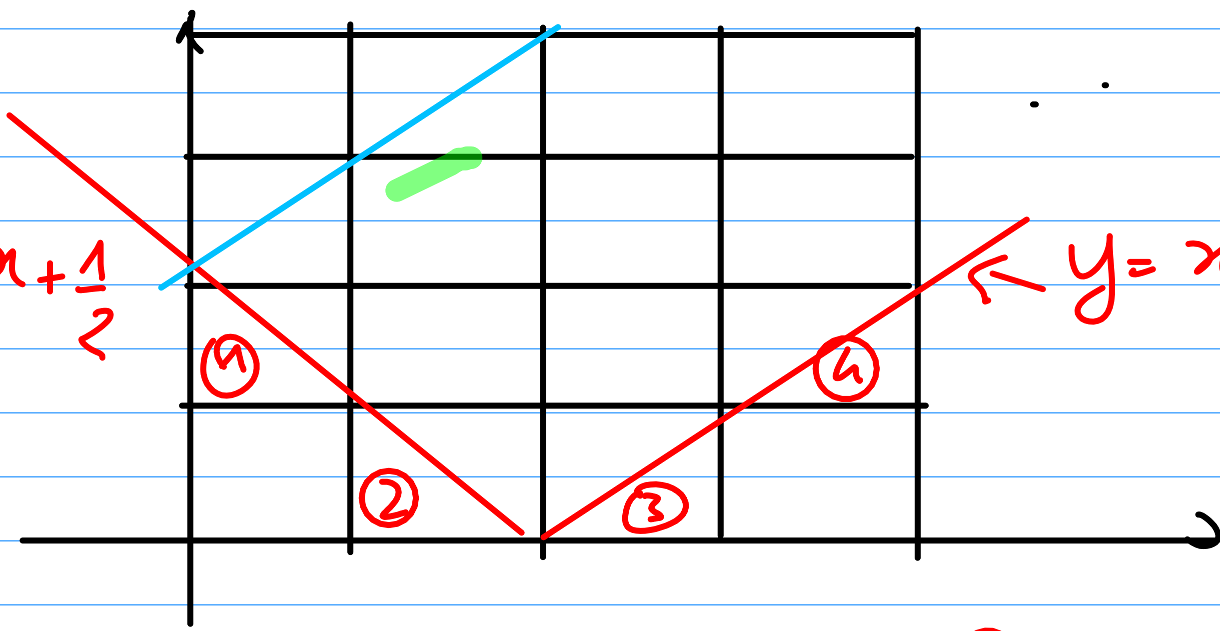
LECTURE 25.03.2020

1) Cantor Dust

$A_0 = [0,1]^2 \subseteq \mathbb{R}^2$



$A_1 = (1) \cup (2) \cup (3) \cup (4)$

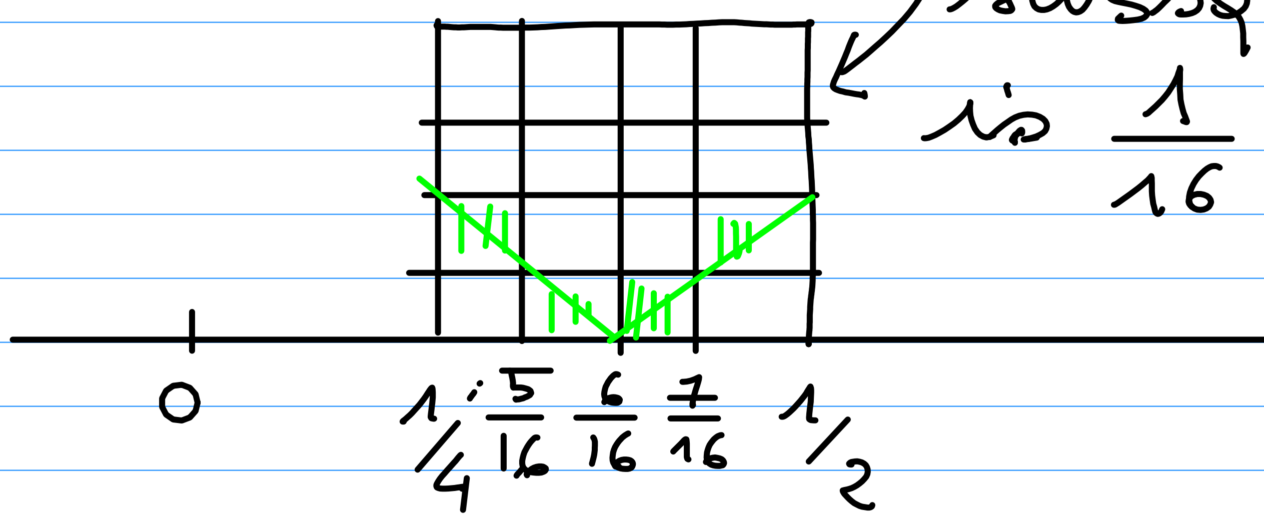
A_2 = UNION of 4^2 subsquares of length $\frac{1}{4^2}$ that have non empty intersection with two lines $y = \pm x \mp \frac{1}{2}$.

