LECTURE 27.05.2020

$f \in C^0(\mathbb{R}^n) \Rightarrow$

$$f(x) = \lim_{r \to 0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} f \, d\mathcal{L}^n$$

$\mu = \mathcal{L}^n$

Thm 5.1.2 (Lebesgue Differentiation Theorem)

$f \in L^1_{loc}(\mathbb{R}^n) \Rightarrow \forall$ a.e. $x \in \mathbb{R}^n$

$$f(x) = \lim_{r \to 0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} f \, d\mathcal{L}^n$$

Actually we will show a stronger property:

Cor. 5.1.7

$f \in L^1_{loc}(\mathbb{R}^n) \Rightarrow \forall$ a.e. $x \in \mathbb{R}^n$

$$\lim_{r \to 0} \fint_{B(x,r)} |f(y) - f(x)| \, d\mathcal{L}^n(y) = 0$$

Recall $\forall E \subseteq \mathbb{R}^n$, $\mathcal{L}^n(E) < \infty$

$$\fint_E f \, d\mathcal{L}^n = \frac{1}{\mathcal{L}^n(E)} \cdot \int_E f \, d\mathcal{L}^n$$

$f \in L^1_{loc}(\mathbb{R}^n) \to$

$$f^*(x) = \sup_{r>0} \fint_{B(x,r)} |f| \, d\mathcal{L}^n$$

HARDY-LITTLEWOOD MAXIMAL FUNCTION

REMARKS

1) $f \in L^1(\mathbb{R}^n)$: $\forall \varepsilon > 0$ $\exists \delta > 0$: $\forall A \subseteq \mathbb{R}^n$ $\mathcal{L}^n(A) < \delta$ A $\mathcal{L}^n$-meas $\Rightarrow \int_A |f| \, d\mathcal{L}^n < \varepsilon$

It follows $\forall r > 0$

$$x \longmapsto \fint_{B(x,r)} |f(y)| \, d\mathcal{L}^n$$

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is continuous: $x_n \to x$

$$\int_{B(x_n,r)} |f| \, d\mathcal{L}^n \to \int_{B(x,r)} |f| \, d\mathcal{L}^n$$

$\Rightarrow f^*(x)$ is lower-semicont

because it is the supremum of

of continuous functions

$\Rightarrow f^*(x)$ is $\mathcal{L}^n$-meas

2) • $f: \mathbb{R}^n \to \mathbb{R}$ $\quad f(x) = c$

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$\Rightarrow f^*(x) = |c| \quad \forall x$

• $f(x) = \frac{1}{x} \chi_{(0,+\infty)}(x)$

$x \in \mathbb{R} \Rightarrow f^*(x) = +\infty$

f is not integrable

in balls containing

the origin

$\forall x \in \mathbb{R}^n \quad B(x, 2|x|)$

contains the origin

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$$\int_{B(x,2|x|)} f(y)\,dy = +\infty$$

$$\Rightarrow f^*(x) \geq \fint_{B(x,2|x|)} |f|\,d\mathcal{L}^n = +\infty$$

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3) $f^* \in L^1(\mathbb{R}^n)$ only when $f \equiv 0$

<u>Sol</u>: $f \neq 0$ $f \in L^1(\mathbb{R}^n)$

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$$\lambda = \int_{B(0,R)} |f|\,d\mathcal{L}^n > 0$$

Take $|x| > R$

$$f^*(x) \geq \fint_{B(x,|x|+R)} |f(y)|\,d\mathcal{L}^n$$

$$\geq \frac{1}{\mathcal{L}^n(B(x,|x|+R))} \underbrace{\int_{B(0,R)} |f(y)|\,d\mathcal{L}^n}_{\circledast}$$

$$B(0,R) \subseteq B(x,|x|+R)$$

If $|y| < R \Rightarrow |y-x| \leq |y|+|x|$

$$\leq R+|x|$$

$$\mathcal{L}^n\left(B(x,|x|+R)\right) = c_n\,(R+|x|)^n$$

$$(*) = c_n\lambda\,\frac{1}{(R+|x|)^n} \underset{|x|>R}{\geq} c_n\,\frac{\lambda}{(2|x|)^n} \notin L^1(\mathbb{R}^n\setminus B(0,R))$$

$\Rightarrow$ If $|x|>R$ then

$$f^*(x) \geq \frac{c_n\lambda}{2^n|x|^n}$$

Since $\dfrac{c_n\lambda}{2^n|x|^n} \notin L^1(\mathbb{R}^n\setminus B(0,R))$

$$\Rightarrow \quad f^*(x) \notin L^1(\mathbb{R}^n)$$

Recall that

$$\int_{\mathbb{R}^n\setminus B(0,R)} \frac{1}{|x|^n}\,d\mathcal{L}^n = \int_R^{+\infty}\frac{1}{\rho^n}\int_{\partial B(0,\rho)} d\sigma\,d\rho$$

$$= \int_R^{+\infty}\frac{1}{\rho^n}\cdot\rho^{n-1}\,d\rho = \int_R^{+\infty}\frac{1}{\rho}\,d\rho = \log\rho\,\Big|_R^{+\infty} = +\infty$$

