LECTURE 18.03.2020

LEBESGUE - STIELTJES MEASURE in $\mathbb{R}$

$F: \mathbb{R} \to \mathbb{R}$ increasing and left-cont: $\forall x_0 \in \mathbb{R}$

$$\lim_{x \to x_0^-} F(x) = F(x_0)$$

$$\lambda_F([a,b)) = \begin{cases} F(b) - F(a) & \text{if } a < b \\ 0 & \text{otherwise} \end{cases}$$

$\boxed{a, b \in \mathbb{R}}$. For $A \subseteq \mathbb{R}$

$$\Lambda_F(A) = \inf\left\{ \sum_{k=1}^{\infty} \lambda_F([a_k, b_k)), \; A \subseteq \bigcup_{k=1}^{\infty} [a_k, b_k) \right\}$$

Λ_F is called Lebesgue-Stieltjes meas. generated by F.

$\boxed{P.S}$ F is increasing and right-cont

$$\lambda_F((a, b])$$

Λ_F is Borel Measure (1st part of Thm 1.6.4)

<u>TODAY</u>

<u>Λ_F is Borel regular</u>

$\forall A \subseteq \mathbb{R} \ \exists B \subseteq \mathbb{R} : A \subseteq B$
B Borel and $\Lambda_F(A) = \Lambda_F(B)$

<u>Proof</u>

Suppose w.l.o.g $\Lambda_F(A) < +\infty$
(if $\Lambda_F(A) = +\infty \Rightarrow B = \mathbb{R}$)

$\forall J \in \mathbb{N} : \left([a_K^J, b_K^J) \right)_{K \in \mathbb{N}}$

$$A \subseteq \underbrace{\bigcup_{K=0}^{\infty} [a_K^J, b_K^J)}_{} = B_J$$

and $\displaystyle\sum_{K=0}^{\infty} \lambda_F\left([a_K^J, b_K^J)\right) \leq \Lambda_F(A) + \frac{1}{J}$.

$$\boxed{A \subseteq B = \bigcap_{J=1}^{\infty} B_J}$$

<u>B is Borel</u>

$$\Lambda_F(A) \leq \Lambda_F(B) \leq \Lambda_F(B_J) \leq \sum_{k=0}^{\infty} \lambda_F\left(\left[a_k^J, b_k^J\right)\right) \leq \underbrace{\Lambda_F(A) + \frac{1}{J}}_{\to \Lambda_F(A)}$$

Let $J \to +\infty$

$$\Rightarrow \Lambda_F(A) = \Lambda_F(B) \quad \square$$

Thm 1.6.5

For $a < b$ it holds

$$\Lambda_F([a,b)) = F(b) - F(a) = \lambda_F([a,b))$$

Proof $\quad a, b \in \mathbb{R}$ $\boxed{a < b}$

- $\Lambda_F([a,b)) \leq \lambda_F([a,b)) = F(b) - F(a)$
- "$\geq$" $\Lambda_F[a,b) \geq \lambda_F[a,b)$

$$[a,b) \subseteq \bigcup_{k=0}^{\infty}\left([a_k, b_k)\right)$$

F cont. from the left $\Rightarrow \forall \varepsilon > 0$
$\exists\, \delta, \delta_k > 0$ such that

$$F(b) - F(b-\delta) < \varepsilon$$

$$F(a_k) - F(a_k - \delta_k) < \varepsilon/_{2^k} \quad k \geq 0$$

We observe that

$$[a, b-\delta] \subseteq \bigcup_{k=0}^{\infty} (a_k - \delta_k, b_k)$$

- $\underline{a_k - \delta_k < b_{k-1}}$ $\quad \forall k \geq 1 \Rightarrow F(a_{k-\delta_k}) < F(b_{k-1})$

