LECTURE 22.05.2020

Prop 4.3.7 (NO PROOF)

$f \in C_c^k(\mathbb{R}^n)$ $k \geq 1$, $g \in L^1_{loc}(\mathbb{R}^n)$

Then $f * g \in C^k(\mathbb{R}^n)$ and

$$D^\alpha(f * g) = D^\alpha f * g \quad \forall\, |\alpha| \leq k$$

$\alpha = (\alpha_1, \dots, \alpha_n)$ $\quad \alpha_i \in \mathbb{N}$

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}$$

$f \in C_c^\infty(\mathbb{R}^n)$, $g \in L^1_{loc}(\mathbb{R}^n)$

$f * g \in C^\infty(\mathbb{R}^n)$.

4.4. Application: Solution of the Laplace Equation

$u \in C^2(\mathbb{R}^n)$

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial^2 x_i} = \mathrm{trace}(D^2 u)$$

$$= \mathrm{div}(Du)$$

$$Du = (\partial_{x_1} u, \dots, \partial_{x_n} u) = \nabla u$$

Given $f \in C_c^\infty(\mathbb{R}^n)$

find $u \in C^2(\mathbb{R}^n)$:

$$-\Delta u = f \text{ in } \mathbb{R}^n$$

First of all we would like to find a sort of "representation formula" of $u \in C_c^\infty(\mathbb{R}^n)$ in terms of its Δu

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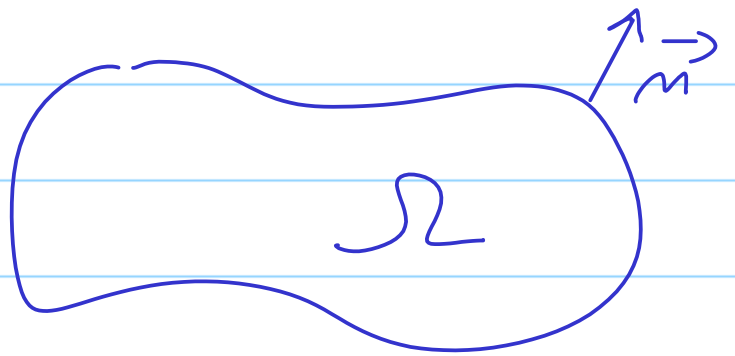
Thm 4.4.4 (Gauss Thm)

Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded open set, $\varphi, \psi \in C^2(\bar{\Omega})$. Then

1) $$\int_\Omega \varphi \Delta \psi - \psi \Delta \varphi \, dx = \int_{\partial\Omega} \varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n} \, d\sigma$$