Precisely:

1 - bis

We take each subsquare ①, ②, ③, ④ and we repeat the same procedure as before: we zoom at the subsquare ②:

The length of each new subsquare is $\frac{1}{16}$



One considers in this case the lines: $y = x - \frac{3}{8}$ $\quad y = -x + \frac{3}{8}$

Define $A = \bigcap_{k=1}^{\infty} A_k$

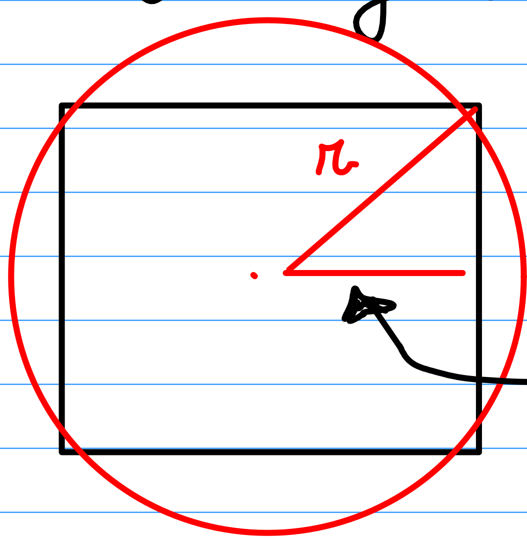
CLAIM $\dim_H(A) = 1$

$$\dim_H(A) = \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\}$$
$$= \sup\{s \geq 0 : \mathcal{H}^s(A) = +\infty\}$$

Proof

$$\frac{1}{2} \leq \mathcal{H}^1(A) \underset{(*)}{\leq} \frac{\sqrt{2}}{2}$$

(*) : A_k = UNION of 4^k squares of length $\frac{1}{4^k}$



$\frac{1}{2} \cdot \frac{1}{4^k}$

$$r = \left[\left(\frac{1}{2}\frac{1}{4^k}\right)^2 2\right]^{1/2} = \frac{\sqrt{2}}{2} \cdot \frac{1}{4^k}$$

The 4^k balls of radius $4^{-k}\frac{\sqrt{2}}{2}$ covers A_k and hence A.

Let $\delta > 0$ and k_0 be such that $r_{k_0} = 4^{-k_0}\frac{\sqrt{2}}{2} < \delta$

$\boxed{\forall k \geq k_0}$ $\Rightarrow r_k = 4^{-k}\frac{\sqrt{2}}{2} < \delta$

$$\mathcal{H}^1_\delta(A) \leq \mathcal{H}^1_\delta(A_k) \leq \sum_{j=1}^{4^k} r_j$$

$$= \sum_{j=1}^{4^k}\left(4^{-k}\frac{\sqrt{2}}{2}\right) = \cancel{4^k}\,\frac{\sqrt{2}}{2}\cdot \cancel{4^{-k}}$$

