Lecture Analysis 3, 9.11.2020

Last week (2.11.2020) :

In the lecture of 2.11.2020 we have seen the representation of a function $f: \mathbb{R} \to \mathbb{R}$ by a real Fourier integral.

Precisely if

- f is piecewise continuous in every bounded interval,
- f has right-hand derivative and left-hand derivative at any point.
- f is absolute integrable:

$$\int_{-\infty}^{+\infty} |f(x)|\, dx < +\infty$$

Then

1) $f(x) = \int_0^{+\infty} [A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)]\, d\omega$

if x is a point of continuity

2) $$\frac{f(x^+) + f(x^-)}{2} =$$

$$= \int_0^{+\infty} [A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)]\, d\omega$$

if x is a DISCONTINUITY POINT for f

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(y) \cos(\omega y)\, dy$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(y) \sin(\omega y)\, dy$$

AS A CONSEQUENCE OF THE ABOVE FORMULAS WE COMPUTED

$$\int_0^{+\infty} \frac{\sin y}{y}\, dy = \frac{\pi}{2}$$

PLAN

1. EXERCISE
2. COMPLEX FORM of FOURIER INTEGRAL
3. FOURIER TRANSFORM & INVERSE FOURIER TRANSFORM
4. EXAMPLES + PROPERTIES
5. CHAPTER 4: PDES (maybe)

EXERCISE

1) Compute the Fourier Integral of $f(x) = e^{-k|x|}$ $x \in \mathbb{R}$ and $k > 0$

2) $\int_0^{+\infty} \frac{\cos(\omega x)}{k^2+\omega^2}\, d\omega = e^{-kx}\frac{\pi}{2k}$ $\quad x>0$

<u>Solution</u>

Observe $f(x)$ is an even function

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-k|x|}\cos(\omega x)\,dx$$

$$= \frac{2}{\pi}\int_0^{+\infty} e^{-kx}\cos(\omega x)\,dx$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-k|x|}\sin(\omega x)\,dx = 0$$

$$\Rightarrow f(x) = \int_0^{+\infty} A(\omega)\cos(\omega x)\,dx \qquad \forall x\in\mathbb{R}$$

$$A(\omega) = \frac{2}{\pi}\int_0^{+\infty} e^{-kx}\cos(\omega x)\,dx$$

$$= \underbrace{\frac{2}{\pi}\frac{\sin(\omega x)}{\omega}e^{-kx}\Big|_0^{+\infty}}_{=0} + \frac{2}{\pi}\frac{1}{\omega}\int_0^{+\infty}\sin(\omega x)(+k)e^{-kx}\,dx$$

$$= \frac{2}{\pi}\frac{k}{\omega}\left(-\frac{\cos(\omega x)}{\omega}\right)e^{-kx}\Big|_0^{+\infty} + \frac{2k}{\pi\omega^2}\int_0^{+\infty}(+\cos(\omega x))(-k)e^{-kx}\,dx$$

$$= \frac{2k}{\pi\omega^2} - \frac{2k^2}{\pi\omega^2}\int_0^{+\infty}\cos(\omega x)\,e^{-kx}\,dx$$

$$= \frac{2k}{\pi w^2} - \frac{k^2}{w^2} A(w)$$

$$A(w) = \frac{2k}{\pi w^2} - \frac{k^2}{w^2} A(w)$$

$$A(w)\left(1 + \frac{k^2}{w^2}\right) = \frac{2k}{\pi w^2}$$

$$A(w) = \frac{2k}{\pi \cancel{w^2}} \cdot \frac{\cancel{w^2}}{w^2+k^2} = \frac{2k}{\pi(w^2+k^2)}$$

$$A(w) = \frac{2}{\cancel{\pi}} \int_0^{+\infty} e^{-kx} \cos(wx)\, dx = \frac{2}{\cancel{\pi}} \frac{k}{w^2+k^2}$$