where $\vec{n}$ is the outward normal vector to $\partial\Omega$ and

$$\frac{\partial\varphi}{\partial n} = D\varphi \cdot \vec{n}$$



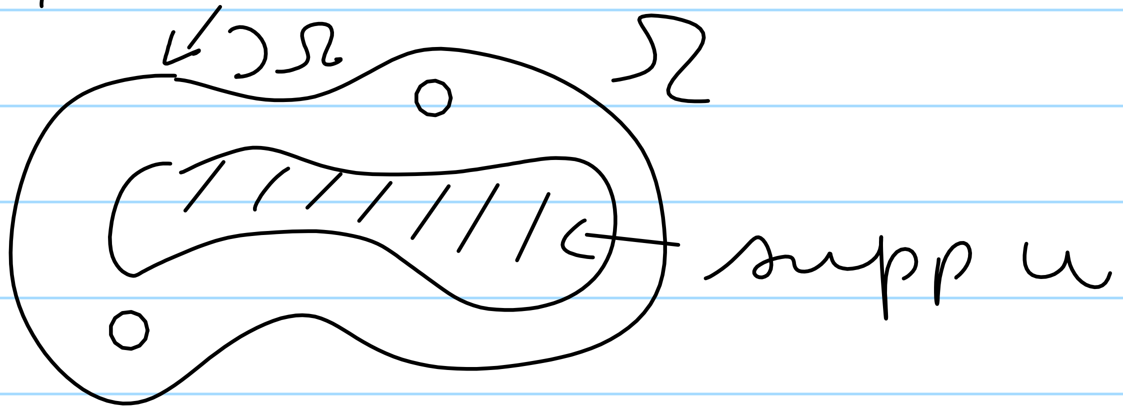
If $u \in C_c^2(\mathbb{R}^n)$

$$\mathrm{supp}(u) = \overline{\{x : u(x) \neq 0\}} \subseteq \Omega$$

Apply 1) by choosing

$\psi = u$, $\varphi \in C^2(\mathbb{R}^n)$

$$\frac{\partial\psi}{\partial n} = 0, \quad \psi = 0$$



1) becomes

$$-\int_{\mathbb{R}^n} \varphi \Delta u \, dx = -\int_{\mathbb{R}^n} u \Delta \varphi \, dx$$

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Find φ that solves in a suitable sense (in a distributional sense)

$$-\Delta\varphi = \delta_0 \leftarrow$$

δ_0 is the Dirac Delta distribution

$$\delta_0 : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$$
$$g \longmapsto \delta_0(g)$$

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$$:= \int_{\mathbb{R}^n} \delta_0\, g \, dx = g(0)$$

$$\int_{\mathbb{R}^n} -\Delta\varphi\, g = \int_{\mathbb{R}^n} \delta_0\, g = g(0)$$

We expect such a representation:

$$u(x) = \varphi * (-\Delta u)$$

$$\boxed{n = 1} \qquad \Delta\varphi = \varphi'' \quad 7$$

$$\varphi(x) = -\frac{1}{2}|x| \qquad x \in \mathbb{R}$$

$$\varphi'(x) = \begin{cases} -\frac{1}{2} & x > 0 \\ \frac{1}{2} & x < 0 \end{cases}$$

$$\varphi''(x) = 0 \qquad \forall\, x \neq 0$$

$$(2)\quad u(x) = -\, u''(x) * \left(-\frac{1}{2}|x|\right)$$

Proof of (2)

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$$-u''(x) * \varphi(x) = -\int_{-\infty}^{+\infty} \varphi(x-y)\, u''(y)\, dy$$

$$= -\lim_{\varepsilon \to 0} \left[\int_{-\infty}^{x-\varepsilon} \varphi(x-y)\, u''(y)\, dy + \int_{x+\varepsilon}^{+\infty} \varphi(x-y)\, u''(y)\, dy \right]$$

$$= -\lim_{\varepsilon \to 0} \Big[\underbrace{u'(y)\, \varphi(x-y)}_{(2)} \Big|_{-\infty}^{x-\varepsilon}$$

$$+ \int_{-\infty}^{x-\varepsilon} u'(y)\, dy\, \varphi'(x-y)\, dy +$$

$$\underbrace{u'(y)\,\varphi(x-y)}_{b)}\Big|_{x+\varepsilon}^{+\infty} + \int_{x+\varepsilon}^{+\infty} u'(y)\,\overset{9}{\varphi'}(x-y)\,dy \Big] =$$

a) $u'(x-\varepsilon)\,\varphi(\varepsilon) - \text{"}\overbrace{u'(-\infty)\,\varphi(x+\infty)}^{=0}\text{"}$

$$\lim_{\varepsilon\to 0} u'(x-\varepsilon)\,\varphi(\varepsilon) = 0$$

because $\varphi(0)=0$

b) the same computation

$$= -\lim_{\varepsilon\to 0}\Big[\,u(y)\,\varphi'(x-y)\Big|_{-\infty}^{x-\varepsilon} +$$

$$+\int_{-\infty}^{x-\varepsilon} \underbrace{u(y)\,\overset{10}{\varphi''}(x-y)}_{=0}\,dy$$

$$+\;u(y)\,\varphi'(x-y)\Big|_{x+\varepsilon}^{+\infty}$$

$$+\int_{x+\varepsilon}^{+\infty} \underbrace{u(y)\,\varphi''(x-y)\,dy}_{=0}\;\Big] =$$

$$= -\lim_{\varepsilon\to 0}\Big[u(x-\varepsilon)\,\varphi'(\varepsilon) - u(x+\varepsilon)\,\varphi'(-\varepsilon)\Big]$$

$$= \lim_{\varepsilon\to 0}\Big[\frac{1}{2}\,u(x-\varepsilon) + \frac{1}{2}\,u(x+\varepsilon)\Big]$$

