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Lecture 8.05.2020

$$f \longmapsto \|f\|_{L^p(\Omega,\mu)}$$

$1 \le p \le +\infty$

2nd part of the proof of Theorem 3.7.5:

LEMMA 3.7.13

$L^p(\Omega,\mu)$ is complete:

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Let $(f_k)_k$ be a Cauchy sequence in $L^p(\Omega,\mu)$. Then there is $f \in L^p$ such that $\|f_k - f\|_{L^p} \to 0$ as $k \to +\infty$

Proof of LEMMA

1) $p = +\infty$

$\forall \varepsilon > 0 \ \exists k_\varepsilon > 0$:

$\forall \ell, m \geq k_\varepsilon$

$$\| f_\ell - f_m \|_{L^\infty} < \varepsilon \quad (*)$$

For $\forall\, \ell, m \geq 0$

$$E_{\ell,m} = \{ x \in \Omega : | f_\ell(x) - f_m(x) | > \| f_\ell - f_m \|_{L^\infty} \}$$

$\mu(E_{\ell,m}) = 0 \quad \forall\, \ell, m$

$$E_0 = \bigcup_{\ell,m=0}^{\infty} E_{\ell,m}$$

$\Rightarrow \quad \mu(E_0) = 0$

Take $x \in \Omega \setminus E_0 \Rightarrow$

$x \notin E_{\ell,m} \quad \forall\, \ell, m$

$$\Rightarrow \quad | f_\ell(x) - f_m(x) | \leq \| f_\ell - f_m \|_{L^\infty}$$

If we take $\ell, m \geq k_\varepsilon$

we have (because of $(*)$)

$$| f_\ell(x) - f_m(x) | < \varepsilon$$

$\Rightarrow \forall x \in \Omega - E_0 \quad (f_k(x))_k$ is CAUCHY SEQUENCE in $\mathbb{R}$

$\mathbb{R}$ is complete $\Rightarrow$

$f_k(x) \to f(x)$ as $k \to +\infty$

$\boxed{\underline{\text{CLAIM 1}}}$: $\|f_k - f\|_{L^\infty} \to 0$

as $k \to +\infty$

Proof of the CLAIM 1

$$|f_\ell(x) - f_m(x)| < \varepsilon \quad (\ast\ast)$$

$\forall x \in \Omega \setminus E_0$, $\forall \ell, m \geq k_\varepsilon$

Let m $(\ast\ast)$ $m \to +\infty$

$\forall$

$$|f_\ell(x) - f(x)| \leq \varepsilon \quad \forall \ell \geq k_\varepsilon$$

$\forall x \in \Omega \setminus E_0$

Since $\mu(E_0) = 0$ it follows that:

$$\boxed{\|f_\ell - f\|_{L^\infty} \leq \varepsilon \quad \forall \ell \geq k_\varepsilon}$$

and

$$\boxed{\begin{aligned}\|f\|_{L^\infty} &\leq \|f_\ell - f\|_{L^\infty} + \|f_\ell\|_{L^\infty} \\ &\leq \varepsilon + \|f_\ell\|_{L^\infty} < +\infty\end{aligned}}$$

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$\Rightarrow$ $f \in L^\infty$ and $\|f_k - f\|_{L^\infty} \to 0$

as $k \to +\infty$ $\square$

2) $1 \le p < +\infty$

$\forall \varepsilon > 0 \ \exists k_\varepsilon : \ell, m \ge k_\varepsilon$

$$\|f_\ell - f_m\|_{L^p} < \varepsilon \qquad (1)$$

• to construct a convergent subsequence f_{n_k}

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Let's choose in (1)

$\varepsilon = \frac{1}{2^n} \Rightarrow \exists k_n : \ell, m \ge k_n$

$$\|f_\ell - f_m\|_{L^p} < \frac{1}{2^n}$$

You may assume without restriction: $n \mapsto k_n$ is non-decreasing.

