LECTURE 6.05.2020

3.7 L^p spaces $1 \le p \le \infty$

μ Radon measure on $\mathbb{R}^n$

$\Omega \subseteq \mathbb{R}^n$ μ-measurable

$f : \Omega \to \overline{\mathbb{R}}$, μ-meas.

Def 3.7.1

$1 \le p < +\infty$

$$\|f\|_{L^p(\Omega,\mu)} = \underbrace{\left(\int_\Omega |f|^p \, d\mu \right)^{1/p}}$$

$p = +\infty$

$$\|f\|_{L^\infty(\Omega,\mu)} = \mu\text{-}\operatorname{esssup}_\Omega |f(x)|$$

$$= \inf \{ c \in [0,+\infty] : |f(x)| \le c \ \mu\text{-a.e} \}$$

$1 \le p \le +\infty$

$$\mathcal{L}^p(\Omega,\mu) = \{ f : \Omega \to \overline{\mathbb{R}},\ \mu\text{ meas}.\ \|f\|_{L^p(\Omega,\mu))} < +\infty \}$$

Remarks

1) $f \in \mathcal{L}^\infty(\Omega,\mu) \Rightarrow$

$$|f(x)| \leq \|f\|_{L^\infty} \quad \mu\text{.a.e}$$

Proof There is $c_k \downarrow \|f\|_{L^\infty}$ as $k \to +\infty$

$$A_k = \{ x \in \Omega : |f(x)| > c_k \}$$

$$\mu(A_k) = 0 \quad \forall k$$

$$\Rightarrow A = \bigcup_k A_k \Rightarrow \boxed{\mu(A) = 0}$$

$\forall x \in \Omega \setminus A \Rightarrow x \notin A \Rightarrow$

$x \notin A_k \quad \forall k \Rightarrow$

$$|f(x)| \leq c_k \quad \forall k$$

Let $k \to +\infty$ $|f(x)| \leq \|f\|_{L^\infty}$

$\forall x \in \Omega \setminus A \Rightarrow \mu$. a. e

2) $\Omega = (0,1]$

$\mu = \mathcal{L}^1$ Lebesgue meas

$\forall \alpha \in \mathbb{R}$: $\varphi_\alpha(x) = x^\alpha$

$\forall x \in (0,1]$

Find $p \in [1, +\infty)$: $\varphi^\alpha \in \mathcal{L}^p$

<u>Sol</u> $\varphi_\alpha \in \mathcal{L}^p \iff$

$|\varphi_\alpha|^p = x^{p\alpha}$ is $\mathcal{L}^1$-summ.

$\iff$

$$\lim_{\varepsilon \to 0} \int_\varepsilon^1 x^{p\alpha} \, dx < +\infty$$

$\cdot \; p\alpha \neq -1$

$$\lim_{\varepsilon \to 0} \int_\varepsilon^1 x^{p\alpha} \, dx = \lim_{\varepsilon \to 0} \left. \frac{x^{p\alpha+1}}{1+p\alpha} \right|_\varepsilon^1$$

$$= \frac{1}{p\alpha+1} - \underbrace{\lim_{\varepsilon \to 0} \frac{\varepsilon^{p\alpha+1}}{1+p\alpha}}_{= 0 \iff 1+p\alpha > 0} = \frac{1}{p\alpha+1}$$

$$\boxed{p > -\frac{1}{\alpha}}$$

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- $p\alpha = -1$

$$\int_0^1 \frac{1}{x}\,dx = \lim_{\varepsilon \to 0} \log x \Big|_{\varepsilon}^{1} = +\infty$$

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3) $\mathcal{L}^p([0,1], \mathcal{L}^1)$ is not an algebra:

$f, g \in \mathcal{L}^p \not\Rightarrow f \cdot g \in \mathcal{L}^p$

$f(x) = x^{-1/2} \quad g(x) = x^{-1/2}$

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$f, g \in \mathcal{L}^1([0,1], \mathcal{L}^1)$

$\Rightarrow f \cdot g = \frac{1}{x} \notin \mathcal{L}^1$

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4) $f \longmapsto \|f\|_{L^p}$ is NOT a NORM