$= u(x).$

$n = 2 \qquad \varphi(x) = -\frac{1}{2\pi} \log |x|$

$n \geq 3 \qquad \varphi(x) = \frac{1}{(n-2)\alpha_{n-1}} \frac{1}{|x|^{n-2}}$

$\alpha_{n-1} = \mathrm{vol}(S^{n-1})$

$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$

Evans' book

$u(x) = -u''(x) * \varphi$

$-\Delta \varphi = \delta_0 \quad$ in $\mathbb{R}^n$

$-\int_{\mathbb{R}} u''(y) \varphi(x-y) dy$

$= -\lim_{\varepsilon \to 0} \left(\int_{-\infty}^{x-\varepsilon} \text{—} \, dy + \int_{x+\varepsilon}^{+\infty} \text{—} \, dy \right)$

$u \in C_c^\infty(\mathbb{R}) \quad \varphi \in L^1_{loc}$

$\Rightarrow y \mapsto u''(y) \varphi(x-y) \in L^1(\mathbb{R})$

$\forall x \in \mathbb{R}^n$

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$\Rightarrow$ Apply Lebesgue Thm

$$u''(\cdot y)\,\varphi(x-y)\,\chi_{B^c(x,\varepsilon)}(y)$$

$$\longrightarrow u''(y)\,\varphi(x-y) \quad \text{as } \varepsilon \to 0$$

$$|u''(y)\,\varphi(x-y)\,\chi_{B^c(x,\varepsilon)}(y)| \le \underbrace{|u''(y)\,\varphi(x-y)|}_{\in L^1(\mathbb{R}^n)}$$

$$\Rightarrow \int_{\mathbb{R}} u''(y)\,\varphi(x-y)\,\chi_{B^c(x,\varepsilon)}\,dy$$

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$$= \int_{\underbrace{\mathbb{R}\setminus B(x,\varepsilon)}} \varphi(x-y)\,u''(y)\,dy \xrightarrow[\varepsilon \to 0]{}$$

$$\xrightarrow[\varepsilon\to 0]{} \int_{\mathbb{R}} \varphi(x-y)\,u''(y)\,dy$$

Theorem 4.4.2

$f \in C_c^2(\mathbb{R}^n)$. Then

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$u = f * \varphi \in C^\infty(\mathbb{R}^n)$

and solves $-\Delta u = f$.

Proof $\boxed{n \geq 3}$

$$\varphi(x) = \frac{1}{n-2} \frac{1}{\alpha_n} \cdot \frac{1}{|x|^{n-2}}$$

$$= c(n) \frac{1}{|x|^{n-2}}$$

- $\varphi \in L^1_{loc}(\mathbb{R}^n)$
- $\nabla \varphi = c(n) \cdot (2-n) |x|^{2-n-1} \frac{x}{|x|}$

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$$= c(n)(2-n) \frac{1}{|x|^n}$$

- $\Delta \varphi = 0 \quad \forall x \neq 0$

"$-\Delta \varphi = \delta_0$"

$\Rightarrow u(x) = \varphi * (-\Delta u) \quad \forall u \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi(x-y) f(y) \, dy =: u$$

$\Longrightarrow u \in C^\infty(\mathbb{R}^n)$

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$$-\Delta_x u = -\Delta_x \int_{\mathbb{R}^n} \varphi(x-y)\, f(y)\, dy$$

$$\parallel$$

$$= \int_{\mathbb{R}^n} -\Delta_x \varphi(x-y)\, f(y)\, dy$$

$$= \int_{\mathbb{R}^n} \delta_0(x-y)\, f(y)\, dy$$

$$= f(x) \quad \parallel$$

(NOT ALWAYS POSSIBLE!)

