LECTURE 1.04.2020

Thm 2.2.5

$$f(x) \geq \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}(x)$$

$$A_k = \left\{ x \in \Omega : f(x) \geq \frac{1}{k} + \sum_{j=1}^{k-1} \frac{1}{j} \chi_{A_j}(x) \right\}$$

- $\sup \{ x \in \Omega : x \in A_k \} = +\infty$

$\forall n \geq 1 \quad \exists\, k_n \geq n : x \in A_{k_n}$

$$\Rightarrow f(x) \geq \underset{0 \swarrow}{\frac{1}{k_n}} + \sum_{j=1}^{k_n - 1} \frac{1}{j} \chi_{A_j}(x) \quad (*)$$

$k_n \to +\infty$ as $n \to +\infty$

If we let $n \to +\infty$ in $(*)$

$$f(x) \geq \sum_{j=1}^{\infty} \frac{1}{j} \chi_{A_j}(x)$$

- $\max \{ x \in \Omega : x \in A_k \} = k < +\infty$

$$\underline{x \in A_{k_0}} \Rightarrow f(x) \geq \frac{1}{k_0} + \sum_{j=1}^{k_0 - 1} \frac{1}{j} \chi_{A_j}(x)$$

$$= \sum_{j=1}^{k_0} \frac{1}{j} \chi_{A_j}(x) \quad (**)$$

because $\chi_{A_{k_0}}(x) = 1$

$k \geq k_0 + 1 \quad \chi_{A_k}(x) = 0$

$$(**) = \sum_{j=1}^{\infty} \frac{1}{j} \chi_{A_j}(x) \qquad \square$$

2.4 Convergence in measure.

μ measure in $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ μ-measurable

$f_k : \Omega \to \overline{\mathbb{R}}$, $f : \Omega \to \overline{\mathbb{R}}$

μ-meas., $|f(x)| < +\infty$

μ-a.e

$A = \{x \in \Omega : |f(x)| = +\infty\}$

$\mu(A) = 0$

Def 2.4.1

$(f_k)_k$ converges to f in measure μ, $f_k \xrightarrow{\mu} f$

as $k \to +\infty$ if $\forall \varepsilon > 0$

$$\lim_{k\to+\infty} \mu\{x\in\Omega : |f(x)-f_k(x)|>\varepsilon\} = 0 \Rightarrow f_k \xrightarrow{\mu} f \text{ as } k\to+\infty$$

Q: Relation between convergence in measure and the pointwise convergence and the μ a.e convergence.

Thm 2.4.2

Let $\mu(\Omega) < +\infty$.

If $f_k \to f$ as $k\to+\infty$ μ a.e

Proof: Apply Egoroff's Thm

$\forall \delta>0$ $\exists$ $K_\delta \subseteq \Omega$ compact

$$\mu(\Omega - K_\delta) < \delta$$

$$\lim_{k\to+\infty} \sup_{K_\delta} |f(x)-f_k(x)| = 0$$

$\forall \varepsilon>0$ $\exists k(\varepsilon)$: $\forall x\in K_\delta$

$$|f(x)-f_k(x)| < \varepsilon \quad \forall k \geq k(\varepsilon)$$

For every $k \geq K(\varepsilon)$

$$\{x \in \Omega : |f_k(x) - f(x)| \geq \varepsilon\} \subseteq \Omega - K_\delta$$

$$\Rightarrow \mu\{x \in \Omega : |f_k(x) - f(x)| \geq \varepsilon\} < \delta$$

$$\lim_{k \to +\infty} \mu\{x \in \Omega \; |f_k(x) - f(x)| \geq \varepsilon\} \leq \delta$$

Let $\delta \to 0$ and you get that

$$\lim_{k \to +\infty} \mu\{x \in \Omega : |f_k(x) - f(x)| \geq \varepsilon\} = 0$$

$$\Updownarrow \quad \boxed{\text{Ex}}$$

$$\lim_{k \to +\infty} \mu\{x \in \Omega : |f_k(x) - f(x)| > \varepsilon\} = 0$$

$\square$

REMARKS

1. Thm 2.4.2 does NOT hold if $\mu(\Omega) = +\infty$.

$$f_k(x) = \chi_{(-k,k)}(x)$$

$$f_k(x) \longrightarrow f(x) \equiv 1 \quad \text{in } \mathbb{R}$$

$$\mathcal{L}^1\{x \in \mathbb{R} : |f_k(x) - 1| > \varepsilon\} \geq \mathcal{L}^1\{\in \mathbb{R} : f_k(x) = 0\} \geq \mathcal{L}^1(-k,k)^c = +\infty$$

$$\Rightarrow \quad f_k \overset{\mu}{\not\to} 1 \quad \text{as } k \to +\infty$$