- $[a, b-\delta]$ is compact $\Rightarrow$

$$[a, b-\delta] \subseteq \bigcup_{k=0}^{m} (a_k - \delta_k, b_k) \quad \boxed{m \geq 1}$$

$$\lambda_F([a, b-\delta]) = F(b-\delta) - F(a)$$
$$\leq \underline{\sum_{k=0}^{m} F(b_k) - F(a_k - \delta_k)}$$
$$= \sum_{k=0}^{m} \lambda_F([a_k - \delta_k, b_k))$$

$$F(b-\delta) - F(a) \leq F(b_m) - F(a_0 - \delta_0)$$

$$= \underbrace{F(b_m) - F(a_m - \delta_m)} + F(a_m - \delta_m) - F(a_0 - \delta) \quad (*)$$

you use $a_m - \delta_m < b_{m-1}$

$\Rightarrow F(a_m - \delta_m) < F(b_{m-1})$

$$(*) \leq F(b_m) - F(a_m - \delta_m) + F(b_{m-1}) - F(a_0 - \delta_0)$$

$$= \underbrace{\qquad\qquad}_{1} + \underbrace{F(b_{m-1}) - F(a_{m-1} - \delta_{m-1})}_{2}$$

$$+ \underbrace{F(a_{m-1} - \delta_{m-1})}_{\leq F(b_{m-2})} - F(a_0 - \delta)$$

$$\leq 1 + 2 + F(b_{m-2}) - F(a_0 - \delta)$$

λ_F

$\lambda_F[a,b) \leq \Lambda_F([a,b))$

Therefore

$$F(b)-F(a) = \underbrace{F(b)-F(b-\delta)}_{<\varepsilon} + F(b+\delta)-F(a)$$

$$\leq F(b-\delta)-F(a) + \varepsilon$$

$$\leq \sum_{k=0}^{m} F(b_k)-F(a_k-\delta_k) + \varepsilon$$

$$= \sum_{k=0}^{m} F(b_k)-F(a_k) + \sum_{k=0}^{m} F(a_k)-F(a_k-\delta_k) + \varepsilon$$

$$\leq \sum_{k=0}^{\infty} F(b_k)-F(a_k) + \underbrace{\sum_{k=0}^{\infty} F(a_k)-F(a_k-\delta_k)}_{} + \varepsilon$$

$$= \sum_{k=0}^{\infty} F(b_k)-F(a_k) + \underbrace{\sum_{k=0}^{\infty} \varepsilon/2^k}_{} + \varepsilon \quad (*)$$

$$\sum_{k=0}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon \cdot \sum_{k=0}^{\infty} \frac{1}{2^k} = \varepsilon \cdot \frac{1}{1-\frac{1}{2}} = 2\varepsilon$$

$$= \sum_{k=0}^{\infty} F(b_k)-F(a_k) + 3\varepsilon$$

$$= \sum_{k=0}^{\infty} \lambda_F([a_k,b_k)) + 3\varepsilon$$

Take the inf. over all covering $([a_k, b_k))_k$ of $[a,b)$ and then let $\varepsilon \to 0$

$$F(b) - F(a) \leq \lambda_F([a,b))$$
$$\leq \Lambda_F([a,b))$$

$\square$

- $F(x) = x \Rightarrow \Lambda_F = \mathcal{L}^1$
- $F(x) = 1 \quad x > 0, \quad F(x) = 0 \quad x \leq 0$

$$\Rightarrow \Lambda_F = \delta_0$$

$$\delta_0(A) = \begin{cases} 1 & 0 \in A \\ 0 & \text{otherwise} \end{cases}$$

1.7 Hausdorff Measures

For $s \geq 0$, $\delta > 0$

For $A \subseteq \mathbb{R}^n$

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{k=0}^{\infty} r_k^s : A \subseteq \bigcup_{k=0}^{\infty} B(x_k, r_k), \ 0 < r_k < \delta \right\}$$

$\mathcal{H}^s_\delta(\emptyset) = 0$

$$B(x_k, r_k) = \{ y \in \mathbb{R}^n : |x_k - y| < r_k \}$$

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s, A fixed $\quad \delta \mapsto \mathcal{H}^s_\delta(A)$ is

non increasing: if $\delta_1 < \delta_2$

$$\mathcal{H}^s_{\delta_1}(A) \geq \mathcal{H}^s_{\delta_2}(A)$$

$\boxed{0^0}$

Every covering of A of balls of radius $r_k < \delta_1$ is also a covering of balls of radius $< \delta_2$

$\Rightarrow$ There exists the limit

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A)$$

<u>Def 1.7.1</u>

$\mathcal{H}^s$ is called the s-dimensional Hausdorff measure on $\mathbb{R}^n$.

