Lecture 24.04.2020

Application of Beppo Levi's

Theorem

$g_n : \Omega \to [0, +\infty]$ μ-meas

$$f(x) := \sum_{i=1}^{\infty} g_i(x)$$

Then

$$\int_\Omega f \, d\mu = \sum_{i=1}^{\infty} \int_\Omega g_i \, d\mu$$

Proof

$$f_n(x) = \sum_{i=1}^{n} g_i(x)$$

$f_n \uparrow$, $f_n \geq 0$, μ-meas

$$\underbrace{\lim_n \int_\Omega f_n \, d\mu}_{} = \int_\Omega \lim f_n \, d\mu$$

$$\lim_{n \to +\infty} \int_\Omega \sum_{i=1}^{n} g_i(x) \, d\mu = \lim_{n \to \infty} \sum_{i=1}^{n} \int_\Omega g_i(x) \, d\mu$$

$$= \sum_{i=1}^{\infty} \int_\Omega g_i \, d\mu = \int_\Omega \sum_{i=1}^{\infty} g_i \, d\mu = \int_\Omega f \, d\mu \quad \square$$

Theorem 3.3.5 (Dominated Convergence Theorem / Lebesgue Theorem)

Let $g : \Omega \to [0, +\infty]$ μ-summable

$(f_k)_{k\geq 1}$, $f : \Omega \to \overline{\mathbb{R}}$

μ-measurable and satisfy:

a) $|f_k(x)| \leq g(x)$ μ-a.e $\forall k \geq 1$

b) $f_k \to f$ μ-a.e

Then

$$\int_\Omega |f_k - f| \, d\mu \to 0 \quad \text{as } k \to +\infty$$

$$\Rightarrow \boxed{\lim_{k\to\infty} \int_\Omega f_k \, d\mu = \int_\Omega f \, d\mu}$$

Proof Observe that

a) & b) $\Rightarrow$

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$$|f| = \lim_{k\to+\infty} |f_k| \leq g \quad \mu\text{ a.e}$$

$$\Rightarrow |f_k - f| \leq |f_k| + |f|$$

$$\leq 2g \quad \mu\text{ a.e}$$

$\Rightarrow |f_k - f|$ is μ-summ.

(g is μ summ. by assumption)

$$2g - |f_k - f| = h_k$$

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$$\int_\Omega \liminf_{k\to+\infty} h_k \, d\mu \leq \liminf_{k\to+\infty} \int_\Omega h_k$$

NOTE

$$\liminf_{k\to\infty} h_k = \liminf_{k\to\infty} (2g - |f_k - f|)$$

$= 2g$ because $|f_k - f| \to 0$

as $k \to +\infty$ μ-a.e

$$\int_\Omega 2g \, d\mu \leq \liminf_{k\to+\infty} \int_\Omega 2g - |f_k - f| \, d\mu$$

$$= \int_\Omega 2g + \liminf_{k\to+\infty} \int -|f_k - f| \, d\mu$$

$$= \int_\Omega 2g \, d\mu - \limsup_{k \to +\infty} \int_\Omega |f_k - f| \, d\mu$$

$$\left(\liminf_{k \to \infty} (-\psi_k) = - \limsup_{k \to +\infty} \psi_k \right)$$

$$\Rightarrow \underbrace{\limsup_{k \to +\infty} \int_\Omega |f_k - f| \, d\mu}_{} \leq 0$$

$$\Rightarrow \limsup \int_\Omega |f_k - f| \, d\mu = 0$$

$$\Rightarrow \lim_{k \to +\infty} \int_\Omega |f_k - f| \, d\mu = 0$$

$$\left| \int f_k \, d\mu - \int f \, d\mu \right|$$

$$= \left| \int_\Omega (f_k - f) \, d\mu \right| \leq \int_\Omega |f_k - f| \, d\mu \longrightarrow 0$$

$$\Rightarrow \lim_{k \to +\infty} \int_\Omega f_k \, d\mu = \int_\Omega f \, d\mu \quad \square$$

COROLLARY 3.1.14

$(f_k)_k$, f μ-integrable

$$\lim_{n\to+\infty} \int_\Omega |f_n - f| \, d\mu = 0 \quad (*)$$

$$\Rightarrow \exists \, f_{n_j} \to f \quad \mu\text{-a.e}$$

Alternative Proof

Select a (f_{k_n}) such that

$$\sum_{n=1}^{\infty} \int_\Omega |f_{k_n} - f| \, d\mu < +\infty$$

$\forall n \geq 1 \quad \exists \, k_n \; : \; k \geq k_n$

$$\int_\Omega |f_k - f| \, d\mu < \frac{1}{2^n}$$

$k = k_n$

$$\underbrace{\sum_{n=1}^{\infty} \int_\Omega |f_{k_n} - f| \, d\mu}_{<\infty} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} < +\infty$$