4) There are examples $f \in L^1(\mathbb{R}^n)$ and $f^* \notin L^1_{loc}(\mathbb{R}^n)$ (HINT $f(x) = \chi_{(0,1/2)}(x) \frac{1}{x \log^2 x}$ ← EXERCISE)

4) $f \in L^p \quad p > 1 \Rightarrow f^* \in L^p$

$$\|f^*\|_{L^p} \leq c \|f\|_{L^p}$$

Prop 5.1.6

$f \in L^1(\mathbb{R}^n) \Rightarrow \forall \alpha > 0$

$$\mathcal{L}^n\{x : f^*(x) > \alpha\} \leq \frac{5^n}{\alpha} \|f\|_{L^1}$$

Proof of Thm 5.1.2 /

Corollary 5.1.7

We assume w.l.o.g that $f \in L^1(\mathbb{R}^n)$ (otherwise you consider $g(x) = f(x)\chi_{B(0,1)} \in L^1$)

$\exists f_k \in C^0_c(\mathbb{R}^n)$

$$\|f_k - f\|_{L^1} \to 0 \text{ as } k \to +\infty$$

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$$\left| \overline{\lim_{r\to 0}} \fint_{B(x,r)} f(y) - f(x)\, d\mathcal{L}^n \right|$$

$$\le \overline{\lim_{r\to 0}} \fint_{B(x,r)} | f(y) - f(x) |\, d\mathcal{L}^n$$

$$= \overline{\lim_{r\to 0}} \fint_{B(x,r)} | f(y) - f_k(y) + f_k(y) - f_k(x) + f_k(x) - f(x) |\, d\mathcal{L}^n$$

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$$\le \underbrace{\overline{\lim_{r\to 0}} \fint_{B(x,r)} | f(y) - f_k(y) |\, d\mathcal{L}^n}_{1} + \underbrace{\overline{\lim_{r\to 0}} \fint_{B(x,r)} | f_k(y) - f_k(x) |\, d\mathcal{L}^n}_{2} + \underbrace{\overline{\lim_{r\to 0}} \fint_{B(x,r)} | \underbrace{f_k(x) - f(x)} |\, d\mathcal{L}^n}_{3}$$

$$\le \underbrace{(f - f_k)^*}(x) + \underbrace{\overline{\lim_{r\to 0}} \fint | f_k(y) - f_k(x) |\, d\mathcal{L}^n}_{= 0} + \underbrace{| f_k(x) - f(x) |} \quad (*)$$

$(2) = 0$ because $f_k \in C_c^0(\mathbb{R}^n)$

$\forall \varepsilon > 0$

$$A_\varepsilon = \left\{ x : \overline{\lim_{r \to 0}} \fint_{B(x,r)} |f(y) - f(x)| \, d\mathcal{L}^n > 2\varepsilon \right\}$$

$(*) \Longrightarrow$

$$A_\varepsilon \subseteq \left\{ x : (f - f_k)^*(x) > \varepsilon \right\}$$

$$\cup \left\{ x : |f_k(x) - f(x)| > \varepsilon \right\}$$

$$\mathcal{L}^n(A_\varepsilon) \leq \underbrace{\mathcal{L}^n \left\{ x : (f - f_k)^*(x) > \varepsilon \right\}}_{(**)}$$

$$+ \underbrace{\mathcal{L}^n \left\{ x : |(f_k - f)(x)| > \varepsilon \right\}}_{(***)}$$

$$(**) \leq \frac{5^n}{\varepsilon} \| f_k - f \|_{L^1} \xrightarrow{k \to +\infty} 0$$

(Prop 5.1.2)

$$(***) \leq \frac{1}{\varepsilon} \| f_k - f \|_{L^1} \xrightarrow{k \to +\infty} 0$$

$$\Rightarrow \mathcal{L}^n(A_\varepsilon) = 0$$

$$A = \left\{ x : \overline{\lim} \fint_{B(x,r)} |f(y) - f(x)| \, d\mathcal{L}^n > 0 \right\}$$

$$A = \bigcup_{n=1}^{\infty} A_{1/n}$$

$$\Rightarrow \quad \mathcal{L}^n(A) = 0$$

$$\Rightarrow \quad \forall \text{ a.e. } x \in \mathbb{R}^n$$

$$\overline{\lim_{r \to 0}} \fint_{B(x,r)} |f(y) - f(x)| \, d\mathcal{L}^n = 0$$

NOTE

$$\underbrace{\left| f(x) - \lim_{r \to 0} \fint_{B(x,r)} f(y) \, d\mathcal{L}^n \right|}_{\to\ 0}$$

$$= \left| \lim_{r \to 0} \fint_{B(x,r)} f(x) - f(y) \, d\mathcal{L}^n \right|$$

$$\leq \overline{\lim_{r \to 0}} \fint_{B(x,r)} |f(y) - f(x)| \, d\mathcal{L}^n = 0$$

$$\Rightarrow \lim_{r \to 0} \fint f(y) \, d\mathcal{L}^n = f(x)$$

$$\forall \text{ a.e. } x \in \mathbb{R}^n \qquad \square$$