-1-

Lecture 28.04.2020

COR 3.1.14 (A REMARK)

f, f_k are μ-integrable

$$\lim_{k\to+\infty} \int_\Omega |f_k - f| \, d\mu = 0$$

$\Rightarrow \exists\, f_{k_n} \to f$ μ.a.e

Thm 3.6.5

Let $\mu(\Omega) < +\infty$ and

2

$(f_n)_n$ μ-meas functions

$f_n \xrightarrow{\mu} f$ as $n \to +\infty$.

Assume $\forall n \geq 0$ $|f_n(x)| \leq |g_n(x)|$

$(g_n)_n$ μ-summable satisfying

$$\lim_{n\to+\infty} \int_\Omega |g_n - g| \, d\mu = 0$$

g is μ-summ. function

$$\Rightarrow \lim_{n\to+\infty} \int_\Omega f_n \, d\mu = \int f \, d\mu$$

<u>Proof</u>

$$\lim_{n \to +\infty} \int_\Omega |g_n(x) - g(x)| \, d\mu = 0$$

$\Rightarrow$ apply Vitali's Thm.

$\Rightarrow$ (g_n) is uniformly

μ-summ: $\forall \varepsilon > 0 \ \exists \delta > 0$

$$\mu(A) < \delta \Rightarrow \sup_{n \in \mathbb{N}} \int_A |g_n(x)| \, d\mu < \varepsilon$$

Hence:

$$\int_A |f_n(x)| \, d\mu \le \int_A |g_n(x)| \, d\mu < \varepsilon$$

$\forall A : \mu(A) < \delta$, $\forall n \in \mathbb{N}$

$(f_n)_n$ is unif μ-summ.

By assumption $f_n \xrightarrow{\mu} f$

$\Rightarrow$ Vitali's theorem

implies that

$$\int_\Omega |f_n - f| \, d\mu \to 0 \text{ as } n \to +\infty$$

$$\Rightarrow \lim_{n \to +\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu$$

Thm 3.6.6

$\mu(\Omega) < +\infty$, $(u_k), u: \Omega \to \overline{\mathbb{R}}$

μ - meas. such that

$$\int_\Omega |u|^p < +\infty \quad \int_\Omega |u_k|^p d\mu < +\infty$$

$$\int_\Omega |u_k - u|^p d\mu \to 0 \quad \infty$$
$k \to +\infty$

$1 \leq p < +\infty$.

Then

$$\lim_{k \to +\infty} \int_\Omega u_k \, d\mu = \int_\Omega u \, d\mu$$

Proof

"$p = 1$": $\int_\Omega |u - u_k| \xrightarrow[k \to +\infty]{} 0$

$\Rightarrow \int_\Omega u_k \xrightarrow[k \to +\infty]{} \int_\Omega u \, d\mu$

"$p > 1$"

i) CLAIM 1 $\quad u_k \xrightarrow{\mu} u \quad k \to +\infty$

Proof CLAIM 1 : Fix $\varepsilon > 0$

$$\mu(\{x : |u_k(x) - u(x)| > \varepsilon\})$$

$$= \mu(\{x : |u_k(x) - u(x)|^p > \varepsilon^p\})$$

$$\leq \frac{1}{\varepsilon^p} \int_\Omega |u_k - u|^p \, d\mu$$

Since $\int_\Omega |u_k - u|^p \, d\mu \to 0$ as $\to \infty$

$\Rightarrow |u_k - u| \xrightarrow{\mu} 0$ as $k \to +\infty$

ii) $\boxed{\text{CLAIM 2}}$:

$(u_k)_k$ is UNIT μ-SUMM

Proof of CLAIM 2

$\forall \varepsilon > 0 \ \exists \delta > 0, \ \bar{k} > 0$

such that

$\mu(A) < \delta$

$$\underbrace{\int_A |u|^p \, d\mu < \varepsilon}_{x \mapsto |u|^p(x) \text{ is } \mu\text{-summ.}} \quad \& \quad \int_\Omega |u_k - u|^p d\mu < \varepsilon \quad k \geq \bar{k}$$

$$|u_k|^p \leq (|u| + |u_k - u|)^p$$

$$\leq 2^{p-1} (|u|^p + |u_k - u|^p)$$

$x, y \in \mathbb{R}$ $\quad \forall p \geq 1$

$$|x+y|^p \leq 2^{p-1}\left(|x|^p + |y|^p\right)$$

$p = 1$ $\quad |x+y| \leq |x| + |y|$

$p > 1$

$t \longmapsto |t|^p$ is convex

$$\Rightarrow \left|\frac{x+y}{2}\right|^p \leq \frac{1}{2}|x|^p + \frac{1}{2}|y|^p$$

