Lecture Analysis 3, 5.10.2020

$$\begin{cases} f''(t) + f(t) = \cos(2t) & t > 0 \\ f(0) = 0 \\ f'(0) = 1 \end{cases}$$

$$\mathcal{L}\{f''\}(s) + \mathcal{L}\{f\}(s) = \frac{s}{s^2+4} \leftarrow$$

<u>Recall</u> : $\mathcal{L}\{f''\}(s) = -\underbrace{f'(0)}_{1} - s\underbrace{f(0)}_{=0} + s^2 \mathcal{L}\{f\}(s)$

$$F(s) = \mathcal{L}\{f\}(s)$$

$$-1 + s^2 F(s) + F(s) = \frac{s}{s^2+4}$$

$$\Rightarrow \quad F(s) = \frac{s}{(s^2+4)(s^2+1)} + \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\{F\} = \underbrace{\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s^2+1)}\right\}}_{1} + \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}}_{2}$$

- $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$

- $\frac{s}{(s^2+4)(s+1)} = \frac{As+B}{1+s^2} + \frac{Cs+D}{s^2+4}$

$$= \frac{(As+B)(s^2+4) + (Cs+D)(s^2+1)}{(1+s^2)(s^2+4)}$$

$$s = (A+C)s^3 + (B+D)s^2 + (4A+C)s + 4B + D$$

$$
\begin{array}{lcl}
s^3 & : & A + C = 0 \\
s^2 & : & B + D = 0 \\
s & : & 4A + C = 1 \\
s^0 & : & 4B + D = 0
\end{array}
\implies
\begin{array}{l}
A = -C = \frac{1}{3} \\
B = D = 0
\end{array}
$$

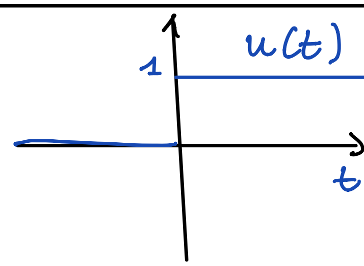
$$F(s) = \frac{1}{3}\,\frac{s}{s^2+1} - \frac{1}{3}\,\frac{s}{s^2+4} + \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\{F\}(t) = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t)$$

$$= \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t) + \sin(t). \quad \blacksquare$$

Heaviside function

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



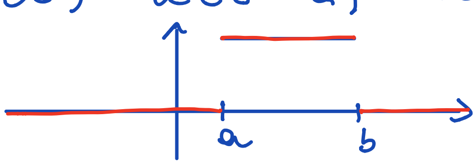
APPLICATIONS

i) Switch a signal on at time "$t = a$" ≥ 0

$$f(t) = u(t-a) = \begin{cases} 1 & t \geq a \\ 0 & \text{otherwise} \end{cases}$$

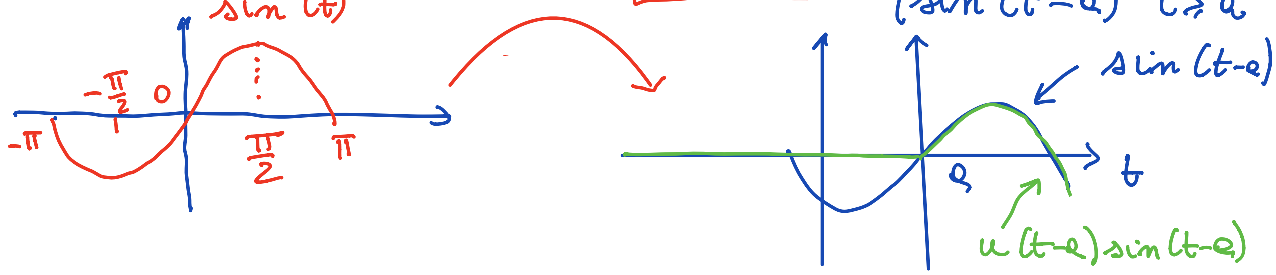
ii) Switch a signal on at "$t = a$" and Switch it off at "$t = b$" $b > a$

$$f(t) = u(t-a) - u(t-b) = \begin{cases} 0 & t < a \\ 1 & a \leq t < b \\ 0 & t > b \end{cases}$$



