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Critical local and nonlocal PDEs and improved Regularity Results

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Contents

Abstract

In this thesis, we contribute to various critical (non-local) Partial Differential Equations by employing a variety of methods often connected to compensation phenomena to account for criticality of the PDEs. Generally speaking, this thesis consists of two main parts: The first part treats the so-called *half-harmonic gradient flow*, the natural L^2-gradient flow associated with the $1/2$-Dirichlet energy as first introduced by Da Lio and Rivière in [21]. We establish existence, uniqueness, regularity and bubbling results for this flow for arbitrary target manifolds and thus generalise the existing theory for harmonic maps to the non-local case ([102], [103], [104]). This generalisation is not unexpected, as the general theory of fractional harmonic maps greatly parallels the developments of the classical harmonic map theory. As as a result, while employing different techniques to account for the intricacies of non-local interactions, the broad ideas are often quite similar. To briefly summarise, we show that existence of solutions with non-increasing energy (which is a property to be expected for smooth solutions) is ensured for a short amount of time and such solutions are even smooth and unique among the set of strong competitors, i.e. there does exist only one solution for which we may make sense of the derivatives in the L^2-sense. We also see that energy might concentrate at specific points characterised by a lower bound on the limes inferior of the localised $1/2$-Dirichlet energy, resulting in the formation of a *half-harmonic bubble*, but at most at finitely many times and one may extend solutions for all times by gluing. This immediately leads us to concentration-compactness-type phenomena typical in the context of conformally invariant PDEs. Indeed, among the most important tools in our proof is a control on the energy concentration for short times. Therefore, it is not surprising that for sufficiently small initial energies, the solutions exist globally, remaining smooth everywhere and uniqueness may be extended to the set of weak competitors. In fact, if the initial energy is sufficiently small, bubbling cannot occur due to the fact that the bubble which would be formed has to absorb a quantum of energy strictly larger than 0.

The second part of the thesis explores critical chirality, a concept explored in Da Lio-Rivière [23] and generalising improved regularity properties that are usually linked to the emergence of anti-symmetric potentials to situations, where such a structure is a-priori not available. The paper Da Lio-Rivière [23] inspired two quite different kinds of compensation results: A Bourgain-Brezis-type inequality leading to a characterisation of Bergmann spaces ([25]) as well as a regularity result for Dirac equations which may be phrased in such a way that it applies to certain systems of divergence-type PDEs ([26]) related to the Clifford derivative. The former relies on revealing improved regularity for a specific iterated operator, using Clifford algebras in order to study such inequalities on arbitrary tori. The latter follows in the footsteps of Da Lio-Rivière [23] and again uses Clifford algebras as a natural, albeit not immediately evident extension of the inclusion $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$ to reveal the possibility for a gauge-approach to the regularity theory in spirit of Rivière [70]. The technique employed, especially regarding the introduction of a suitable gauge operator, forces us to assume that the domain has dimension ≤ 8 due to the properties connected to parallelisability of spheres in dimensions $0, 1, 3, 7$,

but not for higher dimensional spheres.

Zusammenfassung

In dieser Dissertation trägt der Autor zur Untersuchung verschiedener (nicht-lokaler) Partieller Differentialgleichungen bei, indem eine Vielzahl an Methoden im Zusammenhang mit Kompensations-Effekten zur Diskussion kritischer PDEs ausgenutzt werden. Diese Arbeit lässt sich grob in zwei Teile separieren: Im ersten Teil wird der sogenannte *halb-harmonische Gradientenfluss* behandelt, der natürliche L^2-Gradientenfluss im Zusammenhang mit der 1/2-Dirichlet Energie, welche erstmals von Da Lio und Rivière in [21] eingeführt wurde. Wir zeigen Existenz, Eindeutigkeit, Regularität sowie erste Bubbling-Resultate für diesen Gradientenfluss mit Bild in einer beliebigen Mannigfaltigkeit und verallgemeinern somit die existierende Theorie für harmonische Abbildungen ([102], [103], [104]). Diese Verallgemeinerung ist nicht unerwartet, zumal die allgemeine Theorie der gebrochenen harmonischen Abbildungen in groben Zügen der Entwicklung der klassischen Theorie harmonischer Abbildungen folgt. Folglich, obschon die verwendeten Techniken sich zwangsläufig aufgrund des nicht-lokalen Charakters der Gleichung oberflächlich unterscheiden, bleiben die zugrundeliegenden Ideen mehr oder weniger gleich. Kurz zusammengefasst zeigen wir, dass die Existenz von Lösungen für den Fluss mit nicht-wachsender Energie (eine Eigenschaft, welche man von glatten Lösungen erwartet) während eines hinreichend kleinen Zeitfensters garantiert ist und dass diese Lösungen sogar glatt sowie eindeutig unter starken Kompetitoren sind, d.h. eindeutig unter Funktionen, welche ein schwaches Differential in L^2 besitzen. Zudem sehen wir, dass sich die Energie in einzelnen Punkten akkumulieren kann, welche durch eine untere Schranke an die lokale 1/2-Dirichlet Energie charakterisiert werden können. Diese Konzentration führt zur Formierung einer sogenannten *halb-harmonischen Bubble*, wobei solche nur an endlich vielen Punkten entstehen und sich daher die lokalen Lösungen durch "Zusammenkleben" zu einer globalen ergänzen lassen. Das führt uns direkt zu Konzentration-Kompaktheit-Phänomenen, welche typisch für unter konformen Transformationen invarianten PDEs sind. Tatsächlich ist eines der wichtigsten Hilfsmittel in unseren Beweisen eine Abschätzung, welche uns Kontrolle über die Energiekonzentration für kurze Zeiten gewährt. Natürlich zeigt dies auch direkt, dass mit hinreichend kleiner Anfangsenergie auch globale Existenz ohne Singularitäten sowie Eindeutigkeit sogar unter schwachen Kompetitoren gilt. Dies begründet sich in der Tatsache, dass Bubbling unmöglich ist, weil die Formierung einer Bubble ein nicht-verschwindendes Energiequantum konsumieren würde.

Der zweite Teil des Dissertation beschäftigt sich mit kritischer Chiralität, einem Konzept untersucht in Da Lio-Rivière [23] und ein Hilfsmittel in der Verallgemeinerung von verbesserten Regularitätseigenschaften, welche gewöhnlich mit anti-symmetrischen Potentialen assoziert werden, darstellt, man jedoch auf allgemeinere Situationen übertragen möchte. Das Paper Da Lio-Rivière [23] inspirierte zwei ziemlich verschiedene Kompensations-Resultate in diesem Zusammenhang: Eine Bourgain-Brezis-Ungleichung, welche eine Charakterisierung von Bergmann-Räumen erlaubt [25], sowie ein Regularitätsresultat für spezielle Dirac-Gleichungen welches sich mittels geeigneter Formulierung auf gewisse PDE-Systeme bestehend aus Erhaltungsgleichungen im Zusammenhang mit der Clifford-Ableitung anwenden lässt [26]. Ersteres baut auf der verbesserten Integrabilität eines Faltungskerns

im Zusammenhang mit einem iterierten Operator auf und verwendet dazu Clifford-Algebren um die Ungleichungen auf Tori zu beweisen. Letzteres hingegen folgt stärker dem Ansatz aus Da Lio-Rivière [23] und verwendet abermals Clifford-Algebren als die natürlichen, aber nichtsdestotrotz nicht offensichtlichen Verallgemeinerungen der Inklusionsfolge $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$ um einen geeigneten Gauge-Operator zu definieren in Analogie zu Rivière [70]. Die eingesetzte Technik, speziell die Einführung eines geeigneten Gauge-Operators, forcieren eine Einschränkung des Resultats auf Gebiete mit Dimension ≤ 8. Dies begründet sich in gewisser Hinsicht in der Parallelisierbarkeit der Sphären mit Dimension $0, 1, 3$ sowie 7, welche für höher-dimensionale Sphären nicht mehr gilt.

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1 Introduction

Geometric Analysis describes, broadly speaking, the area of mathematics concerned with the intriguing interactions between geometry and analysis, in particular through the manifestation in terms of PDEs naturally associated with, for instance, minimisation of specific energy/area/volume functionals. Many interesting objects and intensely researched topic lie within its scope, such as minimal surfaces, curvature flows or harmonic mappings, just to name a few. In recent times, certain problems in Geometric Analysis, especially suitable geometric variational problems, have found natural counterparts in dimensions where the emerging energies involve non-local quantities, such as the fractional harmonic maps or self-repulsive knot energies. Among the pioneers in this field are Da Lio, Rivière ([21], [22]); Schikorra ([76]); Mazowiecka, Schikorra ([57]); Da Lio, Schikorra ([27], [28]); Millot, Sire ([60]); Blatt, Reiterer, Schikorra ([6]) and many others who have contributed to the theory of critical points of these energies and extended techniques and results from the local theory, for example about harmonic maps, to the non-local case by ingenious arguments and careful investigation of compensation phenomena occurring under these circumstances.

The goal of this thesis is to summarise the author's contributions to the field of Geometric Analysis. These may be naturally split in two parts: Results about the half-harmonic gradient flow and compensation phenomena in a more broad sense. The former results are contained in the sequence of articles [102], [103], [104] and provide existence, uniqueness and regularity results for such non-local gradient flows under a variety of assumptions. Moreover, behaviour in singular times is also studied, ultimately leading to results similar to the ones for the harmonic map flow, relating for instance singularity formation to energy concentration. The key ingredients for establishing the result are a local existence result which is founded in elliptic regularity combined with a maximum principle and a generalisation of the approach by Struwe in [89].

On the other hand, the articles [25], [26] are a collaboration with Da Lio and Rivière. These are concerned with compensation phenomena building on inherent symmetries in a special, but natural system of PDEs. In [25], the author and collaborators study a peculiar compensation phenomena first observed in Da Lio-Rivière [23] during the proof of a generalisation of the main result in Bourgain-Brezis [8], ultimately leading to a surprising characterisation of Bergmann spaces in terms of their boundary values, analogous to the relation between boundary values and Hardy spaces on the disc. Meanwhile in [26], the work in Da Lio-Rivière [23] again provides the foundation and the author and collaborators consider the question whether the results obtained generalise to arbitrary domains. Technical obstructions due to the introduction of quaternions in Da Lio-Rivière [23] as well as the use of natural compact Lie groups in the quaternions require care to adapt the argument, ultimately leading to a surprising restriction on the dimension of the domain. The techniques in both cases rely on compensation phenomena in different guises: Either by explicitly computing a kernel that has better integrability properties than naively anticipated or by gauge techniques ultimately leading to exposing the inner symmetries that allows one to apply Morrey bootstrap techniques.

1.1 Harmonic and Fractional Harmonic Maps

For our discussions in this section, in particular to avoid considerations of boundary behaviour, we usually restrict to the domain being a m-torus $\mathbb{T}^m$:

$$\mathbb{T}^m := \underbrace{S^1 \times \ldots \times S^1}_{m \text{ times}} \simeq \mathbb{R}^m / 2\pi\mathbb{Z}^m,$$

or a circle S^1 equipped with the flat Riemannian metrics, while the smooth target manifold is usually denoted by N and assumed to be closed and isometrically embedded in some $\mathbb{R}^K$, for some sufficiently large $K \in \mathbb{N}$. Naturally, most of the results below may be phrased in a more general setup using arbitrary Riemannian manifolds and involving the associated gradient, working for example with Laplace-Beltrami operators instead of the usual Laplace operator, but we shall usually not pursue the highest degree of generality in favour of clarity of presentation throughout this introduction.

1.1.1 Harmonic Maps - A Classical Object in Geometric Analysis

Among the most prominent and extensively studied variational problems is the minimisation of the Dirichlet energy. It is defined as:

$$E(u) := \frac{1}{2} \int_{\mathbb{T}^m} |\nabla u|^2 dx, \quad \forall u \in H^1(\mathbb{T}^m; \mathbb{R}^n). \tag{1.1}$$

A *weakly harmonic map* $u = (u_1, \ldots, u_n) \in H^1(\mathbb{T}^m; \mathbb{R}^n)$ is a critical point of the Dirichlet energy, i.e. for weakly harmonic maps we require:

$$\frac{d}{d\varepsilon} E(u_\varepsilon)\Big|_{\varepsilon=0} = 0,$$

for all C^1-variations $\varepsilon \mapsto u_\varepsilon$ in $C^\infty(\mathbb{T}^m; \mathbb{R}^n)$ with $u_0 = u$. As usual, aside from maxima and minima, saddle points are also critical points. It is easy to see that such a map satisfies the following PDE:

$$-\Delta u = 0 \quad \text{in } \mathcal{D}'(\mathbb{T}^m), \tag{1.2}$$

in a weak sense, i.e.:

$$\forall \varphi \in C^\infty(\mathbb{T}^m; \mathbb{R}^n): \quad \int_{\mathbb{T}^m} \nabla u \cdot \nabla \varphi \, dx = \sum_{j=1}^n \int_{\mathbb{T}^m} \nabla u_j \cdot \nabla \varphi_j = 0. \tag{1.3}$$

The converse also holds true, therefore any weakly harmonic map is also a critical point for E. By classical elliptic regularity results, one easily deduces that any such weakly harmonic map is actually smooth and satisfies the equation (1.2) in a pointwise sense. The natural properties such as componentwise maximum principles, mean value formulas and so on apply to these critical points as one might expect.

In a next step, one may wonder what happens if we impose further restrictions on the maps u for which we would like to minimise the Dirichlet energy (1.1). A natural restriction would be to require that u assumes values in a given manifold N, leading to the set of competitors now being $H^1(\mathbb{T}^m; N) \subset H^1(\mathbb{T}^m; \mathbb{R}^K)$. This leads to the following notion:

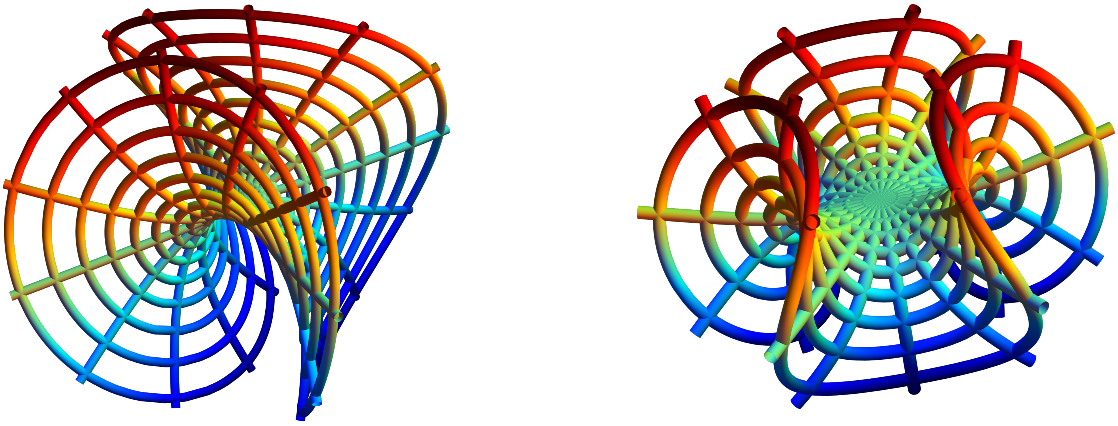

Figure 1.1.1: Two examples of critical points for special cases of Plateau's Problem

Definition 1.1.1.1. *A map* $u \in H^1(\mathbb{T}^m; N)$ *is called (weakly) harmonic, if and only if it is a critical point for the Dirichlet energy* (1.1) *for variations in* $H^1(\mathbb{T}^m; N)$.

Such maps are for example connected to instatons in theoretical physics (Jost [51]), geodesics in Riemannian geometry, minimal surfaces (Plateau's problem, see Figure 1.1.1, if we take discs rather than the torus) and also closely connected to the curvature of the target manifold N, as we shall see. For instance, if one attempts to find a minimal surface with given boundary, the lack of coercivity of the area functional may be circumvented by minimising the Dirichlet energy instead of the area functional and applying the uniformisation theorem (or a generalisation by Morrey, we refer to Rivière [73] for a summary) to find critical points of the area functional by minimising the Dirichlet energy. The key observation is that any critical point of E is automatically also critical for the area functional, as harmonicity and conformality imply vanishing mean curvature, and the two functionals yield the same value (up to a multiplicative constant) when evaluated in critical points. In general, on the other hand, the Dirichlet energy provides only an upper bound for the area. Naturally, the same considerations also apply to geodesics and these considerations were at the heart of the resolution of Plateau's problem by Douglas [31] and Radó [66].

Computing the Euler-Lagrange equation for harmonic maps $u \in H^1(\mathbb{T}^m; N)$ may be achieved by considering a perturbation $\varphi \in H^1(\mathbb{T}^m; \mathbb{R}^N) \cap L^\infty(\mathbb{T}^m)$ and defining a variation of u by:

$$u_\varepsilon := \pi_N\left(u + \varepsilon \cdot \varphi\right)$$

Here π_N denotes the closest point projection to N which is well-defined and smooth in a neigbourhood of N. Thus, if ε belong to a sufficiently small neighbourhood of 0, we have an admissible variation and may compute:

$$0 = \frac{d}{dt} E(u_\varepsilon)\Big|_{\varepsilon=0} = \int_{\mathbb{T}^m} \nabla u \cdot \nabla\left(d\pi_N(u)\varphi\right) dx,$$

which can be rephrased, thanks to $d\pi_N(p)$ being the orthogonal projection to T_pN, as:

$$-\Delta u \perp T_uN \Leftrightarrow -\Delta u = A_N(u)(\nabla u, \nabla u), \tag{1.4}$$

where $A_N(p)$ denotes the second fundamental form of N at the point $p \in N$, introducing curvature to the equation. To be precise, we have:

$$A_N(p)(v,w) := \sum_{j=1}^{K-\dim N} \langle v, d\nu_j(p)w\rangle \nu_j, \quad \forall v, w \in T_pN,$$

where ν_j form a orthonormal basis of the orthogonal complement of T_pN in $\mathbb{R}^K$. Then, the expression $A_N(u)(\nabla u, \nabla u)$ is exactly:

$$A_N(u)(\nabla u, \nabla u) := \sum_{i=1}^{m} A_N(u)(\partial_i u, \partial_i u) = \sum_{i=1}^{m} \sum_{j=1}^{K-\dim N} \langle \partial_i u, d\nu_j(u)\partial_i u\rangle \nu_j.$$

Therefore, if $m = 1$, one may immediately identify the defining equation for geodesics in a Riemannian manifold which are again smooth and intimately connected to the geometry of N. If $N = S^{n-1}$, the equation (1.4) takes a simpler form by observing that the normal space to $S^{n-1} \subset \mathbb{R}^n$ at any point p is spanned by p itself. This leads to the following equation, observing that $\nu(u) := u = Id(u)$ is the normal unit vector:

$$-\Delta u = \sum_{i=1}^{m} \langle \partial_i u, dId(u)\partial_i\rangle u = \sum_{i=1}^{m} \langle \partial_i u, \partial_i u\rangle u = u|\nabla u|^2. \tag{1.5}$$

At this point, it is natural to wonder whether harmonic maps are smooth. From the case of weakly harmonic $u : \mathbb{T}^m \to \mathbb{R}^n$ and the ellipticity of the problem, such a result seems to be reasonable, however the equation is critical for $m = 2$ and in the natural energy space $H^1(\mathbb{T}^m; N)$. Indeed, one can directly verify that:

$$\big|A(u)(\nabla u, \nabla u)\big| \lesssim |\nabla u|^2, \tag{1.6}$$

similar to the case $N = S^{n-1}$. The quadratic non-linearity, however, forces for harmonic maps u:

$$|-\Delta u| \lesssim |\nabla u|^2 \in L^1(\mathbb{T}^m),$$

and for L^1-spaces, Caldéron-Zygmund theory does not apply, meaning that we may only deduce the following Lorentz-regularity form the integrability of the fundamental solution:

$$\nabla u \in L^{(\frac{m}{m-1},\infty)}(\mathbb{T}^m),$$

which is worse than what we had at the beginning. If $m \geq 3$, i.e. when the equation is subcritical, then the regularity is actually strictly worse and it was actually shown by counterexample by Rivière [69] that, if $m \geq 3$, the weak harmonic maps may behave terribly, being discontinuous everywhere. Fortunately, once we assume stationarity (i.e. minimality by among variations in the domain) of a weakly harmonic map and thus have a monotonicity formula at our disposal, one may again prove a suitable partial regularity result yielding regularity outside a small set (small in the sense of Hausdorff measures). On the other end of the spectrum, if $m = 1$, it is known that geodesics are smooth which may be seen by a simple bootstrap.

On the other hand, if $m = 2$, the homogeneity at least remains the same, so hope remains for regularity in this case. To see that the equation is critical in $H^1(\mathbb{T}^2; N)$, let us compare the regularity with the case that $u \in W^{1,p}(\mathbb{T}^2; N)$ with $4 > p > 2$ satisfies (1.4). Then we have, using Caldéron-Zygmund theory and Sobolev embeddings:

$$u \in W^{1,p}(\mathbb{T}^2; N) \Rightarrow -\Delta u \in L^{\frac{p}{2}}(\mathbb{T}^2) \Rightarrow u \in W^{2,\frac{p}{2}}(\mathbb{T}^2; N) \Rightarrow u \in W^{1,\frac{2p}{4-p}}(\mathbb{T}^2; N)$$

If $p > 4$, we actually obtain Hölder continuity of ∇u, otherwise the integrability exponent increases. Thus, in this case, one could bootstrap to arrive at smoothness of the solution. The same technique, due to the considerations above, does not apply to the limiting case $p = 2$. As a spark of hope, one may however see using Morrey estimates that, provided u is continuous and the non-linearity is smoothly dependent on u and ∇u as well as quadratic in its dependence on ∇u, we may still get smoothness by a Morrey-bootstrap.

To further illustrate the difficulties with the quadratic non-linearity, let us consider the PDE:

$$-\Delta u = |\nabla u|^2,$$

for $u : \mathbb{R}^2 \to \mathbb{R}$. By considering $v := e^u$, one may verify directly:

$$-\Delta v = 0 \Leftrightarrow -\Delta u = |\nabla u|^2,$$

and this allows one, as is done explicitly in Rivière [73], to construct examples of solutions u by means of the fundamental solution for the Laplacian which are not continuous. Therefore, no general regularity theory for quadratic non-linearities exist and to deduce smoothness of weakly harmonic maps $u \in H^1(\mathbb{T}^2; N)$, one needs to exploit further structures contained within the equation (1.4).