<u>Thm 1.7.2</u>

For $s \geq 0$ $\mathcal{H}^s$ is a Borel regular measure on $\mathbb{R}^n$.

<u>Proof</u>

i) $\mathcal{H}^1$ is a measure

20.03.2020

Thm $\forall s \geq 0$ $\mathcal{H}^s$ is a Borel regular measure on $\mathbb{R}^n$.

Proof

i) $\mathcal{H}^s$ is a measure

- $\mathcal{H}^s_\delta(\emptyset) = 0 \Rightarrow \mathcal{H}^s(\emptyset) = 0$

ii) $A \subseteq \bigcup_{k=1}^\infty A_k \subseteq \mathbb{R}^n$

$$\mathcal{H}^s(A) \leq \sum_{k=1}^\infty \mathcal{H}^s(A_k)$$

$\forall \delta > 0$ $\mathcal{H}^s_\delta$ is σ-sub-additive:

$$\mathcal{H}^s_\delta(A) \leq \sum_{k=1}^\infty \mathcal{H}^s_\delta(A_k)$$

$$\leq \sum_{k=1}^\infty \mathcal{H}^s(A_k)$$

Let $\delta \to 0$

$$\mathcal{H}^s(A) \leq \sum_{k=1}^\infty \mathcal{H}^s(A_k) \quad \square$$

ii) $\mathcal{H}^s$ is Borel

It is enough to show that $\mathcal{H}^s$ is metric: $\forall A, B \subseteq \mathbb{R}^n$ $\text{dist}(A,B) = \delta_0 > 0 \Rightarrow$

$$\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B)$$

. Take $\delta < \delta_0/4$ and

$$A \cup B \subseteq \bigcup_{k=1}^{\infty} B(x_k, r_k) \qquad 0 < r_k < \delta$$

$\mathcal{A} = \{ B(x_k, r_k) : \quad B(x_k, r_k) \cap A \neq \emptyset \}$

$\mathcal{B} = \{ B(x_k, r_k) : \quad B(x_k, r_k) \cap B \neq \emptyset \}.$

$$A \subseteq \bigcup_{\mathcal{A}} B(x_k, r_k)$$

$$B \subseteq \bigcup_{\mathcal{B}} B(x_k, r_k)$$

$\boxed{\mathcal{A} \cap \mathcal{B} = \emptyset}$

$B(x_k, r_k) \in \mathcal{A} \cap \mathcal{B}$

$B(x_k, r_k) \cap A \neq \emptyset$, $\quad B(x_k, r_k) \cap B \neq \emptyset$

$x \in \qquad\qquad\qquad y \in$

$$\operatorname{dist}(A,B) \le |x-y| \le |x - x_k| + |x_k - y|$$
$$\overset{\shortparallel}{\delta_0} \qquad \le r_k + r_k = 2r_k$$
$$\le 2\delta$$
$$\le \cancel{2}\, \delta_0/4$$

$\rightsquigarrow \quad \delta_0 \le \delta_0/4 \quad \downarrow \quad \square$

$$\mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B) \le \sum_{\mathcal{A}} r_k^s + \sum_{\mathcal{B}} r_k^s$$
$$\quad\ \ \geq \qquad\qquad\qquad\qquad\qquad = \sum_{k=1} r_k^s$$
$$\mathcal{H}^s_\delta(A \cup B)$$

Take the inf. over all coverings of $A \cup B$

$$\mathcal{H}^s_\delta(A \cup B) \le \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B) \le \mathcal{H}^s_\delta(A \cup B)$$

$\boxed{\delta \to 0} \implies \mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B)$

$\square$

· $\mathcal{H}^s$ is regular

$\forall A \subseteq \mathbb{R}^n \quad \exists B \subseteq \mathbb{R}^n$ Borel

$A \subseteq B$ and $\mathcal{H}^s(A) = \mathcal{H}^s(B)$.

