LECTURE 4.03.2020

Thm 1.2.17

K is a covering of X
$\lambda : K \to [0,+\infty]$, $\lambda(\emptyset) = 0$
Then

$$\mu(A) = \inf\left\{ \sum_{j=1}^{\infty} \lambda(K_j) : K_j \in K, \ A \subseteq \bigcup_{j=1}^{\infty} K_j \right\}$$

$\mu : \mathcal{P}(X) \to [0,+\infty]$ is measure.

If $\lambda : \mathcal{A} \to [0,+\infty]$ is pre-measure
($\lambda(\emptyset) = 0$, $\lambda(A) = \sum_{j=1}^{\infty} \lambda(A_j)$ $A \in \bigcup_{j=1}^{\infty} A_j$
$A_j, A \in \mathcal{A}$, $A_{j_1} \cap A_{j_2} = \emptyset$)

$\Rightarrow$ • μ is a measure
• $\mu(A) = \lambda(A)$ $A \in \mathcal{A}$ $\mu_{|\mathcal{A}} = \lambda$
• $\forall A \in \mathcal{A} \Rightarrow \mathcal{A}$ is μ-measurable.

Thm 1.2.21 (UNIQUENESS)

Let $\lambda : \mathcal{A} \to [0,+\infty]$ be a pre-measure σ-finite ($X = \bigcup_k S_k$, $S_k \in \mathcal{A}$, $\lambda(S_k) < +\infty$)

We denote by Σ the σ-algebra of μ-meas. and let $\tilde{\mu} : \mathcal{P}(X) \to [0,+\infty]$ be another measure $\tilde{\mu}_{|\mathcal{A}} = \lambda$

$$\Rightarrow \boxed{\tilde{\mu}_{|\Sigma} = \mu}$$

Remarks

- It is not clear if $\forall A \in \Sigma$ is $\tilde{\mu}$-measurable
- $X = [0,1]$ $\mathcal{A} = \{\phi, X\}$

  $\lambda : \mathcal{A} \to [0,+\infty]$ $\lambda(\phi) = 0$, $\lambda(X) = 1$

  $\to \mu(A) = 1$ if $A \neq \phi$, $\mu(\phi) = 0$

  $$\boxed{\Sigma = \{\phi, X\}}$$

  $\to \mathcal{L}^1$ Lebesgue-measure

  $\mathcal{L}^1([0,1/2]) = 1/2 \neq \mu[0,1/2] = 1$

Proof

i) $\forall A \in \mathcal{P}(X)$

$$\tilde{\mu}(A) \leq \mu(A)$$

$$A \subseteq \bigcup_{k=1}^{\infty} A_k \ , \ A_k \in \mathcal{A} \quad \forall k \in \mathbb{N}$$

$\uparrow$

σ- sub-add. of $\tilde{\mu}$

$$\tilde{\mu}(A) \leq \sum_{k=1}^{\infty} \tilde{\mu}(A_k) = \sum_{k=1}^{\infty} \lambda(A_k)$$

Take the inf over all $(A_k)_k$ and you get

$$\boxed{\tilde{\mu}(A) \leq \mu(A)} \quad (1)$$

ii) $A \in \Sigma$

$$\mu(A) \leq \tilde{\mu}(A).$$

- $\exists S \in \mathcal{A} \quad \lambda(S) < +\infty \quad A \subseteq S$

$$\tilde{\mu}(S \setminus A) \underset{(1)}{\leq} \mu(S \setminus A) \leq \mu(S) = \lambda(S) < +\infty$$

$$\hat{\mu}(A) + \tilde{\mu}(S \setminus A) \underset{(1)}{\leq} \mu(\underset{\mu(A \cap S)}{A}) + \mu(S \setminus A) \underset{A \text{ is } \mu\text{-meas}}{=} \mu(S)$$

$$= \tilde{\mu}(S) = \lambda(S)$$

$$\underset{\text{sub-add of } \tilde{\mu}}{\leq} \tilde{\mu}(S \cap A) + \hat{\mu}(S \setminus A)$$