CLAIM 2: $(f_{k_n})_n$ is convergent in L^p

We write f_n instead of f_{k_n}.

$$\| f_{n+1} - f_n \|_{L^p} < \frac{1}{2^n} \quad \forall n \geq 1$$

Define

$$g_n(x) = \sum_{k=1}^{n} | f_{k+1}(x) - f_k(x) |$$

$n \geq 1$

$g_n \geq 0, \quad g_n \uparrow$

$$\| g_n \|_{L^p} \leq \sum_{k=1}^{n} \| f_{k+1} - f_k \|_{L^p}$$

$$\leq \sum_{k=1}^{n} \frac{1}{2^k} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

$$g_n(x) \to g(x) = \sum_{K=1}^{\infty} | f_{K+1}(x) - f(x) |$$

$$|g_n|^p \uparrow |g|^p$$

$\Rightarrow$ Apply Beppo Levi's Thm

$$\lim_{n \to +\infty} \int_\Omega |g_n|^p_\mu = \int_\Omega |g|^p d\mu$$

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Since $\|g_m\|_{L^p} \leq 1$

$\Rightarrow \|g\|_{L^p} \leq 1$

$\Rightarrow g \in L^p$

Take $l, m \geq 2$ $m \geq l$

$$|f_m(x) - f_l(x)| \leq \qquad (2)$$

$$\leq |f_m(x) - f_{m-1}(x)| + \ldots +$$

$$+ |f_{l+1}(x) - f_l(x)|$$

$$= g_m(x) - g_{l-1}(x) \leq$$

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$g(x) - g_{l-1}(x)$

Since $g_l \to g$ μ-a.e $x \in \Omega$

$\Rightarrow f_m(x)$ is also a

CAUCHY-SEQUENCE in $\mathbb{R}$

$\forall$ μ-a.e $x \in \Omega$

$\Rightarrow f_m(x) \longrightarrow f(x)$ μ-a.e. $x \in \Omega$

CLAIM 3: $f \in L^p$ and

$\|f_m - f\|_{L^p} \to 0$ as $m \to +\infty$

PROOF OF CLAIM 3 13

Let $m \to +\infty$ in (2)

$$|f(x) - f_\ell(x)| \leq$$

$$\leq g(x) - g_{\ell-1}(x)$$

$$\leq 2\, g(x) \qquad (3)$$

$$|f(x)| \leq |f_\ell(x)| + 2\, g(x)$$

$\Rightarrow f \in L^p$ and by

Lebesgue Theorem we

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get that $\| f - f_m \|_{L^p} \to 0$

as $m \to +\infty$

Actually we have

$$h_m(x) = f(x) - f_m(x)$$

$h_m \to 0$ as $m \to +\infty$

$$|h_m(x)| \leq 2\, g(x) \in L^p$$

$$|h_m|^p \leq 2^p g^p(x)$$

$$|h_m|^p \in L^1 \Rightarrow h_m \in L^p$$

$$\Rightarrow \lim_{m \to +\infty} \int_\Omega |h_m|^p d\mu = \int_\Omega \lim_{m \to +\infty} |h_m|^p d\mu = 0$$

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$\boxed{\text{CLAIM 4}}$: $\|f_k - f\|_{L^p} \to 0$ as $k \to +\infty$

Proof of CLAIM 4

$\forall m \in \mathbb{N}$ take $k > k_m$

$$\|f_k - f\|_{L^p} \leq \|f_k - f_{k_m}\|_p + \|f_{k_m} - f\|_{L^p} \leq \frac{1}{2^m} + \|f_{k_m} - f\|_{L^p}$$

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$$\limsup_{k \to +\infty} \|f_k - f\|_{L^p} \leq \lim_{m \to +\infty} \left(\frac{1}{2^m} + \|f_{k_m} - f\|_{L^p} \right) = 0$$

$\Rightarrow \|f_k - f\|_{L^p} \to 0$ as $k \to +\infty$ $\square$

X is separable if $\exists$ $A \subseteq X$ A countable and $\bar{A} = X$

Examples

1) $\overline{\mathbb{Q}} = \mathbb{R} \Rightarrow \overline{Q} = \mathbb{R}$

2) $\overline{X} = (C^0([a,b], \mathbb{R}), \|\cdot\|_{L^\infty})$

The set of polynomials with rational coeff is dense in $\overline{X}$

Thm 3.7.15

Let $1 \le p < +\infty$. Then

i) $L^p(\Omega, \mu)$ is separable

ii) $C_c^0(\Omega, \mu)$ is dense in L^p.