This leads us to the development of two key techniques to overcome the problems discussed before: Hélein's moving frames method Hélein [45] and Rivière's gauge approach [70] building on previous ideas by Uhlenbeck. Namely, the combined efforts a variety of authors, including Béthuel [4], Grüter [41], Hélein [44], Morrey [62], Rivière [70], Shatah [83] and many others ultimately brought about the resolution of the long-standing open problem pertaining to the regularity of weakly harmonic maps by employing compensation phenomena inherent to so-called *2D-jacobians* or *div-curl quatities* summarised and studied first by Wente [101] in the context of prescribed mean curvature equations and later on generalised by Coifman, Lions, Meyer and Semmes [14]. The key estimate is the following:

Theorem 1.1.1.1 ([101])**.** *Let $r > 0$ and $1 \leq p < 2$. Denote by $B_r(0) \subset \mathbb{R}^2$ the ball of radius r around the origin. Then, if $u \in W^{1,p}(B_r(0))$ and $a, b \in W^{1,2}(B_r(0))$ are such that:*

$$-\Delta u = \partial_x a \partial_y b - \partial_y a \partial_x b = \nabla^\perp a \cdot \nabla b \quad \textit{in } \mathcal{D}'(B_r(0)), \tag{1.7}$$

and u has vanishing trace on $\partial B_r(0)$, then u is actually continuous and the following estimate holds:

$$\|u\|_{L^\infty} + \|\nabla u\|_{L^{(2,1)}} + \|\nabla^2 u\|_{L^1} \leq C\|\nabla a\|_{L^2}\|\nabla b\|_{L^2}, \tag{1.8}$$

where the constant C does not depend on u, a, b or r.

The independence of r is an immediate consequence of the scaling properties of the estimate, but crucial in order to establish increased regularity results by Morrey bootstrapping. As hinted at before, such results continue to hold true for scalar products of a divergence-free and a curl-free vectorfields as well as for higher-dimensional jacobians, see Coifman-Lions-Meyer-Semmes [14].

Let us illustrate how the introduction of Theorem 1.1.1.1 applies to the situation at hand in the special case $N = S^{n-1}$. Recalling the PDE (1.5) in this case, Shatah [83] succeeded in rewriting (1.5) in a more suitable form (as a set of conservation laws):

$$\forall i, j \in \{1, \dots, n\} : \quad \operatorname{div}(u_i \nabla u_j - u_j \nabla u_i) = 0 \quad \text{in } \mathcal{D}'(\mathbb{T}^2). \tag{1.9}$$

Indeed, (1.5) is equivalent to (1.9), as for example:

$$\operatorname{div}(u_i \nabla u_j - u_j \nabla u_i) = u_i \Delta u_j + \nabla u_i \cdot \nabla u_j - u_j \Delta u_i - \nabla u_j \cdot \nabla u_i = u_i u_j |\nabla u|^2 - u_j u_i |\nabla u|^2 = 0.$$

The converse follows also by direct considerations. The major advantage of the set of conservation laws (1.9) lies in the following reformulation:

$$\begin{aligned} -\Delta u_i = u_i |\nabla u|^2 &= \sum_{j=1}^n u_i \nabla u_j \cdot \nabla u_j \\ &= \sum_{j=1}^n (u_i \nabla u_j - u_j \nabla u_i) \nabla u_j + \sum_{j=1}^n u_j \nabla u_i \cdot \nabla u_j \\ &= \sum_{j=1}^n (u_i \nabla u_j - u_j \nabla u_i) \nabla u_j =: \sum_{j=1}^n \Omega_{ij} \cdot \nabla u_j, \end{aligned} \tag{1.10}$$

where we used $u \perp \partial_k u$ for $k = 1, 2$, as $u \in S^{n-1}$ and thus the derivatives being tangential to the manifold, while u spans the normal space to TS^{n-1} at $u \in S^{n-1}$. Therefore, we obtain another formulation of (1.5):

$$-\Delta u = \Omega \cdot \nabla u, \tag{1.11}$$

where $\Omega = (\Omega_{ij})_{1 \leq i,j \leq n}$, $\Omega_{ij} := u_i \nabla u_j - u_j \nabla u_i$, is clearly anti-symmetric and divergence-free, thanks to (1.9). As a result, we are in a situation where Theorem 1.1.1.1 applies. We emphasise at this point that while the argument by Shatah suggests that the divergence-freeness of the terms is crucial to the argument, indeed the relevant property is the anti-symmetry of the potential. Namely, in [70], Rivière proved regularity for solutions of conformally invariant PDEs using a change of gauge approach to exhibit 2D-jacobians and thus improved regularity, ultimately settling the question of regularity for weakly harmonic maps as well as a conjecture by Hildebrandt.

To deduce improved integrability for $u \in H^1(\mathbb{T}^2; S^{n-1})$, one now proceeds as follows: Based on the integrability of Ω, choose $r_0 > 0$ such that for all $p \in \mathbb{T}^2$, we have:

$$\int_{B_{r_0}(p)} |\Omega|^2 dx < \varepsilon_0,$$

for some $\varepsilon_0 > 0$ to be fixed. Then we may consider a ball of radius $r < r_0$ around some point $p \in \mathbb{T}^2$ and split u on the ball $B_r(p)$ in a harmonic term and a term with vanishing trace:

$$u = v + w; \quad -\Delta v = 0; \quad u = w \text{ on } \partial B_r(p).$$

Observe that by Theorem 1.1.1.1, one immediately has:

$$\int_{B_{\delta r}(p)} |\nabla w|^2 dx \leq \int_{B_r(p)} |\nabla w|^2 dx \leq C \int_{B_r(p)} |\Omega|^2 dx \int_{B_r(p)} |\nabla u|^2 dx \leq C\varepsilon_0 \cdot \int_{B_r(p)} |\nabla u|^2 dx,$$

while for harmonic v, we have thanks to monotonicity properties for harmonic functions:

$$\int_{B_{\delta r}(p)} |\nabla v|^2 dx \leq \delta^2 \int_{B_r(p)} |\nabla v|^2 dx \leq \delta^2 (2 + 2C\varepsilon_0) \int_{B_r(p)} |\nabla u|^2 dx,$$

where we used in the last step that:

$$\int_{B_r(p)} |\nabla v|^2 dx = \int_{B_r(p)} |\nabla u - \nabla w|^2 dx \leq 2\int_{B_r(p)} |\nabla u|^2 dx + 2\int_{B_r(p)} |\nabla w|^2 dx \leq (2 + 2C\varepsilon_0)\int_{B_r(p)} |\nabla u|^2 dx,$$

again employing the estimate for ∇w. Therefore, choosing $\delta > 0$ and $\varepsilon_0 > 0$ sufficiently small:

$$\int_{B_{\delta r}(p)} |\nabla u|^2 dx \leq 2\int_{B_{\delta r}(p)} |\nabla w|^2 dx + 2\int_{B_{\delta r}(p)} |\nabla v|^2 dx \leq \frac{1}{2}\int_{B_r(p)} |\nabla u|^2 dx,$$

and by iterating this estimate, there is $\alpha > 0$, such that the following Morrey estimate holds:

$$\sup_{p\in\mathbb{T}^2, 0<r<r_0} r^{-2\alpha}\int_{B_r(p)} |\nabla u|^2 dx < +\infty$$

This leads to a similar estimate for $-\Delta u$ by using (1.11) and thus applying Adam's embeddings as in Adams [1], one may deduce:

$$u \in W^{1,q}(\mathbb{T}^2; S^{n-1}),$$

for some $q > 2$. Consequently, one may bootstrap this information to arrive at smoothness of the solution. For general target manifolds N, the proof requires a change of gauge, as divergence-freeness may not be guaranteed, but anti-symmetry still holds, see Rivière [70]. Since anti-symmetry leads to natural gauges in the compact Lie group $SO(K)$ (recall $N \subset \mathbb{R}^K$), the problem is solved by introducing suitable gauge operators, but the argument requires some care, while the general idea behind the improved regularity remains. For example, if u solves:

$$-\Delta u = \Omega \cdot \nabla u,$$

where Ω is anti-symmetric and P is a $SO(K)$-valued map, then one may check the equation solved by $P\nabla u$:

$$-\operatorname{div}(P\nabla u) = -P\Delta u - \nabla P\nabla u = P\Omega P^{-1}\cdot P\nabla u - \nabla PP^{-1}\cdot P\nabla u = -\left(\nabla PP^{-1} - P\Omega P^{-1}\right)\cdot P\nabla u,$$

which now allows us to look for P such that $\nabla PP^{-1} - P\Omega P^{-1}$ is divergence-free and thus renders similar arguments possible.

1.1.2 Fractional Harmonic Maps

So far, we have only talked about harmonic maps and their regularity theory. However, during this thesis, we shall mostly be concerned with *half-harmonic maps* or their more general form, *fractional harmonic maps*, as first introduced by Da Lio and Rivière in [21], see also Schikorra [76].

Firstly, it is important to define what the fractional Laplacian is. It is well-known that the classical Laplacian may be equivalently defined as a Fourier multiplier with the multiplier $|\xi|^2$. Thus, it is natural to define the fractional Laplacian $(-\Delta)^s$ for $s \in \mathbb{R}$ as the operator associated with the Fourier multiplier $|\xi|^{2s}$. Therefore, for Schwartz functions u on $\mathbb{R}^n$ or smooth functions u on $\mathbb{T}^n$, one has:

$$(-\Delta)^s u(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\xi|^{2s}\int_{\mathbb{R}^n} u(y)e^{iy\cdot\xi} dy d\xi, \tag{1.12}$$

or in terms of Fourier series on the n-torus. These operators thus naturally appear in the theory of Bessel-Sobolev spaces $H^s(\mathbb{R}^n)$ or $H^s(\mathbb{T}^n)$ which are defined by:

$$H^s(\mathbb{R}^n) := \Big\{ u : \mathbb{R}^n \to \mathbb{R} \;\Big|\; \left(1 + |\xi|^2\right)^{s/2} \hat{u}(\xi) \in L^2(\mathbb{R}^n) \Big\}, \quad \dot{H}^s(\mathbb{R}^n) := \Big\{ u : \mathbb{R}^n \to \mathbb{R} \;\Big|\; |\xi|^s \hat{u}(\xi) \in L^2(\mathbb{R}^n) \Big\},$$

and analogously in terms of Fourier series instead of the Fourier transform for the higher-dimensional tori.

There is another natural definition for the fractional Laplacian involving singular integrals. If $0 < s < 1$, then there exists a constant $C_{n,s} > 0$, such that:

$$(-\Delta)^s u(x) = C_{n,s} \cdot P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy, \tag{1.13}$$

for sufficiently regular u and similarily for $\mathbb{T}^n$ (in this case, the distance $|x - y|$ is replaced by the natural distance $|e^{ix} - e^{iy}|$). Associated with the fractional Laplacians is the so-called *fractional* or *s-Dirichlet energy* defined by:

$$E_s(u) := \frac{1}{2} \int \int \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dy dx, \tag{1.14}$$

where the domain of integration is either $\mathbb{R}^n$ or $\mathbb{T}^n$. Among the most important properties of the s-Dirichlet energy is its connection with the Bessel-Sobolev seminorm, namely:

$$E_s(u) \sim \frac{1}{2} \int |(-\Delta)^{s/2} u|^2 dx,$$

which renders H^s and $\dot{H}^s$ the natural energy space for E_s, see for instance [63, Prop. 3.4].

For convenience in the discussions that follow, we already introduce the notions of fractional gradient as in Mazowiecka-Schikorra [57]:

$$d_s u(x,y) := \frac{u(x) - u(y)}{|x-y|^s}, \quad |d_s u|^2(x) := \int |d_s u(x,y)|^2 \frac{dy}{|x-y|^n}, \quad |d_s u|(x) = \sqrt{|d_s u|^2(x)}. \tag{1.15}$$

These quantities emerge, for instance, if we try to find a generalisation of Leibniz' rule for fractional Laplacians:

$$(-\Delta)^s (uv) = (-\Delta)^s u \cdot v + u \cdot (-\Delta)^s v - d_s u \cdot d_s v(x), \tag{1.16}$$

where, as with $|d_s u|^2(x)$:

$$d_s u \cdot d_s v(x) := \int \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} dy.$$

Again, the domain of integration above may always be either $\mathbb{R}^n$ or $\mathbb{T}^n$. These quantities are naturally related to s-Dirichlet energies and the fractional Laplacians, rendering them useful tools in the study that will follow.

Fractional Laplacians possess local regularity properties as well, they are elliptic. For example, provided:

$$(-\Delta)^s u = 0 \quad \text{ in } B_r(p), \tag{1.17}$$

then we know that $u \in H^s(\mathbb{T}^n)$ is even smooth inside this ball. This may be seen by considering suitable smooth cut-off functions φ with support in $B_r(p)$. Then we have:

$$(-\Delta)^s(u\varphi) = u(-\Delta)^s\varphi - d_s u \cdot d_s\varphi \in L^2, \tag{1.18}$$

so that $u\varphi \in H^{2s}(\mathbb{T}^n)$. Applying now $(-\Delta)^{s/2}$ to (1.17), taking into account the non-local contributions outside of $B_r(p)$, we may obtain higher and higher regularity for u in smaller and smaller balls, ultimately yielding u smooth in $B_r(p)$. Alternatively and more directly, there is an explicit Poisson kernel which allows one to deduce the very same regularity property. Namely, if $(-\Delta)^s u = 0$ in $B_r(p) \subset \mathbb{R}^n$, then (see Landkof [54]):

$$u(x) = \int_{\mathbb{R}^n \setminus B_r(p)} \frac{u(y)(r^2 - |x|^2)^s}{(|y|^2 - r^2)^s |y - x|^n} dy.$$

One should notice that the non-locality forces us to consider the values everywhere outside of $B_r(p)$ and not merely on the boundary $\partial B_r(p) \simeq r \cdot S^{n-1}$. Indeed, a similar observation may be made by observing that the classical Laplacian is closely connected to Brownian motion (continuous paths), while the fractional Laplacians are related to Lévy processes (allowing jump discontinuities). Using stochastic processes and stopping times thus to solve equations, the requirement to assign values everywhere in the complement of $B_r(p)$ is also clear.

Let us focus our attention back to fractional harmonic maps and also restrict attention to the case $n = 1$ and the circle S^1 and N be an arbitrary closed manifold. We then define:

Definition 1.1.2.1. *A map $u \in H^s(S^1; N)$ is called s-harmonic, if and only if it is a critical point of the s-Dirichlet energy E_s with respect to variations in $H^s(S^1; N)$.*

Of particular importance for us later on will be the case $s = 1/2$, when the energy functional $E_{1/2}$ becomes conformally invariant and thus, by stereographic projection, half-harmonic maps on S^1 and $\mathbb{R}$ are in 1-1-correspondence. Moreover, the conformal invariance of the energy functional relates half-harmonic maps closely to other geometric PDEs with conformal invariance, such as harmonic maps and their associated gradient flow, the Yamabe flow, Yang-Mills equation and the J-holomorphic sphere equation. Solutions to such PDEs exhibit concentration-compactness phenomena, for example leading to improved convergence properties of subsequences of bounded J-holomorphic curves outside of points of concentration, while in points of concentration formation of bubbles is observed after rescaling. We refer to McDuff-Salamon [59] for details on the theory of J-holomorphic curves. In a similar way, the half harmonic gradient flow provides concentration estimates leading to uniform bounds where energy does not concentrate, thus leading to a similar result in this case. Naturally, similar convergence properties also apply to the half-harmonic map, see Da Lio [17].

As we have previously discussed in the context of the Dirichlet energy E, there is an intimate connection to Plateau's problem and minimal surfaces. Such a connection continues to hold for half-harmonic maps. Indeed, these are precisely the boundary values of (branched) immersed free-boundary minimal discs with obstacle N. The connection between minimal surfaces and half-harmonic maps was observed in Millot-Sire [60], Da Lio-Pigati [20] and Da Lio [15] and is built on an analysis of the Hopf differential $H(U)$ given by:

$$H(U) := \left(|\partial_x U|^2 - |\partial_y U|^2\right) - 2i \cdot \langle \partial_x U, \partial_y U \rangle = \langle \partial_z U, \partial_z U \rangle, \tag{1.19}$$

Weak conformality may be obtained by establishing $H(U) = 0$, see Millot-Sire [60], and as we have seen before, this provides the connection between harmonic maps and minimal surfaces.

Similar to the (local) Dirichlet energy, one may wonder what the Euler-Lagrange equation for critical points of the energy $E_{1/2}$ looks like. Using variations $u_\varepsilon := \pi_N (u + \varepsilon \cdot \varphi)$ for φ smooth and ε sufficiently small, such variations are admissible and one sees:

$$0 = \frac{d}{d\varepsilon} E_{1/2}(u_\varepsilon)\big|_{\varepsilon=0} = \int_{S^1} d_{1/2}u \cdot d_{1/2} \left(d\pi_N(u)\varphi\right) dx,$$

which is equivalent, again observing that $d\pi_N(p)$ is the orthogonal projection onto T_pN, to:

$$(-\Delta)^{1/2} u \perp T_u N. \tag{1.20}$$

In Millot-Sire [60] and Mazowiecka-Schikorra [57], the equation (1.20) was rewritten for the target manifold $N = S^{n-1}$ as follows:

$$(-\Delta)^{1/2} u = u|d_{1/2}u|^2. \tag{1.21}$$

At this point, it is worth considering the similarities between (1.21) and (1.5). Both share many common features (quadratic non-linearity in the form $|\nabla u|^2$ and $|d_{1/2}u|^2$, lower regularity by half order, criticality, conformal invariance), but one should also keep in mind the differences (for half-harmonic maps, we are dealing with non-local operators, forcing us to always keep track of effects from far away). Naturally, the same kind of considerations apply verbatim to the domain $\mathbb{R}$ instead of S^1. We mention here that the equation is once more critical and thus Caldéron-Zygmund theory may not be used to deduce regularity by a direct bootstrap argument, as without additional information, $u \in H^{1/2}(S^1)$ only allows for us to deduce $(-\Delta)^{1/2}u = u|d_{1/2}u|^2 \in L^1(S^1)$.

In the case of general target manifolds N, the non-linearity of the equation (1.20) takes also a curvature-like form reminiscent of the second fundamental form that appears in the harmonic map case (1.4). From here, a very natural question demands attention: Are half-harmonic maps actually always smooth? This problem was first studied by Da Lio and Rivière in the papers [21] and [22] where the equation (1.20) was rewritten by means of so-called *Three-term commutator estimates*. These are special linear combinations of terms which combined have better integrability properties than each term on their own, much in the same way as we have previously seen with Theorem 1.1.1.1 where a naive application of Hölder's inequality would only lead to an L^1-estimate instead of the Hardy-estimate. An important example of a Three-term commutator with applications to the half-harmonic map equation is:

$$\mathcal{T} : L^2(\mathbb{R}; \mathbb{R}^m) \times \dot{H}^{1/2}(\mathbb{R}; \mathbb{R}^m) \to \dot{H}^{-1/2}(\mathbb{R}; \mathbb{R}^m), \tag{1.22}$$

$$\mathcal{T}(v, Q) := (-\Delta)^{1/4}(Qv) - Q(-\Delta)^{1/4}v + (-\Delta)^{1/4}Q \cdot v \tag{1.23}$$

While each of the products on its own involved in the definitions of $\mathcal{T}$ does not behave particularly well, Da Lio and Rivière proved in [21] the following inequality:

$$\|\mathcal{T}(v, Q)\|_{\dot{H}^{-1/2}} \lesssim \|v\|_{L^2} \cdot \|Q\|_{\dot{H}^{1/2}}.$$

Such estimates are proven by compensation phenomena found by investigating Littlewood-Paley decompositions, duality arguments and paraproduct decompositions, we refer to Da Lio-Rivière [21] for a complete treatment of the arguments necessary, alternative proofs may be found in Lenzmann-Schikorra [55]. The approach to regularity has been further developed to include higher dimensional domains, other critical exponents and include a fractional-type Wente inequality in Schikorra [76]; Da Lio [16]; Da Lio-Schikorra [28], [27]; Mazowiecka-Schikorra [57]. To draw similarities with the harmonic map equation, we shall focus on the approach to regularity using Wente-type inequalities, namely the following localised estimate which is contained in Proposition 2.4 in Mazowiecka-Schikorra [57]:

Proposition 1.1.2.1 ([57]). *Let $F \in L^p_{od}(\mathbb{R}\times\mathbb{R})$ be such that $\operatorname{div}_s F = 0$, for $s \in (0,1)$ and $p \in (1,\infty)$. Moreover, let $g \in W^{s,p'}(\mathbb{R})$. Then, for any interval $B_r(x_0) \subset \mathbb{R}$ and any $\varphi \in C_c^\infty(B_r(x_0))$, the following estimate:*

$$\int_{\mathbb{R}} \varphi F \cdot d_s g dx \leq C\left(|\varphi|_{BMO} + r^{-1}\|\varphi\|_{L^1}\right)\|F\|_{L^p_{od}(B_{\Lambda r}(x_0)\times B_{\Lambda r}(x_0))}\|d_s g\|_{L^{p'}_{od}(B_{\Lambda r}(x_0)\times B_{\Lambda r}(x_0))}, \tag{1.24}$$

for an uniform constant $\Lambda > 0$.