$$= \tilde{\mu}(A) + \tilde{\mu}(S \setminus A)$$

$$\mu(A) = \mu(S) - \mu(S \setminus A)$$

$$= \tilde{\mu}(S) - \mu(S \setminus A)$$

$$\leq \hat{\mu}(S) - \tilde{\mu}(S \setminus A) \quad \left(\tilde{\mu}(S) \leq \tilde{\mu}(A) + \hat{\mu}(S \setminus A)\right)$$

$$\leq \tilde{\mu}(A)$$

$\blacksquare$

- $X = \bigcup_{k=1}^{\infty} S_k, \quad \lambda(S_k) < +\infty, \quad S_{k_1} \cap S_{k_2} = \emptyset$

$\boxed{A \in \Sigma}$ $\qquad A = A \cap X = \bigcup_{k=1}^{\infty} \underbrace{(A \cap S_k)}_{A_k}$

$A_k := A \cap S_k$

For every $m \geq 0$

$$B_m = \bigcup_{k=1}^{m} A_k \subseteq \bigcup_{k=1}^{m} S_k = S_m$$

$$\Rightarrow \lambda(S_m) < +\infty$$

By step ii)

$$\circledast \quad \tilde{\mu}(B_m) = \tilde{\mu}\left(\bigcup_{k=1}^{m} A_k\right) = \mu\left(\bigcup_{k=1}^{m} A_m\right)$$

From $\circledast$ and by MONOTONICITY of $\tilde{\mu}$:

$$\tilde{\mu}\left(\bigcup_{k=1}^{\infty} A_k\right) \underset{\text{mon.}}{\geq} \tilde{\mu}\left(\bigcup_{k=1}^{m} A_k\right)$$

$$= \mu\left(\bigcup_{k=1}^{m} A_k\right) \qquad B_m = \bigcup_{k=1}^{m} A_k$$

$$\tilde{\mu}(A) \geq \lim_{m \to +\infty} \mu\Big(\underbrace{\bigcup_{k=1}^{m} A_k}\Big) = \mu\left(\lim_{m \to +\infty} \bigcup_{k=1}^{m} A_k\right)$$

$$= \mu(A)$$

$\blacksquare$

1.3 LEBESGUE MEASURE

Def 1.3.1

i) $\forall a = (a_1, .., a_n) \in \mathbb{R}^n$, $b = (b_1, .., b_n) \in \mathbb{R}^n$

$$(a,b) = \begin{cases} \prod_{i=1}^{n} (a_i, b_i) & \text{if } a_i < b_i \ \forall i \\ \emptyset & \end{cases}$$

$[a,b]$, $(a,b]$, $[a,b)$

ii)

$$vol\,(a,b) = vol\,([a,b]) = vol\,(a,b] = vol\,[a,b)$$
$$= \begin{cases} \prod_{i=1}^{n} (b_i - a_i) & a_i < b_i \ \forall i \\ 0 & \text{otherwise.} \end{cases}$$

iii) Elementary set : $I = \bigcup_{k=1}^{m} I_k$

I_k intervals

$$vol\,(I) = \sum_{k=1}^{m} vol\,(I_k).$$

Rmk

- $I = \bigcup_{k=1}^{m} I_k = \bigcup_{l=1}^{p} J_l$

$$\Rightarrow \sum_{k=1}^{m} \mathrm{vol}(I_k) = \sum_{l=1}^{p} \mathrm{vol}(J_l)$$

- The class of elem. sets is an algebra and vol is a pre-measure.

