LECTURE 29.05.2020

$f \in L^1_{loc}(\mathbb{R}^n) \Rightarrow \forall$ a.e

$x \in \mathbb{R}^n$

$$(*) \quad f(x) = \lim_{r \to 0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} f(y)\, d\mathcal{L}^n$$

namely $f(x)$ can be approximated by averages of on balls

in $\mathbb{R}^n$ centered at x with smaller and smaller radii

<u>Def 5.1.8</u>

A point $x \in \mathbb{R}^n$ satisfying $(*)$ is called a Lebesgue point of f.

<u>NOTATION</u> $\mathcal{L}^n(E) = |E|$

Application

Let $E \subseteq \mathbb{R}^n$ be measurable. Then for a.e $x \in E$

$$\lim_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = 1$$

Proof apply Lebesgue differentiation theorem to $f(x) = \chi_E(x)$

(which is clearly $L^1_{loc}(\mathbb{R}^n)$) $\Rightarrow \forall$ a.e $x \in \mathbb{R}^n$

$$\chi_E(x) = \lim_{r \to 0} \fint_{B(x,r)} \chi_E(y)\,dy$$

$$= \lim_{r \to 0} \frac{1}{|B(x,r)|} \cdot |E \cap B(x,r)|$$

Therefore $\forall$ a.e $x \in E$

$$\chi_E(x) = 1 = \lim_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|}$$

It means that as $r \downarrow 0$
$|E \cap B(x,r)|$ gets closer
to $|B(x,r)|$

More precisely $\forall\, 0 < \lambda < 1$
$\exists\, r_0 > 0 : \forall r < r_0$

$$\Rightarrow \frac{|E \cap B(x,r)|}{|B(x,r)|} > \lambda$$

$$\Rightarrow |E \cap B(x,r)| > \lambda\, |B(x,r)|$$

For a.e. $x \in E$, E
occupies a large proportion
of small balls centered
at x.

<u>NOTE</u> $E = \{x\} \Rightarrow$ we
cannot hope for this
set that E occupies
a large proportion of

any balls centered at x!

However the statement of the application still holds for a.e point x of X_E as the set of points of E for which the

statement does not hold is $\{x\}$ which has measure zero

Remark

$f : \mathbb{R} \to \mathbb{R}$ differentiable

$\Rightarrow$ all $x \in \mathbb{R}^n$ are Lebesgue points for f'

By definition we have

$$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x-\varepsilon)}{(x+\varepsilon) - (x-\varepsilon)}$$

$$= \lim_{\varepsilon \to 0} \frac{\int_{x-\varepsilon}^{x+\varepsilon} f'(y)\, dy}{|(x-\varepsilon, x+\varepsilon)|}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} f'(y)\, dy$$

$$B(x,\varepsilon) =]x-\varepsilon, x+\varepsilon[$$

Prop 5.1.6

If $f \in L^1(\mathbb{R}^n)$, then

$\forall \alpha > 0$

$$\mathcal{L}^n\{x : f^*(x) > \alpha\} \leq \frac{5^n}{\alpha} \|f\|_{L^1}$$

$$f^*(x) = \sup_{r>0} \fint_{B(x,r)} |f| \, d\mathcal{L}^n$$

THEOREM 5.1.9 (VITALI'S COVERING THEOREM)

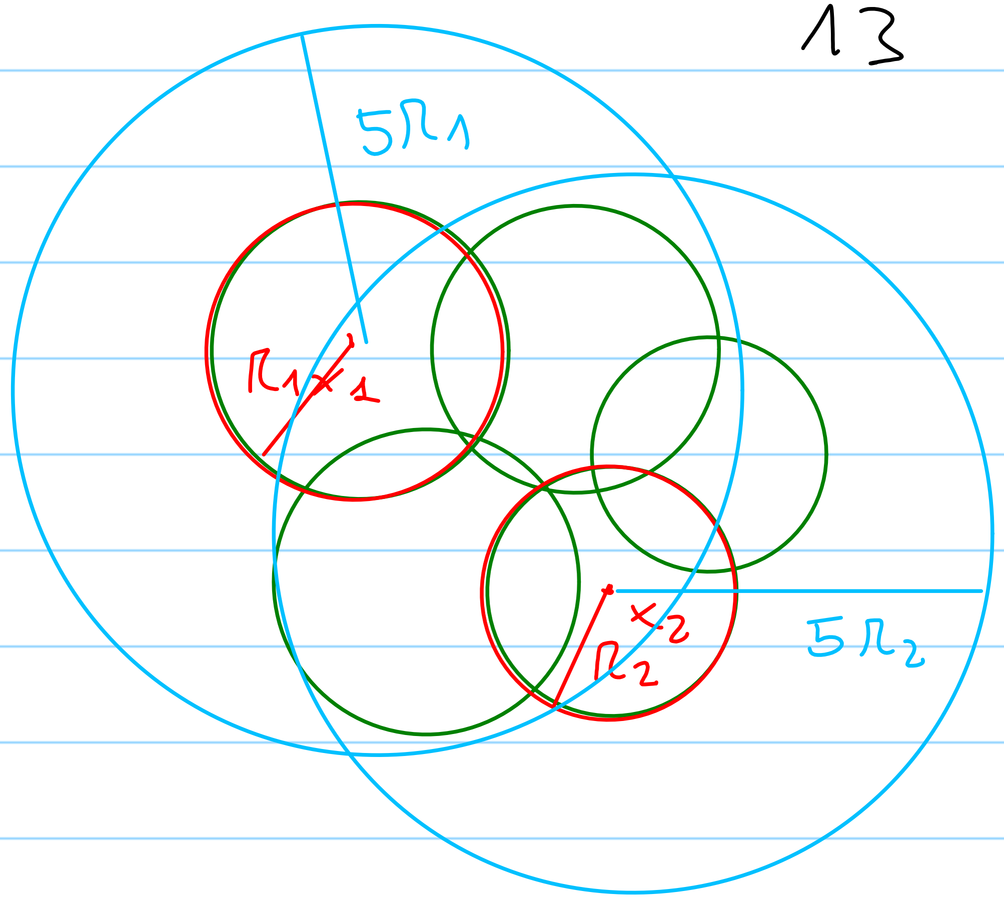
Let $\mathcal{F}$ be a family of non degenerate closed balls $B = B(x,r) \subseteq \mathbb{R}^n$