$$= \frac{\sqrt{2}}{2}$$

Let $\delta \to 0 \Rightarrow \mathcal{H}^1(A) \leq \frac{\sqrt{2}}{2}$

. $\pi : \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \mapsto x$

$\pi(A) = [0,1]$

$A \subseteq \bigcup_{k=1}^{\infty} B(z_k, r_k)$

$z_k = (x_k, y_k) \qquad r_k < \delta$

$$[0,1] = \pi(A) \subseteq \bigcup_{k=1}^{\infty} \pi(B(z_k, r_k))$$

$$= \bigcup_{k=1}^{\infty} (x_k - r_k, x_k + r_k)$$

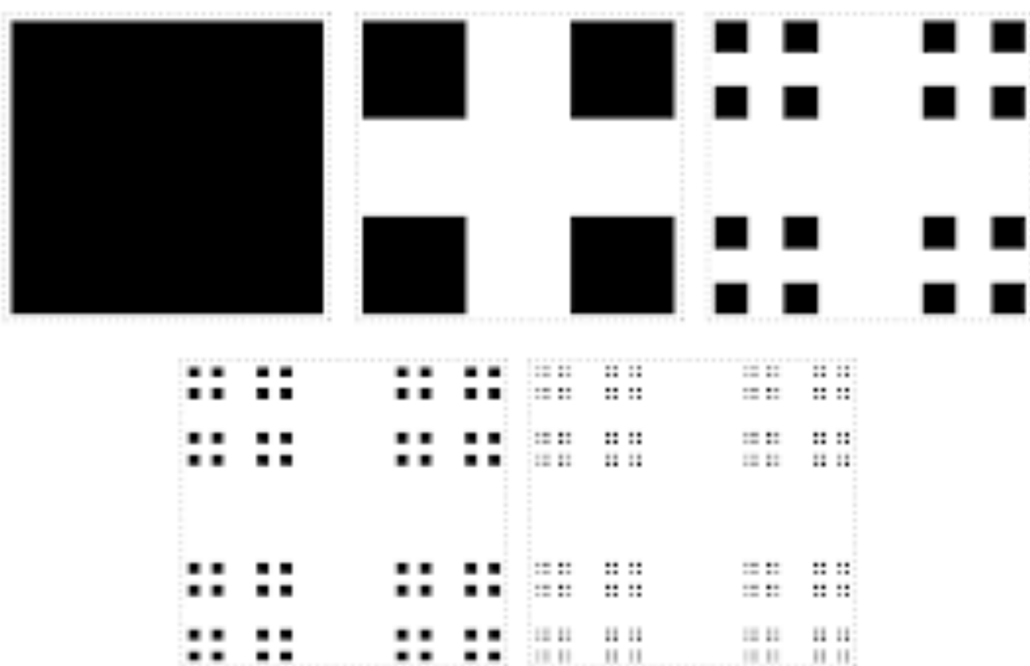
$$\Rightarrow 1 = \mathcal{L}^1[0,1] \leq \sum_1^{\infty} \mathcal{L}^1(x_k - r_k, x_k + r_k)$$