$\|f\|_{L^p} = 0 \Rightarrow |f| = 0$ μ.a.e $\Rightarrow f = 0$ μ.a.e

RECALL

Y a vector space

A norm on Y is a mapping

$$\|\cdot\| : Y \longrightarrow [0,+\infty)$$
$$y \longmapsto \|y\|$$

such that

a) $\|y\| = 0 \Rightarrow y = 0$

b) $\|\alpha y\| = |\alpha|\,\|y\| \quad \forall y \in Y$
$\forall \alpha \in \mathbb{R}$

c) $\|y_1 + y_2\| \leq \|y_1\| + \|y_2\|$

$(Y, \|\cdot\|)$ is NORMED SPACE

$\Rightarrow d(y_1, y_2) = \|y_1 - y_2\|$

RELATION OF EQUIVALENCE

$f, g \in \mathcal{L}^p(\Omega, \mu)$

$f \sim g \iff f = g \ \mu\text{-a.e}$

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$$\boxed{L^p(\Omega,\mu)} = \mathcal{L}^p(\Omega,\mu)/\sim$$

$$= \{ [\varphi] : \varphi \in \mathcal{L}^p(\Omega,\mu) \}$$

$$[\varphi_1] + [\varphi_2]$$

$$f_1 \in [\varphi_1] \quad , \quad f_2 \in [\varphi_2]$$

$$[\varphi_1] + [\varphi_2] = [f_1 + f_2]$$

$$\lambda [\varphi] = [\lambda f] \quad f \in [\varphi]$$

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$$\|[\varphi]\|_{L^p} = \|\varphi\|_{L^p}$$

$$\forall [\varphi] \in L^p(\Omega,\mu)$$

$$\boxed{\|[\varphi]\|_{L^p} = 0}$$

$$\|[\varphi]\|_{L^p} = \|\varphi\|_{L^p} = 0$$

$$\Rightarrow \varphi = 0 \quad \mu.e.e$$

$$\Rightarrow \boxed{[\varphi] = 0}$$

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We will identify $[\varphi]$ with φ.

Thm 3.7.5

The space $L^p(\Omega,\mu)$ is a complete normed space for $1 \le p \le +\infty$.

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Proof

Step 1: $\varphi \longrightarrow \|\varphi\|_{L^p}$ is a norm

i) $\|\varphi\|_{L^p} = 0 \Rightarrow \varphi = 0$

ii) $\|\lambda f\|_{L^p} = \left(\int_\Omega |\lambda f|^p d\mu\right)^{1/p}$

$= |\lambda| \left(\int_\Omega |f|^p d\mu\right)^{1/p}$

iii) $\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$

Preliminary results

LEMMA 3.7.6 (Young Inequality)

Let $1 < p, q < +\infty$ be such that $\boxed{\frac{1}{p} + \frac{1}{q} = 1}$

Then
$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \quad \forall a, b \ge 0$$

Proof

- If $b = 0 \Rightarrow$ ok

- $b > 0$

$$a \overset{\psi}{\longmapsto} ab - \frac{a^p}{p} \quad a \ge 0$$

$$\lim_{a \to +\infty} ab - \frac{a^p}{p} = -\infty$$

$$\Rightarrow \exists\, a^* \ge 0 : \quad a^* b - \frac{{a^*}^p}{p} = \max_{a \ge 0} \psi(a)$$

$$\psi'(a) = b - a^{p-1} = 0$$

$$\Leftrightarrow \quad a^* = b^{1/p-1}$$

$$\psi(a^*) = b^{\frac{1}{p-1}} b - \frac{b^{\frac{p}{p-1}}}{p} =$$

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$$= \left(1-\frac{1}{p}\right) b^{\frac{p}{p-1}} = \frac{1}{q} b^q$$

$$\psi(a) \leq \frac{b^q}{q} \Rightarrow ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$\square$

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p, q : $\frac{1}{p} + \frac{1}{q} = 1$ are called conjugate

Cor. 3.7.7 (Hölder Ineq)

Let $1 \leq p, q \leq +\infty$ be conjugate $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ and $f \in L^p(\Omega, \mu)$, $g \in L^q(\Omega, \mu)$.