BL Theorem $\Rightarrow \displaystyle\int_\Omega \sum_{n=1}^{\infty} |f_{k_n} - f| \, d\mu < +\infty$

$$\Rightarrow \int_\Omega |f| \, d\mu < +\infty$$

$\Rightarrow \quad |f| < +\infty \quad \mu\text{-a.e}$

Hence

$$\sum_{n=1}^{\infty} |f_{k_n} - f| < +\infty \quad \mu\text{-a.e}$$

$$\Rightarrow \lim_{n\to+\infty} |f_{k_n} - f| = 0 \quad \mu\text{-a.e}$$

$$\Rightarrow \lim_{n\to\infty} f_{k_n}(x) = f(x)$$

μ-a.e

$\square$

3.4 Differentiation under integral sign.

Theorem 3.4.1

Let $f : \mathbb{R} \times [0,1] \to \mathbb{R}$:

i) $\forall x \in \mathbb{R}$: $y \mapsto f(x,y)$ is $\mathcal{L}^1$-summ in $[0,1]$

ii) $\dfrac{\partial f}{\partial x}(x,y)$ exists and it is bounded in $\mathbb{R} \times [0,1]$

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Then $\forall x \in \mathbb{R}$

- $y \mapsto \frac{\partial f}{\partial x}(x,y)$ is $\mathcal{L}^1$ - SUMM in $[0,1]$

- $$\frac{d}{dx}\int_{[0,1]} f(x,y)\,dy = \int_{[0,1]} \frac{\partial f}{\partial x}(x,y)\,dy$$

<u>Proof</u> Fix $x \in \mathbb{R}$, $h > 0$

$$\frac{d}{dx}\int_{[0,1]} f(x,y)\,dy =$$

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$$\underbrace{\lim_{h\to 0} \frac{1}{h}\left[\int_{[0,1]} f(x+h,y)\,dy - \int_{[0,1]} f(x,y)\,dy\right]}_{①}$$

$$① = \int_{[0,1]} \frac{f(x+h) - f(x,y)}{h}\,dy$$

Take $h_k \downarrow 0$ as $k \to +\infty$:

$$g_k(x,y) = \frac{f(x+h_k,y) - f(x,y)}{h_k}$$

$$\lim_{k\to+\infty} g_k(x,y) = \frac{\partial f}{\partial x}(x,y)$$

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By Mean Value Theorem

$$|g_k(x,y)| \doteq \left|\frac{\partial f}{\partial x}(x_k, y)\right| \leq \sup_{\substack{x\in\mathbb{R}\\ y\in[0,1]}} \left|\frac{\partial f}{\partial x}(x,y)\right| = C < +\infty$$

$x_k \in (x, x+h_k)$

- g_k are $\mathcal{L}^1$-summ
- $g_k(x,y) \to \frac{\partial f}{\partial x}(x,y)$

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$\left|\frac{\partial f}{\partial x}\right| \leq C \Rightarrow \forall x \in \mathbb{R}$: $y \mapsto \frac{\partial f}{\partial x}(x,y)$ is $\mathcal{L}^1$-summ.

- $|g_k| \leq C$

In our case

$g(x,y) \equiv C < +\infty$

$\forall x$: $y \mapsto C$ is $\mathcal{L}^1$-summ in $[0,1]$: $\int_{[0,1]} C \, dy = C\mathcal{L}^1[0,1] = C$

By Lebesgue Theorem:

$$\lim_{k\to+\infty} \int_{[0,1]} g_k(x,y)\,dy = \int_{[0,1]} \lim_{k\to+\infty} g_k(x,y)\,dy$$

$$\Rightarrow \frac{d}{dx}\int_{[0,1]} f(x,y)\,dy = \int_{[0,1]} \frac{\partial f}{\partial x}(x,y)\,dy$$

$\square$

3.5 Absolute Continuity of Integrals

$f : \Omega \to \overline{\mathbb{R}}$ μ-summ.

For $A \subseteq \Omega$ μ-measurable we set

$$\nu(A) = \int_A f\,d\mu$$

$\boxed{\text{Cor } 3.1.17} \Rightarrow$

$\mu(A) = 0 \Rightarrow \nu(A) = 0$ (1)

EXERCISE

- ν is σ-add. (Prop 3.1.18)
- $f \geq 0$ ν is a RADON MEASURE

$\nu = \mu \llcorner f$

Recall $(\mu \llcorner A)(B) = \mu(A \cap B)$

$\mu \llcorner A = \mu \llcorner \chi_A$

ν, μ, $\overline{\Sigma}_\nu, \overline{\Sigma}_\mu$ the algebra of ν- and μ-meas. sets