- $x_0 \notin \operatorname{supp} f = \overline{\{x : f(x) \neq 0\}}$

$\Rightarrow \forall y \in \operatorname{supp} f \Rightarrow |x_0 - y| \geq r > 0$.

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$$\int_{\mathbb{R}^n} \varphi(x_0-y)\, f(y)\, dy =$$

$$= \int_{\operatorname{supp} f} \varphi(\overbrace{x_0-y}^{\neq 0})\, f(y)\, dy$$

$\Rightarrow$ Apply Thm 3.4.1

$$-\Delta u(x)\big|_{x=x_0} = \int -\Delta \varphi(x_0-y) \cdot f(y)\, dy$$

$$= 0 = f(x_0)$$

$$\boxed{x_0 \in \operatorname{supp} f}$$

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$$-\Delta u(x)\Big|_{x=x_0} = -\Delta\Big|_{x=x_0}\int_{\mathbb{R}^n}\varphi(x-y)f(y)\,dy$$

$$= -\Delta_{|x=x_0}\int_{\mathbb{R}^n}\varphi(y)f(x-y)\,dy$$

$$= \int_{\mathbb{R}^n}\varphi(y)\left(-\Delta_x f(x-y)\right)dy\Big|_{x=x_0}$$

$$\boxed{\Delta_x f(x-y) = \Delta_y f(x-y)}$$

$$= \int_{\mathbb{R}^n}\varphi(y)\left(-\Delta_y f(x-y)\right)dy$$

$$= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B(0,\varepsilon)}\varphi(\varepsilon)\left(-\Delta_y f(x-y)\right)dy$$

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$\Omega = \mathbb{R}^n \setminus B(0,\varepsilon)$ ⇐

$\left(\Omega = B(0,R)\setminus B(0,\varepsilon)\right.$

$\left. f\equiv 0 \text{ on } |x|=R\right)$

Apply Gauss-Theorem on Ω

$$\int_{\mathbb{R}^n\setminus B(0,\varepsilon)}\varphi(y)\left(-\Delta_y f(x-y)\right)dy$$

$=0$ because $|y|>\varepsilon\neq 0$

$$= \underbrace{\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} -\Delta\varphi\, f(x-y)\,dy}$$

$$+\int_{\partial B(0,\varepsilon)}\frac{\partial\varphi}{\partial n}\cdot f(x-y) - \varphi(y)\frac{\partial f(x-y)}{\partial n}\,d\sigma$$

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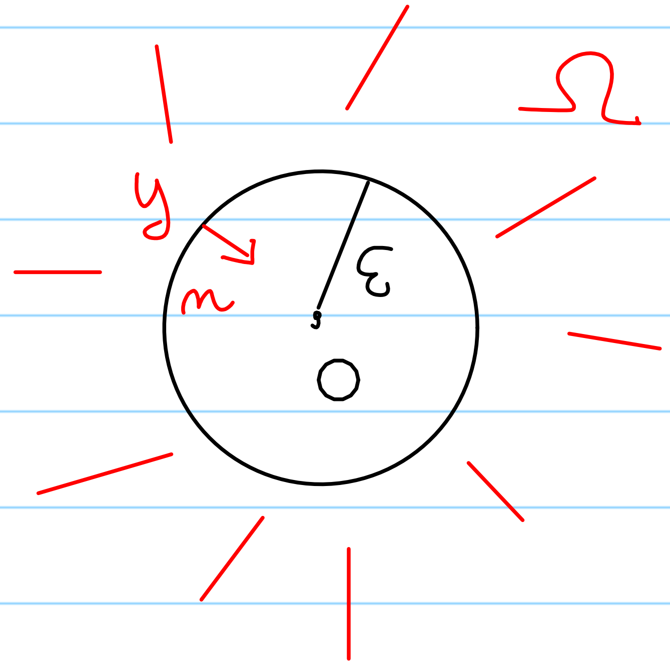