We recall here that the space $L^p_{od}(\mathbb{R}\times\mathbb{R})$ denotes the L^p-space of functions on $\mathbb{R}\times\mathbb{R}$ with respect to the measure $\frac{dydx}{|x-y|}$, satisfying $F(x,y) = -F(y,x)$. To deduce regularity for solutions of (1.21), one imitates Shatah's idea [83], namely one can see:

$$\operatorname{div}_{1/2}\left(u_i d_{1/2} u_j - u_j d_{1/2} u_i\right) = u_i(-\Delta)^{1/2}u_j - u_j(-\Delta)^{1/2}u_i = u_i u_j |d_{1/2}u|^2 - u_j u_i |d_{1/2}u|^2 = 0, \tag{1.25}$$

which is easily verified directly. Thus, rewriting (1.21) yields:

$$\forall j \in \{1,\dots n\} : (-\Delta)^{1/2}u_j = \left(u_j d_{1/2}u_i - u_i d_{1/2}u_j\right) d_{1/2}u_i + T(u) =: \Omega_{ji}\cdot d_{1/2}u_i + T(u), \tag{1.26}$$

where $T(u)$ is a remainder term of better integrability, which does allow for similar estimates as found below. Therefore, the key features of the regularity approach to harmonic map theory are mirrored here as well. Let us sketch the arguments in Mazowiecka-Schikorra [57] in the special and simpler case of $N = S^{n-1}$ to conclude from here:

Given $\varphi \in C_c^\infty(B_r(x_0))$, we take η to be a cut-off function on $B_{\Lambda r}(x_0)$ being equal to 1 on $B_{\Lambda r/2}(x_0)$ with $\Lambda > 2$, we define:

$$\tilde{u} := \eta\left(u - \frac{1}{\Lambda r}\int_{B_{\Lambda r}(x_0)} u dy\right).$$

We shall from now on write $\overline{u}_{B_{\Lambda r}(x_0)}$ for the average above to abbreviate. Now using the splitting $u = \tilde{u} + (u - \tilde{u})$, one has two terms to estimate, the first one being:

$$\begin{aligned}
\int_{\mathbb{R}} \Omega_{ji}\cdot d_{1/2}(u_i - \tilde{u}_i)\varphi_j dx &= \int_{B_r(x_0)}\int_{B_{\Lambda r/2}(x_0)^c} \Omega_{ji}(x,y) d_{1/2}(u_i - \tilde{u}_i)\varphi_j \frac{dydx}{|x-y|} \\
&= \int_{B_r(x_0)}\int_{B_{\Lambda r/2}(x_0)^c} \Omega_{ji}(x,y)(\eta(y)-1)(u(y) - \overline{u}_{B_{\Lambda r}(x_0)})\varphi_j(x)\frac{dydx}{|x-y|} \\
&\lesssim \|\Omega_{ji}\|_{L^2_{od}}\|\varphi\|_{L^2(B_r(x_0))}\cdot\sum_{k=1}^\infty \frac{1}{2^k\Lambda r}\|u - \overline{u}_{B_{\Lambda r}(x_0)}\|_{L^2(B_{2^k\Lambda r}\setminus B_{2^{k-1}\Lambda r}(x_0))} \\
&\lesssim \|\Omega\|_{L^2_{od}}\|(-\Delta)^{1/4}\varphi\|_{L^2}\cdot\sum_{k=1}^\infty \frac{1}{2^k\Lambda r^{1/2}}\|u - \overline{u}_{B_{\Lambda r}(x_0)}\|_{L^2(B_{2^k\Lambda r}\setminus B_{2^{k-1}\Lambda r}(x_0))} \\
&\lesssim \|\Omega\|_{L^2_{od}}\|(-\Delta)^{1/4}\varphi\|_{L^2}\cdot\sum_{k=1}^\infty \frac{k}{2^{k/2}\Lambda^{1/2}}\|(-\Delta)^{1/4}u\|_{L^{(2,\infty)}(B_{2^k\Lambda r}(x_0))}
\end{aligned}$$

$$\lesssim \sum_{k=1}^{\infty} \frac{1}{2^{k\sigma}\Lambda^{\sigma}} \|(-\Delta)^{1/4}u\|_{L^{(2,\infty)}(B_{2^k\Lambda r}(x_0))}, \tag{1.27}$$

where we used the estimates for localisation found in Da Lio-Pigati [20]. In the last line, we have $\sigma \in (0,1/2)$. The second term to estimate follows by first applying fractional Leibniz' rule combined with the fractional Shatah-condition (1.25) and then using Proposition 1.1.2.1 (suppressing the midpoint for the balls in our notation):

$$\begin{aligned}
&\int_{\mathbb{R}} \Omega_{ji} \cdot d_{1/2}\tilde{u}_i \varphi_j dx = -\int_{\mathbb{R}} \Omega_{ji} \cdot d_{1/2}\varphi_j \tilde{u}_i dx \\
&\lesssim \left(|\tilde{u}|_{BMO} + \Lambda^{-1}r^{-1}\|\tilde{u}\|_{L^1}\right) \|\Omega\|_{L^2_{od}(B_{\Lambda 2r}(x_0))} \|d_{1/2}\varphi\|_{L^2_{od}(B_{\Lambda 2r})} \\
&\lesssim \|\Omega\|_{L^2_{od}(B_{\Lambda 2r})} \|d_{1/2}\varphi\|_{L^2_{od}(B_{\Lambda 2r})} \cdot \left(\|(-\Delta)^{1/4}u\|_{L^{(2,\infty)}(B_{\Lambda 2r})} + \sum_{k=1}^{\infty} \frac{1}{2^{k\sigma}\Lambda^{\sigma}} \|(-\Delta)^{1/4}u\|_{L^{(2,\infty)}(B_{\Lambda 2r})} \right) \\
&\lesssim \|\Omega\|_{L^2_{od}(B_{\Lambda 2r})} \|(-\Delta)^{1/4}\varphi\|_{L^2} \cdot \left(\|(-\Delta)^{1/4}u\|_{L^{(2,\infty)}(B_{\Lambda 2r})} + \sum_{k=1}^{\infty} \frac{1}{2^{k\sigma}\Lambda^{\sigma}} \|(-\Delta)^{1/4}u\|_{L^{(2,\infty)}(B_{\Lambda 2r})} \right),
\end{aligned} \tag{1.28}$$

using the localisation of the BMO norm as in Mazowiecka-Schikorra [57] or following Da Lio-Pigati [20] and some $\sigma \in (0,1/2)$. Combining (1.27) and (1.28) (the perturbation term $T(u)$ may be taken care of in a similar manner) and using a duality argument following Mazowiecka-Schikorra [57], one has thus for an appropriately chosen φ:

$$\begin{aligned}
&\|(-\Delta)^{1/4}u\|_{L^{(2,\infty)}(B_r(x_0))} \\
&\lesssim \left| \int_{\mathbb{R}} d_{1/2}u \cdot d_{1/2}\varphi dx \right| + \Lambda^{-\sigma} \|(-\Delta)^{1/2}u\|_{L^{(2,\infty)}(B_{\Lambda 2r}(x_0))} + \sum_{k=1}^{\infty} \frac{1}{2^{k\sigma}\Lambda^{\sigma}} \|(-\Delta)^{1/4}u\|_{L^{(2,\infty)}(B_{2^k\Lambda 2r}(x_0))} \\
&\lesssim \lambda \|(-\Delta)^{1/4}u\|_{L^{(2,\infty)}(B_{\Lambda 2r}(x_0))} + \sum_{k=1}^{\infty} \frac{1}{2^{k\sigma}\Lambda^{\sigma}} \|(-\Delta)^{1/4}u\|_{L^{(2,\infty)}(B_{2^k\Lambda 2r}(x_0))},
\end{aligned} \tag{1.29}$$

where $\lambda := \|\Omega\|_{L^2_{od}(B_{\Lambda r}(x_0)\times B_{\Lambda r}(x_0))} + \Lambda^{-\sigma}$. In the last step, we used the combination of (1.27) and (1.28). Observe that making Λ big and r small also renders λ arbitrarily small. Now arguing using Morrey estimates, which may be obtained by iterating the estimate above allows one to deduce smoothness. For a different approach based on three-commutators, we refer to Da Lio-Pigati [20]. Naturally, one may extend the arguments presented to general target manifolds, but then one has to take care of a change of gauge, see Mazowiecka-Schikorra [57] for details on this additional step.

To conclude this section, it should be mentioned that the techniques employed in the proof of regularity are very flexible and extend to various further problems, such as for other critical exponents (Da Lio-Schikorra [28], [27]), higher dimensional domains and even to self-repulsive knot energies (Blatt-Reiterer-Schikorra [6], Blatt-Reiterer-Schikorra-Vorderobermeier [7] for instance) in an analogous manner. This hints at the naturality of the arguments used in the context of non-local operators as well as potential for further generalisations.

1.2 Half-Harmonic Gradient Flow

Having introduced the notions of harmonic and half-harmonic maps, we are now in a position to turn to the first part of this thesis and the research conducted throughout my doctorate: The properties of the half-harmonic gradient flow.

Once one introduces the notion of (fractional) harmonic maps, one may wonder how to find critical points or approximate them. One may even try to see if any given map in a suitable function space is homotopic to a (fractional) harmonic map, as is motivated for the harmonic gradient flow in Struwe [91] and for example done in the early investigation of the harmonic gradient flow in Eells-Sampson [33] for special cases. To address such questions, it is natural to formulate the associated gradient flow with the energy functional $\mathcal{E}$ which follows the direction of steepest descent, thus intuitively approximating minima, leading to the PDE:

$$\partial_t u = -d\mathcal{E}(u). \tag{1.30}$$

The formulation of these types of problems immediatly necessitate a discussion of existence, uniqueness, regularity and potential formation of singularities, a phenomenon if great interest in general, but specifically in the case of conformally invariant PDEs. This is what we address in the case of the (half-)harmonic gradient flow.

1.2.1 Harmonic Gradient Flow - A Model for later Considerations

Having these kinds of consideration in the back of our mind, we now briefly discuss the harmonic gradient flow with the intent of drawing parallels between the half-harmonic gradient flow and the harmonic gradient flow and identifying the key techniques and arguments to be generalised. The governing equation for the harmonic gradient flow is:

$$\partial_t u - \Delta_M u \perp T_u N, \tag{1.31}$$

which may be transformed into the following expression, in much the similar way as for the harmonic map equation:

$$\partial_t u - \Delta_M u = A_N(u)\left(\nabla u, \nabla u\right). \tag{1.32}$$

Here, $u : [0, T[\times M \to N$ is a map between general closed Riemannian manifolds and Δ_M denotes the Laplace-Beltrami operator on M and A_N the second fundamental form of N embedded in N. To formulate the problem completely, one would also need to assign an intial datum $u(0, \cdot) = u_0 \in H^1(M; N)$.

Some of the first results for the harmonic gradient flow were obtained by Eells and Sampson [33], proving that existence and regularity of solutions in case the target manifold N has non-positive curvature. A general existence result, not relying on special properties of the geometry of N, was later proven by Struwe in the 1980s in [89] for two-dimensional M and also in [90] for higher dimensional M. His approach consisted of establishing uniform estimates for solutions of the harmonic gradient flow in terms of the initial energy as well as the concentration of energy at the starting time, which ultimately also provided control for sufficiently close times afterwards. Using a general existence result for such flows (Hamilton [43]) with smooth boundary data, one then obtains in a sense a concentration-compactness-type property, leading to good convergence outside of the points of concentration, while in points where energy accumulates the formation of a harmonic bubble is observed. To summarise, the following result is proven in Struwe [89] (for convenience, we once more state it for $\mathbb{T}^2$ only, keeping in mind that the result extends to general closed surfaces or also free-boundary versions Li [56]):

Theorem 1.2.1.1 ([89])**.** *Let* $u_0 \in H^1(\mathbb{T}^2; N)$*. Then there exists a solution* $u \in H^1(]0, +\infty[; L^2(\mathbb{T}^2; N))$ *of the harmonic gradient flow:*

$$\partial_t u - \Delta u = A_N(u)(\nabla u, \nabla u) \quad \text{in } \mathcal{D}'(]0, T[\times \mathbb{T}^2), \quad \forall T > 0, \tag{1.33}$$

together with the boundary conditions:

$$u(0, x) = u_0(x), \quad \text{for all } x \in \mathbb{T}^2, \tag{1.34}$$

and satisfying $E(u(t, \cdot)) \leq E(u_0)$ *for all times* $t \geq 0$*. The solution* u *is regular on* $]0, +\infty[\times \mathbb{T}^2$*, except in a finite number of points* (t_k, x_k)*,* $k = 1, \ldots, K$*, for some* $K \in \mathbb{N}$*, where energy concentrates. Additionally,* u *is unique in the class* $\mathcal{E} \subset H^1_{loc}([0, +\infty[\times \mathbb{T}^2)$ *defined by:*

$$\mathcal{E} := \left\{ u \;\middle|\; \exists m \in \mathbb{N}, \exists T_0 = 0 < T_1 < \ldots < T_m < \infty : u \in L^2([T_i, T_{i+1}[; W^{2,2}(\mathbb{T}^2)), \forall i \leq m - 1 \right\}.$$

Finally, there exists a constant $C > 0$ *independent of* u_0*, such that:*

$$K \leq C \cdot E(u_0).$$

Thus, existence, uniqueness of strong solutions (i.e. solutions with sufficient weak regularity to make sense of (1.33) in L^2 outside of bubbling points), regularity and bubbling phenomena due to concentration are all addressed for the harmonic gradient flow. At this point, we would like to draw attention to the naturality of the non-increasing energy assumption: Indeed, if $u : [0, \infty[\times \mathbb{T}^2 \to N$ is any smooth solution of the harmonic gradient flow, then one immediately sees by testing the equation (1.32) against $\partial_t u$, which lies in the tangent space $T_u N$, that for all $T > 0$:

$$\int_0^T |\partial_t u|^2 dx dt + E(u(T)) - E(u(0)) = 0,$$

therefore implying that:

$$E(u(T)) = E(u(0)) - \int_0^T |\partial_t u|^2 dx dt \leq E(u(0)).$$

This shows that one would a-priori expect any meaningful solution to possess this property. The same holds for all gradient flows defined in a similar manner built upon a suitable energy functional. However, as one knows from the linear heat flow, such conditions may be violated for non-physical solutions (see John [49]) and indeed, there exist examples by Topping as in [94] using reverse bubbling as well as by using inner-outer gluing methods by del Pino, Davila and Wei as in [29]. Topping also discussed rates of convergence in [95].

At this point, however, there still are some open questions regarding the harmonic gradient flow: Firstly, it is not clear if bubbling may occur in finite time. In Struwe [89], there are conditions tied to the energy concentration which would force the formation of a bubble, however, no example was provided. This was answered in Chang-Ding-Ye [12], constructing directly examples of initial datums for which energy must accumulate in finite time in a corotational setting. The second question concerns uniqueness of weak solutions, i.e. solutions in the natural energy class. This was answered in Rivière [68] in the small initial energy case and later on extended in Freire [35] for arbitrary initial energies and target spaces. Indeed, the proof in Rivière [68] relies on an absorption argument (while in

Freire [35], Hélein's moving frames method is employed) at almost every fixed time to establish twice weak differentiability in the space directions using Ladyzhenskaya's estimate. It is a manifestation of concentration phenomena, as the smallness condition is tied to non-concentration and immediately also leads to global smooth solutions in this case by the techniques employed in Struwe [89]. The key result, here formulated for arbitrary target manifolds in extension of Rivière [68], is the following (we formulate it on $\mathbb{T}^2$ purely for convenience's sake):

Lemma 1.2.1.1 ([68]). *Let $u \in H^1(\mathbb{T}^2; N)$ and $f \in L^2(\mathbb{T}^2; \mathbb{R}^n)$ be given and assume that u is a solution of the PDE:*

$$-\Delta u = A_N(u)\,(\nabla u, \nabla u) + f. \tag{1.35}$$

Then $u \in H^2(\mathbb{T}^2; N)$.

The main difference in the proof of Lemma 1.2.1.1 in contrast to the one in Rivière [68] is that we need to take care of the change of gauge. However, thanks to results such as Theorem I.3, I.4 in Rivière [70], this is done in a rather direct manner: By observing that $A_N(p)$ maps to the normal space of N at any point p, one finds that (1.35) is equivalent to:

$$-\Delta u_i = \left(A^i_{jk}(u)\nabla u_k - A^j_{ik}(u)\nabla u_k\right)\nabla u_j + f_i, \tag{1.36}$$

using Einstein's summation convention. Thus:

$$-\Delta u = \Omega \cdot \nabla u + f, \tag{1.37}$$

with Ω being antisymmetric. Therefore, Theorem I.4 in Rivière [70] applies, guaranteeing the existence of $A \in L^\infty(B, Gl_n(\mathbb{R})) \cap W^{1,2}(B)$ and $B \in W^{1,2}(B, \mathbb{R}^{n\times n})$ for suitably small balls (where the L^2-norm of Ω is sufficiently small) with:

$$\nabla A - A\Omega = \nabla^\perp B,$$

as well as "good" estimates for A and B in $W^{1,2}$ and for $\operatorname{dist}(A, SO(n))$ in terms of the local L^2-norm of Ω. It should be noted that therefore:

$$\operatorname{div}(A\nabla u) = \nabla A \cdot \nabla u + A\Delta u = \nabla A \cdot \nabla u - A\Omega \cdot \nabla u - Af = \nabla^\perp B \cdot \nabla u - Af.$$

To deduce that $u \in W^{1,q}$ for some $q > 2$, one argues completely analogous to Rivière [70], see also Rivière [73], observing that the L^2-perturbation (it is clear that $Af \in L^2$ again) does not obstruct the argument due to Hölder's inequality. To conclude the argument, one therefore applies the same reasoning as in Rivière [68] to find that $u \in W^{1,4}$, leading immediately by Caldéron-Zygmund theory to $u \in H^2$ and then extend it by smallness to obtain twice weak differentiability of weak solutions to (1.32) and therefore weak uniqueness of solutions with small initial energy.

1.2.2 The Main Results for the Half-Harmonic Gradient Flow

We finally reached the point in our discussion where we are able to address the main equation for the considerations in the first part of this thesis. The equation governing the flow induced by the 1/2-Dirichlet energy is:

$$\partial_t u + (-\Delta)^{1/2} u \perp T_u N, \tag{1.38}$$

for a general closed target manifold N. Once more, the analogy with the harmonic gradient flow is evident. In the case $N = S^{n-1}$, the PDE may be reformulated as:

$$\partial_t u + (-\Delta)^{1/2} u = u|d_{1/2} u|^2, \tag{1.39}$$

drawing, in a certain sense, further connections with the harmonic map equation. In general, a non-local quadratic term emerges on the RHS of (1.38), extending similarities with (1.32) from before:

$$\partial_t u + (-\Delta)^{1/2} u = P.V. \int_{S^1} P(u(x), u(y)) \left(d_{1/2} u(x,y), d_{1/2} u(x,y) \right) \frac{dy}{|x-y|}, \tag{1.40}$$

where P is a non-local substitute of A_N. For the definition of P, we refer to Section 2.2.4.1.

The major technical and obvious difference between (1.32) and (1.40) is of course the introduction of non-local operators. This obstruction also lead to slow progress in the study of the half-harmonic gradient flow. For instance, in the paper Schikorra-Sire-Wang [77], the authors prove a global weak existence result for a broad class of non-local gradient flows associated with Gagliardo-Sobolev seminorms, not just for the 1/2-Dirichlet energy, including energies on domains of dimension ≥ 2. However, the results were restricted to target manifolds with sufficient inherent symmetry such as spheres due to the techniques of discretisation and approximation in the proof, indicating that a satisfactory treatment of existence was still open for the half-harmonic gradient flow. Moreover, the paper Schikorra-Sire-Wang [77] did not address uniqueness at all, leaving this property open for future investigations.

On the other hand, there were already first considerations of infinite time bubbling. In Sire-Wei-Zheng [84], the authors studied bubbling as $t \to \infty$ for the half-harmonic gradient flow in S^1 using the inner-outer gluing method which has also been used extensively in the study of the harmonic gradient flow, see for instance Davila-Del Pino-Wei [29]. This lead to the conclusion that bubbling is possible asymptotically with no restrictions on the number of bubbles that emerge in the limit. Interestingly, the authors formulated a conjecture in the same paper that no bubbling in finite time is possible, a stark contrast to the situation for the harmonic gradient flow, as seen in Chang-Ding-Ye [12]. This conjecture remains, according to our knowledge, open so far.

As a result, even fundamental questions pertaining to the well-posedness of the equation (1.40) were still open and required a different technique. The essence of the approach in this thesis is to generalise the steps in the proof of the corresponding properties for the harmonic gradient flow (for example local control of the concentration of energy), see Struwe [89]. Therefore, the questions reduced basically to three subproblems: Firstly, we have to establish existence of (smooth) solutions for a small time interval, if the initial data is smooth. Such a result for the harmonic gradient flow is fundamental in Struwe [89], as it renders the approximation procedure possible, and can, for example, be found in Hamilton [43]. It should be noticed that here we also need to prove a generalised approximation lemma in $H^{1/2}(S^1; N)$, allowing us to replace arbitrary boundary data by smooth boundary data and passing to the limit. Secondly, we need to generalise the control of the energy concentration found for the harmonic gradient flow. This will ultimately help in establishing uniform estimates over time intervals for Sobolev norms and thus allow for the treatment of general initial data by approximation in $H^{1/2}(S^1; N)$. Additionally, such concentration inequalities are also intimately connected to concentration-compactness phenomena, quantifying the rate of accumulation. Lastly, we would like to deduce statements about the concentration of energy and potential bubbling, potentially to address the conjecture by Sire, Wei and Zheng. To summarise the results in [102], [103] and [104], we have the following:

Theorem 1.2.2.1 ([102], [103], [104])**.** *Let N be any given closed manifold and $u_0 \in H^{1/2}(S^1; N)$. Then there exists a maximal $0 < T := T(u_0) \leq +\infty$ as well as a solution $u : [0, T[\times S^1 \to N \subset \mathbb{R}^n$,*

$u \in L^\infty([0,T[;H^{1/2}(S^1;N)) \cap H^1([0,T[;L^2(S^1;N))$, to the weak formulation of the half-harmonic gradient flow:

$$\partial_t u + (-\Delta)^{1/2} u \perp T_u N, \quad u(0,\cdot) = u_0 \tag{1.41}$$

Moreover, the following properties hold for the solution u:

- ***Monotonicity:*** *The $1/2$-Dirichlet energy is non-increasing along the flow:*

$$\forall 0 \le s \le t < T(u_0): \quad E(u(t)) \le E(u(s)) \le E(u_0) < +\infty. \tag{1.42}$$

- ***Regularity:*** *The solution u is smooth on $]0,T[\times S^1$ and solves the equation (1.41) pointwise.*

- ***Uniqueness:*** *The solution u is unique among competitors in $H^1([0,T[\times S^1)$ with non-increasing energy. The same holds for a global extension by gluing, provided the energy decay holds for all times.*

- ***Small Energy:*** *There exists $\varepsilon > 0$ depending on N, such that:*

$$E_{1/2}(u_0) < \varepsilon \quad \Rightarrow \quad T(u_0) = +\infty \tag{1.43}$$

  In addition, uniqueness holds for such u_0 and the associated solution even within the energy class $L^\infty([0,+\infty[;H^{1/2}(S^1;N)) \cap H^1([0,+\infty[;L^2(S^1;N))$ and for an appropriate subsequence $t_k \to \infty$, we have that $u(t_k,\cdot)$ converges weakly in $H^1(S^1;N)$ to a point.