Assume $\mathcal{H}^s(A) < +\infty$

$\Rightarrow \forall \delta > 0 \quad \mathcal{H}^s_\delta(A) < +\infty$.

For $\delta = \frac{1}{\ell}$ $\quad \ell \geq 1$

$$\bigcup_{k=1}^{\infty} B(x_{k,\ell}, r_{k,\ell}) \supseteq A$$

$$\sum_{k=1}^{\infty} r_{k,\ell} \leq \mathcal{H}^s_{1/\ell}(A) + \frac{1}{\ell}$$

Set $A_\ell := \bigcup_{k=1}^{\infty} B(x_{k,\ell}, r_{k,\ell})$, $A \subseteq A_\ell$

$B = \bigcap_{\ell=1}^{\infty} A_\ell \Rightarrow A \subseteq B$, B Borel

For every ℓ

$$\mathcal{H}^s_{\frac{1}{\ell}}(B) \leq \mathcal{H}^s_{1/\ell}(A_\ell) \leq \sum_{k=1}^{\infty} r_{k,\ell}^s$$

$$\leq \mathcal{H}^s_{1/\ell}(A) + \frac{1}{\ell}$$

Let $\ell \to +\infty$

$$\mathcal{H}^s(A) \underset{\text{MONOT}}{\leq} \mathcal{H}^s(B) \leq \mathcal{H}^s(A)$$

∎

REMARK

- $s=0$ $\mathcal{H}^0$ is the so-called counting measure:

let $k>0$, $E \in \mathbb{R}^n$ $E=\{x_1, x_2, \dots, x_k\}$

$\delta_0 = \inf\{ d(x_i, x_j),\ x_i; x_j \in E \ x_i \neq x_j \}$

Every covering of balls of radius $\delta < \delta_0$ consists at least of k ~~sets~~

$\Rightarrow \mathcal{H}^0_\delta(E) \geq k$

$$\bigcup_{i=1}^{k} B(x_i, \delta) \qquad \delta < \delta_0/4$$

$$r_i = \delta$$

$$\underbrace{\sum_{i=1}^{k} \delta^0}_{} = \sum_{i=1}^{k} 1 = k$$

$\Rightarrow \mathcal{H}^0_\delta(E) = k \leftarrow \forall \delta < \delta_0/4$

Let $\delta \to 0 \Rightarrow \mathcal{H}^0(E) = k = \# E$

If E is infinite then by monotonicity

$\mathcal{H}^0(E) \geq k \quad \forall k \geq 1$

let $k \to +\infty \Rightarrow \mathcal{H}^0(E) = +\infty$

By def $\quad \mathcal{H}^0(\emptyset) = 0$

$\boxed{s=0} \to$ all subsets of $\mathbb{R}^n$ are measurable.

LEMMA 1.7.4

$A \subseteq \mathbb{R}^n \quad 0 \leq s < t < +\infty$

It holds

i) If $\mathcal{H}^s(A) < +\infty \Rightarrow \mathcal{H}^t(A) = 0$

ii) If $\mathcal{H}^t(A) > 0 \Rightarrow \mathcal{H}^s(A) = +\infty$

Proof

i) Let $\mathcal{H}^s(A) < +\infty$. For $\delta > 0$

$$A \subseteq \bigcup_{k=1}^{\infty} B(x_k, r_k) \quad r_k < \delta$$

$$\mathcal{H}^t_\delta(A) \leq \sum_{k=1}^{\infty} r_k^t = \sum_{k=1}^{\infty} r_k^{(t-s)} r_k^s \leq$$

$t-s>0$

$$\le \sum_{k=1}^{\infty} \delta^{t-s} r_k^s = \delta^{t-s} \sum_{k=1}^{\infty} r_k^s$$

$$\mathcal{H}_\delta^t(A) \le \underbrace{\delta^{t-s}}_{\to 0} \mathcal{H}_\delta^s(A)$$

