Lecture 15.05.2020

CHAPTER 4: MEASURES AND MULTIPLE INTEGRALS

$f \in C^0([a,b]\times[c,d])$

$\Rightarrow f$ is R-integrable

$$\iint_{[a,b]\times[c,d]} f \, d\mathcal{L}^2 = \int_c^d \left(\int_a^b f \, dx \right) dy$$

X, Y

Def 3.1.1

μ measure on X

ν measure on Y

$(\mu \times \nu) : \mathcal{P}(X \times Y) \to [0, +\infty]$

$\forall S' \subseteq X \times Y$

$(\mu\times\nu)(S) = \inf \{ \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i)$

$S \subseteq \bigcup_{i=1}^{\infty} A_i \times B_i$, $A_i \subseteq X$ μ-meas

$B_i \subseteq Y$ ν-meas $\forall i \in \mathbb{N} \}$

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$\mu \times \nu$ is called Product Measure of μ and ν

Remark

1) $A \subseteq X$ μ-meas, $B \subseteq Y$ ν-meas

$$\lambda(A \times B) = \mu(A)\nu(B)$$

$\Rightarrow$ extend λ to the the algebra of

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finite disjoint unions of such rectangles $A \times B$

$\Rightarrow$ λ is a pre-measure

$\mu \times \nu$ is just the Carathéodory extension of λ

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2) $\underline{X} = \underline{Y} = \mathbb{R}^2$ $\mu = \nu = \mathcal{L}^1$

$f : \mathbb{R}^2 \to \mathbb{R}$ $\mathcal{L}^2$-summ

$$\int_{\mathbb{R}^2} |f(x,y)| \, d\mathcal{L}^2 < +\infty$$

Q:

$$\int_{\mathbb{R}} |f(x,y)| \, d\mathcal{L}^2 \overset{?}{=} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)| \, d\mathcal{L}^1(x) \right) d\mathcal{L}^1(y)$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)| \, d\mathcal{L}^1(y) \right) d\mathcal{L}^1(x)$$

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Necessary conditions

i) $\forall \bar{y} \in \mathbb{R}$ $x \mapsto |f(x,\bar{y})|$ is $\mathcal{L}^1$-meas

ii) $y \mapsto \int_{\mathbb{R}} |f(x,y)| \, d\mathcal{L}^1(x)$ is $\mathcal{L}^1$ measurable and $\mathcal{L}^1$-summable

EXAMPLE

P non-measurable subset of $\mathbb{R}$ which is contained in $[0,1]$

$E = P \times Q$

- E is $\mathcal{L}^2$-measurable

$E \subseteq [0,1] \times Q$

and $\mathcal{L}^2([0,1] \times Q) =$

$\mathcal{L}^1[0,1] \times \mathcal{L}^1(Q) = 0$

$\Rightarrow \mathcal{L}^2(E) = 0 \Rightarrow$

E is $\mathcal{L}^2$-meas.

- $f(x,y) = \chi_E(x,y) = \chi_P(x)\chi_Q(y)$

Let $\bar{y} \in \mathbb{R}$ be fixed

$$f(x,\bar{y}) = \chi_P(x)\chi_Q(\bar{y})$$

$$= \begin{cases} \chi_P(x) & \bar{y} \in Q \\ 0 & \text{otherwise} \end{cases}$$

If $\bar{y} \in Q$ then

$x \longmapsto f(x, \bar{y})$ is NOT

$\mathcal{L}^1$- meas $\Rightarrow$ it has

not sense $\int_{\mathbb{R}} f(x, \bar{y})\, d\mathcal{L}^1(x)$

But $x \longmapsto f(x, \bar{y})$ is

$\mathcal{L}^1$- meas for $\mathcal{L}^1$-a.e

$\bar{y} \in \mathbb{R}$

Therefore we can

define

$$y \longmapsto \int_{\mathbb{R}} f(x, y)\, d\mathcal{L}^1(x)$$

Theorem 4.1.5 (Fubini)