Then $f \cdot g \in L^1(\Omega, \mu)$

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}$$

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__Proof__ " $p \leq q$ "

- $p = 1 \Rightarrow q = +\infty$

$$|fg| = |f|\,|g| \leq |f|\,\|g\|_{L^\infty(\Omega,\mu)}$$

$$\int_\Omega |fg|\,d\mu \leq \int_\Omega |f|\,\|g\|_{L^\infty(\Omega,\mu)}\,d\mu$$

$$= \|g\|_{L^\infty}\,\|f\|_{L^1}$$

- $1 < p < +\infty$

- If $\|f\|_{L^p} = 0$ or $\|g\|_{L^q} = 0$

$\Rightarrow$ it is trivial

(If $\|f\|_{L^p} = 0 \Rightarrow f = 0$

μ.a.e $\Rightarrow |fg| = 0$ μ.a.e

$$\Rightarrow \int_\Omega |fg|\,d\mu = 0$$

- $\|f\|_{L^p} \neq 0$ $\quad \|g\|_{L^q} \neq 0$

$$\tilde{f} = \frac{|f|}{\|f\|_{L^p}}, \quad \hat{g} = \frac{|g|}{\|g\|_{L^q}}$$

$$\tilde{f}\tilde{g} \leq \frac{\tilde{f}^p}{p} + \frac{\tilde{g}^q}{q}$$

$$\int_\Omega \tilde{f}\hat{g}\,d\mu \leq \frac{1}{p}\int_\Omega \tilde{f}^p + \frac{1}{q}\int_\Omega \tilde{g}^q = 1$$

$$\int_\Omega \tilde{f}\,\tilde{g}\,d\mu \le 1$$

$$\Rightarrow \underbrace{\frac{1}{\|f\|_{L^p}\|g\|_{L^q}}} \int_\Omega |fg|\,d\mu \le 1$$

$$\Rightarrow \int_\Omega |fg|\,d\mu \le \|f\|_{L^p}\|g\|_{L^q}$$

$\square$

COR. 3.7.8

i) $f \in L^p$, $g \in L^q$

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \le 1$$

$\Rightarrow fg \in L^r$ and

$$\|fg\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}$$

ii) If $\mu(\Omega) < +\infty$

$$L^s(\Omega,\mu) \subseteq L^r(\Omega,\mu)$$

$1 \le r < s \le +\infty$

<u>Proof</u>

i) $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \Leftrightarrow 1 = \frac{1}{\left(\frac{p}{r}\right)} + \frac{1}{\left(\frac{q}{r}\right)}$

$\frac{p}{r}$ and $\frac{q}{r}$ are conjugate

$f \in L^p \Rightarrow |f|^r \in L^{p/r}$

$g \in L^q \Rightarrow |g|^r \in L^{q/r}$

Apply H.I to $|f|^r$ and $|g|^r$:

$$\int_\Omega |fg|^r d\mu = \int_\Omega \underbrace{|f|^r}\underbrace{|g|^r} d\mu$$

$$\leq \left(\int_\Omega |f|^{r\frac{p}{r}} d\mu\right)^{r/p} \left(\int_\Omega |g|^{r\frac{q}{r}} d\mu\right)^{r/q}$$

$$\Rightarrow \left(\int_\Omega |fg|^r d\mu\right)^{1/r} \leq \|f\|_{L^p} \|g\|_{L^q}$$

$\square$

ii) $\mu(\Omega) < +\infty$

$g = \chi_\Omega \in L^{\frac{rs}{s-r}}(\Omega)$

$$\boxed{\frac{1}{r} = \frac{1}{s} + \frac{s-r}{rs}}$$

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$f \in L^s \Rightarrow f \in L^r \quad r < s$

$$\int_\Omega |f|^r d\mu = \int_\Omega |f|^r \chi_\Omega^r(x)\, d\mu$$

$$\leq \|f\|_{L^s} \, \|\chi_\Omega\|_{L^{\frac{rs}{s-r}}}$$

apply i) $p = s$, $q = \frac{rs}{s-r}$

<u>Ex</u> $\mu(\Omega) < +\infty$

$1 \leq r < s \leq +\infty$

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$L^s(\Omega, \mu) \subsetneq L^r(\Omega, \mu)$

- $\Omega = (0, 1/2)$

$$f(x) = \frac{1}{(-\log x)^2 x^{1/r}} \in L^r$$

but $f \notin L^s$ with $s > r$

- $\Omega = (0, 1)$

$f(x) = -\log x \in L^p((0,1))$

$\forall p > 1$ but $f \notin L^\infty((0,1))$

$L^\infty(\Omega) \subsetneq L^p(\Omega)$

$\boxed{\underline{\text{COR } 3.7.10}}$ (MINKOWSKI INEQUALITY)