$\underline{\underline{\delta \to 0}} \Rightarrow \mathcal{H}^t(A) = 0$ $\blacksquare$

ii) If $\mathcal{H}^t(A) > 0$ then $\mathcal{H}^s(A) = +\infty$

EXAMPLE

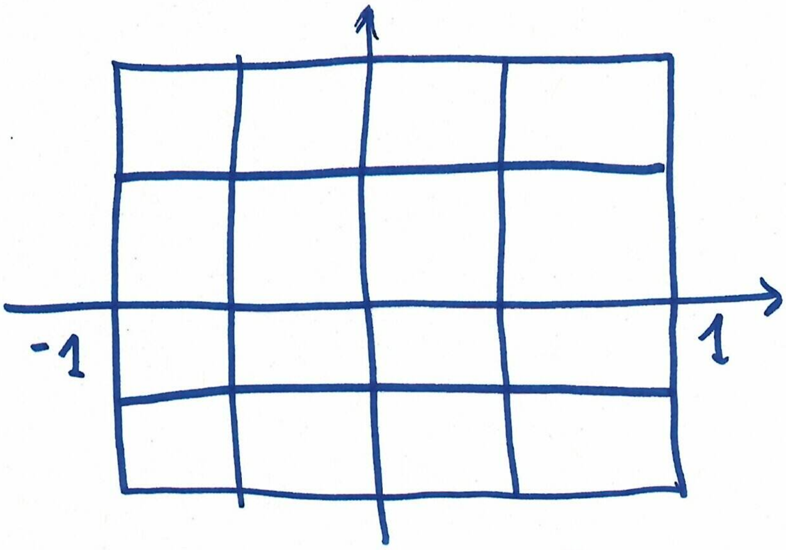
$Q = [-1,1]^n \subseteq \mathbb{R}^n$.

$$2^{-n} \mathcal{L}^n(Q) \le \mathcal{H}^n(Q) \le 2^{-n} n^{\frac{n}{2}} \mathcal{L}^n(Q)$$

__Proof__ Let $\delta > 0$ and take $k > 0$

$r = \sqrt{n}\, 2^{-k-1} < \delta$

$Q \to$ subcubes Q_ℓ of edge length 2^{-k} $1 \le \ell \le 2^{(k+1)n}$.



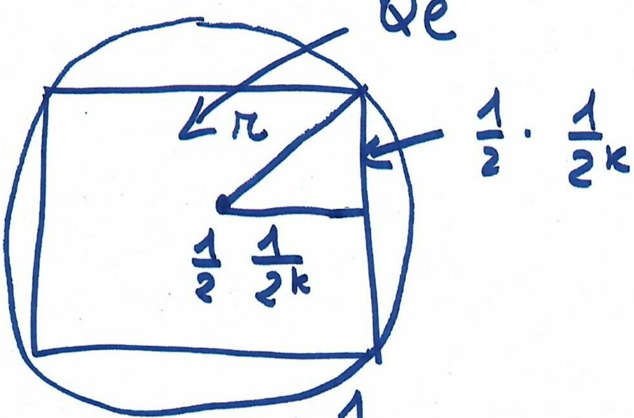
$n = 2$

$k = 1$

$\frac{1}{2^k}$

$l = 2^4 = 16$

$= 2^{(1+1)2}$



$$r = \left(\sum_{i=1}^{n} \left(\frac{1}{2\, 2^k} \right)^2 \right)^{1/2} = \frac{1}{2^{k+1}} \, n^{1/2} < \delta$$

$$\mathcal{H}^n_\delta (Q) \leq \sum_{1 \leq l \leq 2^{(k+1)n}} r^n$$

$$= n^{n/2} \frac{1}{\cancel{2^{(k+1)n}}} \cdot 2^{\cancel{(k+1)}\, n}$$

$$= n^{n/2}\, 2^n\, 2^{-n} = n^{n/2}\, 2^{-n} \mathcal{L}^n(Q)$$

Conversely

If $(B(x_k, r_k))_k$ $r_k < \delta$ is covering of $B(0,1)$

$$w_n = \mathcal{L}^n(B(0,1)) \leq \sum_{k=1}^{\infty} \underbrace{\mathcal{L}^n(B(x_k, r_k))}$$

