LECTURE 22.04.2020

LAST TIME

LEMMA 1.1.16

$f : \Omega \to \overline{\mathbb{R}}$ μ-summable

$\Omega_1 \subseteq \Omega$ μ-measurable

$\Rightarrow$ $f_1 = f|_{\Omega_1}$, $f_2 = f\chi_{\Omega_1}$

μ-summable

$$\int_{\Omega_1} f_1 \, d\mu = \int_{\Omega} f_2 \, d\mu$$

(see Proof of LEMMA 3.1.1 in Struwe's notes)

COR. 1.1.17

$f : \Omega \to \overline{\mathbb{R}}$ μ summable

$\Omega_1 \subseteq \Omega$ μ-measurable

$\mu(\Omega_1) = 0$

$$\int_{\Omega_1} f \, d\mu = 0$$

Proof $f\chi_{\Omega_1} = 0$ μ-a.e

$\Rightarrow \int_{\Omega} f\chi_{\Omega_1} \, d\mu = 0$

By LEMMA 1.1.16

$$\int_{\Omega_1} f \, d\mu = 0$$

$\square$

Prop 1.1.18

$f : \Omega \to \overline{\mathbb{R}}$ μ-SUMMABLE

$\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2$

$$\Rightarrow \int_{\Omega} f \, d\mu = \int_{\Omega_1} f \, d\mu + \int_{\Omega_2} g \, d\mu$$

Proof $f = f_1 + f_2$

$f_1 = f \chi_{\Omega_1}$, $f_2 = f \chi_{\Omega_2}$

f_1, f_2 are μ-summ.

$$\int_{\Omega} (f_1 + f_2) \, d\mu = \int_{\Omega} f \, d\mu$$

$$= \int_{\Omega} f \chi_{\Omega_1} d\mu + \int_{\Omega} f \chi_{\Omega_2} d\mu$$

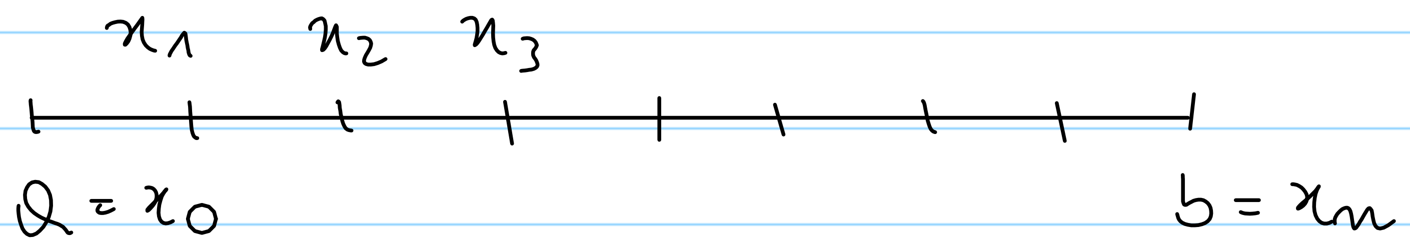
$$= \int_{\Omega_1} f \, d\mu + \int_{\Omega_2} f \, d\mu$$

$\square$

1.2. Comparison between Lebesgue and Riemann Integral

$I = [a, b] \subseteq \mathbb{R}$

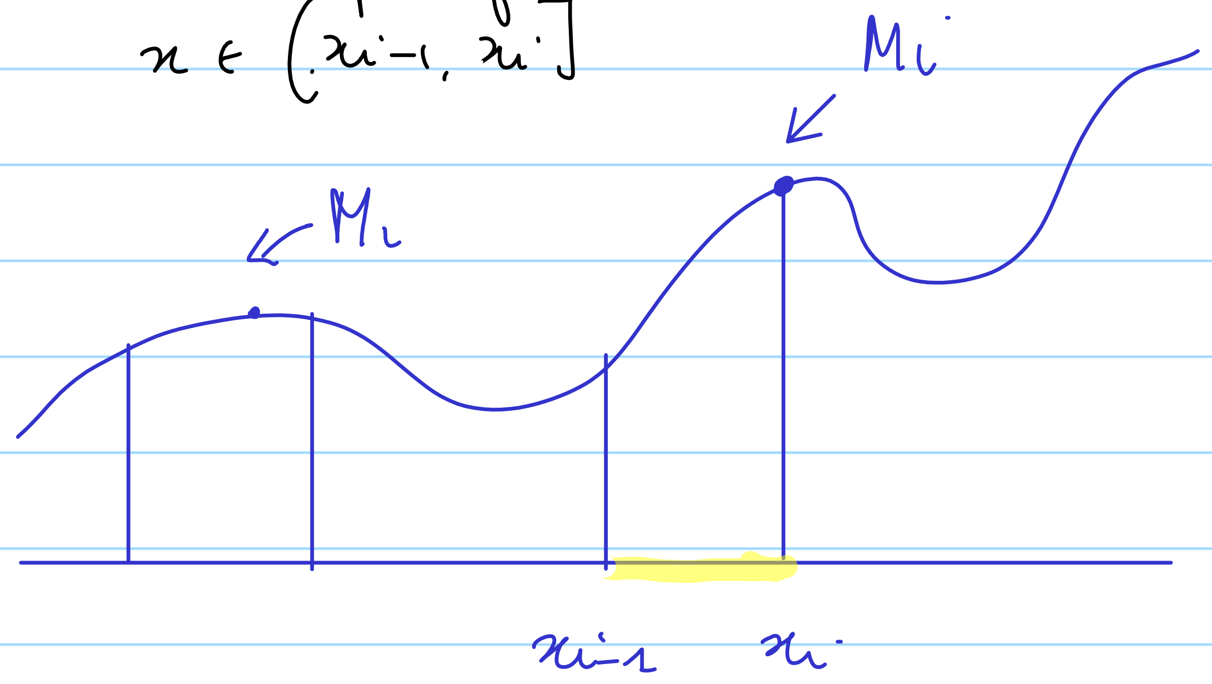
$P = \{ a = x_0, x_1, \dots, x_n = b \}$