- ***Bubbling:*** *It holds:*

$$\limsup_{t\to T} \varepsilon(R;u,t) \ge \varepsilon, \quad \forall R \in]0,1/2[, \tag{1.44}$$

  where:

$$\varepsilon(R;u,T) := \sup_{x\in S^1, t\in[0,T]} \frac{1}{2}\int_{B_R(x)} |(-\Delta)^{1/4} u(t)|^2 dx, \tag{1.45}$$

  and after suitable rescaling, the sequence of reparametrisations converge weakly to a half-harmonic map, i.e. the formation of bubbles at time T may be observed.

The solution on $[0,T[$ may be extended to all of $[0,+\infty[$ by gluing solutions, leading to finitely many times $0 < T_1 < T_2 < \ldots < T_k < +\infty$ where singularities, i.e. bubbling, may occur, with the number of singular times being bounded by:

$$k \le \frac{E_{1/2}(u_0)}{\sigma(N)}, \tag{1.46}$$

where σ is defined by:

$$\sigma(N) := \inf \left\{ E_{1/2}(v) \mid v : S^1 \to N \text{ is half-harmonic and non-constant} \right\} > 0, \tag{1.47}$$

with the convention $\inf \emptyset = +\infty$.

The statements in Theorem 1.2.2.1 are very similar to the ones in Theorem 1.2.1.1. The treatment of $N = S^{n-1}$ except for the bubbling and global existence may be found in [102], the general case (again without bubbling and global existence) in [103] and finally, in [104] we study the local concentration behaviour of solutions and establishes existence of global weak solutions by two different means.

Since the publication of the three papers [102], [103], [104], other authors contributed to the theory of half-harmonic maps: In Hyder-Segatti-Sire-Wang [47], the authors consider a different type of half-harmonic gradient flow with the same critical points, but relying on extensions solving the homogeneous heat equation. The governing equation is:

$$(\partial_t - \Delta)^{1/2} u \perp T_u N. \tag{1.48}$$

They use caloric extensions, Ginzburg-Landau approximation as well as parabolic estimates relying on monotonicity properties of special integral quantities involving the fundamental solution of the heat equation. Thus, they are able to treat domains of arbitrary dimension (finding that, up to the exclusion of lower dimensional subsets, regularity holds), but the flow does not allow for an immediate interpretation of energy minimisation as is possible with the half-harmonic gradient flow we study. Moreover, the solution constructed is not proven to be unique in an appropriate sense. It should also be emphasised that in Hyder-Segatti-Sire-Wang [47], the operator considered is a fractional power of the heat operator, bringing non-locality also to the time derivative.

In Struwe [93], the author considers the harmonic extensions of the boundary values in order to treat the half-harmonic gradient flow by means of local operators. The approach heavily relies on the Dirichlet-to-Neumann interpretation (see Caffarelli-Silvestre [11]) of the half-Laplacian $(-\Delta)^{1/2}$ to formulate a flow for the boundary values and enables the author to find a result very similar to Theorem 1.2.2.1. However, the draw-back of the approach presented there, at least for now, lies in the need for parallelizability of the target manifold, since a signed distance function is used in the arguments. Thus, our approach, while heavily employing fractional methods, remains the most general one and especially significant, as it does not need to put any technical assumptions on the target manifold. Furthermore, in Struwe [93], the author connects the flow to critical point theory and flows for Plateau's problem based on the work in Struwe [87], Jost-Struwe [50], Chang-Liu [13] among others. We refer to Struwe [93] for a detailed exposition and relations with the half-harmonic gradient flow.

To conclude this introduction to the half-harmonic gradient flow, we would like to mention some open problems. For instance, it is still open whether bubbling in finite time can occur, a related conjecture may be found in Sire-Wei-Zheng [84]. Moreover, one may try to apply the techniques to the gradient flow for different Gagliardo-seminorms as energies and higher dimensional domains, extending potentially the results in Schikorra-Sire-Wang [77] even further. The flexibility of the methods associated with fractional harmonic maps for various different critical exponents suggests that similar extensions for the associated flows may also be obtained. Lastly, in a recent line of research, Schikorra, Blatt, Reiterer and Vorderobermeier generalised regularity considerations by Freedman and He for Möbius energies for knots to O'Hara (Blatt-Reiterer-Schikorra [6]) energies and so-called tangent point energies (Blatt-Reiterer-Schikorra-Vorderobermeier [7]). Their approach is based on reformulating the associated self-repulsive knot energies in terms of the normalised derivative, leading to a connection with half-harmonic maps and the techniques associated with their regularity theory. Blatt considered the gradient flow for the Möbius energy in [5], potential generalisations for other types of self-repulsive knot energies may be interesting to consider in the future.

1.3 Compensation Phenomena and Dirac's equation

Next, we switch gears and consider a different problem related to compensation results similar to Rivière [70] and Da Lio-Rivière [23]. Namely, in Da Lio-Rivière [23], the authors prove that provided

$S \in \dot{W}^{1,2}(\mathbb{R}^2, O(n))$ with $S^2 = Id$, any weak solution $u \in L^2(\mathbb{R}^2;\mathbb{R}^n)$ of the PDE:

$$\operatorname{div}(S\nabla u) = \sum_{k=1}^{n}\sum_{\alpha=1}^{2} \partial_{x_\alpha}\left(S_{jk}\partial_{x_\alpha} u_k\right) = 0, \tag{1.49}$$

automatically has some regularity:

$$u \in \bigcap_{p<2} W^{1,p}(\mathbb{R}^2;\mathbb{R}^n). \tag{1.50}$$

As one would expect, the underlying reason for such a result lies in hidden compensation phenomena that one would like to exploit. Such a regularity result is intimately connected with a concentration-compactness result for weakly converging sequences of such solutions u. The assumptions on S are motivated by the existence of a counterexample to the regularity result in Jin-Maz'ya-Van Schaftingen [48], provided $S \in W^{1,2}(B_1(0); Sym(2))$ only, so no orthogonality for the matrices S. Therefore, to arrive at the regularity properties outlined above, more structure of the matrix S has to be available.

The regularity result for (1.49) may be equivalently stated for equations of the form:

$$\frac{\partial f}{\partial z} = \Omega \overline{f}, \tag{1.51}$$

where the imaginary part of $\partial_{\overline{z}}\Omega$ vanishes and $f \in L^2(\mathbb{R}^2;\mathbb{C}^2)$, $\Omega \in L^2(\mathbb{R}^2; so(2)\otimes\mathbb{C})$. Indeed, one of the main novelties in Da Lio-Rivière [23] lies in the possibility of rewriting (1.49) as (1.51). It should be observed that Rivière [70] treats the same equation, after replacing $\overline{f}$ by f, so the result in Da Lio-Rivière [23] allows for the discussion of new types of potentials.

Among the first projects in my doctorate was the investigation of results similar to the one presented above for domains of higher dimension. Natural obstacles immediately arose due to the use of quaternions $\mathbb{H}$[1] as well as the introduction of a suitable gauge group, the unit quaternions. As the proof critically relies on these techniques, finding alternative algebras was a crucial first point, leading to the introduction of *Clifford algebras*. Moreover, during the process of investigating higher-dimensional analogues of (1.49), two quite different results emerged in my work together with Da Lio and Rivière, resulting in [25], [26]: A characterisation of Bergmann spaces in terms of their boundary values ([25]) and a regularity result for solutions to Dirac's equation ([26]). We shall go into some more detail in the next two subsections on each of these results:

1.3.1 Bergmann-Bourgain-Brezis Inequality

When attempting to bring (1.49) into a more suitable form for a change of gauge, one first observes that:

$$\operatorname{div}(S\nabla u) = 0 \Rightarrow \nabla^\perp v = S\nabla u, \tag{1.52}$$

by using Poincaré's lemma. Here, we use the short hand notation:

$$\nabla^\perp v := \begin{pmatrix} -\partial_y v \\ \partial_x v \end{pmatrix},$$

[1] The quaternions form a non-commutative field which is isomorphic as a real vector space to $\mathbb{R}^4$. Denoting by $1, i, j, k$ the standard basis, the quaternionic product is then determined by the identities (defining 1 to be the multiplicative identity):

$$i^2 = j^2 = k^2 = ijk = -1.$$

Naturally, $\mathbb{C} \subset \mathbb{H}$ holds with the natural identifications. The fact that $S^3 \subset \mathbb{H} \simeq \mathbb{R}^4$ is a Lie group follows by noticing that the standard Euclidean norm is multiplicative with respect to the quaternionic product.

to denote the rotation, naturally applied in each component separately. Notice that therefore:

$$\nabla^{\perp} v = S\nabla u = \nabla(Su) - \nabla S \cdot u \in W^{-1,2} + L^1(\mathbb{R}^2; \mathbb{R}^n). \tag{1.53}$$

Indeed, to define $f := u + iv$ and arrive at a more manageable equation, it is therefore crucial to establish $f \in L^2(\mathbb{R}^2; \mathbb{R}^n)$. Thus, it suffices to check:

$$v \in L^2. \tag{1.54}$$

Fortunately, there is a result due to Bourgain and Brezis, see Bourgain-Brezis [8] and Da Lio-Rivière [23] for its generalisation without periodicity assumption, which guarantees that such a v may be chosen, establishing integrability of f.

The inequality by Bourgain and Brezis also allows one to deduce (by duality, if $n = 2$) that for any $h \in L^n(\mathbb{R}^n)$ periodic with vanishing integral over $\mathbb{R}^n$ (denoted by $L^n_{\#}(\mathbb{R}^n)$, there exists a continuous solution $v \in \dot{W}^{1,n} \cap L^\infty(\mathbb{R}^n)$ of the divergence system:

$$\operatorname{div} v = h.$$

Particularily interesting is the fact that there is no bounded linear right inverse $K : L^n_{\#}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ to the divergence operator, implying that the solution found depends non-linearily on h (again, see [8]). The main idea is that one defines the operator $T : C^0 \cap H^1(\mathbb{T}^2) \to L^2_{\#}(\mathbb{T}^2)$ given by $Tf := \operatorname{div} f$ and notices that its dual operator satisfies:

$$T^* : L^2_{\#} \to \mathcal{M} + H^{-1}(\mathbb{T}^2), \quad u \mapsto T^*(u) := \nabla u \tag{1.55}$$

Here, $\mathcal{M} = \mathcal{M}(\mathbb{T}^2)$ denotes the dual space of $C^0(\mathbb{T}^2)$, i.e. the Banach space of signed Radon measures. Bourgain-Brezis' inequality then states:

$$\|u\|_{L^2} \lesssim \|\nabla u\|_{H^{-1}+L^1} = \|T^* u\|_{\mathcal{M}+H^{-1}} \tag{1.56}$$

By the closed range theorem, this immediately shows that T has closed range and the image actually agrees with the annihilator of the kernel of T^*. Since the latter is trivial (note that $\nabla u = 0$ implies that u is constant with mean 0, thus $u = 0$), surjectivity follows immediately.

Such a result is remarkable, as Sobolev embeddings are too weak to find such a solution. Results for Hodge decompositions in critical exponents were also obtained in Bourgain-Brezis [9]. Moreover, in Bourgain-Brezis [8], the construction of such a solution is shown to lead to a non-linear solution operator for the divergence problem, as mentioned above. This highlights the delicacy of the problem. Naturally, one may immediately wonder if Bourgain-Brezis' inequality and similar kinds of estimates could be extended to other situations and spaces, for example more general Hilbert-Sobolev spaces. Such ideas have been pursued, for instance, in Maz'ya [58] and Mironescu [61]. Furthermore, the following fractional version of the result in Bourgain-Brezis [8] is the focus of [25]:

Theorem 1.3.1.1 ([25]). *Let $n \geq 1$ be a natural number, $u \in \mathcal{D}'(\mathbb{T}^n)$. If we have that:*

$$(-\Delta)^{n/4} u, \mathcal{R}_j (-\Delta)^{n/4} u \in L^1 + \dot{H}^{-n/2}(\mathbb{T}^n), \quad \forall j \in \{1, \ldots, n\}, \tag{1.57}$$

where $\mathcal{R}_j$ are the Riesz operators[2]*, then $u - ⨍_{\mathbb{T}^n} u dx \in L^2_{\#}(\mathbb{T}^n)$ together with the estimate:*

$$\left\| u - ⨍_{\mathbb{T}^n} u dx \right\|_{L^2} \leq C \left(\|(-\Delta)^{n/4} u\|_{L^1 + \dot{H}^{-n/2}} + \sum_{j=1}^{n} \|\mathcal{R}_j (-\Delta)^{n/4} u\|_{L^1 + \dot{H}^{-n/2}} \right), \tag{1.58}$$

[2]The Riesz operators $\mathcal{R}_j$ for $j \in \{1, \ldots, n\}$ are just the Riesz transforms with respect to the variable x_j and may be defined as the Fourier multiplier operators associated with the multiplier $m(\xi) := i\xi_j / |\xi|$.

where $C > 0$ is independent of u.

The main idea in proving (1.58) is the introduction of suitable operators on Clifford algebras:

$$Dv := (-\Delta)^{n/4}\left(Id + \sum_{j}^{n} e_j \mathcal{R}_j\right) v, \quad \overline{D}v := (-\Delta)^{n/4}\left(Id - \sum_{j}^{n} e_j \mathcal{R}_j\right) v, \tag{1.59}$$

where e_j is the standard basis of $\mathbb{R}^n$ considered as elements in the universal Clifford algebra and satisfying the identities:

$$e_i e_j + e_j e_i = 2\delta_{ij}, \quad \forall i, j \in \{1, \dots, n\}, \tag{1.60}$$

where $\delta_{ij} = 1$, if $i = j$, and 0 else. One may see by direct computations that the Fourier symbol of D^2 is precisely:

$$2i \cdot |m|^n \left(\sum_{j=1}^{n} e_j \frac{m_j}{|m|}\right) = 2i \cdot |m|^{n-1} m. \tag{1.61}$$

The inverse Fourier symbol exists and can be computed explicitly. It corresponds to the convolution with a suitable bounded function on $\mathbb{T}^n$. This ultimately allows us to obtain better integrability properties and find the L^2-estimate for $u - \fint_{\mathbb{T}^n} u dx$.

Interestingly enough, Theorem 1.3.1.1 is closely connected to the Bergmann space $\mathcal{A}(\mathbb{D})$ on the unit disc $B_1(0) =: \mathbb{D} \subset \mathbb{R}^2$. To fix notation, one defines:

$$\mathcal{A}(\mathbb{D}) := \left\{ f : \mathbb{D} \to \mathbb{C} \mid f \text{ holomorphic and } \|f\|_{L^2(\mathbb{D})} < \infty \right\}. \tag{1.62}$$

This is a Banach space and related to the Hardy space $\mathcal{H}^1(\mathbb{D})$ via the embedding $\mathcal{H}^1(\mathbb{D}) \subset \mathcal{A}(\mathbb{D})$. The Hardy space may be described in terms of its boundary values only[3], so it is natural to wonder whether an analogous characterisation for Bergmann spaces is available as well. Indeed, such a characterisation is a consequence of Theorem 1.3.1.1 in the case $n = 1$:

Theorem 1.3.1.2 ([25])**.** *A holomorphic function $f : \mathbb{D} \to \mathbb{C}$ belongs to the Bergmann space $\mathcal{A}(\mathbb{D})$ if and only if:*

$$\|f\|_{L^1 + H^{-1/2}(S^1)} := \limsup_{r \to 1^-} \left\| f\left(re^{i\theta}\right) \right\|_{L^1 + H^{-1/2}(S^1)} < +\infty. \tag{1.63}$$

Additionally, it even holds:

$$\|f\|_{L^2(\mathbb{D})} \leq C \|f\|_{L^1 + H^{-1/2}(S^1)}, \tag{1.64}$$

for some constant $C > 0$ independent of f.

It even holds that Theorem 1.3.1.2 is equivalent to Theorem 1.3.1.1 in the case $n = 1$. This equivalence is the second main result in [25], its proof shall be explained in Section 3.1.

[3] In fact, we have for holomorphic functions f:

$$f \in \mathcal{H}^1(\mathbb{D}) \Longleftrightarrow \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\varphi})| d\varphi < +\infty.$$

In this case, $f(re^{i\varphi})$ converges pointwise almost everywhere and in L^1 to $f(e^{i\varphi})$ and the L^1-norm of these boundary values are just the Hardy norm of the holomorphic f.

Since the publication of [25], there has been an effort to use similar ideas in another context. In Yong-Yanqi-Zipeng [105], the authors introduce the notion of holomorphic stability for pairs of function spaces and use the results and the proof in [25] to conclude that the Hardy and Bergmann space on the unit disc form such a holomorphic stable pair. We refer the interested reader to the paper Yong-Yanqi-Zipeng [105], as we shall not pursue this line of thought in the present thesis further.

1.3.2 Compensation Results for Dirac's Equation

Let us now turn to compensation phenomena for special PDEs not involving antisymmetric potentials. Before we turn to the results found in [26], let us give a brief overview of the key ideas in Da Lio-Rivière [23]:

For simplicity, let us first consider the case of two-dimensional codomains as in Da Lio-Rivière [23]. Namely, let $u \in L^2(\mathbb{R}^2;\mathbb{R}^2)$ and $S \in \dot{W}^{1,2}(\mathbb{R}^2; O(2) \cap Sym(2))$, such that:

$$\operatorname{div}(S\nabla u) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{1.65}$$

The first step is to apply Poincaré's Lemma combined with a Bourgain-Brezis inequality to find $v \in L^2(\mathbb{R}^2;\mathbb{R}^2)$, such that:

$$\nabla^{\perp} v = S\nabla u. \tag{1.66}$$

Now, finding a matrix $Q \in \dot{W}^{1,2}(\mathbb{R}^2; SO(2))$ under a smallness assumption on $\|\nabla S\|_{L^2}$, such that:

$$Q^{-1}SQ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =: S_0,$$

one may introduce $f := S_0Qu + iQv$. We emphasise here that S_0 could also be the identity matrix up to a sign change. However, this case is trivial, as then also $S = \pm Id$ and thus we are only dealing with u harmonic, for which obvious elliptic regularity properties would immediately prove regularity. Thus, the case considered here corresponds to the interesting case.

The next observation lies in studying the equation solved by f. Due to the emergence of $\nabla^{\perp} v$ and ∇u in (1.66), we may state the equation in terms of the complex derivative for the function $f = S_0Qu + iQv$:

$$\partial_z f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_z \alpha \cdot \overline{f}. \tag{1.67}$$

Here, α is the angle of rotation associated with Q and lies in $\dot{W}^{1,2}(\mathbb{R}^2;\mathbb{R})$, see [23, Thm. III.6]. Writing $f = (f_1, f_2)$, introducing the complex unit $j \in \mathbb{H}$ and defining:

$$\mathfrak{f} := f_1 + f_2 j, \tag{1.68}$$

we now get the more natural formulation (here ∂_L is just ∂_z acting on functions from the left[4]):

$$\partial_L \mathfrak{f} = -\partial_z \alpha j \cdot \mathfrak{f}. \tag{1.69}$$

We notice that in the case of (1.67), the assumption $\operatorname{Im}(\partial_{\bar{z}}\partial_z \alpha) = \operatorname{Im}(\Delta\alpha) = 0$ is always satisfied, keeping in mind that α is real-valued. In the last step above, several properties of the quaternions are

[4]The distinction between ∂_L and ∂_z is made, since in the former i is considered as an element in $\mathbb{H}$, while in the latter it is considered an element in $\mathbb{C}$.

exploited such as non-commutativity of the complex units, the relation to complex conjugation and the relation $ij = k$. At this point, despite starting with an equation like (1.65) without an antisymmetric potential providing compensation properties, we have reformulated the main equation and obtained (1.69), which now is characterised by a potential which takes values in the Lie algebra associated with the unit quaternions, leading to the possibility of a change of gauge approach analogous to Rivière [70]. The steps from here are then natural, but slightly technical: One needs to define an appropriate gauge operator, absorb ill-behaved terms and establish a suitable Morrey decrease to conclude, details may be found in Da Lio-Rivière [23].

In [26], together with Da Lio and Rivière, we continued pursuing this line of thought and pushing the ideas used further. While many compensation phenomena involving antisymmetric potentials (for example also for fractional PDEs as in the previously mentioned Da Lio-Rivière [22], Mazowiecka-Schikorra [57]) have been explored since Rivière [70], a lack of such a potential (i.e. potentials with values not in the real Lie algebra of antisymmetric matrices) still poses problems in the regularity theory of critical PDEs. In fact, while (1.67) involves a potential, but it does not stem from a compact Lie algebra, obstructing the gauge construction at this level.