$y \in \partial\Omega = \partial B(0,\varepsilon)$

$\vec{n}(y) = -\frac{y}{|y|}$

$$\varphi(y) = \frac{1}{(n-2)\alpha_{n-1}}\,\frac{1}{|y|^{n-2}}$$

$$\nabla\varphi(y) = \frac{1}{n-2}\cdot\frac{1}{\alpha_{n-1}}\cdot(2-n)\,|y|^{2-n-1}\,\frac{y}{|y|}$$

$$\frac{\partial\varphi}{\partial n}(y) = \left(-\frac{1}{\alpha_{n-1}}\,\frac{y}{|y|^{n}}\right)\cdot\left(-\frac{y}{|y|}\right)$$

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$$= \frac{|y|^2}{|y|\,|y|^n}\,\frac{1}{\alpha_{n-1}} = \frac{1}{\alpha_{n-1}}\cdot\frac{1}{|y|^{n-1}}$$

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a) $$\lim_{\varepsilon\to 0}\int_{\partial B(0,\varepsilon)} \overbrace{\frac{1}{\alpha_{n-1}}\cdot\frac{1}{|y|^{n-1}}}\, f(x-y)\,dy$$

b) $$\lim_{\varepsilon\to 0}\int_{\partial B(0,\varepsilon)} -\varphi(y)\,\frac{\partial f}{\partial n}(y-y)\,d\sigma$$

$$|b)| \leq \lim_{\varepsilon\to 0}\|\nabla f\|_{L^\infty}\int_{\partial B(0,\varepsilon)} c(n)\,\frac{1}{|y|^{n-2}}\,d\sigma$$

$(*)$

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$$= \lim_{\varepsilon \to 0} \|\nabla f\|_{L^\infty} c(n) \frac{1}{\varepsilon^{n-2}} \cdot \alpha_{n-1} \varepsilon^{n-1}$$

$$= 0$$

d) $$= \lim_{\varepsilon \to 0} \frac{1}{\alpha_{n-1}\varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} f(x-y)\,dy$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\alpha_{n-1}\varepsilon^{n-1}} \int_{\partial B(x,\varepsilon)} f(y)\,dy$$

$$\alpha_{n-1}\varepsilon^{n-1} = \mathcal{L}^n(\partial B(x,\varepsilon))$$

$$= f(x) \qquad \square$$

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CHAPTER 5

Differentiation of measures

5.1 Differentiation of Lebesgue measure

i) $f : \mathbb{R} \to \mathbb{R}$ cont. at x_0

$$F(x) = \int_{x_0}^{x} f(t)\,dt$$

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$$F'(x_0) = \lim_{h\to 0} \frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt = f(x_0)$$

ii) $f : \mathbb{R}^n \to \mathbb{R}$ cont. at $x_0 \in \mathbb{R}^n$

$$f(x_0) = \lim_{r\to 0} \frac{1}{\mathcal{L}^n(B(x_0,r))} \int_{B(x_0,r)} f(y)\,dy$$

(EXERCISE)

Q: Do similar formula hold for $f \in L^1_{loc}(\mathbb{R}^n)$

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or even if we replace $\mathcal{L}^n$ by a general Radon measure

Thm 5.1.2 (LEBESGUE Diff. Thm)

$f \in L^1_{loc}(\mathbb{R}^n) \Rightarrow$ for $\mathcal{L}^n$-a.e $x \in \mathbb{R}^n$

$$\lim_{r\to 0} \fint_{B(x_0,r)} f \, d\mathcal{L}^n = f(x)$$

NOTATION

Given $E \subseteq \mathbb{R}^n$, $\mathcal{L}^n(E) < +\infty$

$$⨍_E f \, d\mathcal{L}^n = \frac{1}{\mathcal{L}^n(E)} \int_E f \, d\mathcal{L}^n$$