Let μ, ν be Radon measures on $X = \mathbb{R}^k$, $Y = \mathbb{R}^l$, $\mu \times \nu$ the product measure on $\mathbb{R}^n$, $n = l + k$. Then

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1) $\forall A \subseteq \overline{X}$ μ-meas

$\forall B \subseteq \overline{Y}$ ν-meas

$A \times B$ is $\mu \times \nu$-meas

and

$$(\mu \times \nu)(A \times B) = \mu(A)\,\nu(B)$$

2) $S \subseteq \overline{X} \times \overline{Y}$ be

$(\mu \times \nu)$-meas. and

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$(\mu \times \nu)(S) < +\infty$.

Then $S_y = \{x \in \overline{X} : (x,y) \in S\}$

is μ-meas $\forall \nu$-a.e. y (*)

and

(**) $y \longmapsto \mu(S_y) = \int_{\overline{X}} \chi_{S_y}(x)\,d\mu$

is summable and

$$(\mu \times \nu)(S) = \int_Y \mu(S_y)\,d\nu =$$

$$= \int_Y \left(\int_{\overline{X}} \chi_{S_y}(x)\,d\mu \right) d\nu$$

it holds for

$$S_x = \{ y \in \overline{Y} : (x,y) \in S \}$$

3) $\mu \times \nu$ is Rado

4) $f : \overline{X} \times \overline{Y} \longrightarrow \overline{\mathbb{R}}$

$(\mu \times \nu)$ - meas.. Then

- $x \longmapsto f(x, \bar{y})$ is μ-summable for ν e.e $\bar{y} \in \overline{Y}$
- $y \longmapsto f(\bar{x}, y)$ is

ν-summable for μ-e.e $\bar{x} \in \overline{X}$ and

$$y \longmapsto \int_{\overline{X}} f(x,y)\, d\mu$$

$$x \longmapsto \int_{\overline{Y}} f(x,y)\, d\nu$$

are resp. ν- and $|\mu$- summable

Moreover

$$\int_{X\times Y} f \, d\mu\times\nu = \int_Y \left(\int_X f(x,y)\, d\mu \right) d\nu$$

$$= \int_X \left(\int_Y f(x,y)\, d\nu \right) d\mu .$$

$\square$

REMARKS

1) A set S satisfying the $(*)$ and $(**)$ in the statement of Fubini's theorem is NOT nec. $(\mu\times\nu)$-meas.

Ex: $X = Y = \mathbb{R}$, $\mu = \nu = \mathcal{L}^1$

$A \subseteq [0,1]$ NOT meas.

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$S' = \{(x,y) : |x - \chi_A(y)| < \frac{1}{2}$

$y \in [0,1]\}$

S is NOT $(\mu\times\nu)$-meas.

but $\forall y \in [0,1]$, S_y is

$\mathcal{L}^1$-meas. and $\mu(S_y) = 1$

$y \longmapsto \mu(S_y)$ is

$\mathcal{L}^1$-summ.

2) $f : \overline{X} \times \overline{Y} \to \mathbb{R}^+$

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is $\mu\times\nu$-meas., then

f is $\mu\times\nu$-summable

if one of the two iterated

integrals exists finite

(Tonelli's THEOREM)

3) It may happen that

the two iterated integrals

exist without f being

$\mu\times\nu$ - meas $(f = \chi_{S'})$

or $\mu\times\nu$ - summ $\left(f(x,y) = \frac{\sin y}{x}\right.$

$y \in [0, 2\pi]$, $x \in (0, 1]$

$$\int_0^{2\pi} \frac{\sin y}{x}\, dy = 0$$

$$\Rightarrow \int_0^1 \int_0^{2\pi} \frac{\sin y}{x}\, dy\, dx = 0)$$

Applications

1. Show that $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ is not $\mathcal{L}^2$-summable in $[0,1]\times[0,1]$.