In the paper [26], we contribute to the theory of critical PDEs with no antisymmetric potential by considering the particular PDE:

$$\partial_L \mathfrak{f} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \hat{\mathfrak{f}}, \tag{1.70}$$

for $\mathfrak{f} \in L^{m/(m-1)}(\mathbb{R}^m; C\ell_{m-1} \times C\ell_{m-1})$ and $\beta \in W^{1,m/2}(\mathbb{R}^m; \operatorname{span}_{\mathbb{R}}\{e_0, e_1, \dots, e_{m-1}\})$, with part of the rotation of β vanishing:

$$\partial_i \beta_j - \partial_j \beta_i = 0, \quad \forall i, j \in \{1, \dots, m-1\},$$

here using $(x_0, x_1, \dots, x_{m-1}) \in \mathbb{R}^m$. Here, we write $C\ell_{m-1}$ to denote the universal Clifford algebra over $\mathbb{R}^{m-1}$, a powerful and often useful generalisation of the complex numbers as well as the quaternions. To define the Clifford algebra, one takes $\mathbb{R}^{m-1}$ (indeed, any vector space V equipped with a quadratic form Q) and defines the multiplication on the standard basis (or on any orthonormal basis for the corresponding bilinear form B associated with Q) elements as follows:

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad \forall i, j \in \{1, \dots, m-1\}, \tag{1.71}$$

and add a neutral element 1. It is important to be aware that Clifford algebras are associative, but generally not commutative. From the very definition, we already have $e_1 e_2 = -e_2 e_1$. Thus, the space $\mathbb{R}^m = \mathbb{R} \oplus \mathbb{R}e_1 \oplus \dots \oplus \mathbb{R}e_{m-1}$ is naturally a subspace of $C\ell_{m-1}$. Among the most prominent applications of Clifford algebras is their intimate connection to geometry (for instance, quantities such as the scalar product, cross product and others appear naturally in this context), the extension of decompositions of the Laplacian into first order operators similar to $\partial_z \partial_{\overline{z}} = \Delta$ by means of Dirac operators and the definition of the *Spin groups*, the double-covering spaces of the special orthogonal groups. We refer interested readers to Hamilton [42] for the basic theory and applications to gauge theory and physics as well as to Gilbert-Murray [38] for applications more closely related to mathematical analysis, including for example an extension of the identification of Hardy spaces by means of holomorphic functions. The most well-known examples of Clifford algebras include:

$$C\ell_0 = \mathbb{R}, \quad C\ell_1 = \mathbb{C}, \quad C\ell_2 = \mathbb{H} \tag{1.72}$$

Due to the close connection with complex numbers and quaternions, the appearance of Clifford algebras should come as no surprise in the formulation of suitable generalisations of the regularity results in Da Lio-Rivière [23].

The hat operator $\mathfrak{f} \mapsto \hat{\mathfrak{f}}$ is the *grade involution*, a natural generalisation of the complex conjugation to arbitrary Clifford algebras[5] with the property:

$$\widehat{ab} = \hat{a}\hat{b}, \quad \forall a, b \in C\ell_{m-1}.$$

The crucial link between the grade involution and the complex conjugation in the argument of Da Lio-Rivière [23] follows by:

$$\forall a \in C\ell_{m-1} : \hat{a}e_m = e_m a,$$

which is nothing but an iterated application of (1.71). As a result, arguing completely analogous to Da Lio-Rivière [23], one notices that the extension to the higher-dimensional Clifford algebra $C\ell_m$ by means of adding the basis element e_m allows us to rewrite this equation in the following way:

$$\partial_L \mathfrak{g} = -\beta e_m \cdot \mathfrak{g},$$

where $\mathfrak{g} := \mathfrak{f}_1 + \mathfrak{f}_2 e_m$. From here on, one proceeds quite similar to Da Lio-Rivière [23], however there are some obstructions in the argument related to the construction of a suitable gauge operator and the improved regularity properties. The former results in the main theorem only holding on domains of dimension ≤ 8, as in higher dimensions the construction of a gauge operator we employ no longer applies. The reason lies in the fact, that the definition hinges on parallelisability of the sphere S^7 by choice of a constant coefficient elliptic differential operator of first order, or to be more precise, the existence of the octonions, by virtue of defining the gauge operator to include an elliptic operator with symbol that realises an orthogonal frame. For higher dimensions, no sufficiently nice division algebra exists and this limits the techniques used to the range of domains specified. The final result may be stated in the following form for codomains of dimension 2:

Theorem 1.3.2.1 ([26])**.** *Let $m \leq 8$. $\beta = (\beta_0, \cdots \beta_{m-1}) \in W^{1,m/2}_{loc}(\mathbb{R}^m, \mathrm{span}_{\mathbb{R}}\{e_0, \cdots, e_{m-1}\})$ with*

$$\forall\, i, j = 1 \cdots m-1 \qquad \partial_{x_i}\beta_j - \partial_{x_j}\beta_i = 0\ . \tag{1.73}$$

Let $\mathfrak{f} \in L^{m/m-1}(\mathbb{R}^m, C\ell_{m-1} \times C\ell_{m-1})$ be a solution of

$$\partial_L \mathfrak{f} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \hat{\mathfrak{f}}. \tag{1.74}$$

Then $\mathfrak{f} \in L^q_{loc}(\mathbb{R}^m, C\ell_{m-1} \times C\ell_{m-1})$ for all $q < \infty$. □

This result is discussed in Section 3.2 for the case $m = 3, 4$ and extended in the last section, Section 3.3, to domains of dimension $m \leq 8$ as well as for arbitrary codomains.

From here, there are several natural open problems associated with the PDE one could investigate. Firstly, the motivation to study the phenomenon of critical chirality in Da Lio-Rivière [23] came partly from the close connection with a divergence-type PDE. Such a connection is not immediately possible

[5] While this is one potential generalisation of complex conjugation, this does not lead to the quaternionic conjugation on $C\ell_2 \simeq \mathbb{H}$. Identifying $e_1 = i, e_2 = j$, then:

$$a + b\widehat{i + c}j + dk = a + be_1 \widehat{+ ce_2} + de_1e_2 = a - be_1 - ce_2 + de_1e_2.$$

Indeed, quaternionic conjugation satisfies $\overline{qp} = \bar{p}\bar{q}$, so it inverts the order of multiplication.

for the case of higher-dimensional domains. At best, one can hope for a system of divergence-type PDEs, as the emergence of the principal automorphism of the Clifford algebra immediately forces us to control half of the component functions, which leads to a less natural system of PDEs. It would be interesting to see, if an alternative approach exists, allowing a treatment of the natural divergence-type PDE:

$$\operatorname{div}(S\nabla u) = 0,$$

also for higher dimensional domains.

On the other hand, the restriction to domains of dimension ≤ 8 might be of technical nature, i.e. inherent to the approach chosen and not a property of the equation itself. Thus, investigating alternative approaches could lead to the discovery of further compensation phenomena and might generalise our results to arbitrary domains.

Lastly, the Dirac equation studied in [26] appears to be closely connected, at least for 3D domains, to Lorentz gauges via the vanishing rotation assumption. However, so far, a more explicit and general connection to physics has eluded the study of the authors. We are convinced that the naturality and connection in the special case of 3D domains hints at a physical interpretation of the PDE, but this has to be studied in some future work.

1.4 Outline of the Thesis

In the current chapter, we have explored the context of the papers [25], [26], [102], [103], [104] included in the subsequent sections in more detail and put them into a general context. Next, in Chapter 2, the author's contribution to the half-harmonic gradient flows are collected. To be precise, in Section 2.1, the paper [102] is contained dealing with the case of spherical target manifolds, in Section 2.2 we have [103] generalising the previous results to arbitrary closed target manifolds, and lastly, Section 2.3 treats [104] and thus introduces a first investigation into the formation of singularities along the flow and potentially laying the foundation for future work in this area. In Chapter 3, we discuss the contributions to compensation phenomena inspired by Da Lio-Rivière [23] as briefly introduced above. Section 3.1 discusses the results in [25] leading to characterisations of Bergmann spaces in terms of their boundary values, while Section 3.2 provides a discussion of results pertaining to improved regularity for the Dirac equation, see [26]. Additionally, in Section 3.3, we expand on the discussion provided in [26] by including some previously unpublished results. To be more specific, we treat the case of general codomains and give some details in the case of domains of dimension ≤ 8.

2 Half-Harmonic gradient flow [102], [103], [104]

2.1 The Case of Spherical Targets [102]

In this section, we study the half-harmonic gradient flow with values in spheres. Similar to the harmonic gradient flow and (fractional) harmonic maps, this case is simpler from a technical point of view, as the equations and compensation phenomena are easier to find. In particular, we prove local existence of the flow, uniqueness in a strong sense as well as smoothness of the solution. In case the initial energy is small, we are even able to show that solutions exist globally and are unique in the energy class, a result which relies on an ingenious trick by Rivière.

2.1.1 Introduction

In this paper, we shall study gradient flows associated with the half-harmonic map equation, in particular questions pertaining to uniqueness, regularity and convergence as $t \to +\infty$ of solutions of the fractional harmonic gradient flow in $S^{n-1} \subset \mathbb{R}^n$. In [89] and [90], Struwe studied global existence and uniqueness for the gradient flow associated with the classical harmonic map equation both in dimension 2 as well as higher dimensions. Recall that harmonic maps are critical points of the standard Dirichlet energy which is defined for all maps $u : M \to N \subset \mathbb{R}^n$ in $H^1(M; N)$ by:

$$E(u) := \frac{1}{2} \int_M g^{\alpha\beta}(x) \gamma_{ij}(u(x)) \frac{\partial u^i}{\partial x_\alpha}(x) \frac{\partial u^j}{\partial x_\beta}(x) dx,$$

where $(M, g), (N, \gamma)$ smooth Riemannian manifolds, $u = (u^1, \dots, u^n)$ and employing Einstein's summation convention. In case $M = \Omega \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ are isometrically embedded in $\mathbb{R}^m$ and $\mathbb{R}^n$ and equipped with the Riemannian metrics induced by the standard scalar product, this reduces to:

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx$$

In domains of dimension 2, Struwe actually showed that up to a bubbling process at finitely many points, the number of which can be bounded by the initial energy, there exists a unique regular solution for all times. To be more precise, Struwe proved the following for the target manifold $N = S^{n-1}$ (a completely analogous result holds for general N):

Theorem 2.1.1.1 (Theorem 1, p.98, [68])**.** *Let $\Omega \subset \mathbb{R}^2$ as well as $u_0 \in H^1(\Omega; S^{n-1}), \gamma \in C^\infty(\partial\Omega; S^{n-1})$. Then there exists a solution $u \in H^1(]0, +\infty[; L^2(\Omega))$ of the harmonic gradient flow:*

$$\partial_t u - \Delta u = u|\nabla u|^2 \quad \text{in } \mathcal{D}'(]0, T[\times\Omega), \quad \forall T > 0, \tag{2.1}$$

together with the boundary conditions:

$$u(t,x) = \gamma(x), \qquad \text{for all } t \geq 0, x \in \partial\Omega \tag{2.2}$$
$$u(0,x) = u_0(x), \qquad \text{for all } x \in \Omega, \tag{2.3}$$

and satisfying $E(u(t,\cdot)) \leq E(u_0)$ *for all times* $t \geq 0$*. The solution* u *is regular on* $]0,+\infty[\times\overline{\Omega}$*, except in a finite number of points* (t_k, x_k)*,* $k = 1,\ldots,K$*, for some* $K \in \mathbb{N}$*. Additionally,* u *is unique in the class* $\mathcal{E} \subset H^1_{loc}([0,+\infty[\times\overline{\Omega})$ *defined by:*

$$\mathcal{E} := \Big\{u \;\Big|\; \exists m \in \mathbb{N}, \exists T_0 = 0 < T_1 < \ldots < T_m < \infty : u \in L^2([T_i, T_{i+1}[; W^{2,2}(\Omega)), \forall i \leq m-1\Big\}$$

Finally, there exists a constant $C > 0$ *independent of* u_0*, such that:*

$$K \leq C \cdot E(u_0)$$

A minor drawback of this result is the additional regularity requirement in the definition of $\mathcal{E}$ needed to ensure uniqueness. However, in [68], Rivière managed to remove this condition for solutions in the *energy class* and $N = S^{n-1}$, provided the initial energy is sufficently small. Solutions in the energy class actually refers to solutions u which lie merely in $H^1(]0,+\infty[; L^2(\Omega)) \cap L^\infty([0,+\infty[; H^1(\Omega))$ satisfying the inequality $E(u(t,\cdot)) \leq E(u_0)$. This approach exploited integrability by compensation phenomena inherent to the structure of the harmonic map equation, namely Wente's estimate. To be precise, the following was proven in Rivière [68]:

Theorem 2.1.1.2 (Theorem 2, p.99, [68])**.** *There exists* $\varepsilon > 0$*, such that for every* $u_0 \in H^1(\Omega; S^{n-1})$ *with:*

$$E(u_0) < \varepsilon,$$

existence of a unique solution of (2.1), (2.2), (2.3) *in* $H^1_{loc}([0,+\infty[\times\overline{\Omega})$ *satisfying* $E(u(t,\cdot)) \leq E(u_0)$ *for almost every time* $t \geq 0$ *is guaranteed. The solution* u *is in fact regular in* $]0,+\infty[\times\overline{\Omega}$*.*

A key point in the proof is the smallness of the energy that allows us to deduce slightly better regularity for the trace $u(t,\cdot)$ at a.e. fixed time. One should notice that if $\varepsilon > 0$ is sufficiently small, in Struwe's result, Theorem 2.1.1.1, the possibility of bubbling could be excluded, hence establishing global regularity. Later, in [35], Freire was able to remove the small energy restriction and prove a general uniqueness result in the energy class for arbitrary N. He did so by employing Hélein's moving frame technique in the context of the harmonic gradient flow.

Our goal is to generalize the approach by Rivière in [68] to the non-local framework and thus to the half-harmonic gradient flow.

In analogy to harmonic maps, we may say that a map $u : S^1 \to N \subset \mathbb{R}^n$ is weakly 1/2-harmonic, if it is a critical point of the following energy:

$$E_{1/2}(u) := \frac{1}{2}\int_{S^1} |(-\Delta)^{1/4}u|^2 dx, \tag{2.4}$$

with respect to variations in the following set:

$$H^{1/2}(S^1; N) := \big\{v \in H^{1/2}(S^1; \mathbb{R}^n) \;\big|\; u(x) \in N, \text{ for a.e. } x \in S^1\big\}$$

For convenience's sake, we shall abbreviate $E_{1/2}$ by E throughout the paper. Observe that the criticality condition implies that for every $\Phi \in \dot{H}^{1/2}(S^1;\mathbb{R}^n) \cap L^\infty(S^1)$, in particular all $\Phi \in C^\infty(S^1;\mathbb{R}^n)$, we have:

$$\frac{d}{dt} E_{1/2}\left(\pi(u + t\Phi)\right)\Big|_{t=0} = 0, \tag{2.5}$$

where π is the orthogonal closest-point projection to N, which is defined in a sufficiently small neighbourhood of N and smooth due to N being smooth. As we shall see, this condition is equivalent to:

$$d\pi(u)(-\Delta)^{1/2} u = 0 \quad \text{ in } \mathcal{D}'(S^1), \tag{2.6}$$

which is sometimes also stated informally in the following form, observing that $d\pi(x)$ is the orthogonal projection to $T_x N$ for every $x \in N$:

$$(-\Delta)^{1/2} u \perp T_u N$$

In our case of interest, $N = S^{n-1}$, this could be restated as:

$$u \wedge (-\Delta)^{1/2} u = 0 \quad \text{ in } \mathcal{D}'(S^1).$$

It is clear that, in order to study the regularity of 1/2-harmonic maps, the first step lies in the reformulation of (2.6). Naturally, corresponding definitions for $\mathbb{R}$ instead of S^1 are possible.

In fact, the regularity and reformulations were first studied by the authors in Da Lio-Rivière [21], only the domain being $\mathbb{R}$ instead of S^1, the same paper where 1/2-harmonic maps were first introduced. Since Da Lio-Rivière [21], several extensions have been considered in Da Lio [16]; Schikorra [76]; Da Lio-Schikorra [28], [27]; Da Lio-Pigati [20]. The regularity of 1/2-harmonic maps relies on the following compensation phenomena discovered in Da Lio-Rivière [22]: If $\Omega \in L^2_{loc}(\mathbb{R}; so(m)), v \in L^2_{loc}(\mathbb{R};\mathbb{R}^m)$ and $f \in L^p_{loc}(\mathbb{R};\mathbb{R}^m)$, where $1 \leq p \leq 2$ satisfy

$$(-\Delta)^{1/4} v = \Omega \cdot v + f \quad \text{ in } \mathcal{D}'(\mathbb{R}),$$

then $(-\Delta)^{1/4} v \in L^p_{loc}(\mathbb{R})$, i.e. $v \in \dot{W}^{1/2,p}(\mathbb{R})$. This phenomena is based on the existence of special operators satsifying improved integrability properties due to compensation. One such operator is, for instance, given by the so-called *three-term commutator*:

$$\mathcal{T} : L^2(\mathbb{R};\mathbb{R}^m) \times \dot{H}^{1/2}(\mathbb{R};\mathbb{R}^{m\times m}) \to \dot{H}^{-1/2}(\mathbb{R};\mathbb{R}^m),$$

defined by:

$$\mathcal{T}(v, Q) := (-\Delta)^{1/4}(Qv) - Q(-\Delta)^{1/4} v + (-\Delta)^{1/4} Q \cdot v$$

It is proven in Da Lio-Rivière [21] that:

$$\|\mathcal{T}(v, Q)\|_{\dot{H}^{-1/2}} \lesssim \|Q\|_{\dot{H}^{1/2}} \|v\|_{L^2}$$

We also refer to Lenzmann-Schikorra [55] for an overview of different types of commutator estimates. Recently in Mazowiecka-Schikorra [57], inspired also by Millot-Sire [60], the authors recast integrability by compensation for fractional operators and commutator estimates in a "classical local way", by applying the notions of fractional divergences and fractional gradients, see Section 2.2 for their definitions. In particular, they succeeded in recasting the integrability by compensation in terms of the following non-local result reminiscent of the result by Coifman, Lions, Meyer and Semmes [14]:

Lemma 2.1.1.1 (Theorem 2.1, [57]). *Let $s \in (0,1)$ and $p \in (1,\infty)$. For $F \in L^p_{od}(\mathbb{R}\times\mathbb{R})$ and $g \in \dot{W}^{s,p'}(\mathbb{R})$, where p' denotes the Hölder dual of p, we assume that $\operatorname{div}_s F = 0$. Then $F \cdot d_s g$ lies in the Hardy space $\mathcal{H}^1(\mathbb{R})$ and we have the estimate:*

$$\|F \cdot d_s g\|_{\mathcal{H}^1(\mathbb{R})} \lesssim \|F\|_{L^p_{od}(\mathbb{R}\times\mathbb{R})} \cdot \|g\|_{\dot{W}^{s,p'}(\mathbb{R})}.$$

Lemma 2.1.1.1 has permitted the authors in [57] to show in an alternative way the regularising effect of non-local systems with anti-symmetric potentials.

In this paper, we are going to study the gradient flow associated with the energy $E_{1/2}$ introduced above, referred to as the fractional or 1/2-harmonic gradient flow. Namely, we shall study solutions u of the following non-local PDE on $[0,+\infty[\times S^1$ taking values in the sphere $S^{n-1} \subset \mathbb{R}^n$:

$$d\pi(u)\left(u_t + E'_{1/2}(u)\right) = d\pi(u)\left(u_t + (-\Delta)^{1/2}u\right) = 0, \tag{2.7}$$

with $u(0,\cdot) = u_0$ for some initial datum $u_0 \in H^{1/2}(S^1; S^{n-1})$. As in the case of fractional harmonic maps, a first step would be to rephrase the fractional harmonic flow and we shall obtain in the paper the reformulation:

$$u_t + (-\Delta)^{1/2}u = u|d_{1/2}u|^2, \tag{2.8}$$

where u satisfies $u(0,\cdot) = u_0$. The notation used shall be introduced later on in the paper, however we emphasise that the RHS of the equation is closely related to the 1/2-harmonic map equation. It should be noted that the formulation (2.8) mirrors some of the features found in the local case and builds upon the formulation of fractional harmonic maps in Mazowiecka-Schikorra [57]. This equation will be derived later on in the paper.

One might ask what is known for the half-harmonic gradient flow (2.7), (2.8). For example, in Schikorra-Sire-Wang [77], the authors studied and proved the existence of a solution to the half-harmonic gradient flow assuming the map takes values in a sufficiently nice target manifold, i.e. a closed homogeneous space such as the space of interest $N = S^{n-1}$. In fact, they consider for $0 < s < 1$ and $1 < p < +\infty$ the energy functional:

$$E_{s,p}(u) := \frac{1}{p}\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dxdy,$$

where $\Omega \subset \mathbb{R}^m$ is smooth and bounded, and study the fractional gradient flow equation given informally by:

$$d\pi(u)\left(\partial_t u + E'_{s,p}(u)\right) = 0, \tag{2.9}$$

for closed manifolds $N \subset \mathbb{R}^n$ and the closest point projection π, showing existence of an appropriate candidate for general N and verifying that the constructed candidate is a solution, provided N is a homogeneous space. Their methods involve approximations by a piecewise minimization process and immediately yield, in contrast to the techniques employed by Struwe, a global existence result. We highlight that provided $p = 2$ and $s = 1/2$, we recover the fractional harmonic gradient flow in (2.7), and consequently (2.8), which we will be studying, thus complementing the treatment in Schikorra-Sire-Wang [77] in the case S^{n-1}. We mention that using S^1 instead of a bounded interval $\Omega \subset \mathbb{R}$ does not obstruct the proof presented in Schikorra-Sire-Wang [77], as all arguments carry over immediately, therefore the existence result continues to hold true for the domain S^1, at least for closed, homogeneous target manifolds.

Nevertheless, the nature of the argument in Schikorra-Sire-Wang [77] does not allow for a uniqueness statement or provide an analysis of possible types of blow-ups in (in)finite time. Questions regarding blow-ups were studied for example in Sire-Wei-Zheng [84] where the authors exhibit that only blow-ups in infinite time may occur for certain initial data and conjecture that the same might hold in general.

Our main result in this paper will be the following:

Theorem 2.1.1.3. *Let $u_0 \in H^{1/2}(S^1; S^{n-1})$ be any initial data. There exists $\varepsilon > 0$, such that if:*

$$\|(-\Delta)^{1/4} u_0\|_{L^2(S^1)} \leq \varepsilon,$$

then there exists a unique energy class solution $u : \mathbb{R}_+ \times S^1 \to S^{n-1} \subset \mathbb{R}^n$ of the weak fractional harmonic gradient flow:

$$u_t + (-\Delta)^{1/2} u = u |d_{1/2} u|^2,$$

satisfying $u(0, \cdot) = u_0$ in the sense $u(t, \cdot) \to u_0$ in L^2, as $t \to 0$. Moreover, the solution fulfills the energy decay estimate:

$$\|(-\Delta)^{1/4} u(t)\|_{L^2(S^1)} \leq \|(-\Delta)^{1/4} u_0\|_{L^2(S^1)}.$$

In fact, $u \in C^\infty(]0, \infty[\times S^1)$ and for an appropriate subsequence $t_k \to \infty$, the sequence $u(t_k)$ converges weakly in $H^1(S^1)$ to a point.