Remarks

1) $C_c^0(\Omega, \mu) = \{ f \in C^0(\Omega, \mu) \;\; \text{supp} f \subseteq \Omega \text{ compact} \}$

$$\operatorname{supp} f = \overline{\{x : f(x) \neq 0\}}$$

2) In general the Thm 3.7.15 does not hold for $p = +\infty$

i) $\Omega = [0,1]$, $\mu = \mathcal{L}^1$

$\forall t \geq 0$ $\quad f_t(x) = \chi_{[0,t]}(x)$

$s \neq t$ $\quad \| f_t - f_s \|_{L^\infty} = 1$

Suppose by contradiction $(e_k)_k$ dense in $L^\infty([0,1], \mathbb{R})$

$$L^\infty([0,1], \mathbb{R}) \subseteq \bigcup_{k=1}^{\infty} B(e_k, 1/2)$$

$$B(e_k, 1/2) = \{ g \in L^\infty([0,1]) \; \| g - e_k \|_{L^\infty} < 1/2 \}$$

<u>NOTE</u>: $f_{t_1} \neq f_{t_2}$ $\quad (t_1 \neq t_2)$

$\Rightarrow$ f_{t_1} and f_2 belong to two different balls

<u>Otherwise</u> : $\|f_{t_1} - f_{t_2}\|_{L^\infty} < 1$ }

$\mathbb{N} \longrightarrow [0,1]$

For $k \geq 1 \longrightarrow t(k)$ such

that $f_{t(k)} \in B(e_k, 1/2)$

Such a map is surjective ↯

ii) Ω is compact,

$C(\Omega)$ is closed in

$L^\infty(\Omega)$:

$f \in \overline{C(\Omega)}$ $\exists f_n \in C(\Omega)$

. $\|f_n - f\|_{L^\infty(\Omega)} \to 0$ as $n \to +\infty$

$\Rightarrow f \in C(\Omega)$

$\Rightarrow \overline{C(\Omega)}^{\|\,\|_{L^\infty}} = C(\Omega)$

NB <u>cannot</u> be that

$\overline{C_c^0(\Omega)}^{\|\,\|_{L^\infty}} = L^\infty(\Omega)$!

Proof of Theorem 3.7.15

We give the proof in the case $\boxed{\Omega = \mathbb{R}^m}$

- $E = \left\{ \sum_{k=1}^{N} e_k \chi_{Q_k},\ e_k \in \mathbb{Q} \right.$ and Q_k is a cube of a dyadic dec of $\mathbb{R}^m$ of length $\left. 2^{-\ell}\ \ell \geq 0 \right\}$

E is countable

CLAIM 1 : E is dense in L^p

Proof of CLAIM 1

- $f \geq 0$

$$f_k = \sum_{j=1}^{k} \frac{1}{j} \chi_{A_j}(x)$$

A_j are μ-measurable sets defined:

$$A_j = \left\{ x \in \Omega : f(x) \geq \frac{1}{j} + \sum_{m=1}^{j-1} \frac{1}{m} \chi_{A_m} \right\}$$

Since f is in L^p we have that $\mu(A_j) < +\infty$ and by Lebesgue Thm

$$\| f_k - f \|_{L^p} \to 0 \quad \text{as } k \to +\infty$$

$$\left(f_k \uparrow f \ \mu\text{.a.e} \quad |f_k| \le |f| \in L^p \right)$$

$$\Rightarrow \ |f_k|^p \le |f|^p \in L^1 \ \ldots \Big)$$

$\Rightarrow$ it is enough to show

the density of E in the case when $f = \chi_A$

A is μ - measurable and $\mu(A) < +\infty$

$\boxed{\underline{\text{CLAIM 2}}}$: $\chi_A \in \overline{E}$

$\forall$ μ - measurable set $A : \mu(A) < +\infty$

<u>Proof of CLAIM 2</u> . . .

$$\mu(A) = \inf_{G \supseteq A \text{ open}} \mu(G)$$

$\exists\, G_k$ open $\quad \mu(G_k \setminus A) < \frac{1}{k}$

$\chi_{G_k} - \chi_A \in \{0,1\}$

$$\int_\Omega |\chi_{G_k} - \chi_A|^p = \int_\Omega |\chi_{G_k \setminus A}|^p$$

$$= \mu(G_k \setminus A) < \frac{1}{k}$$

$\Rightarrow$ we may assume that A is open $\mu(A) < +\infty$

$\Rightarrow A = \bigcup_{l=1}^{\infty} I_l \quad I_l$ disjoint dyadic cubes

$$\mu(A) = \sum_{l=1}^{\infty} \mu(I_l) < +\infty$$

$$g_n = \chi_{\bigcup_{l=1}^{n} I_l} \to g = \chi_A \in L^p$$
$$n \to \infty$$

$$\Rightarrow \|g_m - \chi_A\|_{L^p} \to 0$$

as $m \to +\infty$

But $g_m \in E$ and we conclude when $f \geq 0$

. $f = f^+ - f^-$ ←

ii) $C_c^0(\mathbb{R}^n)$ is dense in L^p

Truncation operator

For $m \geq 1$

$$T_m(x) = \begin{cases} x & |x| \leq m \\ m \dfrac{x}{|x|} & |x| > m \end{cases}$$