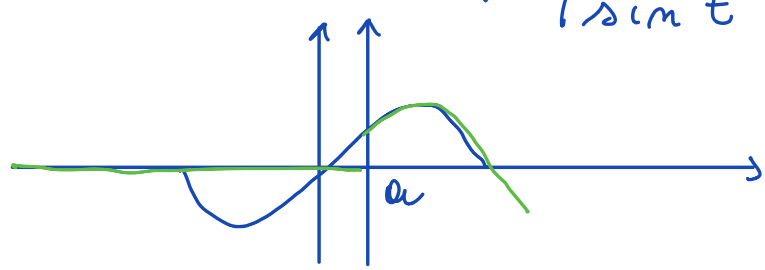
iii) Switch on a sine signal at "$t=a$"

$$f(t) = \sin(t-a)\,\underline{u(t-a)} = \begin{cases} 0 & t < a \\ \sin(t-a) & t \geq a \end{cases}$$



iv) CONNECT AN ALREADY RUNNING SINE SIGNAL AT TIME "$t=a$"

$$f(t) = \sin(t)\,u(t-a) = \begin{cases} 0 & t < a \\ \sin t & t \geq a \end{cases}$$



"t - shifting property"

$$(*) \quad \mathcal{L}\{f(t-a)\,u(t-a)\}(s) = e^{-as}\mathcal{L}\{f\}(s)$$

$$\Updownarrow$$

$$\mathcal{L}^{-1}\{e^{-as}\mathcal{L}\{f\}\}(t) = f(t-a)\,u(t-a)$$

REMARK

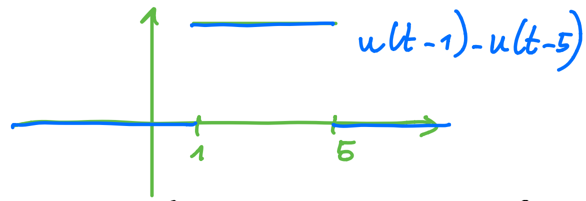
Take in $(*)$ $\quad f \equiv 1$

$$\mathcal{L}\{u(t-a)\}(s) = e^{-as}\mathcal{L}\{1\}(s)$$

$$= \frac{e^{-as}}{s}$$

EXERCISE

$$\begin{cases} f''(t) + f(t) = u(t-1) - u(t-5) \\ f(0) = 0 \\ f'(0) = 0 \end{cases}$$



$$\mathcal{L}\{f''\} + \mathcal{L}\{f\} = \mathcal{L}\{u(t-1)\} - \mathcal{L}\{u(t-5)\}$$

$$s^2\mathcal{L}\{f\} - s\underbrace{f(0)}_{=0} - \underbrace{f'(0)}_{=0} + \mathcal{L}\{f\} = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}$$

$$s^2 F(s) + 1 \cdot F(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}$$

$$F(s) = \frac{1}{(s^2+1)s} e^{-s} - \frac{e^{-5s}}{s(s^2+1)} = *$$

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} = \mathcal{L}\{1 - \cos(t)\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \quad , \quad \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$$

$$* \quad F(s) = e^{-s}\left[\mathcal{L}\{1 - \cos(t)\}\right] - e^{-5s}\left[\mathcal{L}\{1 - \cos t\}\right]$$

$$= \underbrace{\mathcal{L}\{u(t-1)(1-\cos(t-1))\}}_{\text{t-shifting property } t=1}$$

$$- \underbrace{\mathcal{L}\{u(t-5)(1-\cos(t-5))\}}_{\text{t-shifting property } t=5}$$

$$= \mathcal{L}\{\underbrace{u(t-1)(1-\cos(t-1)) - u(t-5)(1-\cos(t-5))}\}$$

$$\Rightarrow f(t) = u(t-1)(1-\cos(t-1)) - u(t-5)(1-\cos(t-5))$$

"t - shifting property"

$$\circledast \quad \mathcal{L}\{f(t-a)\,u(t-a)\}(s) = e^{-as}\,\mathcal{L}\{f\}(s)$$

$$\Updownarrow$$

$$\mathcal{L}^{-1}\{e^{-as}\,\mathcal{L}\{f\}\}(t) = f(t-a)\,u(t-a)$$

"Proof of $\circledast$"