By energy class solution, we mean that u possesses the following regularity:

$$u \in L^\infty(\mathbb{R}_+; H^{1/2}(S^1)), \quad u_t \in L^2(\mathbb{R}_+; L^2(S^1)).$$

The general strategy behind the proof of Theorem 2.1.1.3 is the following: First, following the arguments in Struwe [89] for uniqueness, we show that uniqueness holds for slightly more regular solutions than those in energy class. Namely, we require in addition that $u \in L^2_{loc}(\mathbb{R}_+; H^1(S^1))$ and this improved regularity assumption combined with Sobolev-type embeddings for Triebel-Lizorkin spaces yields uniqueness in this class of functions. Then we establish, following the approach in Rivière [68], that energy class solutions with monotone decreasing 1/2-energy in time and sufficiently small initial energy are actually slightly more regular and satisfy the condition $u \in L^2_{loc}(\mathbb{R}_+; H^1(S^1))$. This gain in regularity crucially relies on the structure of an anti-symmetric potential hidden in the harmonic map equation and changes of gauge as in Rivière's seminal work [70] adapted in a non-local framework and manifested in non-local Wente-type estimates like Lemma 2.1.1.1 found in Mazowiecka-Schikorra [57]. Indeed, the emergence of an anti-symmetric potential and the resulting benefits are more apparent for S^{n-1} than for general manifolds, since in this case, the potential is even 1/2-divergence-free, a property which is in general only obtained after a change of gauge, cf. Rivière [70]. The vanishing 1/2-divergence actually leads to slightly better integrability properties of the potential and hence the improvement in regularity, see Da Lio-Rivière [22], Da Lio-Pigati [20] and Mazowiecka-Schikorra [57].

To be precise, the following regularity result will be the key point to derive uniqueness for small-energy solutions in the energy class:

Proposition 2.1.1.1. *Let u satisfy the following regularity assumptions:*

$$u \in L^\infty(\mathbb{R}_+; H^{1/2}(S^1)); \quad u_t \in L^2(\mathbb{R}_+; L^2(S^1))$$

Moreover, assume u solves the half-harmonic gradient flow equation (2.8). *Then for almost every time $t > 0$, we have:*

$$u(t) \in H^1(S^1).$$

Combining this with a fractional Ladyzhenskaya inequality and sufficiently small energy will show $u \in L^2_{loc}(\mathbb{R}_+; H^1(S^1))$, analogous to Rivière [68].

Proving smoothness of the solution relies on bootstrap techniques and a local regularity result from Hamilton [43] adapted to the non-local setting. Indeed, we shall use the local Inversion Theorem in order to prove existence and regularity of solutions to the flow assuming the boundary data is smooth. The resulting solution will be smooth by using results from Hieber-Prüss [46] on parabolic PDEs and maximal estimates for heat flows using operator semigroups. Then, using a generalisation of a Lemma by Schoen-Uhlenbeck [81] (our proof following the presentation in Struwe [92]) and the extension of the harmonic map flow as presented in Struwe [89] in the case of the half-harmonic map flow, we deduce regularity in general and for all times, provided the initial energy is sufficently small. The ideas follow more or less Struwe [89] and we indicate the most significant changes by establishing the key estimates. Lastly, convergence is obtained just like in Struwe [91] for the harmonic map flow.

We would like to point out that we could have chosen the formulation of the fractional harmonic map equation introduced in Da Lio-Rivière [21]. However, we did choose the formulation in (2.8) for its analogy with (2.1), which also inspired the current investigation into half-harmonic gradient flows.

Some of the main technical difficulties we will encounter in the course this paper will concern the translation of results for the real line $\mathbb{R}$ into results for the unit circle S^1 and working with Triebel-Lizorkin spaces over S^1. Regarding the former difficulties, some of results of this type may be obtained by an extension procedure, others by changes of variables involving the stereographic projection which connect the 1/2-Laplacian on the circle to the one on the real line, see e.g. Da Lio Da Lio [15], Da Lio-Pigati [20], Millot-Sire [60], Da Lio-Martinazzi-Rivière [19]. Both approaches seem to be necessary, as there are advantages to both of them. Many of the results derived by such procedures can also be obtained directly using Triebel-Lizorkin spaces. Once all these ingredients are introduced, the proof is based on the arguments found in Struwe [89] as well as Da Lio-Pigati [20], Mazowiecka-Schikorra [57], Rivière [68].

In future work, the author plans to investigate uniqueness and regularity of solutions to the fractional harmonic gradient flow with small initial energy in an arbitrary closed manifold $N \subset \mathbb{R}^n$ and then to expand our considerations to solutions with arbitrary initial energy. Some bubbling phenomena are expected to be observable in this case, so the more delicate analysis of this will be carried out in a future paper. A paper dealing with uniqueness and regularity in the general setting of an arbitrary closed manifold $N \subset \mathbb{R}^n$ is already in preparation by the author ([103]).

Let us present an outline of the paper: In Section 2, we introduce some of the most important notions and structures for our proofs. In particular, this includes Triebel-Lizorkin spaces on S^1 and the fractional Wente-type Lemma 2.1.2.1. Then, in Section 3, we turn to establishing our main result. First, in Section 3.1, we show the equivalence between (2.7) and (2.8). Then, uniqueness is treated in Section 3.2 following the presentation in Rivière [68] and Struwe [89], regularity in Section 3.3 by a bootstrap trick and using the techniques and results in Hamilton [43], Hieber-Prüss [46], Struwe [89] and finally, we discuss convergence properties in Section 3.4 following the presentation in Struwe [91] in the case of the harmonic map flow. The Appendices complement the presentation and add some technical details.

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this paper.

2.1.2 Preliminaries

We briefly introduce some of the most important notions employed throughout this paper. These concern the fractional Laplacian, fractional divergences and gradients as well as a Wente-type result for fractional div-curl-structures as seen in Mazowiecka-Schikorra [57].

2.1.2.1 Fractional Laplacian and Triebel-Lizorkin Spaces

In this section, we introduce the Triebel-Lizorkin spaces on the unit circle $S^1 \subset \mathbb{R}^2$ and recall some properties of the fractional Laplacian. Much of the current presentation is due to Prats [64], Prats-Saksman [65], Schikorra-Wang [79] and Schmeisser-Triebel [80].

Let us recall the following first: $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ is equipped with a natural distance function given by:

$$\begin{aligned} |x-y|^2 &= |e^{ix} - e^{iy}|^2 = |e^{i(x-y)} - 1|^2 \\ &= (\cos(x-y) - 1)^2 + \sin(x-y)^2 = 2 - 2\cos(x-y) \\ &= 4\sin\left(\frac{x-y}{2}\right)^2, \end{aligned} \tag{2.10}$$

so we have:

$$|x-y| = 2\left|\sin\left(\frac{x-y}{2}\right)\right|$$

We shall tacitly use this distance function, whenever we are working over S^1. Moreover, we define for any $f : S^1 \to \mathbb{R}$:

$$\mathcal{D}_{s,q}(f)(x) := \left(\int_{S^1} \frac{|f(x) - f(y)|^q}{|x-y|^{sq}} \frac{dy}{|x-y|}\right)^{1/q},$$

for all $1 \leq q < \infty$ and $0 < s < 1$. Then:

$$\|f\|_{\dot{W}^{s,(p,q)}(S^1)} := \|\mathcal{D}_{s,q}(f)(x)\|_{L^p(S^1)},$$

for every $1 \leq p \leq \infty$. If $p = q$, these spaces correspond to the usual homogeneous Gagliardo-Sobolev spaces $\dot{W}^{s,p}(S^1)$. For a presentation of the operator $\mathcal{D}_{s,q}$ and its main properties, we refer to Schikorra-Wang [79] and the references therein.

We denote by $\mathcal{D}'(S^1)$ the collection of distributions on S^1 and sometimes denote by $\mathcal{D}(S^1)$ the space $C^\infty(S^1)$ of smooth functions. Let us from now on denote by $\hat{f}(k)$ the k-th Fourier coefficient of f, for all $f \in \mathcal{D}'(S^1)$:

$$\hat{f}(k) := \frac{1}{2\pi}\langle f, e^{-ikx}\rangle = \frac{1}{2\pi} f\left(e^{-ikx}\right), \quad \forall k \in \mathbb{Z}$$

One may also introduce the Triebel-Lizorkin spaces for S^1, denoted by $F^s_{p,q}(S^1)$ in the following way for all $s \in \mathbb{R}$, $p, q \in [1, \infty[$:

$$F^s_{p,q}(S^1) := \left\{ f \in \mathcal{D}'(S^1) \;\middle|\; \|f\|_{F^s_{p,q}} < +\infty \right\}$$

Here we write:

$$\|f\|_{F^s_{p,q}} := \left\| \left\| \left(\sum_{k\in\mathbb{Z}} 2^{js}\varphi_j(k)\hat{f}(k)e^{ikx} \right)_{j\in\mathbb{N}} \right\|_{l^q} \right\|_{L^p(S^1)},$$

for a partition of unity $(\varphi_j)_{j\in\mathbb{N}}$ consisting of smooth, compactly supported functions on $\mathbb{R}$ satisfying:

$$\operatorname{supp}\varphi_0 \subset B_2(0), \quad \operatorname{supp}\varphi_j \subset \{x\in\mathbb{R} \mid 2^{j-1} \leq |x| \leq 2^{j+1}\}, \forall j \geq 1$$

as well as:

$$\forall k \in \mathbb{N} : \sup_{j\in\mathbb{N}} 2^{jk}\|D^k\varphi_j\|_{L^\infty} \lesssim 1$$

The Triebel-Lizorkin spaces on S^1, and more generally on the n-torus, possess an analogous theory to the classical case of these spaces on $\mathbb{R}^n$, see Schmeisser-Triebel [80], Chapter 3. In particular, Sobolev embeddings continue to hold (Schmeisser-Triebel [80] Section 3.5.5), identifications with classical spaces such as $L^p(S^1)$ (Schmeisser-Triebel [80] Section 3.5.4) and duality results (Schmeisser-Triebel [80] Section 3.5.6). We shall use the properties of these spaces throughout this paper and shall refer to the given reference for details. The homogeneous spaces may be defined as well by omitting the Fourier coefficient of 0th-order and adapting the notions accordingly.

In Prats-Saksman [65] or Schikorra-Wang [79], the authors prove the following result:

Theorem 2.1.2.1 (Theorem 1.2, [65])**.** *Let $s\in(0,1)$, $p,q \in]1,\infty[$ and $f\in L^p(\mathbb{R})$. Then:*

(i) We know $\dot{W}^{s,(p,q)}(\mathbb{R}^n) \subset \dot{F}^s_{p,q}(\mathbb{R}^n)$ together with:

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{W}^{s,(p,q)}(\mathbb{R}^n)}. \tag{2.11}$$

(ii) If $p > \frac{nq}{n+sq}$, then we also have the converse inclusion together with:

$$\|f\|_{\dot{W}^{s,(p,q)}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \tag{2.12}$$

The constants depend on s,p,q,n.

As seen in Prats-Saksman [65] and Schikorra-Wang [79] and by using the properties in Schmeisser-Triebel [80], Triebel [97] for periodic functions, we can similarily discover the following equivalence with Triebel-Lizorkin spaces for all $1<q<\infty$ and $1<p<\infty$:

$$\dot{W}^{s,(p,q)}(S^1) = \dot{F}^s_{p,q}(S^1), \tag{2.13}$$

with equivalence of the corresponding seminorms, provided $p > \frac{q}{1+sq}$. We shall prove the part of the identification that we will be using over and over, i.e. the second part of Theorem 2.1.2.1, in Appendix B. If $s=1/2$ and $q=2$, then $p>1$ is the requirement in Theorem 2.1.2.1 for the equality of $\dot{F}^{1/2}_{p,2}$ and $\dot{W}^{1/2,(p,2)}$ to hold. It should be observed that while $\dot{F}^s_{p,2}(S^1) \subset \dot{W}^{s,p}(S^1) = \dot{W}^{s,(p,p)}(S^1)$ for $p\geq 2$, there does not hold equality except for $p=2$. The arguments for the domain S^1 carry out in complete analogy to the case treated in Theorem 1.4 of Schikorra-Wang [79], where all the spaces are introduced over $\mathbb{R}$ and $\mathbb{R}^n$, by using the theory in Schmeisser-Triebel [80]. One just has to observe that the maximal function estimates used are also available on S^1, see Section 3.3.5 and 3.4 in Schmeisser-Triebel [80], enabling the very same arguments to work. Therefore, Theorem 2.1.2.1

continues to hold for S^1. We shall sometimes omit mention of the domain, if it is clear from the context.

On S^1, the fractional s-Laplacian is defined as a Fourier multiplier operating on Fourier series:

$$\widehat{(-\Delta)^s f}(k) = |k|^{2s}\hat{f}(k),$$

for every $k \in \mathbb{Z}$ and all $0 < s < 1$. In particular, this can also be phrased as a principal value:

$$(-\Delta)^{1/2} f(x) = C \cdot P.V. \int_{S^1} \frac{f(x) - f(y)}{|x-y|^2} dy,$$

where $C > 0$ denotes some constant, similar formulas with less explicit kernels exist for $0 < s < 1$ as seen in Gaia [36] and Roncal-Ral Stinga [74]. We also refer to them for further details. By the Fourier multiplier properties, fractional Laplacians interact in a natural way with Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(S^1)$, as is usual for this type of function spaces. This means that it induces an isomorphism:

$$(-\Delta)^s : \dot{F}^{t+2s}_{p,q} \to \dot{F}^t_{p,q},$$

for all $p, q \in (1, \infty)$ and $t, t+2s \in \mathbb{R}$, see Schmeisser-Triebel [80] Section 3.6.3 and the proof of the analogous statement in the case $\mathbb{R}^n$.

In analogy, the s-Laplacian can be defined on $\mathbb{R}$ as a Fourier multiplier using the Fourier transform rather than the Fourier series and leads again to an object which can also be characterised by a similar principal value. We omit the details, as the formulas are virtually the same as for the circle.

2.1.2.2 Fractional Gradients and Divergences

We present some of the notions introduced and studied in Mazowiecka-Schikorra [57]: We denote by $\mathcal{M}_{od}(\mathbb{R} \times \mathbb{R})$ the collection of measurable functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with respect to the measure $\frac{dxdy}{|x-y|}$ and we do the same for S^1 instead of $\mathbb{R}$ on the domain-side. If both domains are possible, we shall merely denote this space by $\mathcal{M}_{od}$. For a measurable function $f : \mathbb{R} \to \mathbb{R}$ or $f : S^1 \to \mathbb{R}$, we define for $0 \leq s < 1$ the fractional s-gradient as follows:

$$d_s f(x, y) = \frac{f(x) - f(y)}{|x-y|^s} \in \mathcal{M}_{od},$$

and the corresponding s-divergence by means of duality. It is immediately clear, but nevertheless useful to observe:

$$d_s f(y, x) = -d_s f(x, y)$$

Observe that by duality, for $F \in \mathcal{M}_{od}(\mathbb{R} \times \mathbb{R})$ or $F \in \mathcal{M}_{od}(S^1 \times S^1)$, we define for every φ smooth and compactly supported on $\mathbb{R}$ or just smooth on S^1 in the latter case:

$$\operatorname{div}_s F(\varphi) := \int \int F(x, y) d_s\varphi(x, y) \frac{dxdy}{|x-y|}$$

This quantity is hence defined merely in a distributional sense. Lastly, we denote for $F, G \in \mathcal{M}_{od}$ over $\mathbb{R}$ or S^1:

$$F \cdot G(x) := \int F(x, y) G(x, y) \frac{dy}{|x-y|}$$

If $F = G$, we also write:

$$F \cdot F(x) = |F|^2(x) \Rightarrow |F|(x) := \sqrt{F \cdot F(x)}$$

Therefore, we immediately have:

$$\||d_s f|\|_{L^p(S^1)} = \|f\|_{\dot{W}^{s,(p,2)}(S^1)},$$

which hints at an intimate connection between Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(S^1)$ and fractional gradient d_s, under some technical conditions on s, p, q. We highlight that for some constant $C_s \in \mathbb{R}$ depending on s:

$$(-\Delta)^s f = C_s \operatorname{div}_s d_s f,$$

which is particularily useful for the weak formulation of PDEs involving non-local operators. This equation is to be understood in the following sense:

$$C_s \int d_s f \cdot d_s g(x) dx = \int (-\Delta)^s f \cdot g dx = \int (-\Delta)^{s/2} f \cdot (-\Delta)^{s/2} g dx,$$

for the domains S^1 and $\mathbb{R}$. Lastly, the following identity, sometimes referred to as fractional Leibniz' rule, is often useful:

$$d_s (fg)(x, y) = d_s f(x, y) g(x) + f(y) d_s g(x, y)$$

This identity can be verified by directly inserting the definition.

In general, we may also introduce $L^p_{od}(S^1 \times S^1)$ or $L^p_{od}(\mathbb{R} \times \mathbb{R})$ as the collection of measurable functions, such that the following norm is finite:

$$\|F\|_{L^p_{od}} := \left(\int \int |F(x, y)|^p \frac{dy dx}{|x - y|} \right)^{1/p},$$

for $1 \leq p < \infty$. The space $L^\infty_{od}(S^1 \times S^1)$ and $L^\infty_{od}(\mathbb{R} \times \mathbb{R})$ could be introduced in the usual manner.

One of the main results we shall be using later on in an appropriately modified formulation is the following non-local Wente-type result:

Lemma 2.1.2.1 (Theorem 2.1, [57])**.** *Let $s \in (0, 1)$ and $p \in (1, \infty)$. For $F \in L^p_{od}(\mathbb{R} \times \mathbb{R})$ and $g \in \dot{W}^{s,p'}(\mathbb{R})$, where p' denotes the Hölder dual of p, we assume that* $\operatorname{div}_s F = 0$. *Then $F \cdot d_s g$ lies in the Hardy space $\mathcal{H}^1(\mathbb{R})$*[1] *and we have the estimate:*

$$\|F \cdot d_s g\|_{\mathcal{H}^1(\mathbb{R})} \lesssim \|F\|_{L^p_{od}(\mathbb{R}\times\mathbb{R})} \cdot \|g\|_{\dot{W}^{s,p'}(\mathbb{R})}.$$

If, for example $s = 1/2$ and $p = p' = 2$, then we may also conclude that $F \cdot d_s g \in H^{-1/2}(\mathbb{R})$ using the embedding of $\dot{H}^{1/2}(\mathbb{R})$ into $BMO(\mathbb{R})$. The estimate continues to hold in a similar manner. Similarly, we may deduce the following for the domain S^1:

[1] We briefly recall that the Hardy space $\mathcal{H}^1(\mathbb{R})$ is the subspace of $L^1(\mathbb{R})$-functions such that:

$$M_\Phi(f)(x) := \sup_{t>0} |\Phi_t * f|(x) \in L^1(\mathbb{R}),$$

where Φ is a Schwartz function on $\mathbb{R}$ with $\int \Phi dx = 1$ and $\Phi_t(x) = 1/t \cdot \Phi(x/t)$. Alternative characterisations using boundary values of harmonic maps, as the dual of $BMO(\mathbb{R})$ and by the theory of function spaces exist. Hardy spaces are of interest, as they remedy some of the issues that appear when working with L^1-functions.

Lemma 2.1.2.2. *For $F \in L^2_{od}(S^1 \times S^1)$ and $g \in \dot{H}^{1/2}(S^1)$, we assume that* $\operatorname{div}_{1/2} F = 0$. *Then $F \cdot d_{1/2} g$ lies in the space $H^{-1/2}(S^1)$ and we have the estimate:*

$$\|F \cdot d_{1/2} g\|_{H^{-1/2}(S^1)} \lesssim \|F\|_{L^2_{od}(S^1 \times S^1)} \cdot \|g\|_{\dot{H}^{1/2}(S^1)}.$$

The proof of this result is postponed to Appendix B.

2.1.3 The Fractional Harmonic Flow with Values in S^{n-1}

This section is devoted to the proof of our main result. For convenience's sake, we restate it once more:

Theorem 2.1.3.1. *Let $u_0 \in H^{1/2}(S^1; S^{n-1})$ be any initial data. There exists $\varepsilon > 0$, such that if:*

$$\|(-\Delta)^{1/4} u_0\|_{L^2(S^1)} \leq \varepsilon,$$

then there exists a unique energy class solution $u : \mathbb{R}_+ \times S^1 \to S^{n-1} \subset \mathbb{R}^n$ of the weak fractional harmonic gradient flow:

$$u_t + (-\Delta)^{1/2} u = u |d_{1/2} u|^2,$$

satisfying $u(0, \cdot) = u_0$ and the energy decay estimate:

$$\|(-\Delta)^{1/4} u(t)\|_{L^2(S^1)} \leq \|(-\Delta)^{1/4} u_0\|_{L^2(S^1)}.$$

In fact, u is even smooth and for an appropriate subsequence $t_k \to \infty$, the sequence $u(t_k)$ converges weakly in $H^1(S^1)$ to a point.

We observe that existence is already clear due to the result in Schikorra-Sire-Wang [77]. Therefore, it remains to check uniqueness, regularity and convergence for $t \to \infty$. We shall treat each of these three different aspects in a separate subsection.

