LECTURE 27.03.2020

CHAPTER 2: MEASURABLE FUNCTIONS

PRELIMINARY NOTIONS.

$$X, Y \neq \emptyset$$

$$\varphi : X \longrightarrow Y$$

PRE-IMAGE OF A

$$A \subseteq Y$$

$$\varphi^{-1}(A) = \{x \in X : \varphi(x) \in Y\}$$
$$= \{\varphi \in A\}$$

Properties

i) $$\varphi^{-1}(A^c) = \left(\varphi^{-1}(A)\right)^c \quad \forall A \subseteq Y$$

ii) $A, B \in \mathcal{P}(Y)$

$$\varphi^{-1}(A \cap B) = \varphi^{-1}(A) \cap \varphi^{-1}(B)$$

iii) $\{A_k\} \in \mathcal{P}(Y)$

$$\varphi^{-1}\left(\bigcup_{k=1}^{\infty} A_k\right) = \bigcup_{k=1}^{\infty} \varphi^{-1}(A_k)$$

$\Rightarrow (Y, \mathcal{F}, \mu)$ is a measure space $\Rightarrow \varphi^{-1}(\mathcal{F})$ is σ - algebra in X

2.2 DEFINITIONS AND BASIC PROPERTIES

μ measure on $\mathbb{R}^n$,
$\Omega \subseteq \mathbb{R}^n$ μ - measurable

Def 2.2.1 $f : \Omega \to [-\infty, +\infty] = \overline{\mathbb{R}}$ is called μ-measurable if

i) $f^{-1}(-\infty)$, $f^{-1}(+\infty)$ are μ-measurable

ii) $f^{-1}(U)$ is μ-meas. $\forall\, U \subseteq \mathbb{R}$ open.

REMARK 1

The condition ii) is equivalent

iii) $f^{-1}(B)$ is μ-meas $\forall\, B \subseteq \mathbb{R}$ Borel

Proof of ii) $\Leftrightarrow$ iii)

iii) $\Rightarrow$ ii) trivial

ii) $\Rightarrow$ iii)

$$\mathcal{G} = \{ B \subseteq \mathbb{R} : f^{-1}(B) \text{ is } \mu\text{-measurable} \}.$$

$\mathcal{G}$ is a σ-algebra that contains the open sets $\Rightarrow$ $\mathcal{G}$ contains the σ-algebra of Borel sets $\square$

iv) $f^{-1}(-\infty, a)$ $\forall a \in \mathbb{R}$ is μ-measurable

Proof

The equivalence between

iii) and iv) follows from the fact σ-algebra generated by $\{(-\infty, a),\ a \in \mathbb{R}\}$ coincides with $\mathcal{B}(\mathbb{R})$.

REMARK 2

We consider $\overline{\mathbb{R}} = [-\infty, \infty]$ with topology generated by the open sets of $\mathbb{R}$ and the neighborhoods $[-\infty, a)$ and $(a, +\infty]$, $a \in \mathbb{R}$ of $\{-\infty\}, \{+\infty\}$.

$f : \Omega \to \overline{\mathbb{R}}$ is μ-meas

$\Longleftrightarrow$

v) $f^{-1}(U)$ is μ-meas $\forall\, U \subseteq \overline{\mathbb{R}}$ open

or $\iff$

vi) $f^{-1}([-\infty, a))$ is μ-meas $\forall\, a \in \mathbb{R}$.

Proof

vi) $\Rightarrow$ i)

Observe that

$$f^{-1}\{-\infty\} = \bigcap_{k=1}^{\infty} f^{-1}\{\underbrace{[-\infty, k)}\}$$

$$f^{-1}\{+\infty\} = \Omega \setminus \underbrace{\bigcup_{k=1}^{\infty} f^{-1}\{[-\infty, k)\}}$$

"$f(x) = +\infty$" $\iff$ $f(x) \geq k \;\; \forall k \geq 1$

vi) $\Rightarrow$ ii)

$$f^{-1}(-\infty, a) = f^{-1}[-\infty, a) \setminus f^{-1}\{-\infty\}.$$

$\square$

REMARK 3

$\underbrace{f : \Omega \longrightarrow \mathbb{R}}_{\mu\text{-meas}}$, $\underbrace{g : \mathbb{R} \longrightarrow \mathbb{R}}_{\text{cont.}}$

$\Rightarrow$ $g \circ f$ is μ-meas.

