LECTURE 3.04.2020

CHAPTER 3: INTEGRATION

μ RADON MEASURE on $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ μ-meas

$f : \Omega \to [-\infty, +\infty] = \overline{\mathbb{R}}$

UPPER INTEGRAL

$$\overline{\int_\Omega} f \, d\mu = \inf \left\{ \int_\Omega g \, d\mu, \ g \text{ is } \mu\text{-integrable, simple funct } g \geq f \ \mu\text{-a.e.} \right\}$$

LOWER INTEGRAL

$$\underline{\int_\Omega} f \, d\mu = \sup \left\{ \int_\Omega h \, d\mu, \ h \text{ is } \mu\text{-integrable, simple function, } h \leq f \ \mu\text{-a.e.} \right\}$$

If f is μ-measurable and

$$\overline{\int_\Omega} f \, d\mu = \underline{\int_\Omega} f \, d\mu$$

$\Rightarrow$ f is μ-integrable

$$\int_\Omega f \, d\mu = \overline{\int_\Omega} f \, d\mu = \underline{\int_\Omega} f \, d\mu$$

CLAIM : $\underline{\int_\Omega} f\, d\mu \le \overline{\int_\Omega} f\, d\mu$

$\Uparrow$

Proof of CLAIM

$g \le f \le h \quad \mu.a.e$

g, h simple, μ-int. functions

$\int_\Omega g\, d\mu \le \int_\Omega h\, d\mu \leftarrow$

$\Downarrow$

$\Rightarrow \sup_g \int g\, d\mu \le \inf_h \int h\, d\mu$

$\uparrow$

$g(y) = \sum_{i=1}^{\infty} a_i \chi_{A_i}(y) \leftarrow$

A_i are μ-meas

$A_i \cap A_J = \emptyset$ if $i \ne J$

$\boxed{\bigcup_{i=1}^{\infty} A_i = \Omega} \leftarrow$

$a_i \in \overline{\mathbb{R}}$

$h(y) = \sum_{J=1}^{\infty} b_J \chi_{B_J}(y)$

B_J μ-meas, $B_J \cap B_K = \emptyset$

$J \ne K$ $\quad \boxed{\bigcup_{i=1}^{\infty} B_J = \Omega} \leftarrow$

$$A_i = \bigcup_{j=1}^{\infty} B_j \cap A_i \, ; \quad B_j = \bigcup_{i=1}^{\infty} B_j \cap A_i$$

$$g(y) = \sum_{i,j=1}^{\infty} a_i \, \chi_{A_i \cap B_j}(y)$$

$$h(y) = \sum_{i,j=1}^{\infty} b_j \, \chi_{A_i \cap B_j}(y)$$

If $g \leq h \Rightarrow a_i \leq b_j$ whenever $\underline{A_i \cap B_j} \neq \emptyset$

Therefore

$$\int_\Omega g \, d\mu = \sum_{i,j=1}^{\infty} a_i \mu(A_i \cap B_j)$$

$$\leq \sum_{i,j} b_j \, \mu(A_i \cap B_j)$$

$$= \int_\Omega h \, d\mu . \qquad \square$$

Prop 3.1.6

$f : \Omega \to [0, +\infty]$ μ-meas

$\Rightarrow f$ is μ-integrable.

Proof We can assume

w.l.o.g $\int_\Omega f \, d\mu < +\infty$ (*)

[If $\underline{\int_\Omega} f \, d\mu = +\infty \Rightarrow$

$\overline{\int_\Omega} f \, d\mu = +\infty \Rightarrow$

f is μ - integrable]

If ⊛ holds

$\Rightarrow f(x) < +\infty$ μ e e

(EXERCISE)

$C = \{x \in \Omega : f(x) = +\infty\}$ $\mu(C) = 0$

- <u>CASE 1</u> $\mu(\Omega) < +\infty$

For $\varepsilon > 0$ we set

$A_k = \{x \in \Omega : k\varepsilon \leq f(x) < (k+1)\varepsilon\}$

$\boxed{k \geq 0}$

<u>Define</u>:

$$e(x) = \varepsilon \sum_{0}^{\infty} k \chi_{A_k}(x)$$

$$g(x) = \varepsilon \sum_{0}^{\infty} (k+1) \chi_{A_k}(x)$$

$\bigcup_{k=0}^{\infty} A_k \subseteq \Omega$, $A_{k_1} \cap A_{k_2} = \emptyset$