2. The converse of Thm 2.4.2 does not hold.

Take $\Omega = [0,1) \subseteq \mathbb{R}$, $\mu = \mathcal{L}^1$, for $k \geq 1$ define

$$f_k(x) = \chi_{\left[\frac{k-2^n}{2^n}, \frac{k+1-2^n}{2^n}\right)}$$

For every $k \geq 1$ the index n is choosen $2^n \leq k < 2^{n+1}$

$\boxed{\text{CLAIM 1}}$

$$f_k \xrightarrow{\mu} 0 \quad \text{as } k \to +\infty$$

Proof of CLAIM 1

$$\{f_k(x) > 0\} = \{f_k(x) = 1\}$$

$$= \left[\frac{k-2^m}{2^m}, \frac{k+1-2^m}{2^m}\right)$$

$$\mathcal{L}^1\left[\frac{k-2^m}{2^m}, \frac{k+1-2^m}{2^m}\right) = \frac{1}{2^m}$$

$$\mathcal{L}^1\{f_k(x) \geq \varepsilon\} \leq \mathcal{L}^1\{f_k(x) > 0\}$$

$$\leq \frac{1}{2^m} < \frac{2}{2^k} \quad (*)$$

From $(*)$ it follows that

$\forall \varepsilon > 0$

$$\lim_{k \to +\infty} \mathcal{L}^1\{f_k(x) \geq \varepsilon\} = 0$$

CLAIM 2

$$f_k \not\to 0 \quad k \to +\infty \quad \mu\text{-a.e}$$

Proof of CLAIM 2

$\forall x \in \Omega, \; m \geq 0, \; k \geq$, 12

$$f_k(x) = \begin{cases} 1 & \text{if } k = [2^n x] + 2^n \\ 0 & \text{for the other } k \in \{2^n, 2^{n+1}-1\} \end{cases}$$

$$f_k(x) = 1 \iff$$

$$\frac{k-2^n}{2^n} \leq x < \frac{k+1-2^n}{2^n}$$

$$\Updownarrow$$

$$k \leq 2^n x + 2^n < k+1$$

$$\Rightarrow \quad k = [2^n x + 2^n] = [2^n x] + 2^n$$

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$\forall s \in \mathbb{R}$

$$[s] = \sup\{n \geq 0 . \ n \leq s\}$$

$$[s] \leq s < [s] + 1$$

$f_k(x) \to 0$ if $\forall \varepsilon > 0 \ \exists K_{\varepsilon, x} > 0$

$\forall k \geq K_{\varepsilon, x}$ $0 \leq f_k(x) < \varepsilon$

But $\forall \bar{k}$ we can always

find $k = [2^n x] + 2^n \geq \bar{k}$

for some $n = n(\bar{k}, x)$

$f_k(x) \equiv 1$, so it cannot be $f_k(x) < \varepsilon$ $\downarrow$

$\square$

Theorem 2.4.4

Let $f_k \xrightarrow{\mu} f$. Then there exists a sub-sequence $f_{k_m} \longrightarrow f$ μ-a.e as $m \to +\infty$.

Proof Since $f_k \xrightarrow{\mu} f$: $\forall m \geq 1$ $\exists\, k_m \in \mathbb{N}$

$$\mu\left\{x \in \Omega : |f(x) - f_k(x)| > \frac{1}{2^m}\right\} < \frac{1}{2^m}$$