$\Rightarrow$ If $I = \bigcup_{k=1}^{\infty} I_k$ $\quad I_k, I$ are el. sets and $I_{k_1} \cap I_{k_2} = \emptyset$

$$\Rightarrow \boxed{\mathrm{vol}(I) = \sum_{k=1}^{\infty} \mathrm{vol}(I_k)}$$

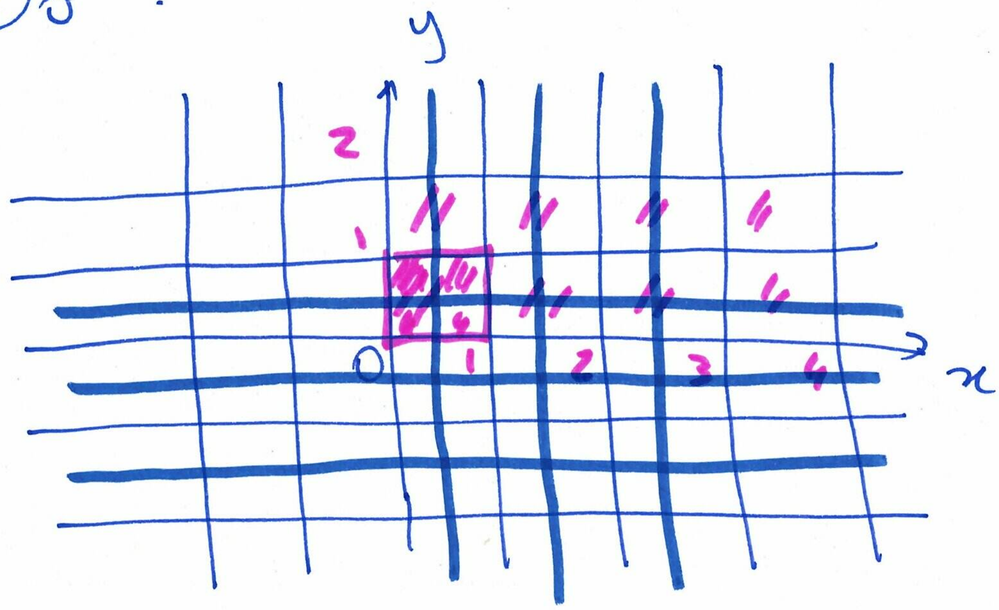
Dyadic decomposition

For $k \in \mathbb{N}$

$$\mathcal{D}_k = \left\{ \prod_{i=1}^{M} \left[\frac{a_i}{2^k}, \frac{a_i+1}{2^m} \right), \; a_i \in \mathbb{Z} \right\}$$

$k = 0$

$\mathcal{D}_0$:



$\mathcal{D}_k$ of cubes such that each cube in $\mathcal{D}_k$ has length 2^{-k} and is the union of 2^n disjoint cubes in $\mathcal{D}_{k+1}$.

$$\underbrace{\{ Q : Q \in \mathcal{D}_k \quad k = 0, 1, 2, \dots \}}_{\text{DYADIC CUBES}}$$

LEMMA $\forall A \subseteq \mathbb{R}^n$ open is the countable union of disjoint dyadic ~~cu~~ cubes.

1) $\forall\, k \geq 0 \qquad \mathbb{R}^n = \bigcup_{Q \in \mathcal{D}_k} Q$

2) $Q \in \mathcal{D}_k$, $P \in \mathcal{D}_h \quad h \leq k$

$\Rightarrow Q \subseteq P$ or $P \cap Q = \emptyset$

3) $Q \in \mathcal{D}_k \qquad \mathrm{vol}(Q) = 2^{-kn}$

Proof of LEMMA

Let $A \subseteq \mathbb{R}^n$ open $\neq \emptyset$.

. So is the collection of all cubes in $\mathcal{D}_0$ which are contained in V.