Proof $A \subseteq \mathbb{R}^n$ open

$$(g \circ f)^{-1}(A) = (f^{-1} \circ g^{-1})(A) =$$

$$= f^{-1}\Big(\underbrace{g^{-1}(A)}_{\text{"}B\text{ is open in }\mathbb{R}}\Big)$$

$\Rightarrow f^{-1}(B)$ is μ-meas because f is μ-meas. $\square$

THEOREM 2.2.4

(Properties of Measurable functions)

i) $f, g : \Omega \longrightarrow \mathbb{R}$ μ-meas

$\Rightarrow f+g,\ f\cdot g,\ |f|, |g|,$

$\max(f,g), \min(f,g),$

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$\frac{f(x)}{g(x)}$ (if $g(x) \neq 0$) and

$\operatorname{sign}[f(x)], \operatorname{sign}(g(x))$

are μ-measurable

ii) $f_k : \Omega \to \overline{\mathbb{R}}$ μ-meas

$\inf_k f_k$, $\sup_k f_k$, $\liminf_{k \to +\infty} f_k$,

$\limsup_{k \to +\infty} f_k$ are μ-meas

RECALL

$$\operatorname{sign} f(x) = \begin{cases} \frac{f(x)}{|f(x)|}, & f(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof

$f : \Omega \to \mathbb{R}$ is μ-meas

iff $f^{-1}(-\infty, a)$ is μ-meas

$\forall a \in \mathbb{R}$

μ-meas

• $f, g : \Omega \to \mathbb{R}$

$$(f+g)^{-1}(-\infty, a) = \bigcup_{\substack{r+s<a \\ r,s\in\mathbb{Q}}} f^{-1}(-\infty, r) \cap g^{-1}(-\infty, s)$$

$$\bullet\ (f^2)^{-1}(-\infty, a) = \begin{cases} \emptyset & \text{if } a \leq 0 \\ f^{-1}(-\infty, \sqrt{a}) \setminus f^{-1}(-\infty, -\sqrt{a}] & a \geq 0 \end{cases}$$

$\Rightarrow f^2$ is μ-meas

$x \in (f^2)^{-1}(-\infty, a) \Leftrightarrow \qquad f^2(x) < a$

If $a \leq 0 \Rightarrow \emptyset$

If $a > 0$

$-\sqrt{a} < f(x) < \sqrt{a}$

$$f \cdot g = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$$

is μ - measurable.

- $g(x) \neq 0$

$$\left(\frac{1}{g}\right)^{-1}(-\infty, a) = \begin{cases} g^{-1}\left(\frac{1}{a}, 0\right) & \text{if } a<0 \\ g^{-1}(-\infty, 0) & \text{if } a=0 \\ g^{-1}(-\infty, 0) \cup g^{-1}\left(\frac{1}{a}, +\infty\right) & a>0 \end{cases}$$

- $a < 0$: $x \in \left(\frac{1}{g}\right)^{-1}(-\infty, a)$

$\Leftrightarrow \quad \frac{1}{g(x)} < a \Leftrightarrow g(x) > \frac{1}{a}$

and $g(x) < 0 \Leftrightarrow$

$x \in g^{-1}\left(\frac{1}{a}, 0\right)$

- $a = 0$ $\quad x \in \left(\frac{1}{g}\right)^{-1}(-\infty, 0)$

$$\frac{1}{g(x)} < 0 \iff g(x) < 0$$

$\iff x \in g^{-1}(-\infty, 0)$

$\Rightarrow \frac{1}{g}$ is μ-meas, $\frac{f(x)}{g(x)}$ is μ-meas

- Recall $s^+ = \max\{s, 0\}$
$s^- = \max\{-s, 0\}$

$s \longmapsto s^+$, $s \to s^-$ are continuous (ANALYSIS I)

$\Rightarrow f^+, f^-$ are continuous

$|f| = f^+ + f^-$

$\max(f,g) = f + (g-f)^+$

$\min(f,g) = f - (g-f)^-$

- $\mathrm{sign}(f(x))$ is μ-meas.

$$x \mapsto \mathrm{sign}(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\mathrm{sign}(x)$ is cont on $\mathbb{R} \setminus \{0\}$

$(\mathrm{sign})^{-1}(U) \quad U \subseteq \mathbb{R}$ open

$= V \subseteq \mathbb{R}^n$ open set.