$$= \mathcal{L}^n(B(0, r_k))$$

$$= r_k^n \cdot \underbrace{\mathcal{L}^n(B(0,1))}_{w_n}$$

$$= w_n \sum_{k=1}^{\infty} r_k^n$$

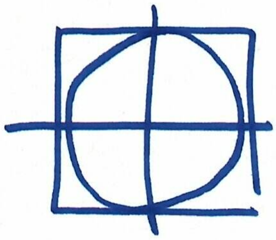
$$\Rightarrow \boxed{\sum_{k=1}^{\infty} r_k^n \geq 1}$$

$$w_n = \frac{\pi^n}{\Gamma\left(1+\frac{n}{2}\right)}$$

$$\cancel{\Gamma\left(1+\frac{n}{2}\right)} = \Gamma(t) = \int_0^{+\infty} x^{t-1} e^{-x} \, dx$$

$t > 0$

$$\mathcal{H}^n_\delta(Q) \geq \mathcal{H}^n_\delta(B(0,1))$$



$$= \inf\Big\{ \sum_{k=1}^{\infty} r_k^n \;\; B(0,1) \subseteq \bigcup_{k=1}^{\infty} B(x_k, r_k),\; 0 < r_k < \delta \Big\}$$

$$\geq 1$$

$\Rightarrow$ let $\delta \to 0$

$$\mathcal{H}^n(Q) \geq 1 = 2^{-n} 2^n = 2^{-n} \mathcal{L}^n(Q)$$

<u>Remark</u>

1. If $A \subseteq \mathbb{R}^n$ $\mathcal{L}^n$-meas. $\Rightarrow \mathcal{L}^n(A) = \omega_n \mathcal{H}^n(A)$

2. $\forall A \subseteq \mathbb{R}^n$ if $\underline{s > n} \Rightarrow \mathcal{H}^s(A) = 0$

$\mathbb{R}^n = \bigcup_{l=1}^{\infty} Q_l$ $\quad Q_l$ <u>dyadic cubes</u>

$0 < \mathcal{H}^n(Q_l) < +\infty \Rightarrow \mathcal{H}^s(Q_l) = 0$

$\Rightarrow \mathcal{H}^s(\mathbb{R}^n) = 0 \Rightarrow \mathcal{H}^s(A) = 0$

Def The Hausdorff dimension of a set $A \subseteq \mathbb{R}^n$ is defined to be

$$\dim_{\mathcal{H}}(A) = \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\}$$

Remark

1) $\dim_{\mathcal{H}}(A) = \sup\{s \geq 0 : \mathcal{H}^s(A) = +\infty\}$.
(Ex)

2) $A \subseteq \mathbb{R}^n$ $\quad \mathcal{H}^s(A) = 0 \ \forall s > n$

$\Rightarrow \dim_{\mathcal{H}}(A) \leq n$

3) $\underline{s_0 = \dim_{\mathcal{H}}(A)}$: $\quad \mathcal{H}^t(A) = 0 \ \forall t > s_0$

$\mathcal{H}^t(A) = +\infty \ \forall t < s_0$

Actually: $\mathcal{H}^t(A) < +\infty$ $\ \cancel{\forall}\ \exists t < s_0$

$\Rightarrow \mathcal{H}^{\tilde{t}}(A) = 0 \ \forall \tilde{t} > t$

$t < \tilde{t} < s_0$

$t \ (\text{---} \ \tilde{t} \ \text{---}) \ s_0$

In general $0 \leq \mathcal{H}^{s_0}(A) \leq +\infty$ and $\dim_H(A)$ is NOT an integer.

Even if $k = \dim_H(A)$ and

$$0 < \mathcal{H}^k(A) < +\infty$$

A need NOT to be a "k-dimensional surface".

