LECTURE 20.05.2020

COROLLARY 4.3.4

$1 \le p, q \le +\infty$

i) $f \in L^1(\mathbb{R}^m)$, $g \in L^p(\mathbb{R}^m)$

$\Rightarrow f * g \in L^p(\mathbb{R}^m)$ and

$$\| f * g \|_{L^p} \le \| f \|_{L^1} \| g \|_{L^p}$$

ii) $f \in L^p$, $g \in L^q$

$$1 \le \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$$

Then $f * g \in L^r(\Omega)$ and

$$\| f * g \|_{L^r} \le \| f \|_{L^p} \| g \|_{L^q}$$

Proof

i) • $p = 1 \Rightarrow$ see Thm 4.3.3

• $1 < p < +\infty$, $\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$

$$\int_{\mathbb{R}} | f * g |^p dx = \int_{\mathbb{R}^m} \left| \int_{\mathbb{R}^m} f(x-y) g(y) dy \right|^p dx$$

$\le$

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$$\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |f(x-y)|\,|g(y)|\,dy \right)^p dx$$

$$|f(x-y)| = |f(x-y)|^{\frac{1}{q}} |f(x-y)|^{\frac{1}{p}}$$

$$\frac{1}{q} + \frac{1}{p} = 1$$

From the case $p = 1$ we deduce that for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$

$$y \longmapsto |f(x-y)|\,|g(y)|^p$$

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is $\mathcal{L}^n$-summable $\circledast$

$$\left(|f(x-y)|^{\frac{1}{p}} \in L^{1?}, \ |g| \in L^p \right)$$

$$\Rightarrow |g|^p \in L^1$$

$$|f|\,|g|^p \in L^1 \ (\text{Thm } 4.3.3)$$

$\Rightarrow \circledast$

We also have

$$y \longmapsto |f(x-y)|^{\frac{1}{p}} |g| \in L^p$$

Moreover $|f(x-y)|^{\frac{1}{q}} \in L^q$

From Hölder inequality we get

$$|f(x-y)|\,|g(y)| = \underbrace{|f(x-y)|^{\frac{1}{q}}}_{\in L^q}\ \underbrace{|f(x-y)|^{\frac{1}{p}}|g(y)|}_{\in L^p}$$

$$\in L^1(\mathbb{R}^n)$$

$$\int_{\mathbb{R}^n} |f*g|^p\,dx \le \int_{\mathbb{R}^n}\left(\int |f(x-y)|^{\frac{1}{q}}\,|f(x-y)|^{\frac{1}{p}}|g(y)|\,dy\right)^p dx$$

$$\le \int_{\mathbb{R}^n}\left(\left(\int_{\mathbb{R}^n} |f(x-y)|^{\frac{1}{q}q}\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^n} |f(x-y)|^{\frac{1}{p}\cdot p}|g(y)|^p dy\right)^{1/p}\right)^p dx$$

$$= \int_{\mathbb{R}^n}\left(\|f\|_{L^1}^{\frac{1}{q}}\left(|f|*|g|^{p}\right)^{1/p}\right)^p dx$$

$$= \|f\|_{L^1}^{\frac{p}{q}}\ \big\||f|*|g|^p\big\|_{L^1(\mathbb{R}^n)}$$

$$= \|f\|_{L^1}^{\frac{p}{q}}\ \|f\|_{L^1}\ \||g|^p\|_{L^1} =$$

$$= \underbrace{\|f\|_{L^1}^{\frac{p}{q}+1}\ \|g\|_{L^p}^{p}}$$

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$$\Rightarrow \|f*g\|_{L^p} = \left(\int_{\mathbb{R}} |f*g|^p dx\right)^{1/p}$$

$$\leq \left(\|f\|_{L^1}^{\frac{p}{q}+1} \|g\|_{L^p}^p\right)^{1/p}$$

$$= \|f\|_{L^1} \|g\|_{L^p}$$

- $p = +\infty \Rightarrow$ it is consequence of Hölder Inequality

ii) $p \leq q$

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Since $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \leq 1 + \frac{1}{q}$

$\Rightarrow \frac{1}{q} \geq \frac{1}{r} \Rightarrow q \leq r$

- $r = +\infty \Rightarrow$ it follows from Hölder Inequality
- $p = 1 \Rightarrow p = q$ (it follows from i)
- $1 < p \leq q < r < +\infty$

$$\int_{\mathbb{R}^n} |f(x-y)|\,|g(y)|\,dy$$

$$= \int_{\mathbb{R}^n} |f(x-y)|^{1-\frac{p}{r}}\,|g(y)|^{1-\frac{q}{r}}\,|f(x-y)|^{\frac{p}{r}}\,|g(y)|^{\frac{q}{r}}\,dy$$

Observe that

* $|f(x-y)|^{1-\frac{p}{r}} \in L^s$

$$s = \frac{p}{1-\frac{p}{r}} = \frac{rp}{r-p}$$

(because $f \in L^p$ $\Rightarrow$ $|f|^r \in L^{p/r}$)

* $|g(y)|^{1-\frac{q}{r}} \in L^t$

$$t = \frac{q}{1-\frac{q}{r}} = \frac{qr}{r-q} \Leftarrow q \neq r$$

* $|f(x-y)|^{p/r}\,|g(y)|^{\frac{q}{r}} \in L^1$

From Thm 4.3.3

$$|f|^p * |g|^q \in L^1$$

$$y \longmapsto |f(x-y)|^{p} \; |g(y)|^{q} \in L^1$$

$$\Rightarrow |f(x-y)|^{\frac{p}{r}} \; |g(y)|^{\frac{q}{r}} \in L^r$$

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = \frac{1}{r} + \frac{r-p}{rp} + \frac{r-q}{rq} =$$

$$= \cancel{\frac{1}{r}} + \frac{1}{p} - \cancel{\frac{1}{r}} + \frac{1}{q} - \frac{1}{r} = 1$$

