LECTURE 8.04.2020

$f : \Omega \to \overline{\mathbb{R}}$ μ-mea.

- f is μ-integreble if

$$\int_{\underline{\Omega}} f\, d\mu = \int_{\Omega} f\, d\mu = \int_{\Omega} f\, d\mu$$

$\boxed{\text{EX}}$: f is μ-integreble iff either $\int_\Omega f^+ d\mu < +\infty$ or $\int_\Omega f^- d\mu < +\infty$

$$\int_\Omega f\, d\mu = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu$$

- f is μ-summeble if $\int_\Omega |f|\, d\mu < +\infty$

THEOREM 3.1.15

$f, g : \Omega \to \overline{\mathbb{R}}$ μ-SUMMABLE

$\lambda \in \mathbb{R}$

1) $f+g$ is μ-SUMMABLE

$$\int_\Omega f+g\, d\mu = \int_\Omega f\, d\mu + \int_\Omega g\, d\mu$$

2) λf is μ - SUMMABLE

$$\int_\Omega \lambda f \, d\mu = \lambda \int_\Omega f \, d\mu$$

Proof

1st step: show 1) & 2)

for simple functions

(μ - summable)

$$f(x) = \sum_{k=1}^{\infty} a_k \chi_{A_k}(x)$$

$$g(x) = \sum_{j=1}^{\infty} b_j \chi_{B_j}(x)$$

- $A_{k_1} \cap A_{k_2} = \emptyset \quad k_1 \neq k_2$

$$\bigcup_{k=1}^{\infty} A_k = \Omega$$

- $B_{j_1} \cap B_{j_2} = \emptyset \quad j_1 \neq j_2$

$$\bigcup_{j=1}^{\infty} B_j = \Omega$$

W.l.o.g: $(A_k)_k = (B_J)$

Otherwise:

$$f(x) = \sum_{J,k} a_k \chi_{A_k \cap B_J}(x)$$

$$g(x) = \sum_{J,k} b_J \chi_{A_k \cap B_J}(x)$$

1st PROPERTY

By assumption we have:

$$\int_\Omega |f|\, d\mu \overset{①}{=} \sum_k |a_k| \mu(A_k) < +\infty$$

$$\int_\Omega |g|\, d\mu \overset{②}{=} \sum_k |b_k| \mu(A_k) < +\infty$$

$$f+g = \sum_{k=1}^{\infty} (a_k + b_k) \chi_{A_k}(x)$$

$$\int_\Omega f+g \, d\mu \text{ "="} \sum_{k=1}^{\infty} (a_k+b_k)\mu(A_k) = ⊛$$

① + ② $\Rightarrow$ $\left|\sum_{k=1}^{\infty} a_k \mu(A_k)\right| < +\infty$

$\left|\sum_{k=1}^{\infty} b_k \mu(A_k)\right| < +\infty$

$$\circledast = \sum_{k=1}^{\infty} a_k \mu(A_k) + \sum_{k=1}^{\infty} b_k \mu(A_k)$$

$$< +\infty$$

$$\Rightarrow f+g \ \mu \text{ integrable}$$

$$\int_\Omega (f+g)\,d\mu = \int_\Omega f\,d\mu + \int_\Omega g\,d\mu$$

$\boxed{\text{CLAIM 1}}$: $\int_\Omega |f+g|\,d\mu < +\infty$

$$\int_\Omega |f+g|\,d\mu = \sum_{k=1}^{\infty} |a_k+b_k|\,\mu(A_k)$$

$$\leq \sum_{k=1}^{\infty} \left[|a_k|\,\mu(A_k) + |b_k|\,\mu(A_k) \right]$$

$$= \sum_{k}^{\infty} |a_k|\,\mu(A_k) + \sum_{k=1}^{\infty} |b_k|\,\mu(A_k)$$

$$< +\infty$$

$\boxed{\text{REMARK}}$

The property 1) holds for simple μ-meas. functions under weaker assumptions.