$\Omega = \bigcup_{k=0}^{\infty} A_k \cup C$ $\quad k_1 \neq k_2$

$$e(x) \leq f(x) \leq g(x) \quad \mu\text{.a.e. } x \in \Omega$$

$$\int_\Omega e(x)\,d\mu \leq \underline{\int_\Omega} f(x)\,d\mu$$

$$\leq \overline{\int_\Omega} f(x)\,d\mu$$

$$\leq \int_\Omega g(x)\,d\mu$$

$$= \int_\Omega e(x)\,d\mu +$$

$$\varepsilon \sum_{k=0}^{\infty} k\,\mu(A_k)$$

$$+ \ \varepsilon \underbrace{\int_\Omega \sum_{k=0}^{\infty} 1\, \chi_{A_k}(x)}$$

$$= \int_\Omega e(x)\,d\mu + \varepsilon \sum_{k=0}^{\infty} \mu(A_k)$$

$$= \int_\Omega e(x)\,d\mu + \varepsilon\, \mu(\underbrace{\cup A_k}_{\subseteq\, \Omega})$$

$$\leq \int_\Omega e(x) + \varepsilon\, \mu(\Omega)$$

$$\int_\Omega e(x)\,d\mu \leq \underline{\int_\Omega} f\,d\mu \leq \overline{\int} f\,d\mu \leq$$

$$\leq \int_\Omega e(x)\,d\mu + \varepsilon\mu(\Omega)$$

Let $\varepsilon \to 0$

$$\Rightarrow \int_{\Omega} f \, d\mu = \overline{\int_{\Omega}} f \, d\mu$$

<u>CASE 2</u> $\mu(\Omega) = +\infty$

$\mathbb{R}^n = \bigcup_{l=1}^{\infty} Q_l$ $\quad Q_l$ dyadic cubes

$\mu(Q_l) < 1 \quad \forall l$

$$\Rightarrow \Omega = \bigcup_{l=1}^{\infty} \underbrace{Q_l \cap \Omega}_{} =$$

$$= \bigcup_{l=1}^{\infty} \Omega_l \qquad \mu(\Omega_l) < +\infty$$

$\forall \varepsilon > 0 \quad \exists \, e_l, g_l : \Omega_l \to \overline{\mathbb{R}}$

$$e_l \le f \le g_l \quad \mu\text{-a.e. in } \Omega_l$$

$$\int_{\Omega} e_l \, d\mu \le \int_{\Omega} g_l \, d\mu \le \int_{\Omega} e_l \, d\mu + \frac{\varepsilon}{2^l} \mu(Q_l)$$

$$\le \int_{\Omega} e_l \, d\mu + \frac{\varepsilon}{2^l}$$

<u>IDEA</u> to sum up over l

$$e(x)=\sum_{l=1}^{\infty} e_l(x)\chi_{\Omega_l}(x), \quad g(x)=\sum_{l=1}^{\infty} g_l(x)\chi_{\Omega_l}(x)$$

are simple μ-integrable functions.

$$\underbrace{\sum_{l=1}^{\infty} e_l\chi_{\Omega_l}(x)}_{e(x)} \le f(x) \le \underbrace{\sum_{l=1}^{\infty} g_l\chi_{A_l}}_{g(x)} \quad \mu\text{.a.e } \Omega$$

$$\int_\Omega e(x)\,d\mu = \sum_{l=1}^{\infty}\int_{\Omega_l} e_l\,d\mu$$

$$\int_\Omega f(x)\,d\mu = \sum_{l=1}^{\infty}\int_{\Omega_l} g_l\,d\mu$$

$$\le \sum_{l=1}^{\infty}\left(\int_{\Omega_l} e_l\,d\mu + \frac{\varepsilon}{2^l}\right)$$

$$= \sum_{l=1}^{\infty}\int_{\Omega_l} e_l\,d\mu + \sum_{l=1}^{\infty}\frac{\varepsilon}{2^l}$$

$$= \int_\Omega e(x)\,d\mu + \varepsilon$$

$$\sum_{1}^{\infty}\frac{1}{2^l} = 1 \,!$$

$$\int_\Omega e(x)\,d\mu \le \underline{\int_\Omega} f\,d\mu$$

$$\le \overline{\int_\Omega} f\,d\mu \le \int_\Omega g(x)\,d\mu \le$$

$$\le \int_\Omega e(x)\,d\mu + \varepsilon$$

Let $\varepsilon \to 0$ and get

$$\underline{\int_\Omega} f\,d\mu = \overline{\int_\Omega} f\,d\mu \quad \square$$

