lim_{x \to x_0^+} F(x)$$

$a, b \in \mathbb{R}$

$$\lambda_F([a,b)) = \begin{cases} F(b) - F(a) & \text{if } a < b \\ 0 & \text{otherwise} \end{cases}$$

Λ_F is the INDUCED measure of λ_F

$$\Lambda_F(A) = \inf \left\{ \sum_{k=1}^{\infty} \lambda_F([a_k, b_k)) \;\; A \subseteq \bigcup_{k=1}^{\infty} [a_k, b_k) \right\}$$

Λ_F is a Borel or even Borel regular?

Def 1.6.1

A measure μ on $\mathbb{R}^n$ is called METRIC if $\forall$ $A, B \subseteq \mathbb{R}^n$:

$\mathrm{dist}(A,B) = \inf\{|a-b| : a \in A, b \in B\} > 0$

we have $\mu(A \cup B) = \mu(B) + \mu(A)$

Thm 1.6.2

A metric measure μ on $\mathbb{R}^n$ is Borel

Proof [$F \subseteq \mathbb{R}^n$ closed $\Rightarrow$ F is μ-meas]

$\forall B \subseteq \mathbb{R}^n$:

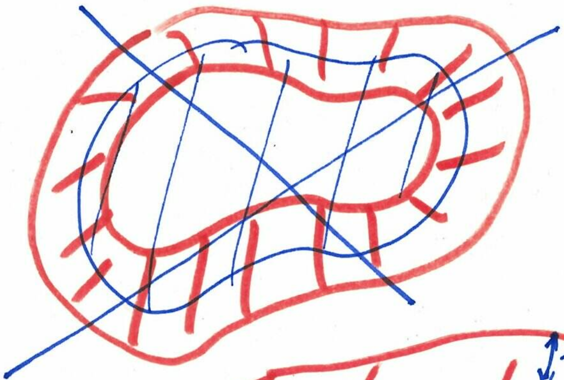
$\mu(B) \underset{\uparrow}{\geq} \mu(B \cap F) + \mu(B \setminus F)$

- If $\mu(B) = +\infty$

- $\mu(B) < +\infty$.