$\forall\, k \geq k_m$

By hyp: $\forall \varepsilon > 0$

$$\lim_{k \to +\infty} \mu\{|f(x) - f_k(x)| > \varepsilon\} = 0$$

You choose $\varepsilon = \frac{1}{2^m}$, $m \geq 1$

We define

$$A_m = \left\{x \in \Omega : |f_{k_m}(x) - f(x)| > \frac{1}{2^m}\right\}$$

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$\forall h \geq 1$

$$E_h = \bigcup_{n \geq h} A_n$$

$$\mu(E_h) \leq \sum_{n=h}^{\infty} \mu(A_n)$$

$$\leq \sum_{n=h}^{\infty} \frac{1}{2^n} = \frac{1}{2^h} \cdot \frac{1}{1-\frac{1}{2}}$$

$$= 2^{1-h}$$

Observe

$E_{h+1} \subseteq E_h$

$$\mu(E_1) \leq 2^{1-1} = 1 < +\infty$$

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$$\lim_{h \to \infty} \mu(E_h) = \mu\left(\bigcap_{h=1}^{\infty} E_h\right)$$

$$= \lim_{h \to +\infty} 2^{1-h} = 0$$

$E = \bigcap_{h=1}^{\infty} E_h$, $\mu(E) = 0$

CLAIM: $f_{k_n}(x) \to f(x)$ in $\Omega \setminus E$ (this means that $f_{k_n} \to f$ μ.a.e since $\mu(E) = 0$)

$$x \in \Omega \setminus E \iff x \notin \bigcap_{h=1}^{\infty} E_h$$

$$\exists\, \bar{h} \geq 1 : x \notin E_{\bar{h}} = \bigcup_{n=\bar{h}}^{\infty} A_n$$

$$\Rightarrow x \notin A_n \quad \forall n \geq \bar{h}$$

$$\boxed{\Rightarrow |f(x) - f_{k_n}(x)| \leq \frac{1}{2^n} \quad \forall n \geq \bar{h}}$$

$$\Rightarrow f_{k_n} \to f \text{ in } \Omega \setminus E \text{ as } n \to +\infty$$

$\square$

CHAPTER 3: INTEGRATION

μ Radon measure on $\mathbb{R}^n$

$\Omega \subseteq \mathbb{R}^n$ μ-measurable

Def 3.1.1

$g : \Omega \to \overline{\mathbb{R}}$ is simple if its image is countable

NOTATION

$$f^+ = \max(f, 0) = \begin{cases} f(x) & f \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f^- = \max(-f, 0)$$

$$f = f^+ - f^-, \quad |f| = f^+ + f^-$$

Def 3.1.2

$g : \Omega \to [0, +\infty]$

simple, μ-measurable and $\boxed{\geq 0}$.

$$\int_\Omega g \, d\mu = \sum_{0 \leq y \leq +\infty} y \, \mu(g^{-1}\{y\})$$

$\leq +\infty$

Def 3.1.3

$g : \Omega \to \overline{\mathbb{R}}$ simple, μ-meas, either $\int_\Omega g^+ d\mu < +\infty$ or $\int_\Omega g^- d\mu < +\infty$

We call g a μ-integrable simple function and define

$$\int_\Omega g\, d\mu = \int_\Omega g^+ d\mu - \int_\Omega g^- d\mu =$$

Therefore

$$\int_\Omega g\, d\mu = \sum_{-\infty \leq y \leq +\infty} y \mu(g^{-1}\{y\}).$$

Def 3.1.4

$f : \Omega \to \overline{\mathbb{R}}$. We define

i) $\overline{\int_\Omega} f\, d\mu = \inf \{ \int_\Omega g\, d\mu : g$ is a μ-integrable simple functions $g \geq f$ μ-a.e$\}$

$\overline{\int_\Omega} f\,d\mu$ is called the <u>upper integral of f</u>

2)

$\underline{\int_\Omega} f\,d\mu = \sup\left\{\int_\Omega g\,d\mu : g \text{ is a } \mu\text{-integrable simple functions: } g \le f \ \mu\text{-a.e}\right\}$

$\underline{\int_\Omega} f\,d\mu$ is called the

<u>lower integral of f</u>

Def 1.5. $f : \Omega \to \overline{\mathbb{R}}$ μ-measurable is called μ-integrable if $\overline{\int_\Omega} f\,d\mu = \underline{\int_\Omega} f\,d\mu$.

In this case we write

$$\int_\Omega f\,d\mu = \overline{\int_\Omega} f\,d\mu = \underline{\int_\Omega} f\,d\mu$$

- $\underline{\int_\Omega} f \, d\mu \leq \overline{\int_\Omega} f \, d\mu$
- A function f is "μ-integrable" provided the integral exists even if is equal to $+\infty$ or $-\infty$ $\boxed{||||}$