$f : I \longrightarrow \mathbb{R}$ be bounded

$$S(P, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$



$$s(P, f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$$

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$\mathcal{P}$ = set of all partitions of I

$$\overline{R\int_a^b} f(x)\,dx = \inf\{S(P,f) \mid P\in\mathcal{P}\}$$

$$\underline{R\int_a^b} f(x)\,dx = \sup\{s(P,f) \mid P\in\mathcal{P}\}$$

f is Riemann integrable

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if $\overline{R\int_a^b} f(x)\,dx = \underline{R\int_a^b} f(x)\,dx$

$$=: \int_a^b f(x)\,dx$$

Let $\varphi(x) = \sum_{i=1}^{n} c_i \, \chi_{(x_{i-1},x_i]}(x)$ (*)

$\{x_i\}$ is a partition of I

and $c_i \in \mathbb{R}$

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$$\int_{[a,b]} \varphi(x)\, d\mathcal{L}^1 = \sum_{i=1}^{n} c_i \mathcal{L}^1([x_{i-1}, x_i]) = \sum_{i=1}^{n} c_i (x_i - x_{i-1})$$

$[a,b]$

$$S(P,f) = \int_{[a,b]} \overline{\varphi}(x)\, d\mathcal{L}^1$$

$$\overline{\varphi}(x) = \sum_{i=1}^{n} M_i \, \chi_{[x_{i-1}, x_i]}$$

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$$M_i = \sup_{[x_{i-1}, x_i]} f(x)$$

$$s(P,f) = \int_{[a,b]} \underline{\varphi}(x)\, d\mathcal{L}^1$$

$$\underline{\varphi}(x) = \sum_{i=1}^{n} m_i \cdot \chi_{[x_{i-1}, x_i]}$$

$$m_i = \inf_{[x_{i-1}, x_i]} f(x)$$

$$\underline{\varphi}(x) \leq f(x) \leq \overline{\varphi}(x)$$

$\forall x \in]$

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$$R\overline{\int_a^b} f(x)\,dx = \inf\left\{ \int_{[a,b]} \overline{\varphi}(x)\,d\mathcal{L}^1, \quad \overline{\varphi} \text{ is as in } ⊛ \quad \overline{\varphi} \geq f \text{ in } I=[a,b] \right\}$$

$$R\int_{\underline{a}}^b f(x)\,dx = \sup\left\{ \int_{[a,b]} \underline{\varphi}(x)\,d\mathcal{L}^1 \quad \underline{\varphi} \text{ is as in } ⊛ \quad \underline{\varphi} \leq f \text{ in } I=[a,b] \right\}$$

Prop 1.2.2 $f : [a,b] \to \mathbb{R}$ bdd and R-integrable

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Then f is $\mathcal{L}^1$-integrable

$$R\int_a^b f(x) = \int_{[a,b]} f\,d\mathcal{L}^1$$

Proof

$$R\int_{\underline{a}}^b f(x)\,dx = \sup\left\{ \int_{[a,b]} \underline{\varphi}(x)\,d\mathcal{L}^1 \quad \underline{\varphi} = \sum_{i=1}^{d} c_i \chi_{(x_{i-1},x_i]} \quad \underline{\varphi} \leq f \text{ in } I \right\}$$

$$\leq \sup\left\{ \int_{[a,b]} g\,d\mu \quad g \text{ is } \mathcal{L}^1\text{-integrable simple function } g \leq f\ \mathcal{L}^1\text{-a.e. } I \right\}$$