$= V \cup \{0\} \quad V \subseteq \mathbb{R} \setminus \{0\}$, open

$\Rightarrow (\text{sign})^{-1}(U)$ is a Borel set

$$(\text{sign}(f))^{-1}(U) = (f^{-1} \circ \text{sign}^{-1})(U)$$

$$= f^{-1}(\underbrace{\text{sign}^{-1}(U)}_{\text{Borel set}})$$

$$= B$$

$\Rightarrow f^{-1}(B)$ is μ-meas

$\Rightarrow \text{sgn}(f)$ is μ-meas.

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ii) $f_k : \Omega \to [-\infty, +\infty]$

$$(\inf_k f_k)^{-1}[-\infty, a) = \bigcup_k f_k^{-1}[-\infty, a)$$

$$\left(\sup_k f_k\right)^{-1}[-\infty, q) = \bigcup_{\ell=1}^{\infty} \bigcap_{k=1}^{\infty} f_k^{-1}\left[-\infty, q-\frac{1}{\ell}\right)$$

$$x \in \sup_k f_k^{-1}[-\infty, q) \iff$$

$$\sup_{k \geq 1} f_k(x) < q$$

$$\exists\, \ell \geq 1 \quad \sup_{k \geq 1} f_k(x) < q - \frac{1}{\ell}$$

$$\Rightarrow \quad f_k(x) < q - \frac{1}{\ell} \quad \forall k \geq 1$$

$$\Rightarrow \quad x \in \underline{\bigcup_{\ell=1}^{\infty} \bigcap_{k=1}^{\infty} f_k^{-1}\left[-\infty, q-\frac{1}{\ell}\right)}$$

<u>Conversely</u> : $x \in$

$$x \in \bigcap_{k=1}^{\infty} f_k^{-1}\left[-\infty, q-\frac{1}{\ell}\right)$$

for some $\ell \geq 1$

$$\Rightarrow f_k(x) < a - \frac{1}{\ell} \quad \forall k \geq 1$$

$$\Rightarrow \sup_{k \geq 1} f_k(x) \leq a - \frac{1}{\ell} < a$$

$$\Rightarrow x \in \left(\sup_{k \geq 1} f_k\right)^{-1} [-\infty, a).$$

- $$\liminf_{k \to \infty} f_k = \sup_{m \geq 1} \inf_{k \geq m} f_k$$

$$\limsup_{k \to \infty} f_k = \inf_{m \geq 1} \sup_{k \geq m} f_k$$

$\square$

$A \subseteq \mathbb{R}^n$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

χ_A is the characteristic function of A

χ_A is μ - meas $\Leftrightarrow$ A is μ - measurable

Def SIMPLE FUNCTION

$$f(x) = \sum_{i=1}^{m} a_i \chi_{A_i}(x), \quad a_i \in \mathbb{R}$$

If A_i are μ - meas then f is called μ - meas simple function

f is a μ-meas simple function $\iff$ f is μ-meas and range f is at most countable.

Thm 2.2.6

Let $f : \Omega \to [0, +\infty]$ be μ-meas. Then there are μ-meas. sets $A_k \subseteq \Omega$ such that

$$\boxed{f(x) = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}(x)}$$

PROOF

$$A_1 = \{x : f(x) \geq 1\} = f^{-1}[-1, +\infty]$$

$\boxed{k > 1}$

$$A_k = \left\{ x \in \Omega : f(x) \geq \frac{1}{k} + \sum_{j=1}^{k-1} \frac{1}{j} \chi_{A_j}(x) \right\}$$

$\boxed{\text{CLAIM}}$

$$f(x) \overset{\geq}{\underset{\leq}{=}} \underbrace{\sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}(x)}$$

Proof of CLAIM

" $\geq$ "

- $\sup \{ k \geq 1 \; ; \; x \in A_k \} = +\infty$

Up to a subsequence $k_n \to +\infty$ as $n \to +\infty$ we have

$$f(x) \geq \frac{1}{k_n} + \sum_{j=1}^{k_n - 1} \frac{1}{j} \chi_{A_j}(x) =$$

We let $k_n \to +\infty$ and get

$$f(x) \geq \sum_{j=1}^{\infty} \frac{1}{j} \chi_{A_j}(x).$$

- $k_0 = \sup\{k \geq 1 : x \in A_k\} < +\infty$

$$\Rightarrow f(x) \geq \frac{1}{k_0} + \sum_{j=1}^{k_0-1} \frac{1}{j} \chi_{A_j}(x)$$

$$= \sum_{j=1}^{k_0} \frac{1}{j} \chi_{A_j}(x)$$

$$= \sum_{j=1}^{\infty} \frac{1}{j} \chi_{A_j}(x)$$

because for $j \geq k_0 + 1$

$\chi_{A_j}(x) = 0$ $\blacksquare$

" $\leq$ "