4. $A \subseteq \mathbb{R}^n$ open $\Rightarrow$ $\boxed{\dim_H(A) = n}$

- $\dim_H(A) \leq n$
- ~~A~~ A open $\Rightarrow$ A is $\mathcal{L}^n$-meas and $\mathcal{L}^n(A) = \omega_n \mathcal{H}^n(A)$

  Since A open $\Rightarrow$ $\exists B$ ball $B \subseteq A$ $\Rightarrow$ $\mathcal{L}^n(A) \geq \mathcal{L}^n(B) > 0$

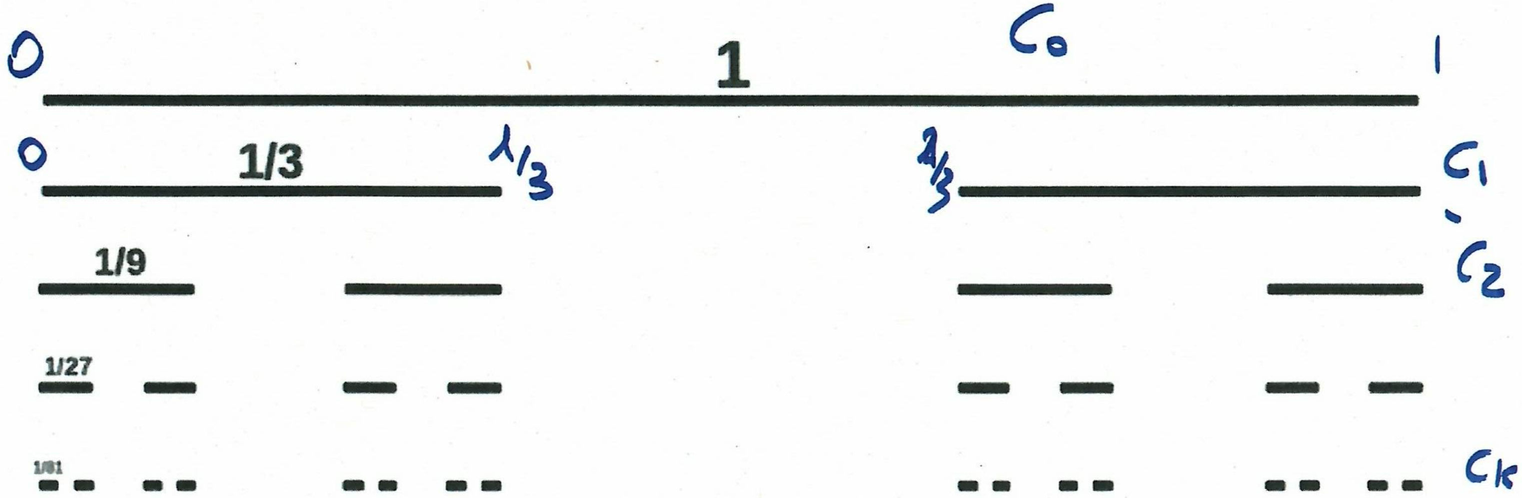
  $\Rightarrow \mathcal{L}^n(A) > 0 \Rightarrow \mathcal{H}^n(A) > 0$

  $\Rightarrow \mathcal{H}^s(A) = 0 \quad \forall s > n$

$\dim_H(A) \geq n$

$\Rightarrow \boxed{\dim_H(A) = n}$

Triadic Cantor Set



$$C = \lim_{k \to \infty} C_k = \bigcap_{k=1}^{\infty} C_k \qquad \mathcal{L}^1(C) = 0$$

$$\dim_H C = \frac{\ln 2}{\ln 3} = s_0 < 1$$

and $\underbrace{2^{-s_0-1} < \mathcal{H}^{s_0}(C) <}\ 2^{-s_0}$

Ex 5 of Serie 5

Heuristics

- $\forall s > 0 \ \forall \lambda > 0 \ \forall A \subseteq \mathbb{R}^n$

$$\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$$

$\boxed{Ex}$

- If there is $d \geq 0$:

$$\mathcal{H}^d(C) \in (0, +\infty)$$

then since C is the union of two copies of $\frac{C}{3}$

$$\cancel{\mathcal{H}^d(C)} = 2\, \mathcal{H}^d\left(\frac{C}{3}\right)$$

$$= \frac{2}{3^d} \cancel{\mathcal{H}^d(C)}$$

$$\frac{2}{3^{\alpha}} = 1 \iff 3^{\alpha} = 2$$

$$\alpha = \log_3 2 = \frac{\ln 2}{\ln 3}$$

$$\ln a = \log_e a$$

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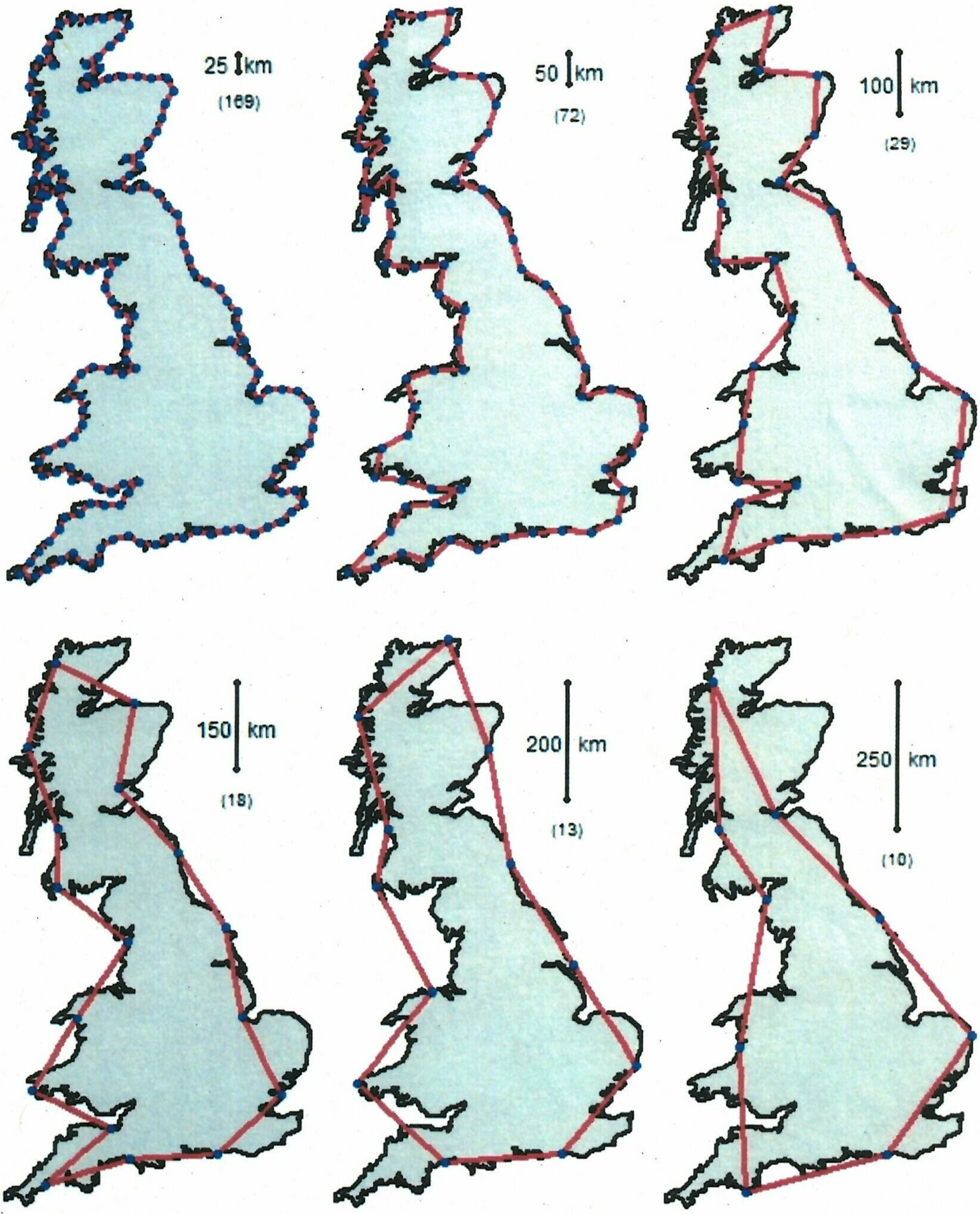
2) <u>CANTOR DUST</u>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Benoit Mandelbrot.

The fractal geometry of nature