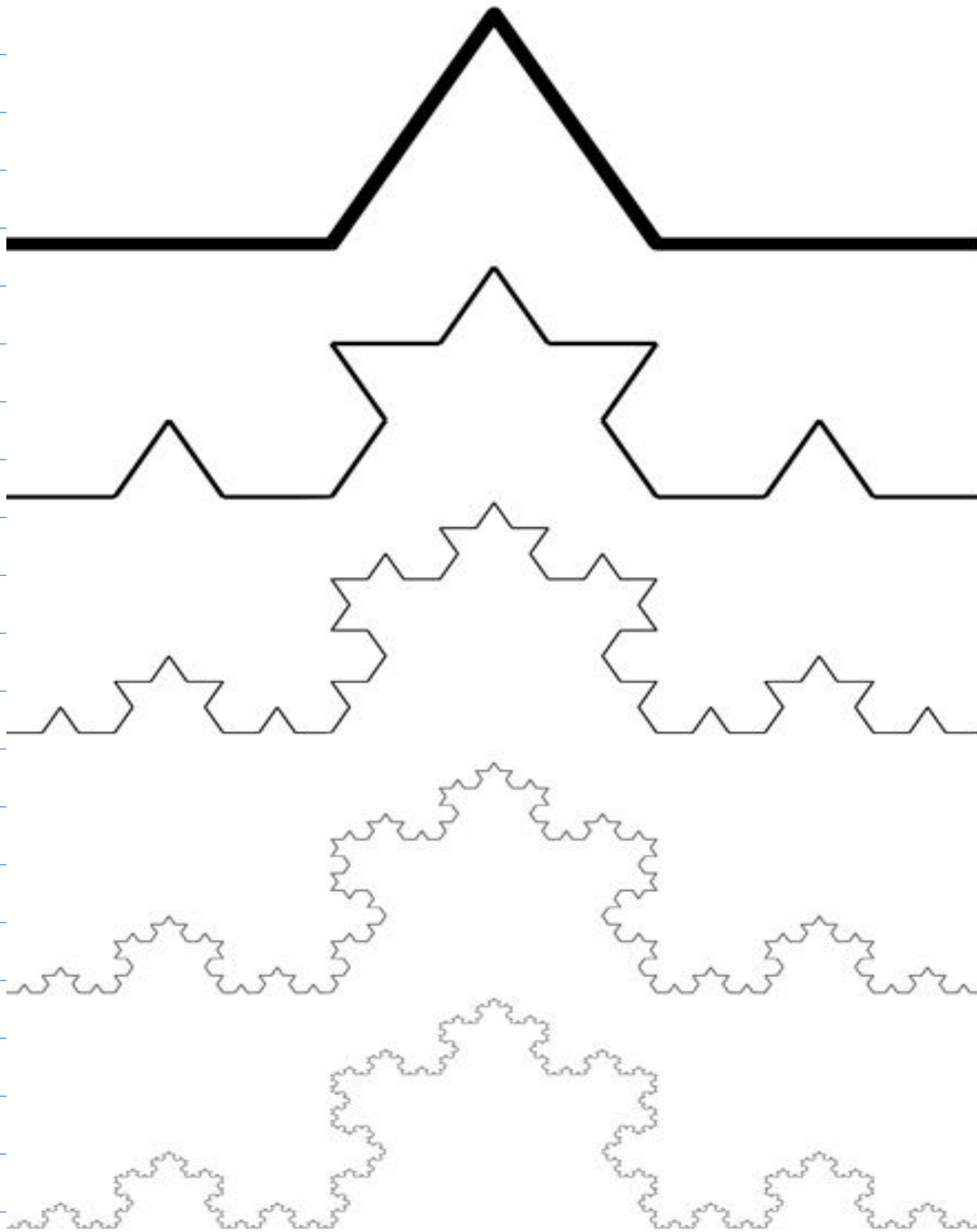
$$= \sum_1^{\infty} 2 r_k$$

$$\sum_{k=1}^{\ell} r_k \geq \frac{1}{2}$$

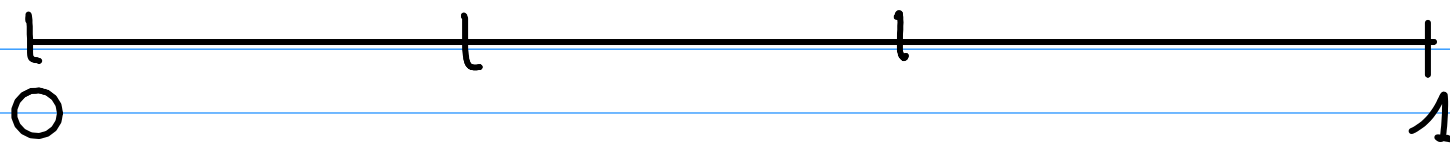
$$\Rightarrow \mathcal{H}^1_\delta(A) \geq \frac{1}{2}$$