Def 3.5.2

Let ν, μ be two measure such that

$\Sigma_\nu \subseteq \overline{\Sigma}_\mu$ and

satisfy (1). ν is called "absolutely continuous" with respect to μ and

we write $\boxed{\nu << \mu}$

<u>Theorem 3.5.3</u>

Let $f : \Omega \to \overline{\mathbb{R}}$ μ-summ

Then $\forall \varepsilon > 0$ $\exists \delta > 0$ such

that:

$\forall A \subseteq \Omega$ μ-meas:

$\mu(A) < \delta \Rightarrow$

$$\int_A |f| \, d\mu < \varepsilon$$

($A \mapsto \int_A |f| \, d\mu$ is "continuous")

<u>Proof</u> We argue by contradiction:

$\exists \varepsilon > 0$, $\exists$ sequence $(A_k) \subseteq \Omega$

μ-meas $\boxed{\mu(A_k) < \frac{1}{2^k}}$

such that

$$\int_{A_k} |f|\,d\mu \geq \varepsilon \quad \forall k \geq 1$$

For $\ell \geq 1$ we set

$$B_\ell = \bigcup_{k=\ell}^{\infty} A_k$$

- $B_{\ell+1} \subseteq B_\ell$
- $\mu(B_\ell) \leq \sum_{k=\ell}^{\infty} \mu(A_k)$
$$\leq \sum_{k=\ell}^{\infty} \frac{1}{2^k} = 2^{1-\ell}$$
- $\mu(B_1) \leq 1$

$\Rightarrow$

$$\lim_{\ell\to\infty} \mu(B_\ell) = \mu\Big(\underbrace{\bigcap_{\ell=1}^{\infty} B_\ell}_{=:A}\Big) = 0$$

<u>because</u> $\lim_{\ell\to\infty} \mu(B_\ell) \leq$

$$\leq \lim_{\ell\to\infty} \frac{1}{2^{\ell-1}} = 0$$

<u>Now</u> $g_\ell = |f|\,\chi_{B_\ell}$

- $|g_\ell| \le |f|$ (it is μ-summ)
- $g_\ell \xrightarrow[\ell \to +\infty]{} |f|\chi_A$

$\Rightarrow$ Lebesgue's Theorem

$$\underline{\lim_{\ell\to+\infty} \int_\Omega g_\ell \, d\mu} = \underbrace{\int_\Omega \lim_{\ell\to+\infty} g_\ell \, d\mu}_{=0}$$

$> \varepsilon$

<u>observe that</u>

1) $\displaystyle \lim_{\ell\to+\infty} \int_\Omega g_\ell \, d\mu = \lim_{\ell\to+\infty} \int_\Omega |f| \chi_{B_\ell} d\mu$

$\displaystyle = \lim_{\ell\to+\infty} \int_{B_\ell} |f| \, d\mu$

2) $\displaystyle \int_\Omega \lim g_\ell \, d\mu = \int_A |f| \, d\mu$

<u>But</u> • $\displaystyle \int_A |f| \, d\mu = 0$

because $\mu(A) = 0$

and $|f|$ is μ-summ

- $\displaystyle \int_{B_\ell} |f| \, d\mu \ge \int_{A_k} |f| \, d\mu > \varepsilon$

$\Downarrow$ $k \ge \ell$

$\displaystyle \lim_{\ell\to+\infty} \int_{B_\ell} |f| \, d\mu \ge \varepsilon$

$\square$

3.6: VITALI'S THEOREM

$(f_k)_k$, $f: \Omega \to \overline{\mathbb{R}}$ μ summ

Def 3.6.1

$(f_k)_k$ is UNIFORMLY μ-SUMMABLE if

$\forall \varepsilon > 0 \ \exists \delta > 0: \ \forall k \geq 0$

$\forall A \subseteq \Omega$ μ-meas

$\mu(A) < \delta \Rightarrow \int_A |f_k| \, d\mu < \varepsilon$

Theorem 3.6.2 (Vitali's Theorem)

$(f_k), f: \Omega \to \overline{\mathbb{R}}$ μ-summ.

$\mu(\Omega) < +\infty$

The following two conditions are equivalent

1. $f_k \xrightarrow{\mu} f$ as $k \to +\infty$ and (f_k) is unif.