$$\Rightarrow \int_0^{+\infty} e^{-kx} \cos(wx)\, dx = \frac{k}{w^2+k^2}$$

$$e^{-k|x|} = \int_0^{+\infty} A(w) \cos(wx)\, dw$$

$$= \int_0^{+\infty} \frac{2}{\pi} \frac{k}{w^2+k^2} \cos(wx)\, d\, w$$

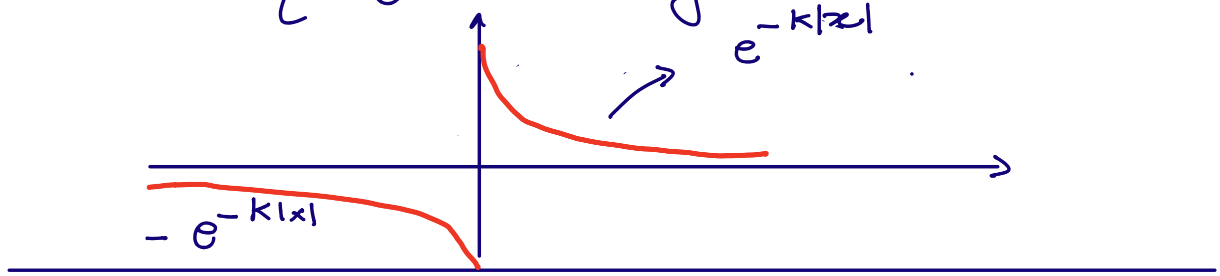
$\Rightarrow$ For every $x \in \mathbb{R}$

$$e^{-k|x|} \frac{\pi}{2k} = \int_0^{+\infty} \frac{\cos wx}{w^2+k^2}\, dw$$

$\boxed{\text{EXERCISE:}}$ $\displaystyle\int_0^{+\infty} \frac{w \sin(wx)}{w^2+k^2}\, dw$

HINT:

$$f(x) = \begin{cases} e^{-kx} & \text{if } x \geq 0 \\ -e^{+kx} & \text{if } x < 0 \end{cases}$$



Complex Fourier integral

$$f(x) = \int^{+\infty} [A(w)\cos(wx) + B(w)\sin(wx)]\,dw$$

$$= \frac{1}{\pi}\int_0^{+\infty} \left(\int_{-\infty}^{+\infty} f(y)\cos(wy)\,dy\right)\cos(wx) +$$

$$+ \left(\int_{-\infty}^{+\infty} f(y)\sin(wy)\,dy\right)\sin(wx)\,dw$$

$$= \frac{1}{\pi}\int_0^{+\infty}\left(\int_{-\infty}^{+\infty} f(y)\left[\cos(wy)\cos(wx) + \sin(wy)\sin(wx)\right] dy\right) dw \quad (*)$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$\alpha = wy$

$\beta = wx$

$$(*) = \frac{1}{\pi}\int_0^{+\infty}\left(\int_{-\infty}^{+\infty} f(y)\cos(wx - wy)\,dy\right) dw$$

$\boxed{\text{Remark}}$

$$\omega \longmapsto \underbrace{\int_{-\infty}^{+\infty} f(y)\cos(\omega x - \omega y)\,dy}_{= F(\omega)}$$

$$F(-\omega) = \int_{-\infty}^{+\infty} f(y)\cos(\underbrace{-\omega x + \omega y}_{-(\omega x - \omega y)})\,dy$$

$$= \int_{-\infty}^{+\infty} f(y)\cos(\omega x - \omega y) = F(\omega)$$

$$\Rightarrow \int_{-\infty}^{+\infty} F(\omega)\,d\omega = 2\int_{0}^{+\infty} F(\omega)\,d\omega$$

$$\int_{0}^{+\infty} F(\omega)\,d\omega = \frac{1}{2}\int_{-\infty}^{+\infty} F(\omega)\,d\omega$$

$$(*) = \frac{1}{\pi}\int_{0}^{+\infty}\left(\int_{-\infty}^{+\infty} f(y)\cos(\omega x - \omega y)\,dy\right)d\omega$$

$$= \frac{1}{\pi}\int_{0}^{+\infty} F(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\,d\omega$$

$$\omega \longmapsto G(\omega) = \int_{-\infty}^{+\infty} f(y)\sin(\omega x - \omega y)\,dy$$

G is ODD

$$\Rightarrow \int_{-\infty}^{+\infty} G(\omega)\,d\omega = 0$$

$$f(x) = \frac{1}{2\pi}\left(\int_{-\infty}^{+\infty} F(\omega)\,d\omega + i\int_{-\infty}^{+\infty} G(\omega)\,d\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(y)\underbrace{\left[\cos(\omega x - i\omega y) + i\sin(\omega x - i\omega y)\right]}_{e^{i[\omega x - \omega y]}}\,dy\right) d\omega$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(y)\, e^{i\omega x} e^{-i\omega y}\,dy\right) d\omega$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(y)\, e^{-i\omega y}\,dy\right) e^{i\omega x}\,d\omega$$