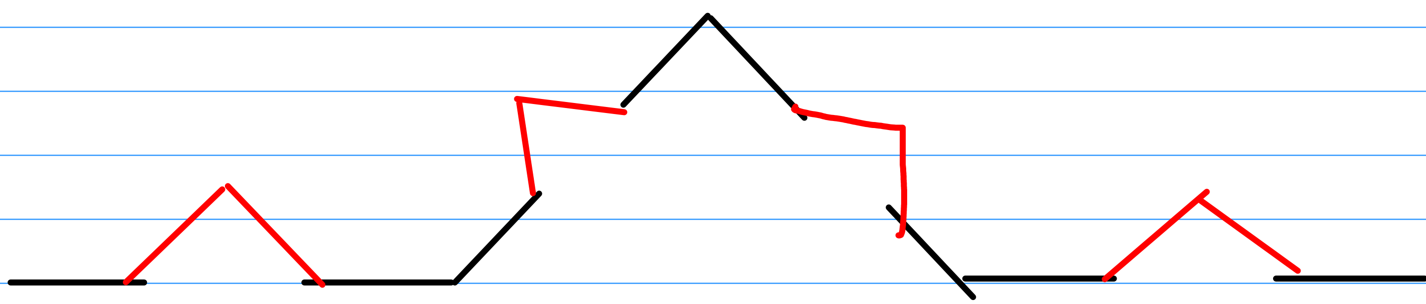
Let $\delta \to 0 \Rightarrow \mathcal{H}^1(A) \geq \frac{1}{2}$ $\square$





Koch curve





1st step $\rightarrow$ $\frac{4}{3}$

2nd step $\rightarrow$ $\left(\frac{4}{3}\right)^2$

nth step $\longrightarrow$ $\left(\frac{4}{3}\right)^n$

$\Rightarrow$ The length of the Koch curve is infinite

1.8 RADON MEASURES

<u>Def</u> A measure μ on $\mathbb{R}^n$ is called a Radon measure if μ is Borel regular and $\mu(K) < +\infty$ $\forall K \subseteq \mathbb{R}^n$ compact.

EXAMPLES

1) $\mathcal{L}^n$ is Radon measure

Every $K \subseteq \mathbb{R}^n$ compact is contained in a pluri-interval Q

$\mathcal{L}^n(Q) < +\infty \Rightarrow \mathcal{L}^n(K) < +\infty$

2) $\mathcal{H}^s$ $s < n$ is NOT RADON MEASURE:

$K = Q = [-1,1]^n$

$$0 < \mathcal{H}^n(Q) < +\infty,$$

(see EXAMPLE 1.7.5 in my lecture notes)

$$\Rightarrow \quad \mathcal{H}^s(Q) = +\infty \quad s < n$$

3) μ is Radon measure and $A \subseteq \mathbb{R}^n$ μ_meas
$\Rightarrow$ $\mu|_A$ is RADON measure as well.

$$\mu|_A(B) = \mu(B \cap A)$$

(EXERCISE)

Regularity properties

Thm 1.8.2

Let μ be a Radon measure on $\mathbb{R}^n$

i) $\forall A \subseteq \mathbb{R}^n$:

$$(*)\quad \mu(A) = \inf\{\mu(G) ;\ A \subseteq G,\ G \text{ open}\}$$

ii) $\forall A \subseteq \mathbb{R}^n$ μ - measurable:

$$(**)\quad \mu(A) = \sup\{\mu(F) :\ F \subseteq A,\ F \text{ compact}\}$$

LEMMA 1.8.3

$\forall B$ Borel set of $\mathbb{R}^n$
$\forall \varepsilon > 0 \ \exists G \supseteq B$ G open

$$\mu(G \setminus B) < \varepsilon$$

(No proof → see Serie 6)

Proof of Thm 1.8.2

i) If $\mu(A) = +\infty$ then
⊛ is trivial

Suppose $\mu(A) < +\infty$

. A is Borel : $\forall \varepsilon > 0 \ \exists$
G open , $G \supseteq A$

$$\mu(G \setminus A) < \varepsilon$$

Since A is Borel (in particular μ-measurable)

$$\mu(G) = \mu(G \cap A) + \mu(G - A)$$
$$= \mu(A) + \mu(G - A)$$

$$\Rightarrow \mu(G) \leq \mu(A) + \varepsilon$$

$$\Rightarrow \mu(A) = \inf \{ \mu(G) : G \supseteq A \text{ open} \}$$

- <u>A arbitrary set</u>.