For instance:

if $\int_\Omega f^+ \, d\mu < +\infty$ and $\int_\Omega g^+ \, d\mu < +\infty$ or

$\int_\Omega f^- \, d\mu < +\infty$ and $\int_\Omega g^- \, d\mu < +\infty$

$\Rightarrow f+g$ is μ - integrable

$$\int_\Omega f+g \, d\mu = \int_\Omega f \, d\mu + \int_\Omega g \, d\mu$$

(in general $f+g$ will not be μ-summ)

$\boxed{2^{nd} \text{ property}}$

$$f(x) = \sum_{k=1}^{\infty} a_k \chi_{A_k}(x)$$

$$\lambda f(x) = \sum_{k=1}^{\infty} \lambda a_k \chi_{A_k}(x)$$

$$\int_\Omega \lambda f(x) \, d\mu = \sum_{k=1}^{\infty} \lambda a_k \mu(A_k)$$

$$= \lambda \sum_{k=1}^{\infty} a_k \mu(A_k)$$

By assumption

$$\left|\sum a_n \mu(A_n)\right| < +\infty$$

$$\Rightarrow \int_\Omega \lambda f(x)\,d\mu \text{ exists}$$

and it is equal to

$\lambda \int_\Omega f(x)\,d\mu$. Moreover

$$\int_\Omega |\lambda f|\,d\mu = |\lambda| \sum_{k=1}^{\infty} |a_n| \mu(A_n) < +\infty$$

$\Rightarrow \lambda f$ is μ-SUMMABLE

2nd-step : f, g μ-summable

- By Thm 2, 2.5 $f+g$ is μ-measurable

$\boxed{\text{CLAIM 2}}$: $f+g$ is μ-integrable

Proof of CLAIM 2

Let $\varepsilon > 0$, choose $f_\varepsilon, g_\varepsilon, f^\varepsilon, g^\varepsilon$ simple μ-integrable functions such that

a) $f_\varepsilon \le f \le f^\varepsilon$ μ.a.e

b) $g_\varepsilon \le g \le g^\varepsilon$ μ.a.e

c) $\int_\Omega f^\varepsilon d\mu \le \int_\Omega f d\mu + \varepsilon$

d) $\int_\Omega f d\mu \le \int_\Omega f_\varepsilon d\mu + \varepsilon$

e) $\int_\Omega g^\varepsilon d\mu \le \int_\Omega g d\mu + \varepsilon$

f) $\int_\Omega g d\mu \le \int_\Omega g_\varepsilon d\mu + \varepsilon$

$\boxed{\text{NOTE}}$

$\underbrace{f_\varepsilon + g_\varepsilon}$, $\underbrace{f^\varepsilon + g^\varepsilon}$ are μ-meas simple functions:

$f_\varepsilon + g_\varepsilon \le f + g \le f^\varepsilon + g^\varepsilon$

Since $\int_\Omega |f| d\mu < +\infty$, $\underbrace{\int_\Omega |g| d\mu}_{<+\infty}$

$\Rightarrow \int_\Omega (f^\varepsilon)^- d\mu < +\infty$ &

$\int_\Omega (g^\varepsilon)^- d\mu < +\infty$

and $\int_\Omega (f_\varepsilon)^+ d\mu < +\infty$

& $\int_\Omega (g_\varepsilon)^+ d\mu < +\infty$

$\Rightarrow f_\varepsilon + g_\varepsilon$ and $f^\varepsilon + g^\varepsilon$

are μ-integrable: we actually have:

$$(f^\varepsilon + g^\varepsilon)^- \leq (f^\varepsilon)^- + (g^\varepsilon)^-$$

$$(f_\varepsilon + g_\varepsilon)^+ \leq f_\varepsilon^+ + g_\varepsilon^+$$

$$\int_\Omega (f^\varepsilon + g^\varepsilon) d\mu = \int_\Omega f^\varepsilon d\mu + \int_\Omega g^\varepsilon d\mu$$

$$\int_\Omega (f_\varepsilon + g_\varepsilon) d\mu = \int_\Omega f_\varepsilon d\mu + \int g_\varepsilon d\mu$$