$$= \underline{\int_{[a,b]}} f \, d\mathcal{L}^1 \le \overline{\int_{[a,b]}} f \, d\mathcal{L}^1$$

$$= \inf \Big\{ \int_{[a,b]} g \, d\mathcal{L}^1 , \ g \text{ is } \mathcal{L}^1\text{- int. simple} \quad g \ge f \ \mathcal{L}^1\text{-a.e. in } I \Big\}$$

$$\le \inf \Big\{ \int_{[a,b]} \overline{\varphi} \, d\mathcal{L}^1 , \quad \overline{\varphi} = \sum_{i=1}^{n} a_i \chi_{(x_{i-1}, x_i]} \quad \overline{\varphi} \ge f \Big\}$$

$$= \underbrace{R \overline{\int_a^b} f(x) \, dx}$$

$\Rightarrow$ f R - integrable $\Rightarrow$

f $\mathcal{L}^1$ - integrable and

$$\int_{[a,b]} f \, d\mathcal{L}^1 = \int_a^b f \, dx \qquad \square$$

EXAMPLE

$f(x) = \chi_{[0,1] \cap \mathbb{Q}}$

$\forall$ P partition of $[0,1]$

$S(P, f) = 1$, $s(P, f) = 0$

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$\Rightarrow \quad \mathcal{R}\overline{\int_0^1} f(x)\,dx = 1$

$\mathcal{R}\underline{\int_0^1} f(x)\,dx = 0$

$\Rightarrow$ f is NOT R-integrable

But. f is $\mathcal{L}^1$-integrable

$f = 0$ $\mathcal{L}^1$-a.e

$\Rightarrow \int_{[0,1]} f \, d\mathcal{L}^1 = 0$

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Another issue

$f_n(x) = \chi_{\{r_1,\dots,r_n\}}(x)$

$\{r_n\}$ is enumeration of $\mathbb{Q}\cap[0,1]$

$f_n(x)$ is R-integrable

and $\int_0^1 f_n(x)\,dx = 0 \leftarrow$

- $f_n(x) \to \chi_{\mathbb{Q}\cap[0,1]}(x)$

NO SENSE

$$\lim_{n\to+\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \chi_{\mathbb{Q}\cap[0,1]}\,dx$$

1.3 Convergence Results

μ RADON MEASURE,

$\Omega \subseteq \mathbb{R}^n$ μ-measurable

Thm 1.3.1 (Fatou's LEMMA)

$f_k : \Omega \to [0,+\infty]$ μ-meas.

Then

$$\int_\Omega \liminf_{k\to+\infty} f_k \, d\mu \leq \liminf_{k\to+\infty} \int_\Omega f_k \, d\mu \qquad (2)$$

Proof

f_k is μ-integrable $\forall k$

$f := \liminf_{k\to+\infty} f_k$ μ-integrable

If $f = 0$ μ.a.e then (2)

holds

It is enough to prove that

$$\int_\Omega g \, d\mu \le \lim_{h \to +\infty} \int_\Omega f_k \, d\mu$$

$\forall g$ μ-int. simple function $g \le f$

$$g(x) = \sum_{j \ge 0}^{\infty} a_j \chi_{A_j}$$

$a_0 = 0 \quad a_j > 0 \quad \forall j \ge 1$

$A_j \cap A_i = \emptyset \quad (i \ne j)$

Fix $0 < t < 1$ and $\forall j, k \ge 1$

$$B_{j,k} = \{x \in A_j : f_\ell(x) > t a_j \ \forall \ell \ge k\}$$

Since $f(x) = \liminf_{k \to +\infty} f_k(x)$

$$A_j = \bigcup_{k=1}^{\infty} B_{j,k}$$

$k \longmapsto B_{j,k}$ is non-decreas.

$B_{j,k} \subseteq B_{j,k+1}$

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$$\lim_{k\to\infty} \mu(B_{j,k}) = \mu(A_j)$$

For $J, k \in \mathbb{N}$:

$$\int_\Omega f_k(x)\, d\mu \geq \sum_{j=1}^{J} \int_{A_j} f_k\, d\mu \geq$$

$$\bigcup_{j=1}^{J} A_j \subseteq \Omega \qquad A_{j_1} \cap A_{j_2} = \phi$$

$$\geq \sum_{J=1}^{J} \int_{B_{J,k}} f_k\, d\mu \geq$$

$$B_{J,k} \subseteq A_J$$

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$$\geq t \sum_{j=1}^{J} a_j\, \mu(B_{j,k})$$

- let $k \to +\infty$

$$\liminf_{k\to+\infty} \int_\Omega f_k\, d\mu \geq t \sum_{j=1}^{J} a_j\, \mu(A_j)$$

(recall $\lim_{k\to+\infty} \mu(B_{j,k}) = \mu(A_j)$)

- let $J \to +\infty$

$$\liminf \int_\Omega f_k\, d\mu \geq t \sum_{J=1}^{\infty} a_J\, \mu(A_J) =$$

$= t \int_\Omega g \, d\mu$

- let $t \to 1$ and we get the conclusion.