Def 3.1.7

i) $f: \Omega \to \overline{\mathbb{R}}$ is μ-summable if it is μ-measurable and

$$\int_\Omega |f|\,d\mu < +\infty$$

ii) $f: \Omega \to \overline{\mathbb{R}}$ is locally μ-summable if $K \subseteq \Omega$ compact, $f|_K$ is μ-summ

$$\int_{\Omega \cap K} |f|\,d\mu < +\infty$$

Prop 3.1.8

i) f μ-summable $\Rightarrow$ f is μ-int.

ii) $f \equiv 0$ μ.a.e $\Rightarrow$ f is μ-integrable and $\int_\Omega f \, d\mu = 0$

Proof

i) $f = f_+ - f_-$

f_+, f_- are μ-integrable

$$0 \le \int_\Omega f_+ \, d\mu \underset{1}{\le} \int_\Omega |f| \, d\mu < +\infty$$

$$0 \le \int_\Omega f_- \, d\mu \le \int_\Omega |f| \, d\mu < +\infty$$

1. $g \le f_+ \Rightarrow g \le |f|$ μ e.e

$f_+ \le |f|$

$$A_+ = \{ g : \ g \le f_+ \}$$
$$\subseteq \{ g : \ g \le |f| \} =: \bar{A}.$$

$$\sup_{A_+} \int g \, d\mu \le \sup_{\bar{A}} \int g \, d\mu$$

$$\sup_{A_+} \int g \, d\mu = \underline{\int_\Omega} f_+ \, d\mu = \int f \, d\mu$$

$$\sup_{\bar{A}} \int g \, d\mu = \underline{\int_\Omega} |f| \, d\mu = \int_\Omega |f| \, d\mu$$

$\forall \varepsilon > 0 \quad 0 \le e_\pm \le f_\pm \le g_\pm$

a) $\begin{cases} e_+ \le f_+ \le g_+ \\ e_- \le f_- \le g_- \end{cases}$

b) $\int e_\pm \, d\mu \le \int f_\pm \, d\mu \le \int_\Omega g_\pm$

$\le \int_\Omega e_\pm \, d\mu + \varepsilon$

(see proof of Prop 3.1.6)

$e = e_+ - g_- \qquad g = g_+ - e_-$

From a),

$\Rightarrow e \le f \le g \quad \mu.a.e$

From b) and def of $\underline{\int_\Omega} f$, $\overline{\int_\Omega} f$

$\int e \, d\mu \le \underline{\int_\Omega} f \, d\mu \le \overline{\int_\Omega} f \, d\mu$

$\le \int_\Omega g \, d\mu \le \int_\Omega e \, d\mu + 2\varepsilon$

$g = g_+ - e_-$

$\int_\Omega g_+ \, d\mu \le \int_\Omega e_+ \, d\mu + \varepsilon$

$\int_\Omega g_- \, d\mu \le \int_\Omega e_- \, d\mu + \varepsilon$

$\Rightarrow -\int_\Omega e_- \le -\int_\Omega g_- + \varepsilon$

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$$\int g\, d\mu = \int (g_+ - e_-)\, d\mu$$

$$\leq \int e_+\, d\mu + \varepsilon - \int g_-\, d\mu + \varepsilon$$

$$= \int (e_+ - g_-) + 2\varepsilon$$

Let $\varepsilon \to 0$ and you

$$\underline{\int_\Omega} f\, d\mu = \overline{\int_\Omega} f\, d\mu .$$

ii) It is trivial : chose

$e = g = 0$: if $f \equiv 0$ μ.a.e

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- $e \equiv 0 \leq f$ μ.a.e

$$\Rightarrow \sup\left\{\int h\, d\mu \quad h \underset{\mu.a.e}{\leq} f\right\} \geq 0$$