The following estimates hold:

i) $\overline{\int_\Omega}(f+g)\,d\mu \leq \int_\Omega f^\varepsilon + g^\varepsilon\,d\mu$

$= \int_\Omega f^\varepsilon\,d\mu + \int_\Omega g^\varepsilon\,d\mu$

$\leq \int_\Omega f\,d\mu + \varepsilon + \int_\Omega g\,d\mu + \varepsilon$

$= \int_\Omega f\,d\mu + \int_\Omega g\,d\mu + 2\varepsilon$

ii) $\underline{\int_\Omega} f\,d\mu \geq \int_\Omega (f_\varepsilon + g_\varepsilon)\,d\mu$

$= \int_\Omega f_\varepsilon\,d\mu + \int_\Omega g_\varepsilon\,d\mu$

$\geq \int_\Omega f\,d\mu + \int_\Omega g\,d\mu - 2\varepsilon$

---

i) & ii) & letting $\varepsilon \to 0$

$$\int_\Omega (f+g)\,d\mu = \int_\Omega f\,d\mu + \int_\Omega g\,d\mu$$

$\square$

CLAIM 3 $f+g$ μ-SUMMABLE

$$|f+g| \le |f| + |g|$$

$$\int_\Omega |f+g|\,d\mu \le \int_\Omega \underbrace{(|f|+|g|)}\,d\mu = \int_\Omega |f|\,d\mu + \int_\Omega |g|\,d\mu < +\infty$$

$\Rightarrow$ $f+g$ is μ-summ $\square$

CLAIM 4 $\lambda \in \mathbb{R}$, λf is μ-integrable

We assume $\lambda > 0$

(exercise $\lambda = 0$, $\lambda < 0$)

Let $\varepsilon > 0$, choose

f_ε, f^ε simple μ-integrable functions such that

$$f_\varepsilon \le f \le f^\varepsilon \quad \mu\text{.a.e}$$

$$\int_\Omega f^\varepsilon \, d\mu \le \int_\Omega f + \varepsilon$$

$$\int_\Omega f \cdot d\mu \le \int_\Omega f_\varepsilon + \varepsilon$$

$$-1 f_\varepsilon \le -1 f \le -1 f^\varepsilon$$

$\boxed{\text{NOTE}}$ $-1 f_\varepsilon$, $-1 f^\varepsilon$ are μ-integrable simple functions. In particular they satisfy

$$\int_\Omega f_\varepsilon^+ \, d\mu \le \int_\Omega f^+ \, d\mu < +\infty$$

$$\int_\Omega (f^\varepsilon)^- \, d\mu \le \int_\Omega f^- \, d\mu < +\infty$$

It follows

$$\int_\Omega (-1 f_\varepsilon)^+ = -1 \int_\Omega f_\varepsilon^+ \le -1 \int_\Omega f^+ \, d\mu$$

$$\int (\lambda f^{\varepsilon})^{-} d\mu = \lambda \int (f^{\varepsilon})^{-} d\mu$$

$$\le \lambda \int_{\Omega} f \, d\mu < +\infty$$

$$\Rightarrow \int_{\Omega} \lambda f^{\varepsilon} d\mu = \lambda \int_{\Omega} f^{\varepsilon} d\mu$$

$$\int_{\Omega} \lambda f_{\varepsilon} \, d\mu = \lambda \int_{\Omega} f_{\varepsilon} \, d\mu$$

i) $\overline{\int_{\Omega}} \lambda f \, d\mu \le \int_{\Omega} \lambda f^{\varepsilon} d\mu$

$$= \lambda \int_{\Omega} f^{\varepsilon} d\mu \le \lambda \int_{\Omega} f \, d\mu + \lambda \varepsilon$$

ii) $\underline{\int_{\Omega}} \lambda f \, d\mu \ge \int \lambda f_{\varepsilon} \, d\mu$

$$= \lambda \int_{\Omega} f_{\varepsilon} \, d\mu \ge \lambda \int_{\Omega} f \, d\mu - \lambda \varepsilon$$