$\boxed{\underline{\text{CLAIM 3}}}$. $f \in L^p$, $\varepsilon > 0$,

$\exists\, g \in L^\infty$, $g \equiv 0$ outside a compact $K \subseteq \mathbb{R}^n$

$$\|f - g\|_{L^p} < \varepsilon$$

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Proof CLAIM 3

$$f_n = \chi_{\overline{B(0,n)}} T_n(f)$$

$f_n \to f$ as $n \to +\infty$

and since $f \in L^p$

$\Rightarrow \|f_n - f\|_{L^p} \to 0$ as $n \to +\infty$

$g = f_n$ for n large enough $\square$

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$\boxed{\textbf{CLAIM 4}}$ Given $g \in L^\infty$

$\operatorname{supp} g \subseteq \mathbb{R}^n$ compact,

$\delta > 0$: $\exists\, g_1 \in C_c^0(\mathbb{R}^n)$

$$\| g - g_1 \|_{L^1} < \delta$$

Proof of CLAIM 4

• $g \geq 0$ $\exists\, g_n \to g$

as $n \to +\infty$

$$\| g_n - g \|_{L^p} \to 0 \text{ as } n \to +\infty$$

$\Rightarrow$ It is enough to prove the claim when

$g = \chi_A$ $\quad$ A is μ-meas and A is bounded

(because supp g is compact)

Find F, G (F closed, G open)

$$\boxed{F \subseteq A \subseteq G}$$

$\mu(G \setminus F) < \delta$

$\overline{G}$ compact

$\Rightarrow$ Apply Urysohn's Lemma

$\exists\, g_1 \in C(\mathbb{R}^n)$:

1) $0 \leq g_1 \leq 1$

2) $g_1 \equiv 0$ in G^c

3) $g_1 \equiv 1$ in F

$\Rightarrow |\chi_A - g_1| \leq |\chi_G - \chi_F|$

$$\int_\Omega |\chi_A - g_1|\, d\mu \quad 35$$

$$\leq \int_\Omega |\chi_G - \chi_F|\, d\mu = \mu(G \setminus F) < \delta \quad \square$$

$$\Rightarrow \exists\, g_1 \in C_c^0(\mathbb{R}^n) : \|g - g_1\|_{L^1(\mathbb{R}^n)} < \delta$$

$$\boxed{\|g_1\|_{L^\infty} \leq \|g\|_{L^\infty}}$$

$$\tilde{g}_1(x) = T_M\, g_1 \qquad M = \|g\|_{L^\infty}$$

$$\Rightarrow \|\tilde{g}_1\|_{L^\infty} \leq M = \|g\|_{L^\infty} \quad 36$$

$$\int_\Omega |g - g_1|^p\, d\mu =$$

$$= \int_\Omega |g - g_1|\, \underbrace{|g_1 - g|^{p-1}}\, d\mu$$

$$\leq \int_\Omega |g - g_1| \left(2^{p-2}\left(|g_1|^p + |g|^p\right)\right) d\mu$$

$$\le 2^{p-2} \, 2 \, \|g\|_{L^\infty}^{p} \, \|g - g_1\|_{L^1}$$

$$= 2^{p-1} \, \|g\|_{L^\infty}^{p} \, \delta$$

$$\Rightarrow \|g - g_1\|_{L^p} \le \underbrace{2^{\frac{p-1}{p}} \, \|g\|_{L^\infty} \, \delta^{1/p}}_{< \varepsilon/2}$$

Given $f \in L^p(\mathbb{R}^n)$:

$\exists\, g$ as claim 3 and

$\exists\, g_1$ as claim 4 :

$$\|f - g\|_{L^p} < \varepsilon/2$$

$$\|g - g_1\|_{L^p} < \varepsilon/2$$

$$\Rightarrow \|f - g_1\|_{L^p} \le \|f - g\|_{L^p} + \|g - g_1\|_{L^p} \le \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\square$