$d_0 = \sup\{\operatorname{diam} B : B \in \mathcal{F}\} < +\infty$

For each $B = B(x,r) \in \mathcal{F}$ denote $\hat{B} = \hat{B}(x, 5r)$.

Then there is a countable family $\mathcal{G} \subseteq \mathcal{F}$ of mutually disjoint balls such that

$$\bigcup_{B \in \mathcal{F}} B \subseteq \bigcup_{B \in \mathcal{G}} \hat{B}$$



Proof

1. $d_0 = \sup\{\mathrm{diam}(B), B \in \mathcal{F}\}$

$\forall J = 1, 2, ..$

$$\mathcal{F}_J = \left\{ B \in \mathcal{F} : \frac{d_0}{2^J} < \mathrm{diam}(B) \leq \frac{d_0}{2^{J-1}} \right\}$$

$$\mathcal{G}_J \subseteq \mathcal{F}_J$$

a) $J = 1$ $\quad \mathcal{G}_1$ is any MAXIMAL disjoint collection of balls in $\mathcal{F}_1$

b) $\mathcal{G}_1, \mathcal{G}_2, .., \mathcal{G}_{k-1}$, we choose $\mathcal{G}_k$ to be any maximal disjoint collection of balls in $\mathcal{F}_K$:

$$\left\{ B \in \mathcal{F}_K \;\middle|\; B \cap B' = \emptyset \quad \forall B' \in \bigcup_{j=1}^{K-1} \mathcal{G}_j \right\}$$

$$\mathcal{G} = \bigcup_{J=1}^{\infty} \mathcal{G}_J$$

$\mathcal{G} \subseteq \mathcal{F}$ is made by disjoint balls

2) $\displaystyle \bigcup_{B \in \mathcal{F}} B \subseteq \bigcup_{B \in \mathcal{G}} \hat{B}$ ←

CLAIM

Let $B \in \mathcal{F} \Rightarrow \exists\, B' \in \mathcal{G}$ such that $B \cap B' \neq \emptyset$ and $B \subseteq \hat{B}'$

Proof of CLAIM

Fix $B \in \mathcal{F} \Rightarrow \exists\, J$ such that $B \in \mathcal{F}_J$

By the maximality of $\mathcal{G}_J$

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There is $B' \in \bigcup_{k=1}^{J} \mathcal{G}_k$

with $B \cap B' \neq \emptyset$

Of $B' \in \bigcup_{k=1}^{\bar{J}} \mathcal{G}_k \subseteq \bigcup_{k=1}^{J} \mathcal{F}_k$

$\text{diam}(B') > \frac{d_0}{2^J}$

(if $B' \in \mathcal{G}_k$ $k < J$

$\text{diam}(B') \underset{\uparrow}{>} \frac{d_0}{2^k} > \frac{d_0}{2^J}$)

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$\text{diam}(B) \leq \frac{d_0}{2^{J-1}}$

$\Rightarrow \text{diam}(B) \leq 2 \frac{d_0}{2^J}$

$< 2\, \text{diam}(B')$

$\Rightarrow B \subseteq \widehat{B'}$.