Solution

1st proof

If $f \in L^1([0,1]\times[0,1])$

then $\mathcal{L}^1$-a.e $x \in [0,1]$

$f(x,\cdot) \in L^1([0,1])$

and $\int_Y f(x,y)\, d\mathcal{L}^1(y) \in L^1(X)$;

for $\mathcal{L}^1$-a.e $y \in [0,1]$.

$f(\cdot,y) \in L^1([0,1])$ and

$\int_X f(x,y)\, d\mathcal{L}^1(x) \in L^1(Y)$

Moreover

$$\int_{[0,1]\times[0,1]} f(x,y)\, d\mathcal{L}^2 = \int_{[0,1]} \left(\int_{[0,1]} f(x,y)\, d\mathcal{L}^1(x) \right) d\mathcal{L}^1(y)$$

$$= \int_{[0,1]} \left(\int_{[0,1]} f(x,y)\, d\mathcal{L}^1(y) \right) d\mathcal{L}^1(x)$$

$$\int_0^1 \underbrace{\left(\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\, dy \right)}_{1} dx$$

$$1 = \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\, dy = \int_0^1 \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2}\, dy$$

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$$= \int_0^1 \frac{1}{x^2+y^2}\,dy + \int_0^1 \underbrace{\frac{-2y^2}{(x^2+y^2)^2}}\,dy$$

$$= \int_0^1 \cancel{\frac{1}{(x^2+y^2)}}\,dy + \qquad \frac{-2y}{(x^2+y^2)^2} = \frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2}\right)$$

$$+ \frac{1}{x^2+y^2}\cdot y\,\Big|_0^1 - \int_0^1 \cancel{\frac{1}{x^2+y^2}}\,dy$$

$$= \frac{1}{x^2+1}$$

$$\Rightarrow \int_0^1 \frac{1}{x^2+1}\,dx = \operatorname{arctg} x\,\Big|_0^1 = \boxed{\frac{\pi}{4}}.$$

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But

$$\int_0^1\left(\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\right)dy = \boxed{-\frac{\pi}{4}}$$

2nd proof

$$\int_{[0,1]\times[0,1]} \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| dx\,dy = +\infty$$

$$\|$$

$$\int_0^1\Big(\underbrace{\int_0^y \frac{y^2-x^2}{(x^2+y^2)^2}\,dx}_{1} + \underbrace{\int_y^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx}_{2}\Big)dy$$

$$\left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| = \begin{cases} \dfrac{x^2-y^2}{(x^2+y^2)^2} & |x|\geq|y| \\[2ex] \dfrac{y^2-x^2}{(x^2+y^2)^2} & |x|\leq|y| \end{cases}$$

$\pm x^2$

$$(1) = \int_0^y \frac{y^2-x^2}{(x^2+y^2)^2}\,dx =$$

$$= \int_0^y \frac{y^2+x^2}{(x^2+y^2)^2}\,dx - \int_0^y \frac{2x^2}{(x^2+y^2)^2}\,dx$$

$$= \int_0^y \cancel{\frac{1}{x^2+y^2}}\,dx + \frac{x}{x^2+y^2}\Big|_0^y -$$

$$-\int_0^y \cancel{\frac{1}{x^2+y^2}}\,dx = \frac{y}{y^2+y^2} = \frac{1}{2y}$$

$\pm x^2$

$$(2) = \int_y^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx = -\int_y^1 \frac{1}{(x^2+y^2)}\,dx$$

$$+ \int_y^1 \frac{2x^2}{(x^2+y^2)^2}\,dx$$

$$\frac{2x}{(x^2+y^2)^2} = -\frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right)$$

$$= -\int_y^1 \cancel{\frac{1}{x^2+y^2}}\,dx - \frac{x}{(x^2+y^2)}\Big|_y^1$$

$$+ \int_y^1 \cancel{\frac{1}{(x^2+y^2)}}\,dx = -\frac{1}{1+y^2} + \frac{1}{2y}$$

$$\int_0^1 ① + ② \, dy = \int_0^1 \left(\frac{1}{2y} + \frac{1}{2y} \right) dy$$

$$+ \int_0^1 - \frac{1}{1+y^2} \, dy =$$

$$= \log|y| \Big|_0^1 - \operatorname{arctg} y \Big|_0^1$$

$$= 0 + \infty - \operatorname{arctg} 1$$

$$= +\infty$$

0

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2. $\int_0^{+\infty} e^{-x^2} dx$