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = \int_0^{+\infty} f(t-a)\underbrace{u(t-a)}_{\substack{u(t-a)=0 \\ \text{if } t<a}} e^{-st}\,dt$$

$$= \int_a^{+\infty} f(t-a)\cdot 1\; e^{-st}\,dt$$

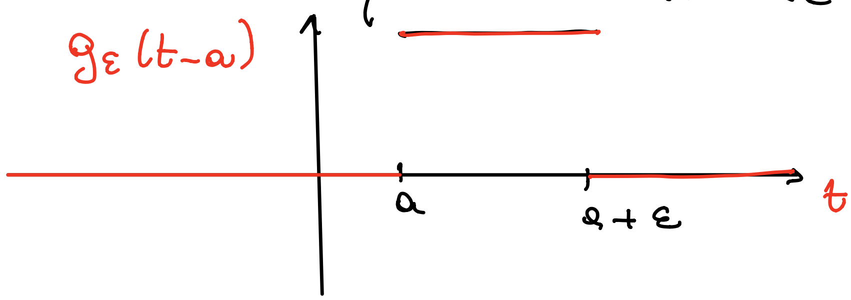
$$\underset{t'=t-a}{=} \int_0^{+\infty} f(t')\,e^{-s(t'+a)}\,dt' =$$

$$= e^{-sa}\int_0^{+\infty} f(t')\,e^{-st'}\,dt' = e^{-sa}\,\mathcal{L}\{f\}(s) \quad \square$$

HOW CAN WE REPRESENT IMPULSES IN MATHEMATICS? (Sect. 2.6 Iozzi's NOTES)

$\varepsilon > 0, \quad a \geq 0$

$$g_\varepsilon(t-a) = \begin{cases} \frac{1}{\varepsilon} & a \leq t \leq a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

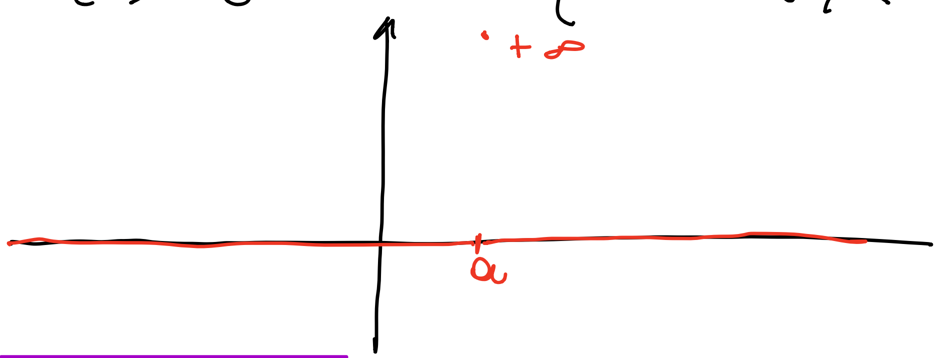


$$g_\varepsilon(t-a) = \frac{1}{\varepsilon} \quad u(t-a) - u(t-(a+\varepsilon))]$$

$$I_\varepsilon = \int_0^{+\infty} g_\varepsilon(t-a)\,dt = \frac{1}{\varepsilon}\int_a^{a+\varepsilon} 1\,dt = \frac{1}{\varepsilon}\cdot\varepsilon = 1$$

Let $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} g_\varepsilon(t-a) = \begin{cases} +\infty & t = a \\ 0 & t \neq a \end{cases}$$