2.1.3.1 The $1/2$-Harmonic Gradient Flow Equation

First, we would like to prove the equivalence of the formulations in (2.7) and (2.8). To do this, we assume that $u \in L^\infty(\mathbb{R}_+; H^{1/2}(S^1))$ and $u_t \in L^2(\mathbb{R}_+; L^2(S^1))$ is a solution of (2.7) such that $u(t,x) \in S^{n-1}$ for almost every $(t,x) \in \mathbb{R}_+ \times S^1$. Therefore, it satisfies the following equation:

$$d\pi(u) \left(u_t + (-\Delta)^{1/2} u \right) = 0,$$

where $\pi : \mathbb{R}_n \setminus \{0\} \to S^{n-1}, x \mapsto x/|x|$ is the closest point projection to S^{n-1}. This means:

$$\int_0^\infty \int_{S^1} \left(u_t + (-\Delta)^{1/2} u \right) d\pi(u) \varphi dx dt = 0, \quad \forall \varphi \in C^\infty_c(\mathbb{R}_+ \times S^1; \mathbb{R}^n)$$

Letting $\varphi \in C^\infty_c(\mathbb{R}_+ \times S^1; \mathbb{R}^n)$, we therefore have, using the notation $d\pi^\perp = Id - d\pi$:

$$\begin{aligned} \int_0^\infty \int_{S^1} \left(u_t + (-\Delta)^{1/2} u \right) \varphi dx dt &= \int_0^\infty \int_{S^1} \left(u_t + (-\Delta)^{1/2} u \right) d\pi^\perp(u) \varphi dx dt \\ &= \int_0^\infty \int_{S^1} u_t \cdot u \tilde{\varphi} + d_{1/2} u \cdot d_{1/2} (u \tilde{\varphi}) \, dx dt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \int_{S^1} d_{1/2}u \cdot d_{1/2}\left(u\tilde{\varphi}\right) dx dt \\
&= \int_0^\infty \int_{S^1}\int_{S^1} d_{1/2}u(t,x,y) \cdot u(t,y) d_{1/2}\tilde{\varphi}(t,x,y) \frac{dydx}{|x-y|} dt \\
&+ \int_0^\infty \int_{S^1}\int_{S^1} d_{1/2}u(t,x,y) d_{1/2}u(t,x,y)\tilde{\varphi}(t,x) \frac{dydx}{|x-y|} dt \\
&= \int_0^\infty \int_{S^1}\int_{S^1} d_{1/2}u(t,x,y) d_{1/2}u(t,x,y)\langle u(t,x), \varphi(t,x)\rangle \frac{dydx}{|x-y|} dt \\
&= \int_0^\infty \int_{S^1} u(t,x)|d_{1/2}u|^2(t,x) \cdot \varphi(t,x) dx dt
\end{aligned} \tag{2.14}
$$

where we used that u_t is a.e. tangential to S^{n-1} (seen by using approximation by convolutions), $d\pi(x)v = v$ for all $v \perp x$ and $x \in S^{n-1}$ and $d\pi(x)x = 0$. The latter was used to write:

$$
d\pi^\perp(u(t,x))\varphi(t,x) = \langle \varphi(t,x), u(t,x)\rangle u(t,x) =: \tilde{\varphi}(t,x)u(t,x),
$$

with $\tilde{\varphi} \in H^{1/2}(S^1;\mathbb{R}) \cap L^\infty(S^1)$ by direct computation. Observe that we implicitely used:

$$
\begin{aligned}
&\int_0^\infty \int_{S^1}\int_{S^1} d_{1/2}u(t,x,y) \cdot u(t,y) d_{1/2}\tilde{\varphi}(t,x,y) \frac{dydx}{|x-y|} dt \\
&\quad = \int_0^\infty \int_{S^1}\int_{S^1} d_{1/2}u(t,x,y) \cdot \frac{u(t,x)+u(t,y)}{2} d_{1/2}\tilde{\varphi}(t,x,y) \frac{dydx}{|x-y|} dt = 0,
\end{aligned} \tag{2.15}
$$

since:

$$
d_{1/2}u(t,x,y) \cdot (u(t,x)+u(t,y)) = \frac{u(t,x)-u(t,y)}{|x-y|^{1/2}} \cdot (u(t,x)+u(t,y)) = \frac{|u(x)|^2 - |u(y)|^2}{|x-y|^{1/2}} = 0,
$$

since $u \in S^{n-1}$ for almost all (t,x). Therefore, we have shown that:

$$
u_t + (-\Delta)^{1/2}u = u|d_{1/2}u|^2 \quad \text{in } \mathcal{D}'(\mathbb{R}_+ \times S^1),
$$

which is the formulation provided in (2.8). This proves the aforementioned equivalence between the two formulations.

2.1.3.2 Uniqueness

The first property we verify is uniqueness. As already mentioned in the introduction, the key idea is to first show uniqueness under slightly better regularity assumptions similar to Struwe [89]. Then, we use the fractional Wente-Lemma 2.1.2.2 and argue similar to Rivière [70] in order to show that energy class solutions of sufficiently small energy actually are slightly more regular and thus the uniqueness result for more regular solutions applies in this situation.

Uniqueness under Higher Regularity Assumptions Let us assume that u, v are two solutions to the fractional gradient flow taking a.e. values in $S^{n-1} \subset \mathbb{R}^n$ and such that the following holds true:

$$
u, v \in L^\infty(\mathbb{R}_+; H^{1/2}(S^1)) \cap L^2_{loc}(\mathbb{R}_+; H^1(S^1)); \quad u_t, v_t \in L^2(\mathbb{R}_+; L^2(S^1)) \tag{2.16}
$$

The local integrability is meant with respect to the domain $[0,\infty[$. It should be noticed that we include more regularity than is actually required/given by the existence result in Schikorra-Sire-Wang [77], which is in agreement with the uniqueness treatment in Struwe [89]. Additionally, it is easy to see thanks to $u, v \in S^{n-1}$ almost everywhere, that:

$$u, v \text{ are bounded.}$$

We assume that they satisfy the gradient flow associated with the 1/2-harmonic map, which we have seen in the previous subsection to be equivalent to:

$$u_t + (-\Delta)^{1/2} u = u|d_{1/2}u|^2, \quad v_t + (-\Delta)^{1/2} v = v|d_{1/2}v|^2, \tag{2.17}$$

together with the boundary condition:

$$u(0, \cdot) = v(0, \cdot) = u_0 \in H^{1/2}(S^1; S^{n-1})$$

By the assumptions, we may evaluate the 1/2-Laplacian for a.e. fixed time t (as $\nabla u(t)$ for almost every fixed time t is in $L^2(S^1)$), which shows that the gradient flow is satisfied in a strong sense by using fractional integration by parts on the weak formulation. Our goal is to prove the following result:

Theorem 2.1.3.2. *Let u, v as above be solutions to the fractional gradient flow with the same initial datum u_0. Assume that we have the following $1/2$-energy decay estimate:*

$$\|(-\Delta)^{1/4} u(t)\|_{L^2(S^1)}, \|(-\Delta)^{1/4} v(t)\|_{L^2(S^1)} \leq \|(-\Delta)^{1/4} u_0\|_{L^2(S^1)}, \quad \forall t \in \mathbb{R}_+$$

Then we may conclude:

$$u = v,$$

i.e. the solutions agree for every time $t > 0$ as well.

It will be clear from the proof of Theorem 2.1.3.2 that it would also suffice to assume:

$$\sup_{t \in \mathbb{R}_+} \|(-\Delta)^{1/4} u(t)\|_{L^2(S^1)} < +\infty,$$

and similarly for v.

Proof. The proof relies on the same ideas as the proof of uniqueness under better regularity provided in Lemma 3.12 in Struwe [89] for the harmonic gradient flow. Therefore, we begin by defining $w := u - v$ and observe:

$$w \in L^\infty(\mathbb{R}_+; H^{1/2}(S^1)) \cap L^2_{loc}(\mathbb{R}_+; H^1(S^1)); \quad w_t \in L^2(\mathbb{R}_+; L^2(S^1)),$$

as well as the initial condition:

$$w(0, \cdot) = 0 \tag{2.18}$$

This is an immediate consequence of the regularity and initial data of u and v. Let us now combine the equations in (2.17) to determine the non-local PDE solved by w:

$$\begin{aligned} w_t + (-\Delta)^{1/2} w &= u_t + (-\Delta)^{1/2} u - v_t - (-\Delta)^{1/2} v \\ &= u|d_{1/2}u|^2 - v|d_{1/2}v|^2 \\ &= (u - v)|d_{1/2}u|^2 + v(|d_{1/2}u|^2 - |d_{1/2}v|^2) \\ &= w|d_{1/2}u|^2 + v(|d_{1/2}u|^2 - |d_{1/2}v|^2) =: R_1 + R_2 \end{aligned} \tag{2.19}$$

If we test (2.19) against w itself, we obtain for any $T \in \mathbb{R}_+$:

$$\begin{aligned}\int_0^T \int_{S^1} w_t \cdot w + (-\Delta)^{1/2} w \cdot w dx dt &= \int_0^T \frac{d}{dt}\left(\frac{1}{2}\|w(t)\|_{L^2(S^1)}\right) dt + \int_0^T \|(-\Delta)^{1/4} w(t)\|_{L^2(S^1)} dt \\ &= \int_0^T \int_{S^1} |w|^2 |d_{1/2} u|^2 + v(|d_{1/2}u|^2 - |d_{1/2}v|^2) \cdot w dx dt \\ &\leq \int_0^T \int_{S^1} |w|^2 |d_{1/2}u|^2 dx dt + \int_0^T \int_{S^1} |w||R_2| dx dt \qquad (2.20)\end{aligned}$$

So, using the fundamental theorem of calculus, we arrive at:

$$\frac{1}{2}\|w(T)\|_{L^2(S^1)} + \int_0^T \|(-\Delta)^{1/4} w(t)\|_{L^2(S^1)} dt \leq \int_0^T \int_{S^1} |w|^2 |d_{1/2}u|^2 dx dt + \int_0^T \int_{S^1} |w||R_2| dx dt \quad (2.21)$$

We emphasise that we used (2.18) in order to evaluate the integral of the derivative at $t = 0$. In order to proceed, we have to investigate the term $|d_{1/2}u|^2 - |d_{1/2}v|^2$ more closely. To do this, let us write for $x \in S^1$ by means of the fundamental theorem:

$$\begin{aligned}|d_{1/2}u|^2(x) - |d_{1/2}v|^2(x) &= \int_0^1 \frac{d}{ds}\left(|d_{1/2}(v + s(u-v))|^2(x)\right) ds \\ &= \int_0^1 \frac{d}{ds}\left(\int_{S^1} \frac{\left(v(x) - v(y) + s(u(x) - v(x) - u(y) + v(y))\right)^2}{|x-y|^2} dy\right) ds \\ &= \int_0^1 \frac{d}{ds}\left(\int_{S^1} \frac{\left(v(x) - v(y) + s(w(x) - w(y))\right)^2}{|x-y|^2} dy\right) ds \\ &= \int_0^1 \int_{S^1} 2\frac{(v(x) - v(y) + s(w(x) - w(y)))(w(x) - w(y))}{|x-y|^2} dy ds \\ &\leq 2\int_0^1 |d_{1/2}((1-s)v + su)|(x) \cdot |d_{1/2}w|(x) ds \\ &\leq C\left(|d_{1/2}u|(x) + |d_{1/2}v|(x)\right) \cdot |d_{1/2}w|(x)\end{aligned}$$

where we used Hölder's inequality and the integrability properties of u, v during the sequence of inequalities above. This implies the following estimate for R_2:

$$\begin{aligned}|R_2|(x) &\leq C|v|(x)\left(|d_{1/2}u|(x) + |d_{1/2}v|(x)\right) \cdot |d_{1/2}w|(x) \\ &\leq C\left(|d_{1/2}u|(x) + |d_{1/2}v|(x)\right) \cdot |d_{1/2}w|(x), \qquad (2.22)\end{aligned}$$

where we implicitely used $|v| = 1$ almost everywhere. By using Cauchy-Schwarz and Young's inequality, we therefore find:

$$\int_0^T \int_{S^1} |w||R_2| dx dt \leq \delta \int_0^T \int_{S^1} |d_{1/2}w|^2 dx dt + C(\delta) \int_0^T \int_{S^1} |w|^2 \left(|d_{1/2}u| + |d_{1/2}v|\right)^2 dx dt,$$

for any $\delta > 0$. Observe that:

$$\int_0^T \int_{S^1} |d_{1/2}w|^2 dx dt \sim \int_0^T \|(-\Delta)^{1/4} w\|_{L^2(S^1)} dt,$$

by direct computations, see Lemma 2.1.3.2 after this proof. Thus, we may choose $\delta > 0$, such that after absorbing and using Cauchy-Schwarz:

$$\begin{aligned}\frac{1}{2}\|w(T)\|_{L^2(S^1)} + \frac{1}{2}\int_0^T \|(-\Delta)^{1/4} w(t)\|_{L^2(S^1)} dt \\ \leq C \int_0^T \int_{S^1} |w|^2 \left(|d_{1/2}u| + |d_{1/2}v|\right)^2 dxdt \\ \leq \tilde{C} \left(\int_0^T \int_{S^1} |w|^4 dxdt\right)^{1/2} \cdot \left(\int_0^T \int_{S^1} \left(|d_{1/2}u| + |d_{1/2}v|\right)^4 dxdt\right)^{1/2} \end{aligned} \tag{2.23}$$

Using Lemma 2.1.3.1 below for each fixed t, we can estimate:

$$\begin{aligned}\int_0^T \int_{S^1} |w|^4 dxdt &\leq C \int_0^T \|w(t)\|^2_{L^2(S^1)} \cdot \left(\|w(t)\|^2_{L^2(S^1)} + \|(-\Delta)^{1/4} w(t)\|^2_{L^2(S^1)}\right) dt \\ &\leq \tilde{C} \left(\sup_{0\leq s\leq T} \|w(s)\|^2_{L^2(S^1)} + \int_0^T \|(-\Delta)^{1/4} w(t)\|^2_{L^2(S^1)} dt\right)^2 \end{aligned} \tag{2.24}$$

Here, $\tilde{C}$ depends on T, which may be chosen sufficiently small, as seen afterwards, by using an iteration process to increase T step by step. We notice that there is no dependence of the constant on the L^2-norm of $w(s)$, as we estimate:

$$\begin{aligned}\int_0^T \|w(t)\|^2_{L^2(S^1)} \cdot \left(\|w(t)\|^2_{L^2(S^1)} + \|(-\Delta)^{1/4} w(t)\|^2_{L^2(S^1)}\right) dt \\ \leq \int_0^T \sup_{0\leq s\leq T} \|w(s)\|^2_{L^2(S^1)} \cdot \left(\sup_{0\leq s\leq T} \|w(s)\|^2_{L^2(S^1)} + \|(-\Delta)^{1/4} w(t)\|^2_{L^2(S^1)}\right) dt \\ = \sup_{0\leq s\leq T} \|w(s)\|^2_{L^2(S^1)} \cdot \left(T \cdot \sup_{0\leq s\leq T} \|w(s)\|^2_{L^2(S^1)} + \int_0^T \|(-\Delta)^{1/4} w(t)\|^2_{L^2(S^1)}\right) dt \\ \leq \tilde{C} \left(\sup_{0\leq s\leq T} \|w(s)\|^2_{L^2(S^1)} + \int_0^T \|(-\Delta)^{1/4} w(t)\|^2_{L^2(S^1)} dt\right)^2 \end{aligned} \tag{2.25}$$

which is precisely the estimate presented above and the dependence on T is benign, i.e. $\tilde{C}$ remains bounded as $T \to 0$. Observe that we used the inequality:

$$\sup_{0\leq s\leq T} \|w(s)\|^2_{L^2(S^1)} \leq \sup_{0\leq s\leq T} \|w(s)\|^2_{L^2(S^1)} + \int_0^T \|(-\Delta)^{1/4} w(t)\|^2_{L^2(S^1)} dt,$$

which is trivially true.

Claim 1: For every $\varepsilon > 0$, there is $T > 0$ small enough, such that:

$$\int_0^T \int_{S^1} \left(|d_{1/2}u| + |d_{1/2}v|\right)^4 dxdt < \varepsilon, \tag{2.26}$$

However, before we prove this claim, we observe that this is indeed sufficient to conclude our proof of Theorem 2.1.3.2, as then we may choose $T > 0$ as in the proof of Lemma 3.12 in Struwe [89] to maximize the $L^2(S^1)$-norm on $[0,T]$, i.e. $\|w(T)\|_{L^2(S^1)} = \sup_{0\leq s\leq T}\|w(s)\|^2_{L^2(S^1)}$. This is possible by continuity of $t\mapsto w(t)$ with respect to the L^2-norm due to the assumptions in (2.16), in particular the integrability of the weak derivative in time-direction u_t, v_t and thus w_t. We refer to Evans [34], Chapter 5.9.2 for details regarding this continuity. Alternatively, (2.16) implies that u, v and therefore also w are in $H^1_{loc}(S^1\times]0,\infty[)$. Thus, by the trace theorem $u(t,\cdot), v(t,\cdot), w(t,\cdot)$ lie in $H^{1/2}(S^1)$ and depend continuously on t, the latter property owing to the continuity of the trace operator.

In addition, for some fixed, but arbitrary, a-priori time $0 < T_0 \leq T$, T as in the claim, the maximum of the L^2-norms at a given time over the interval $[0, T_0]$ must be attained at some $t_0 > 0$, as otherwise we have $w = 0$ for all times $t \leq T_0$, which would also show the desired equality and hence we could restart the argument from T_0 on. Thus, for sufficiently small $\varepsilon > 0$, we may absorb the right hand side in (2.23) into the left hand side, immediately giving the desired result, i.e. $w(T) = 0$ and thus for all $0 \leq t \leq T$. More precisely, from (2.21), (2.23) and a sufficiently small ε, we obtain:

$$\begin{aligned}\frac{1}{2}\|w(T)\|_{L^2(S^1)} &+ \int_0^T \|(-\Delta)^{1/4}w(t)\|_{L^2(S^1)}dt \\ &\leq C\delta\int_0^T \|(-\Delta)^{1/4}w\|_{L^2(S^1)}dt + \tilde{C}(\delta)\sqrt{\varepsilon}\left(\sup_{0\leq s\leq T}\|w(s)\|^2_{L^2(S^1)} + \int_0^T \|(-\Delta)^{1/4}w(t)\|^2_{L^2(S^1)}dt\right),\end{aligned}$$

which for $\delta > 0$ sufficiently small and observing maximality of the choice of T (note that we may choose T smaller than required by (2.26) to ensure that it is also the time with maximal L^2-norm) then becomes:

$$\begin{aligned}\frac{1}{2}\sup_{0\leq s\leq T}\|w(s)\|_{L^2(S^1)} &+ \frac{1}{2}\int_0^T \|(-\Delta)^{1/4}w(t)\|_{L^2(S^1)}dt \\ &\leq \tilde{C}\sqrt{\varepsilon}\left(\sup_{0\leq s\leq T}\|w(s)\|^2_{L^2(S^1)} + \int_0^T \|(-\Delta)^{1/4}w(t)\|^2_{L^2(S^1)}dt\right),\end{aligned}$$

and if $\tilde{C}\sqrt{\varepsilon} = \frac{1}{4}$, we may absorb this contribution into the left hand side to find:

$$\sup_{0\leq s\leq T}\|w(s)\|_{L^2(S^1)} = 0$$

The argument then is completed by the usual connectedness-type argument using convergence $u(t)\to u(t_0)$ for $t\to t_0$ in L^2 and iterating, see Struwe [89]. Indeed, consider the set:

$$I\subset[0,\infty[,\quad I := \left\{t\in[0,\infty[\ \middle|\ w(s) = 0, \forall s\leq t\right\}$$

Clearly, $0\in I$ by construction and hence $I\neq\emptyset$. Moreover, I is open, since if $t\in I$, we may use the arguments above to deduce that $w(s) = 0$ for all $t\leq s < t+\varepsilon$, for some sufficiently small ε. Finally, I is closed, which then shows $I = [0,\infty[$ and finishes the argument. This is clear as:

$$\lim_{s\to t}\|w(s)\|_{L^2} = \|w(t)\|_{L^2},$$

which proves that if all $s < t$ satisfy $s\in I$, then $w(t) = 0$ in $L^2(S^1)$ and hence $t\in I$.

Proof of Claim 1: Now, let us return to (2.26). We shall provide two justifications of this estimate. The first argument postponed to Appendix A relies on some properties found in Da Lio

[15], Da Lio-Martinazzi-Rivière [19] connecting the fractional Laplacian on the circle to the one on the real line. The precise results shall be stated and proven in Appendix A. A different apporach uses Theorem 2.1.2.1 for S^1 directly. Here, we shall just present an outline of the argument:

We observe that it suffices to find corresponding estimates for $d_{1/2}u$ and $d_{1/2}v$ respectively. For these, we have:

$$\begin{aligned}
\int_0^T \int_{S^1} |d_{1/2}u|^4 dxdt &\leq C \int_0^T \|(-\Delta)^{1/4}u(t)\|_{L^4}^4 dt \\
&\leq \tilde{C} \int_0^T \|(-\Delta)^{1/4}u(t)\|_{L^2}^2 \cdot \|(-\Delta)^{1/2}u\|_{L^2}^2 dt \\
&\leq \tilde{C} \sup_{0\leq s\leq T} \|(-\Delta)^{1/4}u(s)\|_{L^2}^2 \cdot \int_0^T \int_{S^1} |(-\Delta)^{1/2}u|^2 dxdt \\
&\leq \tilde{C} \|(-\Delta)^{1/4}u_0\|_{L^2}^2 \cdot \int_0^T \int_{S^1} |(-\Delta)^{1/2}u|^2 dxdt
\end{aligned} \tag{2.27}$$

where we used $u \in H^1(S^1)$, which immediately implies $(-\Delta)^{1/4}u \in H^{1/2}(S^1)$, as well as Lemma 2.1.3.1. We refer to Appendix A for the details on the proof of the estimate using an extension procedure, in particular the first inequality which is actually the only missing step here. Alternatively, directly using the second part of Theorem 2.1.2.1 on the domain S^1, see (2.12), as described in the preliminary section and proven in Appendix B, the first inequality could also be obtained immediately and the rest follows by using Lemma 2.1.3.1. The main difference between these two approaches lies in the use of Theorem 2.1.2.1 either on $\mathbb{R}$ or on S^1, depending on which techniques are used.