$$\Rightarrow |x+y|^p \leq 2^{p-1}\left(|x|^p + |y|^p\right)$$

$x = 1$, $y \in \mathbb{R}$

$$|y+1|^p \leq 2^{p-1}\left(1 + |y|^p\right)$$

$$\log|1+y|^p \leq \log 2^{p-1} + \underline{\log\left(1+|y|^p\right)}$$

$$\underbrace{\log|1+y|^p - \log\left(1+|y|^p\right)} \leq (p-1)\log 2$$

$$y \overset{\varphi}{\longmapsto} \log|1+y|^p - \log\left(1+|y|^p\right)$$

$$\max_{\mathbb{R}} \varphi = (p-1)\log 2 = \log\left(2^{p-1}\right)$$

11

$$\int_A |u_k| \, d\mu$$

$\forall \varepsilon' > 0 \quad \exists \delta' > 0 \quad \exists k' > 0$

$$\sup_{k \geq k'} \int_A |u_k| \, d\mu < \varepsilon'$$

$\mu(A) < \delta'$ $\quad \wedge$

$$\int_A |u_k| \leq \int_A 1 + |u_k|^p \, d\mu$$

$$\leq \mu(A) + 2^{p-1} \int_A (|u|^p + |u_k - u|^p) \, d\mu$$

$= \circledast$

12

$$\left(|u_k|^p = |(u_k - u) + u|^p \leq 2^{p-1} (|u|^p + |u_k - u|^p) \right)$$

$$\circledast = \mu(A) + 2^{p-1} \int_A |u|^p + 2^{p-1} \int_A |u_k - u|^p \, d\mu$$

We know that $\forall \varepsilon > 0 \ \exists \delta > 0 \ \exists \bar{k} > 0$:

$$\int_A |u|^p \, d\mu < \varepsilon, \quad \int_\Omega |u_k - u|^p \, d\mu < \varepsilon$$

$\mu(A) < \delta, \ k \geq \bar{k}$

Choose $\varepsilon > 0$: $2^p \varepsilon < \varepsilon' / 2$,

and $\delta' = \min(\delta, \varepsilon'/2)$

and $k' = \bar{k}$

By these choices we get:

$$\circledast \leq \delta^{1} + 2^{p-1}\varepsilon + 2^{p-1}\varepsilon$$

$$= \delta^{1} + 2^{p}\varepsilon < \frac{\varepsilon^{1}}{2} + \frac{\varepsilon^{1}}{2} = \varepsilon^{1}$$

By combining the fact that:

$u_k \xrightarrow{\mu} u$ as $k \to +\infty$

$\Rightarrow$ VITALI'S THEOREM

gives $\int_\Omega |u_k - u| \, d\mu \to 0$

as $k \to +\infty$ $\square$

LAST WEEK

$f : [a,b] \to \mathbb{R}$ bounded

If f is R-integrable

then it is $\mathcal{L}^1$-integrable

Property (see Ex 3.6.7)

$f : [a, +\infty) \to \mathbb{R}$ $a \in \mathbb{R}$

f is locally R-integrable

$\left(\forall [b,c] \subseteq [a,+\infty) : \int_b^c f\,dx < +\infty\right)$

We say that f is absolutely R-integrable in $[a,+\infty)$ if

$$\lim_{J\to+\infty} \int_a^J |f(x)|\,dx < +\infty$$

$$\parallel$$

$$\int_a^{+\infty} |f(x)|\,dx$$

f is $\mathcal{L}^1$-summable in $[a,+\infty)$ $\Leftrightarrow$ f is absolutely R-integrable and in this case we have

$$\int_{[a,+\infty)} f\,d\mathcal{L}^1 = \int_a^{+\infty} f(x)\,dx$$

$$= \lim_{J\to+\infty} \int_a^J f(x)\,dx$$

EXAMPLE

$f(x) = \frac{\sin x}{x}$ in $[0, +\infty)$

f is locally R-integrable because it is locally bounded

$\left(\lim_{x \to 0^+} \frac{\sin x}{x} = 1 \right)$

$$\int_0^{+\infty} \frac{\sin x}{x}\, dx < +\infty$$

$\uparrow$

$$\int_0^{+\infty} \frac{\sin x}{x}\, dx =$$

$$= \underbrace{\int_0^1 \frac{\sin x}{x}\, dx}_{< +\infty} + \lim_{J \to +\infty} \underbrace{\int_1^J \frac{\sin x}{x}\, dx}$$

$$\int_1^J \frac{\sin x}{x}\, dx = -\frac{\cos x}{x} \Big|_1^J - \int_1^J + \cos x \left(+ \frac{1}{x^2} \right) dx$$

$$= -\frac{\cos J}{J} + \frac{\cos 1}{1} - \underbrace{\int_1^J \frac{\cos x}{x^2}\, dx}$$