($B = B(x_1, r_1)$, $B' = B(x_2, r_2)$)

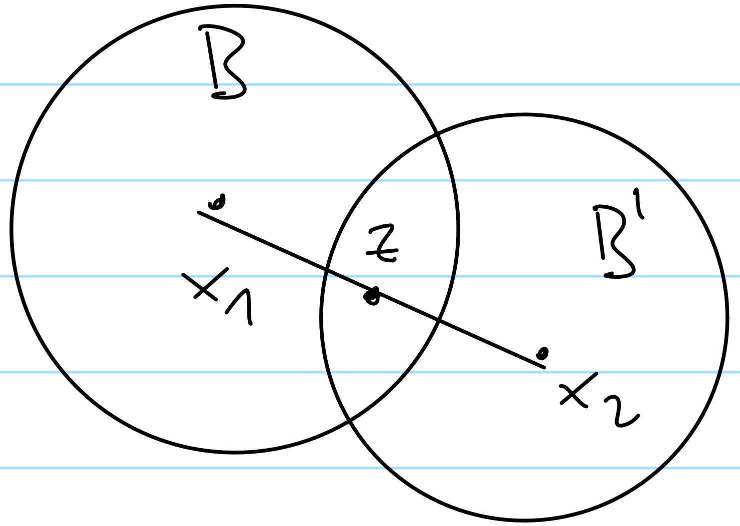
$r_1 \leq 2 r_2$

$$\boxed{B(x_1, r_1) \subseteq B(x_2, 2r_2)}$$

$y \in B(x_1, r_1)$

$|y - x_2| \le |y - x_1| + |x_1 - x_2|$

$\le r_1 + |x_1 - x_2|$ (*)



(*) $\le r_1 + |x_1 - z| + |z - x_2|$

$\le r_1 + r_1 + r_2$

$= 2r_1 + r_2 \le 4r_2 + r_2 = 5r_2$ $\square$

Remark

1) $\mathcal{G}_1 \neq \emptyset$ because

$d_0 = \sup\{\operatorname{diam}(B) : B \in \mathcal{F}\} > 0$

$\mathcal{F} = \bigcup_{J=1}^{\infty} \mathcal{F}_J$

2) If $B \cap B' = \emptyset$

$\forall B' \in \bigcup_{k=1}^{J-1} \mathcal{G}_k \Rightarrow B \cap \mathcal{G}_J \neq \emptyset$

$\Rightarrow$ either $B \in \mathcal{G}_J$ or

$\exists B' \in \mathcal{G}_j : B \cap B' \neq \phi$

3) $\mathcal{G} = \bigcup_{k=1}^{\infty} \mathcal{G}_k$ is a sub-collection of $\mathcal{F}$ made by disjoint balls

Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ we have that $\mathcal{G}$ is countable

Actually the following property holds: any family of pairwise disjoint (nonempty) open sets is countable (Each open set in the family has nonempty intersection with $\mathbb{Q}^n$, no

Two distinct sets in the family can contain the same point of $\mathbb{R}^n$)

Proof of Prop 5.1.6

Let $\alpha > 0$. Set

$A = \{x : f^*(x) > \alpha\}$

For $x \in A$ let us choose $r = r(x) > 0$.

$$\int_{B(x,r)} |f(y)| \, d\mathcal{L}^n \geq \alpha |B(x,r)|$$

If we set $\omega_n = |B(0,1)|$

$$\alpha \underbrace{\omega_n r^n}_{|B(x,r)|} \leq \int_{B(x,r)} |f| \, d\mathcal{L}^n \leq \|f\|_{L^1}$$

$$r = r(x) \leq \left(\frac{\|f\|_{L^1}}{\alpha \omega_n} \right)^{\frac{1}{n}} = r_0$$

$$\mathcal{F} = \{ B(x, r(x)) : x \in A \}$$

$$A \subseteq \bigcup_{B \in \mathcal{F}} B$$

$\Rightarrow$ By Vitaly Covering Theorem:

$$A \subseteq \bigcup_{B \in \mathcal{F}} B \subseteq \bigcup_{B \in \mathcal{G}} \hat{B}$$

$$|A| \leq \sum_{B \in \mathcal{F}} |B| \leq \sum_{B \in \mathcal{G}} |\hat{B}|$$

$$= \sum_{B \in \mathcal{G}} 5^n |B| \leq$$

$$\underbrace{(|B(x, 5r)| = 5^n |B(x,r)|)}$$

$$\leq \frac{5^n}{\lambda} \sum_{B \in \mathcal{G}} \int_B |f| \, d\mathcal{L}^n =$$

$$= \frac{5^m}{\lambda} \int_{\bigcup_{B \in \mathcal{G}} B} |f| \, d\mathcal{L}^m \le \frac{5^m}{\lambda} \|f\|_{L^1(\mathbb{R}^m)}$$

$\square$

Definition (Def 5.1.3 in Struwe Notes)

A Radon measure μ on $\mathbb{R}^m$ satisfies the **doubling property**

if there is $c > 0$ such that

$$\underbrace{\mu(B(x, 2r)) \le c\,\mu(B(x,r))}_{\forall x \in \mathbb{R}^m,\, r > 0}$$

Remark The proof of Prop 5.1.6 and Thm 5.1.8 hold with no big changes in

The case of doubling measures.

$\square$