DEFINITION We define the DELTA DIRAC FUNCTION CENTERED AT $t = a$

$$\delta(t-a) := \lim_{\varepsilon \to 0} g_\varepsilon(t-a)$$

NOTE : δ IS NOT AN ORDINARY FUNCTION

$$I_\varepsilon = \int_0^{+\infty} g_\varepsilon(t-a)\,dt = 1 \qquad \forall \varepsilon > 0$$

$$1 = \lim_{\varepsilon \to 0} I_\varepsilon = \lim_{\varepsilon \to 0} \int_0^{+\infty} g_\varepsilon(t-a)\,dt$$

$$= \int_0^{+\infty} \lim_{\varepsilon \to 0} g_\varepsilon(t-a)\,dt$$

$$= \int_0^{+\infty} \delta(t-a)\,dt$$

PROPERTIES of DIRAC DELTA FUNCTION

1. $\displaystyle\int_0^{+\infty} \delta(t-a)\,dt = 1$

SIFTING PROPERTY

2. $\displaystyle\int_0^{+\infty} \delta(t-a)\,g(t)\,dt = g(a)$

NOTE If $g \equiv 1$ you get 1

3. $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

3 can be deduce from 2 by choosing $g(t) = e^{-at}$

$$\mathcal{L}\{\delta(t-a)\} = \int_0^{+\infty} \delta(t-a)\underbrace{e^{-st}}_{g(t)}dt = e^{-sa}$$

EXAMPLE

$$\begin{cases} f''(t)+3f'(t)+2f(t) = \delta(t-1) \\ f(0) = f'(0) = 0 \end{cases}$$

Sol Apply LT:

$$s^2F(s) - \cancel{sf(0)} - \cancel{f'(0)} + 3sF(s) - \cancel{3f(0)} + 2F(s) = e^{-s}$$

$$(s^2+3s+2)F(s) = e^{-s}$$

$$\Rightarrow \quad F(s) = e^{-s} \cdot \underbrace{\frac{1}{(s^2+3s+2)}}$$

$$\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2} = \mathcal{L}\{e^{-t}\} - \mathcal{L}\{e^{-2t}\}$$
$$= \mathcal{L}\{e^{-t} - e^{-2t}\}$$

$$\Rightarrow F(s) = e^{-s}\mathcal{L}\{e^{-t} - e^{-2t}\}$$

t-shifting property $t=a$

$$= \mathcal{L}\{u(t-1)[e^{-(t-1)} - e^{-2(t-1)}]\}$$

$$\Rightarrow f(t) = u(t-1)[e^{-(t-1)} - e^{-2(t-1)}]$$

(EXERCISE: DRAW f)

CONVOLUTION (Section 2.7 Iozzi's NOTES)

$\mathcal{L}\{f \cdot g\} \neq \mathcal{L}\{f\}\mathcal{L}\{g\}$

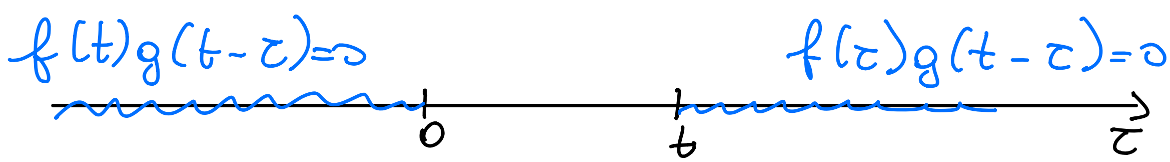
$$(f * g)(t) = \int_0^t \underbrace{f(\tau)\, g(t-\tau)}\, d\tau$$

$f * g$ IS CALLED CONVOLUTION OF f AND g

$$(f*g)(t) = \int_{-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau = \int_0^t f(\tau) g(t-\tau) d\tau$$

NOTE: $f \equiv 0$ if $\tau < 0$

$g(t-\tau) \equiv 0$ if $t - \tau < 0 \Leftrightarrow \tau > t$



Properties

1) $f * g = g * f$ COMMUTATIVE LAW

2) $f * (g + h) = f * g + f * h$ DISTRIBUTION LAW

3) $f * (g * h) = (f * g) * h$ ASSOCIATIVE LAW

4) $f * 0 = 0$

5) $f * 1 \neq f$ 1 is NOT THE NEUTRAL TERM OF CONVOLUTION

6) $f * f \not\geq 0$

7) $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$

Proofs next time !