$$\lim_{J \to +\infty} \frac{\cos J}{J} = 0$$

$$\left| \frac{\cos x}{x^2} \right| \le \frac{1}{x^2} \qquad |\cos x| \le 1$$

$$\lim_{J \to +\infty} \int_1^J \frac{1}{x^2}\, dx < +\infty$$

$$\Rightarrow \lim_{J \to +\infty} \int_1^J \frac{\cos x}{x^2}\, dx < +\infty$$

EXERCISE

$$\int_0^{+\infty} \left| \frac{\sin x}{x} \right| dx = +\infty$$

Another example

$$f(x) = \frac{1}{x} \sin\left(\frac{1}{x}\right) \quad x \in (0, 1]$$

REMARK

f is locally R-integrable on $I = [a, b)$ (or $I = (a, b]$, (a, b)) but it has a discontinuity in $x = b$ (or in $x = a$ or in $x = a$ & $x = b$)

f is $\mathcal{L}^1$-summ on $I = [a, b)$ iff f is absolutely R-integrable in the generalized sense

$$\lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} |f(x)|\,dx < +\infty$$

and in this case

$$\int_{[a,b)} f(x)\,d\mathcal{L}^1 = \int_a^b f(x)\,dx =$$

$$= \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)\,dx. \quad \square$$

Proof Ex 3.6.7 (part 2)

1) $g_n(x) := |f|\chi_{[a,n]}(x)$

$n \geq a$

g_n are $\mathcal{L}^1$-summable because they are R-integrable

$g_n \to |f|\chi_{[a,+\infty)}$ pointwise as $n \to +\infty$

$g_n \geq 0$

Beppo-Levi Theorem:

$$\lim_{n\to+\infty} \int_{\mathbb{R}} g_n \, d\mathcal{L}^1 = \int_{\mathbb{R}} \lim_{n\to+\infty} g_n \, d\mathcal{L}^1$$

$$= \int_{\mathbb{R}} |f| \chi_{[a,+\infty)} (x) \, d\mathcal{L}^1$$

$$= \int_{[a,+\infty)} |f| (x) \, d\mathcal{L}^1$$

f is $\mathcal{L}^1$-summ in $[a,+\infty)$

$\left(\int_{[a,+\infty)} |f| \, d\mu < +\infty\right)$ iff

$$\lim_{n\to+\infty} \int_{\mathbb{R}} g_n \, d\mathcal{L}^1 < +\infty$$

But $\lim_{n\to+\infty} \int_{\mathbb{R}} g_n \, d\mathcal{L}^1$

$$= \lim_{n\to+\infty} \int_{[a,n]} |f| \, d\mathcal{L}^1 =$$

$$= \lim_{n \to +\infty} \int_a^n |f|\, dx < +\infty$$

$$\Leftrightarrow \int_a^{+\infty} |f|\, dx < +\infty$$

$\Leftrightarrow$ f is absolutely R-integrable in the generalized sense

2) $f_n = f \chi_{[a,n]}$

f_n are R-integrable

$\Rightarrow$ f_n are $\mathcal{L}^1$-integrable

$f_n \to f \chi_{[a,+\infty)}$ as $n \to +\infty$

pointwise

$|f_n| \leq |f| \chi_{[a,+\infty)}$ is $\mathcal{L}^1$-summ.

27

Lebesgue Theorem

gives:

$$\lim_{n\to+\infty}\int_{\mathbb{R}} f_n \, d\mathcal{L}^1 =$$

$$= \lim_{n\to+\infty}\int_{[0,n]} f(x)\, d\mathcal{L}^1 =$$

$$= \lim_{n\to+\infty}\int_0^n f(x)\, dx =$$

28

$$\int_{\mathbb{R}} f(x)\, d\mathcal{L}^1 = \int_{[0,+\infty)} f(x)\, d\mathcal{L}^1$$

$$\Downarrow$$

$$\lim_{\to+\infty}\int_0^n f(x)\, dx = \int_0^{+\infty} f(x)\, dx$$

$$= \int_{[0,+\infty)} f(x)\, d\mathcal{L}^1$$

$$f(x) \sim \frac{1}{\sqrt{|x - r_k|}} \qquad r_k \in \mathbb{Q}\cap[0,1]$$

$x \in [0,1]$

$$\lim_{x \to r_k} f(x) = +\infty$$

$$\int_0^1 |f(x)|\, dx =$$

$$\underbrace{\lim_{\varepsilon \to 0} \int_0^{r_k - \varepsilon} |f(x)|\, dx}_{< +\infty} + \underbrace{\lim_{\delta \to 0} \int_{r_k + \delta}^1 |f(x)|\, dx}_{< +\infty}$$