Let $1 \leq p \leq +\infty$ and $f, g \in L^p(\Omega, \mu)$. Then $f+g \in L^p(\Omega, \mu)$ and

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

<u>Proof</u>

- $p = 1$

$$|f+g| \leq |f| + |g|$$

$$\int_\Omega |f+g| \, d\mu \leq \int_\Omega (|f| + |g|) \, d\mu$$

$$= \int_\Omega |f| \, d\mu + \int_\Omega |g| \, d\mu$$

- $p = \infty$

$$|(f+g)(x)| \leq |f(x)| + |g(x)|$$

$$\leq \|f\|_{L^\infty} + \|g\|_{L^\infty}$$

- $1 < p < \infty$

$$|(f+g)(x)|^p \leq 2^{p-1}\left(|f(x)|^p + |g(x)|^p\right)$$

$$\Rightarrow f+g \in L^p$$

$$|f+g|^{p-1} \in L^{\frac{p}{p-1}}$$

$$\|f+g\|_{L^p}^p = \int_\Omega |f+g|^p \, d\mu$$

$$= \int_\Omega |f+g|\,|f+g|^{p-1} \, d\mu$$

$$\leq \underbrace{\int_\Omega |f|\,|f+g|^{p-1} \, d\mu}_{①} + \underbrace{\int_\Omega |g|\,|f+g|^{p-1} \, d\mu}_{②}$$

Apply in ① and ② Hölder inequality between p and $p/p-1$

$$① \leq \left(\int_\Omega |f|^p\right)^{1/p} \left(\int_\Omega |f+g|^{\cancel{p-1}\cdot\frac{p}{\cancel{p-1}}}\right)^{\frac{p-1}{p}} + \left(\int_\Omega |g|^p d\mu\right)^{1/p} \left(\int_\Omega |f+g|^{\cancel{p-1}\,\frac{p}{\cancel{p-1}}}\right)^{\frac{p-1}{p}}$$

$$= \left(\|f\|_{L^p} + \|g\|_{L^p}\right) \|f+g\|_{L^p}^{p-1}$$

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$$\Rightarrow \|f+g\|_{L^p}^{p} \leq \left(\|f\|_{L^p} + \|g\|_{L^p}\right)\left(\|f+g\|_{L^p}^{p-1}\right)$$

$$\|f+g\|_{L^p}^{\cancel{p}} \|f+g\|_{L^p}^{1-\cancel{p}} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

$$\Rightarrow \|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

Ex : 1) $1 \leq p < r < q < \infty$

$$f \in L^p \cap L^q \Rightarrow f \in L^r$$

$$\|f\|_{L^r} \leq \|f\|_{L^p}^{\theta} \|g\|_{L^q}^{1-\theta}$$

$$\frac{1}{r} = \frac{\vartheta}{p} + \frac{1-\vartheta}{q} \quad \vartheta \in (0,1)$$

2) $\mu(\Omega) < +\infty$, $1 \leq p < +\infty$

$f : \Omega \to \overline{\mathbb{R}}$ $\quad f \cdot g \in L^1(\Omega, \mu)$

$\forall g \in L^p(\Omega, \mu) \Rightarrow$

$f \in L^q \quad \forall q \in [1, p')$

<u>Hint</u> Since $\mu(\Omega) < +\infty$

$g = \chi_\Omega \in L^p \Rightarrow f \in L^1$

$\Rightarrow |f|^{\frac{1}{p}} \in L^p$

· Use the hypothesis with

$g = |f|^{\frac{1}{p}}$

$\Rightarrow |f| \cdot |f|^{\frac{1}{p}} \in L^1$

$|f|^{1+\frac{1}{p}} \in L^1 \Rightarrow$

$|f|^{\frac{1}{p}+\frac{1}{p^2}} \in L^p$

· Use the hyp $g = |f|^{\frac{1}{p}+\frac{1}{p^2}}$

$\Rightarrow$ By iteration we get $\forall n \geq 0$:

$$\underline{|f|^{\sum_{i=0}^{n} \frac{1}{p^i}} \in L^1} \Rightarrow \ldots \qquad \square$$