LECTURE 6.03.2020

LEMMA 1.3.4

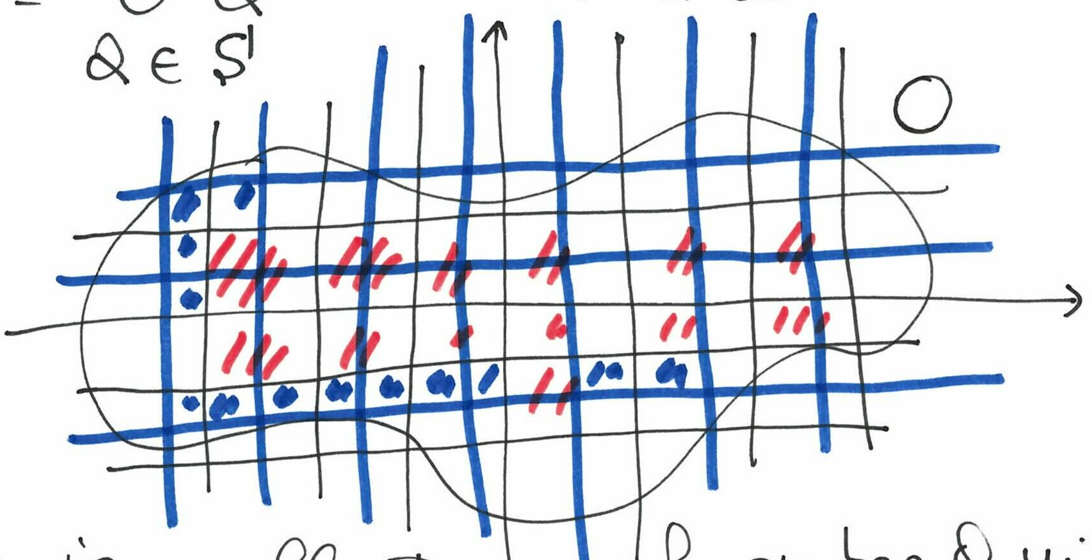
Every open set in $\mathbb{R}^n$ can be written as a countable union of disjoint dyadic cubes.

$$\mathcal{D} = \{ Q \ , \ Q \in \mathcal{D}_k , \ k = 0, 1, 2, \dots \}$$

$\mathcal{D}_k$ is the collection of cubes whose side length 2^{-k} and each cube is the union of 2^n disjoint cubes in $\mathcal{D}_{k+1}$.

Proof: $\mathcal{O} \subseteq \mathbb{R}^n \Rightarrow \exists$

$$\mathcal{O} = \bigcup_{Q \in S'} Q \qquad S' \subseteq \mathcal{D}$$



- S_0 is collection of cubes Q in $\mathcal{D}_0$ $Q \subseteq \mathcal{O}$

- S_1 the cubes in $\mathcal{D}_1$ which lie in O but they are NOT subcubes of any cube in S_0.
- $\forall k \geq 1$, S_k is the set of $Q \in \mathcal{D}_k$ $Q \subseteq O$ but $Q \notin$ any cubes in $S_0, S_1, \dots S_{k-1}$.

$$S = \bigcup_{k=1}^{\infty} S_k$$

It is countable union!

CLAIM $\quad O \overset{\supseteq}{\subseteq} \bigcup_{Q \in S} Q$

$x \in O$. Since O is open $\exists$ a cube Q of side length $= 2^{-N}$ $x \in Q \subseteq O$ ($\rightarrow$ see Carlotto's course)

$Q \in S_N \Downarrow x \in \bigcup_{Q \in S} Q$

$Q \subseteq P \in S_m$, $m < N$ $\Rightarrow x \in \bigcup_{Q \in S} Q$ $\blacksquare$

Def 1.3.5 (LEBESGUE MEASURE)

$\mathcal{L}^n$ is the Carathéodory-Hahn extension of the volume.

$\forall A \subseteq \mathbb{R}^n$

$$\mathcal{L}^n(A) = \inf \left\{ \sum_{k=1}^{\infty} \mathrm{vol}(I_k),\ I_k \text{ INTERVALS},\ A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}$$

From LEMMA 1.3.4 $\quad A = \bigcup_{Q \in S} Q$

A open

$\Rightarrow$ any open set of $\mathbb{R}^n$ is $\mathcal{L}^n$-measurable

$\Rightarrow$ σ-algebra of $\mathcal{L}^n$-measurable sets contains the σ-algebra of Borel sets.

Def 1.3.6

A measure μ on $\mathbb{R}^n$ is called **Borel** if every Borel set is μ-meas.