$$\Rightarrow \underline{\int_\Omega} f\, d\mu \geq 0 \quad (*)$$

- $g \equiv 0 \geq f$ μ a.e

$$\Rightarrow \inf\left\{\int h\, d\mu : h \geq f\right\} \leq 0$$

$$\Rightarrow \overline{\int_\Omega} f\, d\mu \leq 0 \quad (**)$$

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From above

$$\overline{\int_\Omega} f \, d\mu = \underline{\int_\Omega} f \, d\mu$$

$\Rightarrow f$ is μ-integrable

$\square$

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Prop 3.1.9

$f : \Omega \to [0, +\infty]$ μ-meas

i) $\int_\Omega f \, d\mu = 0 \Rightarrow f = 0$ μ-a.e

ii) $\int_\Omega f \, d\mu < +\infty \Rightarrow f(x) < +\infty$ μ-a.e

Proof

i) $A = \{x, \ f(x) > 0\}$

$\mu(A) > 0$

$$A = \bigcup_{k=1}^{\infty} A_k \leftarrow$$

$$A_k = \left\{ x \in \Omega : \ f(x) \geq \frac{1}{k} \right\}$$

$k \geq 1$

$$A_{k+1} \supseteq A_k$$

$$\Rightarrow \mu(A) = \lim_{k \to +\infty} \mu(A_k) > 0$$

$\exists \bar{k} > 0 \ : \quad \mu(A_k) > 0$

$\forall k \geq \bar{k}$

Fix $k > \bar{k}$

$$s(x) = \frac{1}{k} \chi_{A_k}(x)$$

$$s(x) \leq f(x) \quad \mu\text{.e.e}$$

$$\int_\Omega s(x)\,d\mu = \frac{1}{k}\mu(A_k) > 0$$

$$\Rightarrow 0 = \int_\Omega f(x)\,d\mu \geq \int_\Omega s(x)\,d\mu$$

$\downarrow$

<u>hypothesis</u> $= \frac{1}{k}\mu(A_k) > 0$

$\Downarrow$

$$\Rightarrow \mu(A) = 0 \Rightarrow f = 0 \ \mu\text{.e.e}$$

ii) $\{x : f(x) = +\infty\} = B$

$\mu(B) > 0$

$s_n(x) = n \chi_B(x) \quad \forall n \geq 1$

$0 \leq s_n(x) \leq f(x) \quad \forall n \geq 0$

$$\Rightarrow \int_\Omega s_n(x)\, d\mu = n\,\mu(B) \leq \int_\Omega f\, d\mu < +\infty$$

Let $n \to +\infty$ and get a contradiction

($\int_\Omega f\, d\mu < +\infty \Rightarrow f(x) < +\infty$ μ.a.e)

Prop 1.1.10

$f_1, f_2 : \Omega \to \overline{\mathbb{R}}$ μ-integr.

$f_1 \geq f_2$ μ.a.e $\Rightarrow$

$$\int_\Omega f_1\, d\mu \geq \int_\Omega f_2\, d\mu$$

Proof

Let g be simple funct μ integrable:

$g \geq f_1 \ \mu$.a.e $\Rightarrow g \geq f_2 \ \mu$.a.e

$$\int_\Omega f_1 \, d\mu = \overline{\int_\Omega} f_1 \, d\mu = \inf_{g \geq f_1} \int g \, d\mu$$

$$\geq \inf_{g \geq f_2} \int g \, d\mu$$

$$= \overline{\int_\Omega} f_2 \, d\mu = \int_\Omega f_2 \, d\mu \qquad \square$$

Cor 3.1.11

f_1, f_2 μ-integrable

$f_1 = f_2 \ \mu$.a.e $\Rightarrow \int_\Omega f_1 \, d\mu = \int_\Omega f_2 \, d\mu$

Thm 3.1.12 (Tchebychev Inequality)

$f : \Omega \to \overline{\mathbb{R}}$ μ-summable

$\forall a > 0$

$$\mu\{x \in \Omega : |f(x)| \geq a\} \leq \frac{1}{a} \int_\Omega |f| \, d\mu$$

Proof

$f_1 = |f| \qquad f_2 = a \chi_A$

$A = \{x \in \Omega : |f(x)| \geq a\}$

$f_1 \geq f_2 \Rightarrow \int f_1 \, d\mu \geq \int f_2 \, d\mu$

$= a \mu(A)$

$\Rightarrow \mu(A) \leq \frac{1}{a} \int_\Omega |f| \, d\mu$ $\square$

COROLLARY

$f_k, f : \Omega \to \overline{\mathbb{R}}$ integrable

$$\int_\Omega |f_k - f| \, d\mu \to 0$$

$\Rightarrow f_k \to f$ in measure

$\Rightarrow \exists f_{k_n} \to f$ μ.a.e.

<u>Proof</u>

$\forall \varepsilon > 0$

$$\mu\{x \in \Omega : |f_n - f| > \varepsilon\} \leq \frac{1}{\varepsilon}\int_\Omega |f_n - f| \, d\mu \to 0$$

as $n \to +\infty$

$$\Rightarrow \mu\{x \in \Omega . |f_n - f| > \varepsilon\} \to 0$$

as $n \to +\infty$