- $f(x) = +\infty \Rightarrow x \in A_k \ \forall k$

$$\Rightarrow \underbrace{\sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}(x)} = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty$$

- $f(x) = 0 \quad \forall k \quad x \notin A_k$

$\Rightarrow \chi_{A_k}(x) = 0$

$$\Rightarrow \sum_{1}^{\infty} \frac{1}{k} \chi_{A_k}(x) = 0$$

- $\boxed{0 < f(x) < +\infty}$

$\forall k_0 : \quad x \notin \bigcap_{k \geq k_0} A_k$

$\Rightarrow \underline{\exists k \geq k_0} \quad \underline{x \notin A_k}$

$\left(\exists k_0 : \quad x \in \bigcap_{k \geq k_0} A_k \Rightarrow \underline{\chi_{A_k}(x) = 1} \right.$

$\forall k \geq k_0$

$$\Rightarrow +\infty = \sum_{k=k_0}^{\infty} \frac{1}{k} \underbrace{\chi_{A_k}(x)}_{=1} \leq f(x) < +\infty \quad \}$$

Hence $\forall n \geq 1$, $\exists\, k_n \geq n$ such that

$$f(x) - \sum_{j=1}^{k_n - 1} \frac{1}{j} \chi_{A_j}(x) < \frac{1}{k_n} \quad (*)$$

Observe that $k_n \to +\infty$ as $n \to +\infty$. Therefore if we let $n \to +\infty$ in $(*)$ we get

$$f(x) \leq \sum_{j=1}^{\infty} \frac{1}{j} \chi_{A_j}(x)$$

$\square$

Remark

1) $$f_k(x) = \sum_{j=1}^{k} \frac{1}{j} \chi_{A_j}(x)$$

is an increasing sequence converging to $f(x)$ as $k \to +\infty$

2) If $0 \le f(x) \le M$ then

$$\sup_{x \in \Omega} |f(x) - f_k(x)| \to 0$$

as $k \to +\infty$ $\boxed{Ex}$

3) f^+, f^- can be approx as before

$\Rightarrow f = f^+ - f^-$ can be approximated by sequences of simple functions as well. $\blacksquare$

Prop 2.2.8

$f : \Omega \to \mathbb{R}$ be cont.

μ is a Borel meas $\Rightarrow$ f is μ-measurable.

Proof

$U \subseteq \mathbb{R}$ open $\Rightarrow$ $f^{-1}(U)$ is open. Since μ is a Borel meas $\Rightarrow$ $f^{-1}(U)$ is μ meas. $\Rightarrow$ f is μ-meas

$\square$

2.3 Lusin's & Egoroff's Theorem

μ is Radon measure in $\mathbb{R}^n$

Theorem 2.3.1 (Egoroff)

Let $\Omega \subseteq \mathbb{R}^n$ μ-meas

$\mu(\Omega) < +\infty$

$f_k : \Omega \to \overline{\mathbb{R}}$ μ meas

$f : \Omega \to \overline{\mathbb{R}}$ μ-meas

($f(x)$ is finite μ. a.e)

$f_k \to f$ as $k \to +\infty$ μ. a.e in Ω.

$\forall \delta > 0$ $\exists$ $F \subseteq \Omega$ compact

$\mu(\Omega \setminus F) < \delta$

$$\sup_{x \in F} |f_k(x) - f(x)| \to 0 \quad \text{as } k \to +\infty$$

NOTATION

"μ - a.e" : almost everywhere with respect to the measure μ, that is except possibly on $A \subseteq \mathbb{R}^n$ $\mu(A) = 0$

$f(x)$ finite μ a.e

$$\{x \in \Omega : f(x) = \pm\infty\} = A$$

$$\mu(A) = 0$$

$f_k \to f$ μ a.e in Ω

$$B := \{x \in \Omega : f_k(x) \not\to f(x)\}$$

$$\mu(B) = 0$$

Remark

$\mu(\Omega) < +\infty$ is necessary.