$\Rightarrow$ We can apply the generalized Hölder Inequality

$$|f*g|^r = \left(\left|\int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy\right|^r\right)$$

$$\le \left(\int_{\mathbb{R}^n} |f(x-y)|\,|g(y)|\, dy\right)^r$$

$$\le \left(\int_{\mathbb{R}^n} |f(x-y)|^{\left(1-\frac{p}{r}\right)s}\right)^{\frac{r}{s}} \left(\int_{\mathbb{R}^n} |g(y)|^{\left(1-\frac{q}{r}\right)t}\right)^{r/t}$$

$$\left(|f|^p * |g|^q\right) =$$

$$\frac{r-p}{r} \cdot \frac{rp}{r-p} = p$$

$$\left(1-\frac{q}{r}\right) \cdot \frac{rq}{r-q} = q$$

$$\|f\|_{L^p}^{p\frac{r}{s}}\,\|g\|_{L^q}^{q\frac{r}{t}}\left(|f|^p * |g|^q\right)$$

$$\frac{pr}{s} = pr\,\frac{r-p}{rp} \qquad \frac{qr}{t} = \cancel{q}\cancel{r}\cdot\frac{r-q}{\cancel{q}\cancel{r}}$$

$$\left(\int_{\mathbb{R}^n} |f*g|^r\,dx\right)^{1/r} \le \left(\int_{\mathbb{R}^n} \|f\|_{L^p}^{r-p}\,\|g\|_{L^q}^{r-q}\,|f|^p * |g|^q\,dx\right)^{1/r} \; (*)$$

$$\boxed{\begin{array}{l} |f|^p \in L^1 \quad |g|^q \in L^1 \\ \Rightarrow |f|^p * |g|^q \in L^1 \end{array}}$$

$$(*) \le \left(\|f\|_{L^p}^{r-p}\,\|g\|_{L^q}^{r-q}\,\|f\|_{L^p}^{p}\,\|g\|_{L^q}^{q}\right)^{1/r}$$

$$= \|f\|_{L^p}^{1-\cancel{p/r}+\cancel{p/r}}\,\|g\|_{L^q}^{1-\cancel{q/r}+\cancel{q/r}}$$

$$\Rightarrow$$

$$\|f*g\|_{L^r} \le \|f\|_{L^p}\,\|g\|_{L^q}$$

□

Def $1 \le p \le +\infty$.

$f \in L^p_{loc}(\mathbb{R}^n) \iff \forall K \subseteq \mathbb{R}^n$ compact $f \chi_K \in L^p(\mathbb{R}^n)$

NOTE if $f \in L^p_{loc}(\mathbb{R}^n) \Rightarrow$ $f \in L^1_{loc}(\mathbb{R}^n)$ $\forall s < p$.

Prop 4.3.6

$f \in C^0_c(\mathbb{R}^n)$, $g \in L^1_{loc}(\mathbb{R}^n)$

Then $f * g \in C^0(\mathbb{R}^n)$

Proof

$\forall x \in \mathbb{R}^n$ $y \mapsto f(x-y)\,g(y)$ is $\mathcal{L}^n$-summable (easy exercise)

$\Rightarrow (f*g)(x)$ is well defined.

CLAIM if $x_m \to x$, then

$(f*g)(x_n) \to (f*g)(x)$

Proof of CLAIM Let

$x_n \to x$ & $K \subseteq \mathbb{R}^n$ compact

such that $x_n - \operatorname{supp} f \subseteq K$ $\forall n$

$\left(\operatorname{supp} f = \overline{\{x : f(x) \neq 0\}}\right)$

$\Rightarrow f(x_n - y) = 0 \quad \forall n \ \forall y \notin K$

$\Rightarrow f(x - y) = 0 \quad \forall y \notin K$

$(y \notin K \Rightarrow y \notin x_n - \operatorname{supp} f$

$\forall n \quad x_n - y \notin \operatorname{supp} f$ and

$x - y \notin \operatorname{supp} f)$

$\Rightarrow f(x_n - y) = 0$

$|f(x_n - y) - f(x - y)| \leq$

$\leq \omega_f(|x_n - x|) \chi_K(y)$

$\omega_f : \mathbb{R}^+ \to \mathbb{R}^+$

$r \leq s \Rightarrow \omega_f(r) \leq \omega_f(s)$

$\lim_{r \to 0^+} \omega_f(r) = 0$

$$|(f*g)(x_n) - (f*g)(x)|$$

$$= \left| \int_{\mathbb{R}^n} \big(f(x_n - y) - f(x-y) \big) g(y)\, dy \right|$$

$$\leq \int_{\mathbb{R}^n} \omega_f \big(|x_n - \cancel{y} - x + \cancel{y}| \big) \underbrace{\chi_K(y) |g(y)|}\, dy$$

$$\leq \omega_f(|x_n - x|)\, \|g\|_{L^1(K)}$$

Since $\lim_{n \to +\infty} \omega_f(|x_n - x|) = 0$ we can conclude the proof.

Observe that $\chi_K(y)\, g(y) \in L^1(\mathbb{R}^n)$ because $g \in L^1_{loc}(\mathbb{R}^n)$.

$\square$