---

i) & ii) & $\varepsilon \to 0$

$$\int_{\Omega} \lambda f \, d\mu = \lambda \int_{\Omega} f \, d\mu$$

$\boxed{\text{CLAIM 5}}$ λf is μ-SUMM

$$\int_\Omega |\lambda f| \, d\mu = \int_\Omega |\lambda| \, |f| \, d\mu$$

$$= |\lambda| \int_\Omega |f| \, d\mu < +\infty$$

$\Rightarrow$ λf is μ-summable

COR 3.1.16

$f : \Omega \to \overline{\mathbb{R}}$ μ-SUMMABLE

$$\left| \int_\Omega f \, d\mu \right| \leq \int_\Omega |f| \, d\mu$$

Proof

$$-|f|(x) \leq f(x) \leq |f|(x)$$

$\Rightarrow$ By monotonicity

$$\underbrace{\int_\Omega -|f|(x) \, d\mu}_{\circledast} \leq \int_\Omega f(x) \, d\mu \leq \int_\Omega |f| \, d\mu \quad (**)$$

$\lambda = -1$

$$\circledast = -\int_\Omega |f| \, d\mu$$

From $(\ast\ast)$ and $\circledast$ we get the result $\square$

LEMMA 3.1.17

$f : \Omega \to \overline{\mathbb{R}}$ μ-SUMMABLE

$\Omega_1 \subseteq \Omega$ μ-meas

Then $f_1 = f|_{\Omega_1}$, $f \cdot \chi_{\Omega_1}$ are μ-SUMMABLE

$$\int_{\Omega_1} f_1 \, d\mu = \int_\Omega f \chi_{\Omega_1} \, d\mu$$

Proof the first NOTE

f_1 and $f \chi_{\Omega_1}$ are μ-meas functions

1st step : show the result for simple μ-summ functions

$$g(x) = \sum_{i=1}^{\infty} a_i \, \chi_{A_i}(x)$$

$$A_{i_1} \cap A_{i_2} = \emptyset \quad i_1 \neq i_2 \qquad \bigcup_{i=1}^{\infty} A_i = \Omega$$

• $g_1(x) = g|_{\Omega_1} = \sum_{i=1}^{\infty} a_i \, \chi_{A_i \cap \Omega_1}$

$$\int_{\Omega_1} g_1 \, d\mu = \sum_{i=1}^{\infty} a_i \cdot \mu(A_i \cap \Omega_1)$$

g_1 is μ-summable because $\sum_{i=1}^{\infty} |a_i| \, \mu(A_i) < +\infty$

• $g \chi_{\Omega_1} = \sum_{i=1}^{\infty} a_i \underbrace{\chi_{A_i} \chi_{\Omega_1}}_{= \chi_{\Omega_1 \cap A_i}}$

$$\int_{\Omega} g \chi_{\Omega_1} \, d\mu = \sum_{i=1}^{\infty} a_i \, \mu(A_i \cap \Omega_1)$$

2^{nd} step

Let $\varepsilon > 0$ and choose

$$g \leq f \leq h \quad \mu\text{.a.e}$$

g, h μ-integrable simple functions:

$$\int_\Omega f \, d\mu \leq \int_\Omega g \, d\mu + \varepsilon$$

$$\int_\Omega h \, d\mu \leq \int_\Omega f \, d\mu + \varepsilon$$

CLAIM 1

f_1 is μ-integrable

Proof of CLAIM 1

$$0 \leq \overline{\int_\Omega} f_1 \, d\mu - \underline{\int_\Omega} f_1 \, d\mu$$