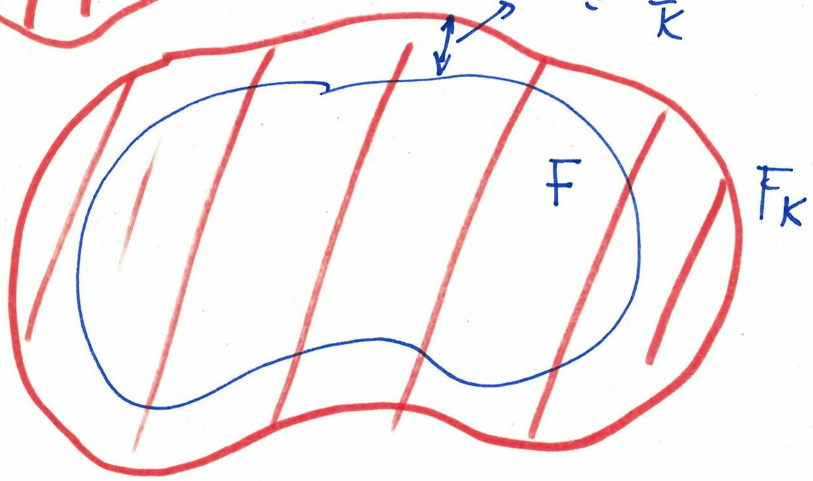
For $k \geq 1$ we define

$$F_k = \left\{ x \in \mathbb{R}^n,\ d(x,F) \leq \frac{1}{k} \right\}$$



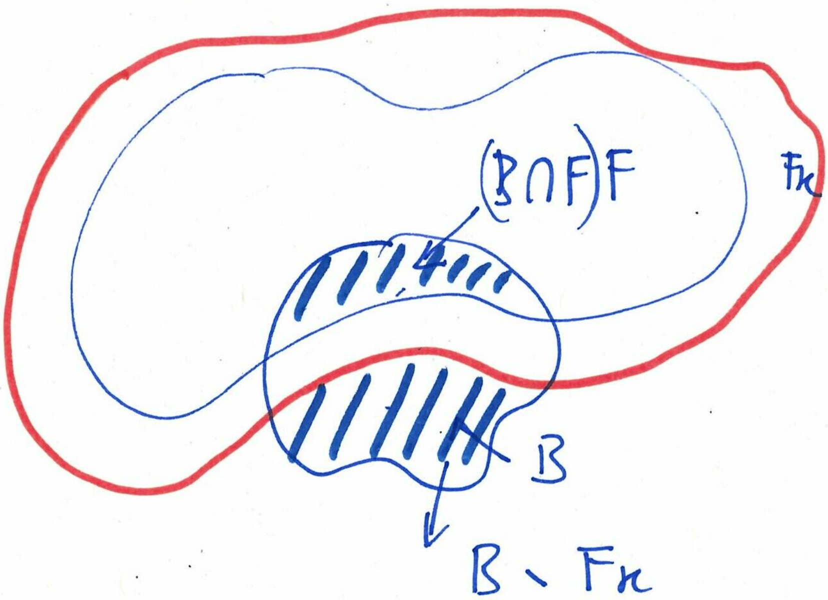
Ex: F_k is closed

$x \mapsto d(x,F)$ $\leq \frac{1}{k}$ $[0, 1/k]$



Since μ is metric and

$$\mathrm{dist}(B \setminus F_k, B \cap F) \geq \boxed{\frac{1}{k} > 0}$$



μ is metric

$$\underbrace{\mu(B\cap F)+\mu(B\setminus F_k)}=\mu\big((B\cap F)\cup(B\setminus F_k)\big)$$

$\leq \mu(B)$

by monotonicity

CLAIM $\lim\limits_{k\to+\infty}\mu(B\setminus F_k)=\mu(B\setminus F)$

If the claim holds then you

$$\mu(B\cap F)+\mu(B\setminus F)\leq\mu(B)$$

Proof of the claim

For $k \geq 1$

$$R_k = (F_k \setminus F_{k+1}) \cap B$$

$$= \left\{ x \in B : \frac{1}{k+1} < d(x, F) \leq \frac{1}{k} \right\}.$$

$$B \setminus F = B \setminus \left(\bigcap_{l=1}^{\infty} F_l\right) = \underbrace{\bigcup_{l=1}^{\infty} (B \setminus F_l)}$$

$$F = \left(\bigcap_{l=1}^{\infty} F_l\right)$$

$$= (B \setminus F_1) \cup \bigcup_{l=1}^{\infty} (B \setminus F_{l+1}) \setminus (B \setminus F_l)$$

$$= (B \setminus F_k) \cup \bigcup_{l=k}^{\infty} \underbrace{(B \setminus F_{l+1}) \setminus (B \setminus F_l)}$$

$$= (B \setminus F_k) \cup \bigcup_{l=k}^{\infty} B \cap (F_l \setminus F_{l+1})$$

$$(B \setminus F_{\ell+1}) \setminus (B \setminus F_\ell) =$$

$$= (B \cap F_{\ell+1}^c) \setminus (B \cap F_\ell^c)$$

$$= (B \cap F_{\ell+1}^c) \cap (B \cap F_\ell^c)^c$$

$$= (B \cap F_{\ell+1}^c) \cap (B^c \cup F_\ell)$$

$$= \underbrace{[(B \cap F_{\ell+1}^c) \cap F_\ell]} \cup [\underbrace{(B \cap F_{\ell+1}^c) \cap B^c}_{\emptyset}]$$

$$= B \cap (F_\ell \cap F_{\ell+1}^c) = B \cap (F_\ell \setminus F_{\ell+1})$$

$$= \boxed{R_\ell}.$$

$$B \setminus F = (B \setminus F_k) \cup \bigcup_{\ell=k}^{\infty} R_\ell$$

$$\mu(B \setminus F) \leq \mu(B \setminus F_k) + \sum_{\ell=k}^{\infty} \mu(R_\ell)$$

$$\boxed{k \to +\infty \Rightarrow \sum_{\ell=k}^{\infty} \mu(R_\ell) = 0}$$

$\sum_{l=k}^{\infty} \mu(R_l)$ is exactly the rest of $\sum_{l=1}^{\infty} \mu(R_l)$