$\Rightarrow$ $\mathcal{L}^n$ is a Borel measure

SOME PROPERTIES

Thm 1.3.7

$\forall A \subseteq \mathbb{R}^n$.

$$\mathcal{L}^n(A) = \inf_{A \subseteq G} \mathcal{L}^n(G) \quad G \text{ open}$$

Proof

"$\leq$" If $A \subseteq G$

$$\Rightarrow \quad \mathcal{L}^n(A) \underset{\text{monot.}}{\leq} \inf_{A \subseteq G} \mathcal{L}^n(G)$$

"$\geq$" $\mathcal{L}^m(A) < +\infty$

~~$\mathcal{L}^m(A) =$~~

$\forall \varepsilon > 0$ $(I_k)_k$ intervals such that $A \subseteq \bigcup_k I_k$ and

$$\sum_{k=1}^{\infty} \underbrace{\mathrm{vol}(I_k)}_{<+\infty} \leq \mathcal{L}^m(A) + \varepsilon$$

For every I_k we choose $\tilde{I}_k \supseteq I_k$ open

$$\mathrm{vol}(\tilde{I}_k) \leq \mathrm{vol}(I_k) + \varepsilon/2^k$$

Set $G = \bigcup_k \tilde{I}_k \Rightarrow G$ is open and

$$\mathcal{L}^m(G) \leq \sum_1^{\infty} \mathrm{vol}(\tilde{I}_k)$$

$$\leq \sum_1^{\infty} \mathrm{vol}(I_k) + \underbrace{\sum_1^{\infty} \varepsilon/2^k}_{\varepsilon}$$

$$\leq \mathcal{L}^m(A) + \varepsilon + \varepsilon$$

$$= \mathcal{L}^m(A) + 2\varepsilon$$

$\Rightarrow \inf_{G \supseteq A} \mathcal{L}^m(G) \leq \mathcal{L}^m(A) + 2\varepsilon$

Let $\boxed{\varepsilon \to 0}$ $\square$

Characterization of $\mathcal{L}^n$- measurable sets

Thm 1.3.8

Let $A \subseteq \mathbb{R}^n$. The following two cond. are equivalent:

i) A is $\mathcal{L}^n$- measurable

ii) $\forall \varepsilon > 0 \; \exists G \supseteq A$ open : $\mathcal{L}^n(G \setminus A) < \varepsilon$

Proof

i) $\Rightarrow$ ii)

- $\mathcal{L}^n(A) < +\infty$

$\forall \varepsilon > 0 \; \exists G \supseteq A$ open :

$$\boxed{\mathcal{L}^n(G) \leq \mathcal{L}^n(A) + \varepsilon}$$

A is $\mathcal{L}^n$- measurable

$\forall B \subseteq \mathbb{R}^n$

$$\mathcal{L}^n(B) = \mathcal{L}^n(B \cap A) + \mathcal{L}^n(B \setminus A)$$

Let us choose $B = G$.

$$\mathcal{L}^n(G) = \mathcal{L}^n(\underbrace{G \cap A}_{A}) + \mathcal{L}^n(G \setminus A)$$

$$\mathcal{L}^n(G \setminus A) = \mathcal{L}^n(G) - \mathcal{L}^n(A) \leq \varepsilon$$

(III)

. $\mathcal{L}^m(A) = +\infty$

$A_k = A \cap [-k,k]^m$ A_k are $\mathcal{L}^m$-meas.

and $A = \bigcup_{k=1}^{\infty} A_k$, $\underline{\mathcal{L}^m(A_k) < +\infty}$

For $\varepsilon > 0$ and $\forall k \geq 1$: $G_k \subseteq A_k$ open set

$\mathcal{L}^m(G_k \setminus A_k) \leq \mathcal{L}^m(G_k) \leq \mathcal{L}^m(A_k) + \varepsilon/2^k$ COUNT

