LECTURE 13.05.2020

Theorem 3.7.21

$f, f_k \; (k \geq 1) \in L^p(\Omega, \mu)$

$1 \leq p < +\infty$.

Then the following two conditions are equivalent:

1) $\| f_k - f \|_{L^p} \to 0$ as $k \to +\infty$

2) $f_k \xrightarrow{\mu} f$ and $\|f_k\|_{L^p} \to \|f\|_{L^p}$ as $k \to +\infty$

Proof

1) $\Rightarrow$ 2)

- $f_k \xrightarrow{\mu} f$ is a consequence of Tchebychev ineq.
- Minkowski Inequality $\Rightarrow$

$$\left| \|f_k\|_{L^p} - \|f\|_{L^p} \right| \leq \underbrace{\|f_k - f\|_{L^p}}_{\downarrow 0}$$

$\Rightarrow \|f_k\|_{L^p} \to \|f\|_{L^p}$ as $k \to +\infty$

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2) $\Rightarrow$ 1)

$$|f_k - f|^p \le 2^{p-1}|f_k|^p + 2^{p-1}|f|^p$$

$(t \mapsto |t|^p$ convex $1 \le p < +\infty)$

$$g_k := 2^{p-1}|f_k|^p + 2^{p-1}|f|^p - |f_k - f|^p \ge 0)$$

Since $f_n \xrightarrow{\mu} f \Rightarrow \exists f_{k_n} \to f$ as $n \to +\infty$ μ-a.e $\Rightarrow$

APPLY FATOU'S LEMMA TO g_{k_n}:

$$\liminf_{n\to+\infty} \int_\Omega g_{k_n} d\mu \ge \int_\Omega \liminf_{n\to+\infty} g_{k_n} \quad (1)$$

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$$\Rightarrow \int_\Omega |f_{k_n}|^p \to \int_\Omega |f|^p d\mu \quad n\to+\infty$$

$$\liminf_{n\to+\infty}\left[\int_\Omega 2^{p-1}|f_{k_n}|^p d\mu + \int_\Omega 2^{p-1}|f|^p d\mu - \int_\Omega |f_{k_n} - f|^p d\mu\right]$$

$$= 2\,2^{p-1}\int_\Omega |f|^p d\mu - \limsup_{n\to+\infty}\int_\Omega |f_{k_n} - f|^p$$

$$\ge \int_\Omega \left(2^{p-1}|f|^p + 2^{p-1}|f|^p\right) d\mu$$

$(f_{k_n} \to f \ \mu\text{.e} \Rightarrow |f_{k_n} - f|^p \to 0 \quad n\to+\infty)$

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$$2^p \cancel{\int_\Omega |f|^p d\mu} - \limsup_{n\to+\infty} \int_\Omega |f_{k_n} - f|^p d\mu$$

$$\geq 2^p \cancel{\int_\Omega |f|^p d\mu}$$

$$\Rightarrow \limsup_{n\to+\infty} \int_\Omega |f_{k_n} - f|^p d\mu = 0$$

$$\Rightarrow \|f_{k_n} - f\|_{L^p} \to 0 \text{ as } n\to+\infty$$

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<u>CLAIM</u>: $\|f_k - f\|_{L^p} \to 0$ as $k \to +\infty$

<u>Proof of CLAIM</u>

Suppose $\exists f_{k_n}$

$$\limsup \int |f_{k_n} - f|^p d\mu > 0$$

$\Rightarrow \exists f_{k_n'} \to f$ μ.a.e

$\Rightarrow \|f_{k_n'} - f\|_{L^p} \to 0$ as $n\to+\infty$

by above arguments

CHAPTER 4: PRODUCT MEASURES AND MULTIPLE INTEGRALS

Ref: Thm 0.0.2 in T. Tao's book

Result 1. (Tonelli's theorem for series)

Let $(x_{n,m})_{n,m\in\mathbb{N}}$ be a doubly infinite sequence of extended nonnegative

reals $x_{n,m}\in[0,+\infty]$.

Then

$$\sum_{(n,m)\in\mathbb{N}^2} x_{n,m} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{n,m} \quad (1)$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{n,m}$$

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ \vdots & & & \end{pmatrix}$$

Proof

"$\le$" $\quad \overline{\sum_{(n,m)\in\mathbb{N}^2}} x_{n,m} \le \overline{\sum_{n=1}^{\infty}}\,\overline{\sum_{m=1}^{\infty}} x_{n,m}$

$F \subseteq \mathbb{N}^2$ finite

$N > 0 \quad F \subseteq \underbrace{\{1,\dots,N\}\times\{1,\dots,N\}}_{A_N}$

$$\overline{\sum_{(n,m)\in F}} x_{n,m} \le \overline{\sum_{(n,m)\in A_N}} x_{n,m}$$

$$= \overline{\sum_{n=1}^{N}}\,\overline{\sum_{m=1}^{N}} x_{n,m} \le$$

$$\le \overline{\sum_{n=1}^{\infty}}\,\overline{\sum_{m=1}^{\infty}} x_{n,m}$$

$$\Rightarrow \overline{\sum_{(n,m)\in\mathbb{N}^2}} x_{n,m} \le \overline{\sum_{n=1}^{\infty}}\,\overline{\sum_{m=1}^{\infty}} x_{n,m}$$

"$\ge$" $\quad \overline{\sum_{n=1}^{\infty}}\,\overline{\sum_{m=1}^{\infty}} x_{n,m} \le \overline{\sum_{(n,m)\in\mathbb{N}^2}} x_{n,m}$

It is enough to show that $\forall N > 0$

$$\underbrace{\overline{\sum_{n=1}^{N}}\,\overline{\sum_{m=1}^{\infty}} x_{n,m}}_{(2)} \le \overline{\sum_{(n,m)\in\mathbb{N}^2}} x_{n,m}$$

Fix $N > 0$

$$\sum_{m=1}^{+\infty} x_{n,m} = \lim_{M \to +\infty} \sum_{m=1}^{M} x_{n,m}$$

(2) can be seen as the

$$\lim_{M \to +\infty} \sum_{n=1}^{N} \sum_{m=1}^{M} x_{n,m}$$

$$\Rightarrow \underbrace{\sum_{n=1}^{N} \sum_{m=1}^{M} x_{n,m}}_{(3)} \leq \sum_{(n,m) \in \mathbb{N}^2} x_{n,m}$$

$$(3) = \sum_{(n,m) \in \{1,..,N\} \times \{1,..,M\}} x_{n,m}$$

$$\leq \sum_{(n,m) \in \mathbb{N}^2} x_{n,m} \quad (x_{n,m} \geq 0)$$

Take first $\lim_{M \to +\infty}$ and $\lim_{N \to +\infty}$ and you get

" $\geq$ " $\square$

It is important that the $x_{n,m} \geq 0$

In the general case we need the additional assumption that

$$\sum_{(n,m)\in\mathbb{N}^2} |x_{n,m}| < +\infty$$

(ABSOLUTE SUMMABILITY)

$\Rightarrow$ Fubini's theorem

for the series.

$\Rightarrow$ you can interchange the two sums.

WARNING: Without the absolute summability or the non-negativity hyp the theorem fails

EXAMPLES

1) $$x_{n,m} = \begin{cases} 1 & n = m \\ -1 & n = m+1 \\ 0 & \text{otherwise} \end{cases}$$

2) $$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -1 & & & \\ \frac{1}{4} & \frac{1}{2} & -1 & & \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \end{pmatrix}$$