μ Borel regular $\Rightarrow$

$\exists B$ Borel $B \supseteq A$ $\mu(B) = \mu(A)$

$$\mu(A) = \mu(B) = \inf\{\mu(U) : B \subseteq U \text{ open}\}$$

$$\geq \inf\{\mu(U) : A \subseteq U \text{ open}\}$$

$\downarrow$

$B \supseteq A$

$$\Rightarrow \mu(A) = \inf\{\mu(U) : A \subseteq U,\ U \text{ open}\}$$

("$\leq$" is always satisfied by monotonicity)

ii) $A \subseteq \mathbb{R}^n$ μ-measurable

- <u>CASE</u> : $\mu(A) < +\infty$

We set $\nu = \mu \llcorner A$

ν is Radon as well.

$\forall \varepsilon > 0 \ \exists G$ open $G \supseteq \mathbb{R}^n \setminus A$

$$\nu\big(G \setminus (\mathbb{R}^n \setminus A)\big) < \varepsilon$$

$$\nu(G) \setminus \overset{\shortparallel}{\nu}(\mathbb{R}^n \setminus A)$$

$$\Rightarrow \quad \nu(G) \le \underbrace{\nu(\mathbb{R}^n \setminus A)}_{=0} + \varepsilon$$

Let $C = \mathbb{R}^n \setminus G$, C is closed and $C \subseteq A$

$$\mu(A \setminus C) = \mu\big(A \cap (\mathbb{R}^n \setminus C)\big)$$
$$= \nu(\mathbb{R}^n \setminus C)$$
$$= \nu(G) < \varepsilon$$

Since C is μ-measurable

$$\mu(A) = \mu(A \cap C) + \mu(A \setminus C)$$
$$= \mu(C) + \mu(A \setminus C)$$
$$\leq \mu(C) + \varepsilon$$

$$\mu(C) \geq \mu(A) - \varepsilon$$

Therefore

$$\mu(A) = \sup\{\mu(C) : C \subseteq A \text{ closed}\}$$

- <u>Case 2</u> $\mu(A) = +\infty$

Define $D_k = \{x : k-1 \leq |x| < k\}$

$A = \bigcup_{k=1}^{\infty} D_k \cap A$ ($D_k \cap A$ are mutually disjoint)

Since μ is a Radon measure $\underline{\mu(D_k \cap A)} \leq$

$\leq \mu(\overline{D_k}) < +\infty$

$\forall k \geq 1$: $\exists$ $C_k \subseteq D_k \cap A$ closed and

$\mu(C_k) \geq \mu(D_k \cap A) - \frac{1}{2^k}$

For every m: $\underbrace{\bigcup_{k=1}^{m} C_k}_{\text{is a closed set}} \subseteq A$

$B_m = \bigcup_{k=1}^{m} C_k$ is increasing sequence

$$\lim_{m\to+\infty} \mu(B_m) = \mu\left(\bigcup_{k=1}^{\infty} C_k\right)$$

$$= \sum_{k=1}^{\infty} \mu(C_k)$$

$$\geq \sum_{k=1}^{\infty} \left[\mu(D_k \cap A) - \frac{1}{2^k}\right]$$

$$= \mu(A) - 1 = +\infty$$

$$\mu(A) \underset{\uparrow}{=} \sup\{\mu(C) : C \subseteq A \text{ closed}\}$$

$$\overline{B}_m = \overline{B}(0, m) = \{x, |x| \leq m\}$$

C closed $\Rightarrow C_m = C \cap \overline{B}_m$ is compact

$$\mu(C) = \mu\left(\bigcup_{k=1}^{\infty} C_k\right)$$

$$= \lim_{m \to +\infty} \mu(C_m)$$

$$\sup\{\mu(K) : K \subseteq A \text{ compact}\}$$

$$\underset{\substack{\text{"}\leq\text{"} \\ \text{"}\geq\text{"}}}{=} \sup\{\mu(C) : C \subseteq A \text{ closed}\}$$

$\square$