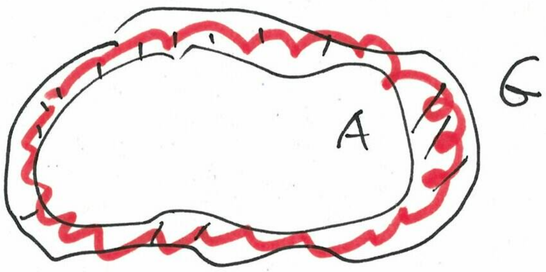
$G = \bigcup_{k=1}^{\infty} G_k$ is open (it is union of open sets)

$$\mathcal{L}^m(G \setminus A) \leq \sum_{k=1}^{\infty} \mathcal{L}^m(G_k \setminus A_k)$$

$$\leq \sum_{k=1}^{\infty} \left[\mathcal{L}^m(G_k) - \mathcal{L}^m(A_k)\right] \leq \underbrace{\sum_1^{\infty} \varepsilon/2^k}_{= \varepsilon}$$

$$\boxed{\begin{aligned} G \setminus A &= \bigcup_{k=1}^{\infty} (G_k \setminus A) \\ &\subseteq \bigcup_{k=1}^{\infty} (G_k \setminus A_k) \quad (A_k \subseteq A) \end{aligned}}$$

$\boxed{///}$



ii) $\Rightarrow$ i)

For $\varepsilon > 0$ let us choose G open

$G \supseteq A$, $\underline{\mathcal{L}^n(G \setminus A) \leq \varepsilon}$

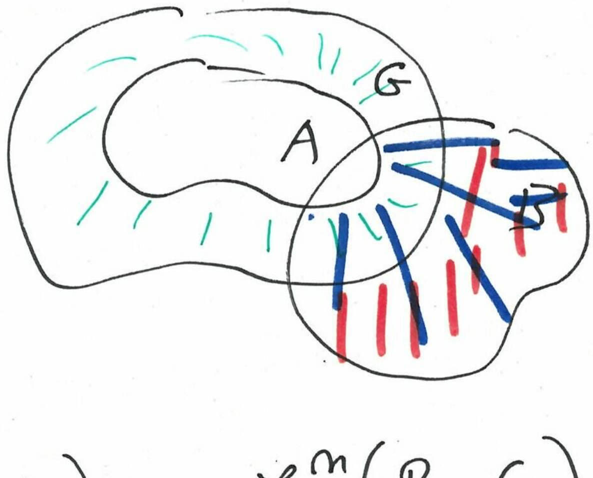
$\Rightarrow$ <u>A is $\mathcal{L}^n$-meas.</u>

G open $\Rightarrow$ G is $\mathcal{L}^n$-meas.

$\forall B \subseteq \mathbb{R}^n$

$$\mathcal{L}^n(B) = \mathcal{L}^n(B \cap G) + \mathcal{L}^n(B \setminus G)$$

$$B \setminus A \subseteq (B \setminus G) \cup (G \setminus A)$$



$$\mathcal{L}^n(B \setminus A) \leq \mathcal{L}^n(B \setminus G) + \mathcal{L}^n(G \setminus A)$$

$$\mathcal{L}^n(B) = \mathcal{L}^n(\underbrace{B \cap G}_{\supseteq B \cap A}) + \mathcal{L}^n(B \setminus G) \quad (*)$$

$$\mathcal{L}^n(B \setminus G) \geq -\mathcal{L}^n(G \setminus A) + \mathcal{L}^n(B \setminus A)$$

$$\mathcal{L}^n(B) \overset{\leq}{\geq} \mathcal{L}^n(B \cap A) + \mathcal{L}^n(B \setminus A) - \mathcal{L}^n(G \setminus A)$$

$$\geq \mathcal{L}^n(B \cap A) + \mathcal{L}^n(B \setminus A) - \varepsilon$$

$\boxed{\text{Let } \varepsilon \to 0}$

COR 1.3.9

$A \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$-meas. $\Leftrightarrow$ it can be approximated from inside and outside: $\forall \varepsilon > 0$ $\exists F, G$ F closed, G open $F \subseteq A \subseteq G$

$$\mathcal{L}^n(G \setminus A) + \mathcal{L}^n(A \setminus F) \leq \varepsilon \quad (1)$$

Proof (1) holds $\Rightarrow$ A is $\mathcal{L}^n$-meas because of previous Thm.