2.3 CONVOLUTION

$\mu = \mathcal{L}^n$, $\Omega \subseteq \mathbb{R}^n$

$1 \le p \le +\infty$ $L^p(\Omega)$, L^p

$L^p(\Omega, \mu)$

LEMMA 4.3.1

$f : \mathbb{R}^n \to \overline{\mathbb{R}}$ $\mathcal{L}^n$-meas.

Then

$$F(x,y) = f(x-y) : \mathbb{R}^{2n} \to \overline{\mathbb{R}}$$

is $\mathcal{L}^{2n}$-meas.

<u>Proof</u>

$F^{-1}([-\infty, a))$ is $\mathcal{L}^{2n}$-meas.

$a \in \mathbb{R}$

$F^{-1}([-\infty, a)) = \{(x,y) : f(x-y) < a\}$

$z = x - y$

$$(*) = \{(x, x-z) : f(z) < a\}$$

$$= T(\mathbb{R}^n \times \{z : f(z) < a\})$$

$T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$

$(x,z) \mapsto (x, x-z)$

T is a Lipschitz (Ex)

cont and $\mathbb{R}^n \times \{z : f(z) < a\} \overset{(*)}{=}$ is $\mathcal{L}^{2n}$-meas

(because $\{z : f(z) < a\}$ is $\mathcal{L}^n$-meas)

LEMMA 4.3.2

$T : \mathbb{R}^m \to \mathbb{R}^m$ be Lipschitz continuous:

$$|T(x) - T(y)| \leq L|x-y| \quad \forall x, y \in \mathbb{R}^m$$

$\Rightarrow$ If A is $\mathcal{L}^1$-measurable then $T(A)$ is $\mathcal{L}^1$-meas

Given $f, g : \mathbb{R}^m \to \overline{\mathbb{R}}$

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy$$

$$= \int_{\mathbb{R}} f(z)\,g(x-z)\,dz$$

$f * g$ is called convolution between f and g

$\Rightarrow f * g = g * f$

Theorem 4.3.3

Let $f, g \in L^1(\mathbb{R}^m)$. Then

$f * g$ is $\mathcal{L}^n$-measurable and $f * g \in L^1(\mathbb{R}^n)$ with

$\boxed{**}$ $\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}$

Proof By Lemma 4.3.3 $F(x,y) = f(x-y)g(y)$ is $\mathcal{L}^{2n}$-meas.

i) $f, g \geq 0$

$\Rightarrow F$ is $\mathcal{L}^{2n}$-integrable.

By Tonelli's Thm F is $\mathcal{L}^{2n}$-summ $\Leftrightarrow$ one of the iterated integrals exists finite.

$$\|f * g\|_{L^1} = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy \right) dx$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-y)\, g(y)\, dx \right) dy$$

$$= \int_{\mathbb{R}} g(y) \left(\underbrace{\int_{\mathbb{R}} f(x-y)\, dx}_{z := x-y} \right) dy$$

$$= \|g\|_{L^1} \|f\|_{L^1} < +\infty$$

$\Rightarrow F(x,y)$ is $\mathcal{L}^{2n}$-summ.

ii) f, g general

From i)

$$|F(x,y)| = |f(x-y)|\,|g(y)| \in L^1(\mathbb{R}^{2n})$$

By Fubini's thm we have $|f| * |g| \in L^1(\mathbb{R}^n)$

$$\|f * g\|_{L^1} = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy \right| dx$$

$$\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)\,g(y)|\,dy\,dx$$

$$= \int_{\mathbb{R}^n} |f| * |g|(x)\,dx$$

$$= \| |f| * |g| \|_{L^1} = \|f\|_{L^1} \|g\|_{L^1}$$

$\square$