<u>EX</u> $\mu = \mathcal{L}^n$ $\Omega = \mathbb{R}^n$

$f_k = -\frac{1}{k^n} \chi_{B_k(0)}$

$\int_\Omega f_k \, d\mu = -\frac{1}{k^n} \mathcal{L}^n(B_k(0))$

$= -\cancel{\frac{1}{k^n}} \cancel{k^n} \omega_1$

$\liminf \int f_k \, d\mu = -\omega_1$

$\liminf f_k(x) = \lim f_k(x) = 0$

$\Rightarrow$ Fatou's Lemma does NOT holds!

<u>THEOREM 1.3.3</u> (Beppo Levi's Theorem / Monotone convergence Theorem)

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$f_k : \Omega \to [0, +\infty]$ μ-meas.

$f_1 \le f_2 \le \dots \le f_k \le \dots$

$$\int_\Omega \lim_{k\to+\infty} f_k(x)\, d\mu = \lim_{k\to+\infty} \int f_k(x)\, d\mu$$

<u>Proof</u>

$\forall\, k \ge 1$

$$f_k(x) \le f(x) = \lim_{k\to+\infty} f_k(x)$$

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$$\int_\Omega f_k(x)\, d\mu \le \int_\Omega f(x)$$

$$\Rightarrow \lim_{k\to+\infty} \int_\Omega f_k(x)\, d\mu \le \int_\Omega f(x)\, d\mu \quad 3$$

By FATOU's LEMMA

$$\int_\Omega \liminf f_k(x) = \int_\Omega \lim f_k(x)\, d\mu$$

$$= \int_\Omega f(x)\, d\mu \le \liminf_{k\to+\infty} \int_\Omega f_k(x)\, d\mu$$

$$= \lim_{k\to+\infty} \int_\Omega f_k(x)\, d\mu \quad \square \quad 4$$

REMARK (Beppo Levi ⇒ Fatou's LEMMA)

$$g_k = \inf_{l \geq k} f_l$$

g_k is a non decreasing sequence such that

$$\lim_{k\to+\infty} g_k = \liminf_{k\to+\infty} f_k$$

$$\forall k \quad \int_\Omega g_k = \int_\Omega \inf_{l \geq k} f_l \, d\mu$$

$$\leq \inf_{l \geq k} \int_\Omega f_l \, d\mu \quad \Leftarrow$$

Beppo Levi's theorem applied to g_k gives:

$$\int_\Omega \lim_{k\to+\infty} g_k \, d\mu = \int_\Omega \liminf_{k\to+\infty} f_k$$

$$= \lim_{k\to+\infty} \int_\Omega g_k \, d\mu \leq \liminf_{k\to+\infty} \int f_k \, d\mu$$

□

REMARK.

Proof of Beppo Levi without using Fatou's

Lemma:

Since (f_n) is non decreasing we have

$$\lim_{n\to+\infty} \int_\Omega f_n \, d\mu \leq \int_\Omega f \, d\mu$$

Let h be a simple function $0 \leq h \leq f$ and $\varepsilon > 0$

$$E_n = \{x \in \Omega : f_n(x) \geq (1-\varepsilon) h\}$$

Since $f_n(x) \to f(x)$ we have $\bigcup_{n=1}^{\infty} E_n = \Omega$ and

$$\int_\Omega f_n \, d\mu \geq \int_{\Omega_n} f_n \, d\mu$$

$$\geq \int_{\Omega_n} (1-\varepsilon) h \, d\mu$$

$$\lim_{n\to+\infty} \int_{\Omega_n} f_n \, d\mu \geq \lim_{n\to+\infty} \int_{\Omega_n} (1-\varepsilon) h$$

$$= \int_\Omega (1-\varepsilon) h \, d\mu$$

$$= \int_\Omega (1-\varepsilon) h \, d\mu$$

Let $\varepsilon \to 0$

$$\lim_{n\to+\infty} \int_\Omega f_n \, d\mu \geq \int_\Omega h \, d\mu$$

Since h is arbitrary

we get

$$\lim_{n\to+\infty} \int_\Omega f_n \, d\mu \geq \int_\Omega f \, d\mu$$

$\square$

Application (EXERCISE)

(g_n) $g_n : \Omega \to [0, +\infty]$

μ - measurable.

$\Rightarrow \sum_{n=1}^{\infty} g_n : \Omega \to [0, +\infty]$

is μ - measurable

and

$$\int_\Omega \sum_{n=1}^{\infty} g_n \, d\mu = \sum_{n=1}^{\infty} \int_\Omega g_n \, d\mu .$$

$\square$