If A is $\mathcal{L}^n$-meas $\Rightarrow$ (1) holds

If A is $\mathcal{L}^n$-meas then A^c is $\mathcal{L}^n$-meas $\forall \varepsilon>0$ $\exists G$: $G \supseteq A^c$, G is open

$$\mathcal{L}^n(G \setminus A^c) < \varepsilon/2 \quad (*)$$

$$G \setminus A^c = G \cap A = A \setminus G^c$$

We set $F = G^c$ is closed

$$(*) \quad \mathcal{L}^n(G \setminus A^c) = \mathcal{L}^n(A \setminus F) < \varepsilon/2$$

$\boxed{\forall \varepsilon>0}$: $\exists \tilde{G}$ open $\tilde{G} \supseteq A$ and $\mathcal{L}^n(\tilde{G} \setminus A) < \varepsilon/2$

$\exists F$ closed $F \subseteq A$

$$\mathcal{L}^n(A \setminus F) < \varepsilon/2$$

$$\Rightarrow \mathcal{L}^n(\tilde{G} \setminus A) + \mathcal{L}^n(A \setminus F) < \varepsilon \quad \square$$

<u>Cor</u> $A \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$-meas $\Leftrightarrow$

$\forall \varepsilon > 0 \;\exists\; \underset{\text{closed}}{\overset{\uparrow}{F}} \subseteq A \subseteq \underset{\text{open}}{\overset{\uparrow}{G}}$

$$\mathcal{L}^n(G \setminus F) < \varepsilon \quad (2)$$

<u>Proof</u>

<u>Observe</u> $\boxed{F \subseteq A \subseteq G}$

$$G \setminus F = (G \setminus A) \cup (A \setminus F)$$

- <u>Suppose (1) holds</u>
- $\mathcal{L}^n(G \setminus F) \leq \mathcal{L}^n(G \setminus A) + \mathcal{L}^n(A \setminus F) < \varepsilon$

$\Rightarrow$ (2) holds

- <u>Suppose (2) holds</u>

$G \setminus A \subseteq G \setminus F \quad (F \subseteq A)$

$A \setminus F \subseteq G \setminus F \quad (A \subseteq G)$

$$\mathcal{L}^n(G \setminus A) + \mathcal{L}^n(A \setminus F) \leq \mathcal{L}^n(G \setminus F) + \mathcal{L}^n(G \setminus F) \leq \varepsilon + \varepsilon = 2\varepsilon \quad \blacksquare$$

COMPARISON BETWEEN JORDAN & LEBESGUE MEASURE

$A \subseteq \mathbb{R}^n$ <u>bounded</u>

A is <u>Jordan measurable</u> if

$\bar{\mu}(A) = \inf \{ \mathrm{vol}(E),$ E elementary set $A \subseteq E \}$

$\underline{\mu}(A) = \sup \{ \mathrm{vol}(E) :$ E elementary set $E \subseteq A \}$.

$E = \bigcup_{k=1}^{m} I_k \quad I_{k_1} \cap I_{k_2} = \emptyset \quad k_1 \neq k_2$

I_k are half-open intervals

$m \geq 1$

$$\boxed{\underline{\mu}(A) = \bar{\mu}(A)}$$

<u>Thm 1.3.11</u>

$A \subseteq \mathbb{R}^n$ be bounded, then

i) $\underline{\mu}(A) \le \mathcal{L}^n(A) \le \overline{\mu}(A)$

ii) If A is Jordan-measurable $\Rightarrow$ A is $\mathcal{L}^n$-measurable and $\mathcal{L}^n(A) = \mu(A)$.

Proof

i)

$$\mathcal{L}^n(A) = \inf\left\{ \sum_{k=1}^{\infty} \mathrm{vol}(I_k) \;\middle|\; A \subseteq \bigcup_{k=1}^{\infty} I_k,\ I_k \text{ half-open intervals} \right\}$$

$$\le \inf\left\{ \sum_{k=1}^{m} \mathrm{vol}(J_k) \;\middle|\; A \subseteq \bigcup_{k=1}^{m} J_k,\ J_k \text{ mutually disj. intervals},\ m \ge 1 \right\}$$

$$= \overline{\mu}(A)$$

$\mathcal{L}^n(A) \ge \underline{\mu}(A)$

For every $E = \bigcup_{k=1}^{m} I_k$, I_k mut. disj.