$f_k(x) = \chi_{[-k,k]}(x) \to f(x) = 1$

as $k \to +\infty$ $\forall x$

$\nexists$ $F \subseteq \mathbb{R}$ compact

such that

$$\sup_F |f_k(x) - f(x)| \to 0$$

$\mu(\mathbb{R} - F) < \delta$

If it is the case $\Rightarrow$

$f_k(x) \equiv 1$ $x \in F$

$$F \subseteq [-k, k]$$

$$[-k,k]^c \subseteq F^c$$

and $\mu [-k,k]^c = +\infty \quad \forall k$

$\Downarrow$

Proof

$f \geq 0$

$\forall i, j \geq 1$

$$C_{ij} = \bigcup_{k=j}^{\infty} \left\{ x \in \Omega : |f_k(x) - f(x)| > \frac{1}{2^i} \right\}$$

$$C_{i,j+1} \subseteq C_{i,j}$$

Since $\mu(C_{i,1}) \leq \mu(\Omega) < +\infty$

$$\lim_{j \to +\infty} \mu(C_{i,j}) = \mu\left(\bigcap_j C_{ij}\right)$$

$$= 0$$

Since $f_{k_j}(x) \to f(x)$ μ.a.e

$$\Rightarrow \mu\left(\bigcap_{j=1} C_{i_j}\right) = 0$$

$\uparrow$

$$x \in \bigcap_{j=1}^{\infty} C_{i_j} \iff x \in C_{i_j} \ \forall j$$

$$\forall j \quad \exists k_j \geq j \quad |f_{k_j}(x) - f(x)| > \frac{1}{2^i}$$

$$\Rightarrow \mu\left(\bigcap_{j=1}^{\infty} C_{i_j}\right) = 0$$

because

$$\bigcap_{j=1}^{\infty} C_{i_j} \subseteq \underbrace{\left\{x \in \Omega : f_k(x) \underset{k \to +\infty}{\not\to} f(x)\right\}}_{\mu(\quad) = 0}$$

$\forall i$: $\exists N(i) > 0$ such that

$$\mu\left(C_{i,N(i)}\right) < \frac{\delta}{2^{i+1}} \quad (*)$$

$$A = \Omega \setminus \bigcup_{i=1}^{\infty} C_{i,N(i)}$$

$$\mu(\Omega \setminus A) = \mu\left(\bigcup_{i=1}^{\infty} C_{i,N(i)}\right)$$

$$\underset{(*)}{\leq} \sum_{i=1}^{\infty} \frac{\delta}{2^{i+1}} = \frac{\delta}{2}$$

$\forall x \in A$ $\forall i$ $\forall k \geq N(i)$

$$|f_k(x) - f(x)| \leq \frac{1}{2^i}$$

$\Rightarrow f_k \to f$ uniformly in A

Apply Thm 1.8.2

$\exists\, F \subseteq A$ compact

$\mu(A \setminus F) < \delta/2$

$$\mu(\Omega \setminus F) \leq \mu(\Omega \setminus A) + \mu(A \setminus F)$$

$$(\Omega \setminus F \subseteq (\Omega \setminus A) \cup (A \setminus F))$$

$$\leq \delta/2 + \delta/2 = \delta$$

$\blacksquare$

CONTINUITY PROPERTY OF MEASURABLE FUNCTIONS

Theorem 2.3.3 (Lusin's Theorem)

Assume $\mu(\Omega) < +\infty$. Let $f : \Omega \to \mathbb{R}$ be μ-meas. and $|f(x)| < +\infty$ μ-a.e. Then $\forall \varepsilon > 0$ $\exists K \subseteq \Omega$ compact such that $\mu(\Omega \setminus K) < \varepsilon$ and $f|_K$ is continuous.

Remark The theorem does not make the stronger assertion that the function f defined on Ω is continuous w.r.t.

$\boxed{\text{EXAMPLE}}$

$\Omega = [0,1] \subseteq \mathbb{R}$, $f = \chi_{[0,1]\setminus\mathbb{Q}}$

f is NOT continuous

but $f|_{[0,1]\setminus\mathbb{Q}} : [0,1]\setminus\mathbb{Q} \to \mathbb{R}$

is continuous.