μ - summable.

ii) $\lim_{k \to +\infty} \int_\Omega |f_k - f| \, d\mu = 0$

Remark

$\mu(\Omega) < +\infty$ is necessary

$\Omega = \mathbb{R}^n$, $\mu = \mathcal{L}^n$,

$f_k = \frac{1}{k^n} \chi_{B(0,k)}$

$f_k \xrightarrow{\mu} 0$ as $k \to +\infty$ ($f_k \to 0$ and $\mu(\Omega) < +\infty$)

(f_k) is UNIF. μ-summ.

$$\int_A f_k \, d\mu = \int_A \frac{1}{k^n} \chi_{B(0,k)} \, d\mathcal{L}^n$$

$$= \frac{1}{k^n} \mathcal{L}^1(B(0,k) \cap A)$$

$$\leq 1 \cdot \mathcal{L}^1(A)$$

$\forall \varepsilon > 0 \Rightarrow$ choose $\delta = \varepsilon$

If $\mathcal{L}^1(A) < \delta = \varepsilon$, then

$$\int_A f_k \, d\mathcal{L}^n < \varepsilon$$

$$\to \int_{\mathbb{R}^n} f_k \, d\mu = \omega_n = \mathcal{L}^n(B(0,1))$$

$$\to \lim_{k \to +\infty} \int_{\mathbb{R}^n} f_k \, d\mu = \omega_n \neq \int_{\mathbb{R}^n} f \, d\mu = 0$$

$\square$

<u>Proof</u> ii) $\Rightarrow$ i)

$$\int_\Omega |f_k - f| \, d\mu \to 0 \text{ as } k \to +\infty$$

$$\Rightarrow f_k \xrightarrow{\mu} f \text{ as } k \to +\infty$$

- (f_k) is u. μ-summable.

$\forall \varepsilon > 0 \; \exists \bar{k} = \bar{k}(\varepsilon) : k \geq \bar{k}$

$$\int_\Omega |f_k - f| \, d\mu < \varepsilon.$$

Let $\delta > 0$ be such that if $A \subseteq \Omega$ μ-meas

$$\mu(A) < \delta \Rightarrow \int_A |f| \, d\mu < \varepsilon$$

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and $\int_A |f_k| \, d\mu < \varepsilon$

$\uparrow \; \forall 1 \leq k \leq \bar{k}$

CLAIM 1

$$\int_A |f_k| \, d\mu < 2\varepsilon \qquad \boxed{k > \bar{k}}$$

Proof of CLAIM 1

$$\int_A |f_k| \, d\mu \leq \int_A |f_k - f| \, d\mu + \int_A |f| \, d\mu$$

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$$\leq \int_\Omega |f_k - f| \, d\mu + \int_A |f| \, d\mu$$

$$\leq \varepsilon + \varepsilon$$

$$\Rightarrow \int_A |f_k| \, d\mu < 2\varepsilon \quad \forall k \geq \bar{k}$$

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i) $\Rightarrow$ ii)

$f_k \xrightarrow{\mu} f$ as $k \to +\infty$ $\quad (f_k)$ is u. μ-summ.

$\exists f_{k_n} \to f$ μ-a.e

as $n \to +\infty$

$\forall \varepsilon > 0 \ \exists \delta > 0 : A \subseteq \Omega$

μ-meas and $\mu(A) < \delta$

$\underline{\int_A |f| \, d\mu < \varepsilon.}$ $\underline{\int_A |f_k| \, d\mu < \varepsilon}$

By Egoroff theorem

$\exists K_\delta$ compact such

that $\mu(\Omega - K_\delta) < \delta$

and

$\sup_{K_\delta} |f_{k_n} - f| \to 0$ as $n \to +\infty$

$\exists \bar{n} : n \geq \bar{n}$

$\sup_{K_\delta} |f_{k_n}(x) - f(x)| < \varepsilon / \mu(\Omega)$

For $n \geq \bar{n}$:

$\int_\Omega |f_{k_n} - f| \, d\mu =$

$$= \underbrace{\int_{K_\delta} |f_{k_n} - f|\, d\mu}_{} + \underbrace{\int_{\Omega \setminus K_\delta} |f_{k_n} - f|\, d\mu}_{}$$

$$\leq \frac{\varepsilon}{\mu(\Omega)} \cdot \underbrace{\mu(K_\delta)}_{\leq 1} + \int_{\Omega \setminus K_\delta} |f_{k_n}|\, d\mu$$

$$+ \int_{\Omega \setminus K_\delta} |f|\, d\mu$$

$$\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

$$\Rightarrow \lim_{n \to +\infty} \int_\Omega |f_{k_n} - f|\, d\mu = 0$$

CLAIM 2:

$$\lim_{k \to +\infty} \int |f_k - f|\, d\mu = 0$$

Proof of CLAIM 2

$$\limsup_{k \to +\infty} \int |f_k - f|\, d\mu > 0$$

Let $f_{k'_n}$ be a subseq. of f_k:

$$\lim_{n \to +\infty} \int |f_{k'_n} - f|\, d\mu =$$

$$= \limsup_{n \to +\infty} \int |f_k - f|\, d\mu > 0$$

Apply the previous arguments $\to$ find $(f_{k''_n})$ subsequence of $(f_{k'_n})$:

$$\lim_{n\to+\infty} \int_\Omega |f_{k''_n} - f| \, d\mu = 0$$

$\square$

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