Periodic case

$$f(x) = \sum_{-\infty}^{+\infty} c_n e^{i\frac{n\pi}{L}x}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(y)\, e^{-i\frac{n\pi}{L}y}\,dy$$

Definition

$f: \mathbb{R} \to \mathbb{R}$ $\qquad \int_{-\infty}^{+\infty} |f(x)|\, dx < +\infty$

Fourier transform of f,

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i\omega x}\, dx$$

Inverse Fourier transform of f

$$\check{f}(x) = \mathcal{F}^{-1}(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\omega)\, e^{i\omega x}\, d\omega$$

NOTE Under the assumption that $\int_{-\infty}^{+\infty} |f(x)|\, dx < +\infty$, $\mathcal{F}(f)$, $\mathcal{F}^{-1}(f)$ are well-defined.

$$|\mathcal{F}(f)(\omega)| = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{+\infty} f(x)\, e^{-i\omega x}\, dx \right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |f(x)|\, \underbrace{|e^{-i\omega x}|}_{=1}\, dx$$

$\left(\cos^2(\omega x) + \sin^2(\omega x)\right)^{1/2} = 1$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |f(x)|\, dx < +\infty$$

$$f \longmapsto \mathcal{F}(f) \longrightarrow \mathcal{F}^{-1}(\mathcal{F}(f)) = f$$

Under the assumptions at the beginning you have

$$f(x) = \mathcal{F}^{-1}(\mathcal{F}(f))(x)$$

if x is a point of continuity of f

and

$$\frac{f(x^+)+f(x^-)}{2} = \mathcal{F}^{-1}(\mathcal{F}(f))(x)$$

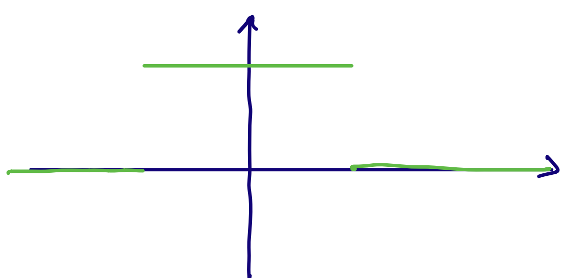
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{+\infty} f(w)e^{-iwy}\,dw\right) e^{iyx}\,dy$$

if x is a point of DISCONTINUITY of f.

EXAMPLE (EX 3.20 IOZZI'S NOTES)

Compute the Fourier Transform of

$$f(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$



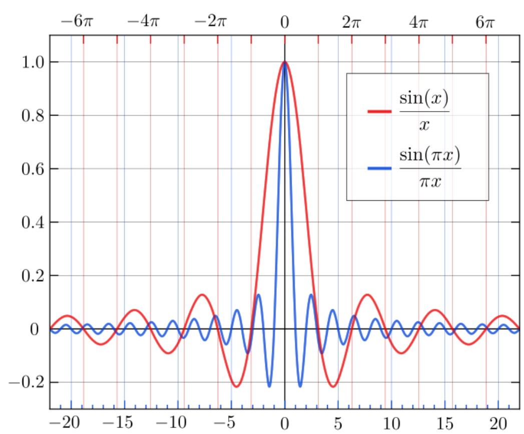
$f(x)$ is CALLED CHARACTERICS function of $[-1,1]$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)e^{-iwx}\,dx =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx}\,dx =$$

$$= \frac{1}{\sqrt{2\pi}} \, \frac{e^{-i\omega x}}{-i\omega} \Bigg|_{-1}^{1} = \frac{1}{\sqrt{2\pi}} \, 2 \left[\frac{e^{i\omega} - e^{-i\omega}}{2i\omega} \right]$$

$$= \sqrt{\frac{2}{\pi}} \, \frac{\sin \omega}{\omega} = \sqrt{\frac{2}{\pi}} \, \operatorname{sinc}(\omega)$$