CLAIM $\sum_{l=1}^{\infty} \mu(R_l) < +\infty$

if i, j $|i-j| \geq 2 \Rightarrow \text{dist}(R_i, R_j) > 0$

$\downarrow$ $\boxed{\text{EX}}$

Since μ is metric it follows by induction that

① $\sum_{k=1}^{m} \mu(R_{2k}) = \mu\left(\bigcup_{k=1}^{m} R_{2k}\right) \leq \mu(B)$

$\text{dist}(R_{2k_1}, R_{2k_2}) > 0$

$k_1 \neq k_2$

② $\sum_{k=0}^{m} \mu(R_{2k+1}) = \mu\left(\bigcup_{k=0}^{m} R_{2k+1}\right) \leq \mu(B)$

$$① + ② \Rightarrow \sum_{k=1}^{m} \mu(R_{2k}) + \sum_{k=0}^{m} \mu(R_{2k+1})$$

$$= \sum_{k=1}^{2m+1} \mu(R_k) \leq 2\mu(B)$$

Let $m \to +\infty \Rightarrow \sum_{1}^{\infty} \mu(R_k) \leq 2\mu(B) < +\infty$ ∎

$$\mu(B \setminus F) \leq \mu(B \setminus F_k) + \underbrace{\sum_{l \geq k}^{\infty} \mu(R_l)}_{\to 0}$$

$$\rotatebox{90}{$\leq$} \Leftarrow \boxed{F \subseteq F_k}$$

$$\mu(B \setminus F_k)$$

$$\Rightarrow \lim_{k \to +\infty} \mu(B \setminus F_k) = \mu(B \setminus F)$$ ∎

Thm 1.6.4 The Lebesgue - Stieltjes measure is Borel regular

Proof

- Λ_F is Borel
- Λ_F is regular.

Λ_F is metric ($\Rightarrow \Lambda_F$ is Borel).

If $A, B \subseteq \mathbb{R}$: $\delta = \text{dist}(A,B) > 0$

$\Rightarrow \Lambda_F(A \cup B) = \Lambda_F(A) + \Lambda_F(B)$.

For $\varepsilon > 0$, choose $a_k, b_k \in \mathbb{R}$

$$\bigcup_{k=1}^{\infty} [a_k, b_k) \subseteq A \cup B$$

$$\sum_{k=1}^{\infty} \lambda_F([a_k, b_k)) \leq \Lambda_F(A \cup B) + \varepsilon.$$

Up to a sub-division of $[a_k, b_k)$ we can suppose that $|b_k - a_k| < \delta/2$

Since $\operatorname{dist}(A,B) > \delta > 0$

$\Rightarrow \forall k \in \mathbb{N}$ $\quad A \cap [a_k, b_k) \neq \emptyset$ or

$B \cap [a_k, b_k) \neq \emptyset$

$\Rightarrow$ The covering $\left([a_k, b_k)\right)_k$ is decomposed in a covering $\mathcal{A}$ of A and a covering $\mathcal{B}$ of B.

$$\Lambda_F(A \cup B) \leq \Lambda_F(A) + \Lambda_F(B)$$

$$\leq \sum_{[a_k,b_k) \in \mathcal{A}} \lambda_F([a_k, b_k)) + \sum_{[a_k,b_k) \in \mathcal{B}} \lambda_F([a_k, b_k))$$

$$= \sum_{k=1}^{\infty} \lambda_F([a_k, b_k))$$

$$\leq \Lambda_F(A \cup B) + \varepsilon$$

Let $\varepsilon \to 0$ $\quad \Rightarrow \Lambda_F(A \cup B) = \Lambda_F(A) + \Lambda_F(B)$

$\blacksquare$

ii) <u>Λ_F is Borel regular</u>

Let $A \subseteq \mathbb{R}$ ' $\Lambda_F(A) < +\infty$

(if $\Lambda_F(A) = +\infty \Rightarrow B = \mathbb{R}$)

For $J \in \mathbb{N}$ we choose $\left([a_k^J, b_k^J)\right)_k$

$$A \subseteq \bigcup_{k=0}^{\infty} [a_k^J, b_k^J)$$

$$\sum_{k=1}^{\infty} \lambda_F\left([a_k^J, b_k^J)\right) \leq \Lambda_F(A) + \frac{1}{J}$$