$$\leq \int_{\Omega_1} h_1 \, d\mu - \int_{\Omega_1} g_1 \, d\mu = \boxtimes$$

$$(h_1 = h|_{\Omega_1}, \; g_1 = g|_{\Omega_1})$$

$$\boxed{X} = \int_\Omega (h_1 - g_1)\, d\mu$$

$$= \int_\Omega \underbrace{(h-g)}_{\geq 0} \underbrace{\chi_{\Omega_1}}_{\leq 1}\, d\mu$$

$$\leq \int_\Omega (h-g)\, d\mu$$

$$= \int_\Omega h\, d\mu - \int_\Omega g\, d\mu$$

$$\leq 2\varepsilon$$

Letting $\varepsilon \to 0 \Rightarrow$

$\int_\Omega f_1\, d\mu$ is μ-integrable $\square$

$\boxed{\text{CLAIM 2}}$ $f\chi_{\Omega_1}$ is μ-integrable

$$0 \leq \overline{\int_\Omega} f\chi_{\Omega_1} - \underline{\int_\Omega} f\chi_{\Omega_1}\, d\mu$$

$$\leq \int_\Omega (h-g)\chi_{\Omega_1}\, d\mu$$

$$\leq \int_\Omega h \, d\mu - \int_\Omega g \, d\mu$$

$$\leq \cancel{\int_\Omega f \, d\mu} + \varepsilon - \cancel{\int_\Omega f \, d\mu} + \varepsilon = 2\varepsilon$$

Let $\varepsilon \to 0$ and get the result

$\boxed{\text{CLAIM 3}}$ $\displaystyle\int_\Omega f_1 \, d\mu = \int_\Omega f \chi_{\Omega_1} \, d\mu$

Proof of CLAIM 3

i) $\displaystyle\int f_1 \, d\mu - \int f \chi_{\Omega_1} \, d\mu$

$$\leq \int_\Omega h_1 \, d\mu - \int_\Omega g \chi_{\Omega_1} \, d\mu$$

$$= \int_\Omega h \chi_{\Omega_1} \, d\mu - \int_\Omega g \chi_{\Omega_1} \, d\mu$$

$$= \int_\Omega (h - g) \chi_{\Omega_1} \, d\mu \leq 2\varepsilon$$

ii)

$$\int_{\Omega_1} f_1 \, d\mu - \int_{\Omega} f \chi_{\Omega_1} \, d\mu$$

$$\geq \int_{\Omega} g_1 \, d\mu - \int_{\Omega} h \chi_{\Omega_1} \, d\mu$$

$$\geq \int_{\Omega} (g - h) \chi_{\Omega_1} \, d\mu$$

$$\geq \int_{\Omega} (g - h) \, d\mu = \int_{\Omega} g \, d\mu - \int_{\Omega} h \, d\mu$$

$$\geq \int_{\Omega} f \, d\mu - \varepsilon - \cancel{\int_{\Omega} f \, d\mu} - \varepsilon = -2\varepsilon$$

By combining i) and ii)

$$\left| \int_{\Omega_1} f_1 \, d\mu - \int_{\Omega} f \chi_{\Omega} \, d\mu \right| \leq 2\varepsilon$$

Let $\varepsilon \to 0$ and get the result. $\square$

Remark

In the proof of LEMMA 3.1.17 we use that

$$\int_\Omega (h \dot{-} g)\, d\mu = \int_\Omega h\, d\mu - \int_\Omega g\, d\mu$$

and

$$\int_\Omega (g \dot{-} h)\, d\mu = \int_\Omega g\, d\mu - \int_\Omega h\, d\mu$$

Actually we have

$$\int_\Omega f\, d\mu \leq \int_\Omega h\, d\mu \leq \int_\Omega f + \varepsilon$$

$$\int_\Omega f\, d\mu - \varepsilon \leq \int_\Omega g\, d\mu \leq \int f$$

Since f is μ-summable we get $\left|\int h\, d\mu\right| < +\infty$ and $\left|\int g\, d\mu\right| < +\infty$

(in particular $h - g$ and $g - h$ are both

μ-integrable and
the linearity holds)