$\forall \; x \neq -1, 1$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sqrt{\frac{2}{\pi}} \, \frac{\sin \omega}{\omega} \, e^{ix\omega} \, d\omega$$

$\boxed{x = +1}$

$$\underbrace{\frac{f(1^+) + f(1^-)}{2}}_{= \frac{1}{2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sqrt{\frac{2}{\pi}} \, \frac{\sin \omega}{\omega} \, e^{i\omega} \, d\omega$$

$\boxed{x = 0}$

$$1 = \frac{1}{\cancel{\sqrt{2}}\sqrt{\pi}} \sqrt{\frac{\cancel{2}}{\pi}} \int_{-\infty}^{+\infty} \frac{\sin \omega}{\omega} \, d\omega$$

$\Rightarrow$

$$\int_{-\infty}^{+\infty} \frac{\sin \omega}{\omega}\, d\omega = \pi$$

$$2\int_{0}^{+\infty} \frac{\sin \omega}{\omega}\, d\omega = \pi \Rightarrow \int_{0}^{+\infty} \frac{\sin \omega}{\omega}\, d\omega = \frac{\pi}{2}$$

$\boxed{\text{EXERCISE}}$

Compute the FT of $e^{-a|x|}$ $a>0$

$$\mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+\omega^2}$$

$$\Rightarrow e^{-a|x|} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+\omega^2} e^{i\omega x}\, d\omega$$

Physical Interpretation

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega.$$

Properties of Fourier Transform

1) $\boxed{\text{LINEARITY}}$

$$\mathcal{F}(\alpha f + \beta g) = \alpha \mathcal{F}(f) + \beta \mathcal{F}(g)$$

$\forall\, \alpha, \beta \in \mathbb{R}$

2) f is continuous, $\lim_{|x|\to+\infty} f(x) = 0$

$$\int_{-\infty}^{+\infty} |f'(x)|\, dx < +\infty$$

$$\mathcal{F}(f')(\omega) = (i\omega)\mathcal{F}(f)$$

$$\Rightarrow \mathcal{L}(f')(s) = s\,\mathcal{L}(f)(s) - f(0)$$

3) $\boxed{\text{TRANSLATION}}$

$$\tau_a f(x) = f(x-a) \qquad a \in \mathbb{R}$$

$$\mathcal{F}(\tau_a f)(\omega) = e^{-i\omega a}\mathcal{F}(f)(\omega)$$

4) $\boxed{\text{DILATION}}$

$$\delta_\lambda f(x) := f(\lambda x)$$

$$\mathcal{F}(\delta_\lambda f)(\omega) = \begin{cases} \frac{1}{\lambda}\mathcal{F}(f)\left(\frac{\omega}{\lambda}\right) & \forall \lambda > 0 \\ -\frac{1}{\lambda}\mathcal{F}(f)\left(\frac{\omega}{\lambda}\right) & \forall \lambda < 0 \end{cases}$$

$$\Rightarrow \mathcal{F}(\delta_\lambda f)(\omega) = \frac{1}{|\lambda|}\mathcal{F}(f)\left(\frac{\omega}{\lambda}\right)$$

EXAMPLE 3.22 Compute $\mathcal{F}(x e^{-x^2})$ by using the fact that $\mathcal{F}(e^{-x^2})(\omega) = \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}}$

$\boxed{\text{SOL:}}$

$$x e^{-x^2} = \frac{d}{dx}\left(-\frac{e^{-x^2}}{2}\right)$$

$$\mathcal{F}\left(x e^{-x^2}\right) = \mathcal{F}\left(\frac{d}{dx}\left(-\frac{e^{-x^2}}{2}\right)\right)$$

$$= (iw)\,\mathcal{F}\left(-\frac{e^{-x^2}}{2}\right)$$

$$= -\frac{iw}{2}\,\mathcal{F}\left(e^{-x^2}\right)$$

$$= -\frac{iw}{2}\cdot\frac{1}{\sqrt{2}}\,e^{-\frac{w^2}{4}}$$

$\boxed{\text{EXERCISE}}$

Compute $\mathcal{F}\left(e^{-\frac{x^2}{16}}\right)$ by using the fact that $\mathcal{F}\left(e^{-x^2}\right) = \frac{1}{\sqrt{2}} e^{-\frac{w^2}{4}}$ and <u>dilation property</u> $\blacksquare$