$E \subseteq A$

$$\mathrm{vol}(E) = \sum_{k=1}^{m} \mathrm{vol}(I_k)$$

$$= \mathcal{L}^n(E) \leq \mathcal{L}^n(A)$$

$$\Rightarrow \quad \underline{\mu}(A) \leq \mathcal{L}^n(A)$$

---

ii) If A is Jordan-meas then $\mathcal{L}^n(A) = \mu(A)$.

It remains to prove that actually A is $\mathcal{L}^n$-measurable.

$\mathcal{L}^n(A) < +\infty$ $\quad A_k = A \cap [-k,k]^n$

Since A is Jordan-meas.
$\forall \varepsilon > 0 \quad \exists\ E_\varepsilon,\ E^\varepsilon$ elementary sets such that

$$E_\varepsilon \subseteq A \subseteq E^\varepsilon \Leftarrow$$

$$\mathrm{vol}(E^\varepsilon) - \varepsilon \leq \mu(A) \leq \mathrm{vol}(E_\varepsilon) + \varepsilon$$

(by def).
We may always assume $G = E^\varepsilon$ is open.

$$\mathcal{L}^m(G \setminus A) = \mathcal{L}^m(E^\varepsilon \setminus A)$$
$$\leq \mathcal{L}^m(E^\varepsilon \setminus E_\varepsilon)$$
$$= \mathrm{vol}(E^\varepsilon \setminus E_\varepsilon)$$
$$= \mathrm{vol}(E^\varepsilon) \setminus \mathrm{vol}(E_\varepsilon)$$
$$\leq 2\varepsilon$$

$\boxed{4}$

<u>Remark</u>

$$\mathbb{Q} \cap [0,1] = \{q_1, q_2, \dots\}$$

<u>Let $\varepsilon > 0$</u>

$$V = \bigcup_{n=1}^{\infty} \left(q_n - \varepsilon/2^n,\ q_n + \varepsilon/2^n\right)$$

<u>V is open</u>

$$\mathcal{L}^1(V) \leq \sum_{n=1}^{\infty} \underbrace{\mathrm{vol}\left(q_n - \varepsilon/2^n,\ q_n + \varepsilon/2^n\right)}_{2\varepsilon/2^n}$$

$$= \sum_{n=1}^{\infty} \frac{2\varepsilon}{2^n} = 2\varepsilon \quad \left(\sum_{n=1}^{\infty} \frac{1}{2^n} = 1\right)$$

$$\overline{\mathbb{Q} \cap [0,1]} = [0,1]$$

$\bar{V} \supseteq [0,1]$

$\bar{\mu}(\bar{V}) = \bar{\mu}(V) \geq \bar{\mu}([0,1]) = \mathrm{vol}[0,1] = 1$

$\underline{\mu}(V) \leq \mathcal{L}^{1}(V) \leq 1 \leq \bar{\mu}(V)$

$\boxed{\mathcal{L}^1(V) \leq 2\varepsilon}$

$\Rightarrow \quad 2\varepsilon < \frac{1}{2} \iff \boxed{\varepsilon < \frac{1}{4}}$

$\underline{\mu}(V) < 1 \leq \bar{\mu}(V)$

$\mathbb{Q} \cap [0,1]$ is NOT Jordan measurable

but it is $\mathcal{L}^1$ meas and

$\mathcal{L}^1((\mathbb{Q} \cap [0,1])) = 0$

$\mathbb{Q} \cap [0,1] = \bigcup_{n=1}^{\infty} \{q_n\}$

$\Rightarrow \mathcal{L}^1((\mathbb{Q} \cap [0,1])) = \sum_{n=1}^{\infty} \mathcal{L}^1\{q_n\} = 0$