We cannot take $\varepsilon = 0$ in Lusin's theorem

<u>Proof</u> $\forall i \geq 1$ $\{B_{ij}\}$ Borel sets of $\mathbb{R}$ such that

$\mathbb{R} = \bigcup_{j\geq 1} B_{ij}$, B_{ij} are mutually disjoint and diam $B_{ij} = \sup\{|x-y| : x,y \in B_{ij}\} < \frac{1}{i}$

$A_{ij} = f^{-1}(B_{ij})$ A_{ij} is μ-meas and mutually disjoint

$$\tilde{\Omega} = \bigcup_{j=1}^{\infty} A_{ij} \quad (\Omega = \tilde{\Omega} \cup f^{-1}(\pm\infty))$$

$$\boxed{\mu(f^{-1}(\pm\infty)) = 0}$$

Since μ is a Radon measure $\forall\, i,j$ $\exists\, K_{ij} \subseteq A_{ij}$ compact such that

$$\mu(A_{ij} \setminus K_{ij}) < \frac{\varepsilon}{2^{i+j}}$$

(Thm 1.8, 2)

$$\mu\Big(\underbrace{\tilde{\Omega} \setminus \bigcup_{j=1}^{\ell} K_{ij}}_{\boxed{\begin{aligned} &= \Big(\bigcup A_{ij}\Big) \setminus \Big(\bigcup K_{ij}\Big) \\ &= \bigcup_{j=1}^{\ell} \Big(A_{ij} \setminus \bigcup K_{ij}\Big) \\ &\subseteq \bigcup_{j=1}^{\ell} \big(A_{ij} \setminus K_{ij}\big) \end{aligned}}}\Big) \le \mu\Big(\bigcup_{j=1}^{\ell} (A_{ij} \setminus K_{ij})\Big) \quad \otimes$$

$$\otimes \le \sum_{j=1}^{\ell} \mu\big(A_{ij} \setminus K_{ij}\big) < \sum_{j=1}^{\ell} \frac{\varepsilon}{2^{i+j}}$$

$$= \frac{\varepsilon}{2^{i}}$$

$\tilde{\Omega} \setminus \bigcup_{j=1}^{n} K_{ij}$ is decreasing

$\mu(\Omega) < +\infty$

$$\Rightarrow \lim_{n \to +\infty} \mu\left(\tilde{\Omega} \setminus \bigcup_{j=1}^{n} K_{ij}\right)$$

$$= \mu\left(\tilde{\Omega} \setminus \bigcup_{j=1}^{\infty} K_{ij}\right)$$

$$\leq \varepsilon/2^i$$

$\forall i \ \exists N(i)$:

$$\mu\left(\tilde{\Omega} \setminus \bigcup_{j=1}^{N(i)} K_{ij}\right) < \varepsilon/2^i$$

$D_i = \bigcup_{j=1}^{N(i)} K_{ij}$ is compact

$\forall i, j$: choose $b_{ij} \in B_{ij}$

$g_i(x) : D_i \longrightarrow \mathbb{R}$

$g_i(x) = b_{ij}$ if $x \in K_{ij}$

$\forall j \leq N(i)$

g_i are continuous

K_{ij} are compact

sets mutually disjoint

$d(K_{ij_1}, K_{ij_2}) > 0 \quad \forall j_1 \neq j_2$

$j_1, j_2 \leq N(i)$

$|f(x) - g_i(x)| < \frac{1}{i} \quad \forall x \in D_i$

because $\operatorname{diam}(B_{ij}) < \frac{1}{i}$

$K = \bigcap_{i=1}^{\infty} D_i$, K is compact

and

$$\mu(\tilde{\Omega} - K) = \mu\left(\tilde{\Omega} \setminus \bigcap_{i=1}^{\infty} D_i\right)$$

$$= \mu\left(\bigcup_{i=1}^{\infty} \tilde{\Omega} \setminus D_i\right)$$

$$\leq \sum \mu(\tilde{\Omega} - D_i)$$

$$\leq \sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon$$

$$\Omega - K = (\tilde{\Omega} - K) \cup (f^{-1}(\pm\infty) \setminus K)$$

$$\mu(\Omega \setminus K) \le \mu(\tilde{\Omega} \setminus K) + \mu\left(f^{-1}(\pm\infty) \setminus K\right)$$

$$\le \varepsilon + 0$$

Since $|f(x) - g_i(x)| < \frac{1}{i}$ $\forall x \in D_i$

$\Rightarrow g_i(x) \to f(x)$ unif. in $K \Rightarrow f_{|K}$ is cont.

(it is the uniform limit of cont. functions)

$\square$