The claim now follows by (2.16) and the L^2-integrability of the 1/2-Laplacian of u. Notice that the supremum is finite due to the assumptions in the statement of the Theorem. □

We now prove some useful results that were invoked in the proof above or motivate the conditions of the main result of this subsection, Theorem 2.1.3.2:

Lemma 2.1.3.1. *Let $u \in H^{1/2}(S^1)$. Then the following estimate holds for some $C > 0$:*

$$\|u\|_{L^4} \leq C\|u\|_{L^2}^{1/2}\|u\|_{H^{1/2}}^{1/2}$$

Proof. By Sobolev embeddings, we immediately find for some $C > 0$:

$$\|u\|_{L^4} \leq C\|u\|_{H^{1/4}}$$

Additionally, we have by definition:

$$\begin{aligned}
\|u\|_{H^{1/4}}^2 &= \sum_{n\in\mathbb{Z}} (1+|n|^2)^{1/4}|\hat{u}(n)|^2 \\
&= \sum_{n\in\mathbb{Z}} (1+|n|^2)^{1/4}|\hat{u}(n)| \cdot |\hat{u}(n)| \\
&\leq \left(\sum_{n\in\mathbb{Z}} (1+|n|^2)^{1/2}|\hat{u}(n)|^2\right)^{1/2} \cdot \left(\sum_{n\in\mathbb{Z}} |\hat{u}(n)|^2\right)^{1/2} \\
&= \|u\|_{H^{1/2}} \cdot \|u\|_{L^2}
\end{aligned}$$

This now yields:

$$\|u\|_{L^4} \leq C\|u\|_{H^{1/4}} \leq C\sqrt{\|u\|_{H^{1/2}} \cdot \|u\|_{L^2}} = C\|u\|_{L^2}^{1/2}\|u\|_{H^{1/2}}^{1/2}$$

Thus, the Lemma is proven. □

We highlight that Lemma 2.1.3.1 continues to be true on $\mathbb{R}$ by using classical rescaling techniques or relying, for example, on Littlewood-Paley theory.

Lemma 2.1.3.2. *It holds the following for every $u \in H^{1/2}(S^1)$:*

$$\int_{S^1} |d_{1/2}u|^2 dx \sim \|(-\Delta)^{1/4}u\|_{L^2(S^1)}^2 \tag{2.28}$$

Proof. Let us observe the following for smooth functions u:

$$\begin{aligned}\int_{S^1} |d_{1/2}u|^2 dx &= \int_{S^1}\int_{S^1} \frac{|u(x)-u(y)|^2}{|x-y|^2} dydx \\ &= \int_{S^1} P.V. \int_{S^1} \frac{(u(x)-u(y))}{|x-y|^2}(u(x)-u(y))dydx \\ &= \int_{S^1} 2P.V. \int_{S^1} \frac{u(x)-u(y)}{|x-y|^2} dy \cdot u(x)dx \\ &= \tilde{C}\int_{S^1} (-\Delta)^{1/2}u(x) \cdot u(x)dx \\ &= \tilde{C}\int_{S^1} |(-\Delta)^{1/4}u(x)|^2 dx \\ &= \tilde{C}\|(-\Delta)^{1/4}u\|_{L^2(S^1)}^2 \end{aligned} \tag{2.29}$$

for some $\tilde{C} > 0$, see also the definition of the fractional Lapacian in section 2. Here, $P.V.$ stands for principal value. For complete rigor, one has to take the integral on a subset of $S^1 \times S^1$ omitting the diagonal and letting the neighbourhood become arbitrarily small to deduce the second equality, to ensure the principal value can be taken and the fractional Laplacian emerges. The statement for general u follows now by approximation. □

Finally, let us motivate the decay assumption on solutions of the fractional gradient flow in Theorem 2.1.3.2:

$$\|(-\Delta)^{1/4}u(t)\|_{L^2(S^1)} \leq \|(-\Delta)^{1/4}u_0\|_{L^2(S^1)}, \quad \forall t \in \mathbb{R}_+$$

It should be noted that this is a "classical" assumption when working with gradient flows, nevertheless we present the idea: To do this, let us assume that u is a smooth solution of the fractional gradient flow. Then, we may test against u_t and find:

$$\int_0^T \int_{S^1} |u_t|^2 + (-\Delta)^{1/2}u \cdot u_t dxdt = \int_0^T \int_{S^1} u|d_{1/2}u|^2 \cdot u_t dxdt = 0, \tag{2.30}$$

where the last equality follows by observing that u assumes values in a sphere, hence the derivative in t-direction will be tangential to the sphere and, as a result, orthogonal to u, implying:

$$u \cdot u_t = 0$$

In addition, we have:

$$\begin{aligned}\int_0^T \int_{S^1} (-\Delta)^{1/2} u \cdot u_t dx dt &= \int_0^T \int_{S^1} (-\Delta)^{1/4} u \cdot (-\Delta)^{1/4} u_t dx dt \\ &= \int_0^T \frac{1}{2} \frac{d}{dt} \left(\int_{S^1} |(-\Delta)^{1/4} u|^2 dx \right) dt \\ &= \frac{1}{2} \|(-\Delta)^{1/4} u(T)\|^2_{L^2(S^1)} - \frac{1}{2} \|(-\Delta)^{1/4} u(0)\|^2_{L^2(S^1)}. \\ &= \frac{1}{2} \|(-\Delta)^{1/4} u(T)\|^2_{L^2(S^1)} - \frac{1}{2} \|(-\Delta)^{1/4} u_0\|^2_{L^2(S^1)} \end{aligned} \tag{2.31}$$

Consequently, this computation shows that in the case of regular solutions:

$$\frac{1}{2} \|(-\Delta)^{1/4} u(T)\|^2_{L^2(S^1)} \leq \frac{1}{2} \|(-\Delta)^{1/4} u(T)\|^2_{L^2(S^1)} + \int_0^T \int_{S^1} |u_t|^2 dx dt = \frac{1}{2} \|(-\Delta)^{1/4} u_0\|^2_{L^2(S^1)} \tag{2.32}$$

This yields the desired boundedness of energy (in fact monotone decay of energy) and thus motivates the assumption we had in Theorem 2.1.3.2. We formulate this in the following slightly imprecise:

Lemma 2.1.3.3. *Let u be a sufficiently regular solution of the $1/2$-harmonic gradient flow as previously defined with $u(0, \cdot) = u_0$. Then the following holds for all $T \geq 0$:*

$$\frac{1}{2} \|(-\Delta)^{1/4} u(T)\|^2_{L^2(S^1)} \leq \frac{1}{2} \|(-\Delta)^{1/4} u_0\|^2_{L^2(S^1)}$$

In fact, the energy $T \mapsto \|(-\Delta)^{1/4} u(T)\|_{L^2(S^1)}$ monotonically decreases in T.

Improved Regularity of the Solution We would like to show how we may obtain the required improvement in regularity for energy-class solutions, i.e. solutions which do a-priori not satisfy a L^2-local bound on the first derivative in space-direction, to the fractional gradient flow (2.17) in a similar manner as in Rivière [68]. The key idea is that we may fix some time t and consider the corresponding equation for fixed time to obtain improved regularity. Namely, we will obtain the following result:

Theorem 2.1.3.3. *Let $u : \mathbb{R}_+ \times S^1 \to S^{n-1} \subset \mathbb{R}^n$ be a solution of the weak fractional harmonic gradient flow* (2.17) *with initial datum $u_0 \in H^{1/2}(S^1)$ and satisfying the following regularity assumptions:*

$$u \in L^\infty(\mathbb{R}_+; H^{1/2}(S^1)); \quad u_t \in L^2(\mathbb{R}_+; L^2(S^1))$$

Then there exists $\varepsilon > 0$ such that among all such u satisfying the smallness condition:

$$\|(-\Delta)^{1/4} u(t)\|_{L^2(S^1)} \leq \varepsilon, \quad \forall t \in \mathbb{R}_+,$$

the solution to the fractional harmonic gradient flow (2.17) *with initial datum u_0 is unique.*

If we assume that the energy is bounded by some sufficiently small $\varepsilon > 0$, then it is sufficient to show that $u(t) \in H^1(S^1)$ for almost every $t \in \mathbb{R}_+$. In fact, the following holds:

Proposition 2.1.3.1. *Let $u : \mathbb{R}_+ \times S^1 \to S^{n-1} \subset \mathbb{R}^n$ be a solution of the weak fractional harmonic gradient flow* (2.17) *with initial datum $u_0 \in H^{1/2}(S^1)$ and satisfying the following regularity assumptions:*

$$u \in L^\infty(\mathbb{R}_+; H^{1/2}(S^1)); \quad u_t \in L^2(\mathbb{R}_+; L^2(S^1)); \quad u(t) \in H^1(S^1) \text{ for a.e. } t \in \mathbb{R}_+$$

Then there exists $\varepsilon > 0$ such that among all such u satisfying the smallness condition:

$$\|(-\Delta)^{1/4}u(t)\|_{L^2(S^1)} \leq \varepsilon, \quad \forall t \in \mathbb{R}_+,$$

the solution to the fractional harmonic gradient flow (2.17) *with initial datum u_0 is unique.*

Proof. To verify this, let us observe that if $u(t) \in H^1(S^1)$ for almost every $t \in \mathbb{R}_+$, we may deduce for a fixed time t:

$$(-\Delta)^{1/2}u(t) = u(t)|d_{1/2}u(t)|^2 - \partial_t u(t)$$

Hence, by standard elliptic estimates for the fractional Laplacian or simply observing that with $\mathcal{R}$ being the Riesz transform, we have:

$$\nabla u(t) = \mathcal{R}\left(u(t)|d_{1/2}u(t)|^2 - \partial_t u(t)\right) \tag{2.33}$$

Keeping in mind that $\mathcal{R}$ is a continuous linear operator on $L^2(S^1)$, we are led to the following estimate:

$$\begin{aligned} \|u(t)\|_{H^1(S^1)}^2 &\leq C\left(\|u(t)\|_{L^2}^2 + \||d_{1/2}u(t)|^2\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2\right) \\ &\leq C\left(1 + \||d_{1/2}u(t)|^2\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2\right), \end{aligned} \tag{2.34}$$

where we used $u(t) \in S^{n-1}$ almost everywhere for almost every time t. It is clear that regarding local L^2-integrability with respect to time, it thus remains to study the following contribution:

$$\||d_{1/2}u(t)|^2\|_{L^2}^2$$

Using the same ideas as in the proof of (2.27) for the uniqueness statement Theorem 2.1.3.2 (which are proved in Appendix A or rely on the second part of Theorem 2.1.2.1 for S^1, see also Appendix B), we may estimate this term by:

$$\||d_{1/2}u(t)|^2\|_{L^2}^2 \leq C'\|u(t)\|_{H^{1/2}}^2\|u(t)\|_{H^1}^2$$

By applying this inequality to $u - \hat{u}(0)$ instead of u, we may replace the $H^{1/2}$- and H^1-norms by the corresponding seminorms:

$$\||d_{1/2}u(t)|^2\|_{L^2}^2 \leq C'\|u(t)\|_{\dot{H}^{1/2}}^2\|u(t)\|_{\dot{H}^1}^2$$

We emphasise that adding a constant to u does not affect the LHS of the estimate above. Therefore, we have the energy term appearing:

$$\||d_{1/2}u(t)|^2\|_{L^2}^2 \leq C'\|(-\Delta)^{1/4}u(t)\|_{L^2}^2\|u(t)\|_{\dot{H}^1}^2 \leq C'\varepsilon \cdot \|u(t)\|_{H^1}^2,$$

where $\varepsilon > 0$ is an a priori energy estimate as in Rivière [68] and we may still choose $\varepsilon > 0$ appropriately. Indeed, if $\varepsilon > 0$ is sufficiently small, for example $\varepsilon \leq 1/(2CC')$, we may absorb this term in the left hand side of (2.34) to arrive at:

$$(1 - CC'\varepsilon) \cdot \|u(t)\|_{H^1(S^1)}^2 \leq \tilde{C}\left(1 + \|\partial_t u(t)\|_{L^2}^2\right) \Rightarrow \|u(t)\|_{H^1} \leq \frac{C}{1 - C'C\varepsilon}\left(1 + \|\partial_t u(t)\|_{L^2}\right), \tag{2.35}$$

which thus yields an estimate for the H^1-norm. We observe that hence, by the integrability properties of $\partial_t u$ and the constant function (which rely on the compactness of S^1):

$$u \in L^2_{loc}(\mathbb{R}_+; H^1(S^1)) \tag{2.36}$$

Thus, we may apply the previous uniqueness statement in Theorem 2.1.3.2 even if we merely know:

$$u \in L^\infty(\mathbb{R}_+; H^{1/2}(S^1)); \quad u_t \in L^2(\mathbb{R}_+; L^2(S^1)); \quad \|(-\Delta)^{1/4}u(0)\|_{L^2(S^1)} \leq \varepsilon,$$

with $\varepsilon > 0$ sufficiently small as above and assuming the energy decrease holds, provided we get increased regularity for $u(t)$. □

In particular, if we assume that the 1/2-energy is non-increasing in time, as seen to be true for smooth solutions to the fractional harmonic gradient flow in Lemma 2.1.3.3, the smallness condition could be rephrased as:

$$\|(-\Delta)^{1/4}u_0\|_{L^2(S^1)} \leq \varepsilon$$

Consequently, all that remains is to deduce H^1-regularity for a.e. fixed time to apply Proposition 2.1.3.1 and deduce Theorem 2.1.3.3. The following Lemma in the spirit of Rivière [68] takes care of this by investigating the regularity for a fixed time $t \in \mathbb{R}_+$:

Lemma 2.1.3.4. *Let $f \in L^2(S^1)$ and assume that $u \in H^{1/2}(S^1)$ solves the following equation:*

$$(-\Delta)^{1/2}u = u|d_{1/2}u|^2 + f. \tag{2.37}$$

Then, we have the following improved regularity property:

$$u \in H^1(S^1).$$

The key point in the proof will be the appearance of an anti-symmetric potential Ω satisfying $\operatorname{div}_{1/2}\Omega = 0$ to which we can apply the non-local Wente-type inequality in Lemma 2.1.2.1 or (2.119) in Appendix B. If we apply the result in Lemma 2.1.3.4 to $f = \partial_t u(t)$ and $u = u(t)$, we may deduce $u(t) \in H^1(S^1)$ for almost every t, given a sufficiently small bound on the 1/2-energy at a given time t. Thus we may derive Theorem 2.1.3.3 by combining the statements in Proposition 2.1.3.1 and Lemma 2.1.3.4.

In addition, let us observe that for given $f \in L^2(S^1)$, this seems to be an optimal result, as any solution $(-\Delta)^{1/2}u = f$ would satisfy $u \in H^1(S^1)$, but no higher regularity can be deduced in general.

Proof. As in Mazowiecka-Schikorra [57], we know that there exists a map $\Omega \in L^2_{od}(S^1 \times S^1; \mathbb{R}^{n\times n})$ depending on u, such that $\Omega^T = -\Omega$ and $\operatorname{div}_{1/2}\Omega = 0$, such that we derive from (2.37):

$$(-\Delta)^{1/2}u = \Omega \cdot d_{1/2}u + T(u) + f, \tag{2.38}$$

where $T(u)$ is as in Mazowiecka-Schikorra [57]. In fact, we have by using the components $u = (u^1, \dots, u^n)$ and Einstein's summation convention:

$$\begin{aligned}
u^i(x)d_{1/2}u^k(x,y)d_{1/2}u^k(x,y) &= u^i(x)d_{1/2}u^k(x,y)d_{1/2}u^k(x,y) - u^k(x)d_{1/2}u^i(x,y)d_{1/2}u^k(x,y) \\
&\quad + u^k(x)d_{1/2}u^i(x,y)d_{1/2}u^k(x,y) \\
&=: \Omega_{ik}(x,y)d_{1/2}u^k(x,y) + u^k(x)d_{1/2}u^i(x,y)d_{1/2}u^k(x,y) \\
&= \Omega_{ik}(x,y)d_{1/2}u^k(x,y) + \frac{1}{2}d_{1/2}u^i(x,y)|d_{1/4}u^k(x,y)|^2 \\
&=: \Omega_{ik}(x,y)d_{1/2}u^k(x,y) + T^i(u)
\end{aligned} \tag{2.39}$$

Thus, the following formula for every $i = 1, \dots n$ holds:

$$T^i(u) := \sum_{k=1}^{n} \int_{S^1} d_{1/2}u^i(x,y)|d_{1/4}u^k(x,y)|^2 \frac{dy}{|x-y|}, \quad T(u) = (T^1(u), \dots, T^n(u)),$$

and moreover:

$$\Omega_{ik}(x,y) := u^i(x)d_{1/2}u^k(x,y) - u^k(x)d_{1/2}u^i(x,y), \quad \forall i,k \in \{1, \dots n\}$$

We introduce the following notion $T(u,v,w) := (T^1(u,v,w), \dots, T^n(u,v,w))$:

$$T^i(u,v,w) := \sum_{k=1}^{n} \int_{S^1} d_{1/2}u^i(x,y)d_{1/4}v^k(x,y)d_{1/4}w^k(x,y) \frac{dy}{|x-y|}, \quad \forall i \in \{1, \dots, n\}, \tag{2.40}$$

and clearly $T(u,u,u) = T(u)$. We have the following estimates, refining the ones already found in Mazowiecka-Schikorra [57]:

Assume that $p > 2$ as well as $u \in \dot{F}^{1/2}_{p,2}(S^1)$ and $v, w \in \dot{H}^{1/2}(S^1)$. Then we have by using Hölder's inequality

$$\begin{aligned}
\|T(u,v,w)\|_{L^{\frac{2p}{p+2}}(S^1)} &\leq \left(\int_{S^1} \mathcal{D}_{1/2,2}(u)\mathcal{D}_{1/4,4}(v)\mathcal{D}_{1/4,4}(w)dx \right)^{\frac{p+2}{2p}} \\
&\lesssim \|u\|_{\dot{W}^{1/2,(p,2)}} \|v\|_{\dot{W}^{1/4,(4,4)}} \|w\|_{\dot{W}^{1/4,(4,4)}} \\
&\lesssim \|u\|_{\dot{F}^{1/2}_{p,2}} \|v\|_{\dot{F}^{1/4}_{4,4}} \|w\|_{\dot{F}^{1/4}_{4,4}} \\
&\lesssim \|u\|_{\dot{F}^{1/2}_{p,2}} \|v\|_{\dot{F}^{1/4}_{4,2}} \|w\|_{\dot{F}^{1/4}_{4,2}} \\
&\lesssim \|u\|_{\dot{F}^{1/2}_{p,2}} \|v\|_{\dot{F}^{1/2}_{2,2}} \|w\|_{\dot{F}^{1/2}_{2,2}} \\
&= \|u\|_{\dot{F}^{1/2}_{p,2}} \|v\|_{\dot{H}^{1/2}} \|w\|_{\dot{H}^{1/2}},
\end{aligned} \tag{2.41}$$

where we used the second part of Theorem 2.1.2.1 for the circle S^1, see also Appendix B. Furthermore, standard embeddings for Triebel-Lizorkin spaces were used in the estimates above. One should notice that:

$$4 > \frac{1 \cdot 4}{1 + \frac{1}{4}4} = 2,$$

meaning that the second part of Theorem 2.1.2.1 applies to $\cdot F^{1/4}_{4,4}(S^1)$. This also implies thanks to the Sobolev-type embedding:

$$\dot{F}^{1/2}_{\frac{p}{p-1},2}(S^1) \hookrightarrow L^{\frac{2p}{p-2}}(S^1), \quad \forall p > 2,$$

that we have an estimate of the following form by (2.41) and using duality of Triebel-Lizorkin spaces:

$$\|T(u,v,w)\|_{\dot{F}^{-1/2}_{p,2}(S^1)} \lesssim \|u\|_{\dot{F}^{1/2}_{p,2}(S^1)} \|v\|_{\dot{H}^{1/2}(S^1)} \|w\|_{\dot{H}^{1/2}(S^1)} \tag{2.42}$$

Moreover, if $u, v, w \in \dot{H}^{1/2}(S^1)$, we also know by first switching x, y and then using Hölder's inequality and Sobolev-type embeddings:

$$\int_{S^1} \int_{S^1} \varphi^i(x) d_{1/2}u^i(x,y)d_{1/4}v^k(x,y)d_{1/4}w^k(x,y) \frac{dydx}{|x-y|}$$

$$
\begin{aligned}
&= \int_{S^1}\int_{S^1} (\varphi^i(x)-\varphi^i(y)) d_{1/2}u^i(x,y) d_{1/4}v^k(x,y) d_{1/4}w^k(x,y) \frac{dydx}{|x-y|} \\
&\lesssim \int_{S^1} \mathcal{D}_{1/2,2}(\varphi)\mathcal{D}_{1/6,6}(u)\mathcal{D}_{1/6,6}(v)\mathcal{D}_{1/6,6}(w)dx \\
&\lesssim \|\varphi\|_{\dot{W}^{1/2,(2,2)}}\|u\|_{\dot{W}^{1/6,(6,6)}}\|v\|_{\dot{W}^{1/6,(6,6)}}\|w\|_{\dot{W}^{1/6,(6,6)}} \\
&\lesssim \|\varphi\|_{\dot{F}^{1/2}_{2,2}}\|u\|_{\dot{F}^{1/6}_{6,6}}\|v\|_{\dot{F}^{1/6}_{6,6}}\|w\|_{\dot{F}^{1/6}_{6,6}} \\
&\lesssim \|\varphi\|_{\dot{H}^{1/2}}\|u\|_{\dot{H}^{1/2}}\|v\|_{\dot{H}^{1/2}}\|w\|_{\dot{H}^{1/2}}, \qquad (2.43)
\end{aligned}
$$

again using Theorem 2.1.2.1 on the circle as well as:

$$6 > \frac{1\cdot 6}{1+\frac{1}{6}6} = 3,$$

justifying the application in this case. This immediately yields:

$$\|T(u,v,w)\|_{\dot{H}^{-1/2}(S^1)} \lesssim \|u\|_{\dot{H}^{1/2}(S^1)}\|v\|_{\dot{H}^{1/2}(S^1)}\|w\|_{\dot{H}^{1/2}(S^1)} \qquad (2.44)$$

Finally, if w is smooth (and similarily for v smooth), we find by similar arguments:

$$\|T(u,v,w)\|_{L^{\frac{2p}{p+2}}} \lesssim \|\nabla w\|_{L^\infty}\|u\|_{\dot{H}^{1/2}}\|v\|_{\dot{F}^{1/2-1/p}_{p,2}} \lesssim \|\nabla w\|_{L^\infty}\|u\|_{\dot{H}^{1/2}}\|v\|_{\dot{H}^{1/2}}, \qquad (2.45)$$

for all $2 \leq p < +\infty$ and therefore, $T(u,v,w) \in L^r(S^1)$ for all $r \in [1,2[$, provided either v or w smooth.

Let us return to (2.38). This may now be rewritten as:

$$(-\Delta)^{1/2}u - T(u,u,u) = \Omega \cdot d_{1/2}u + f, \qquad (2.46)$$

Letting $v = u - \fint_{S^1} u = u - \hat{u}(0)$, we thus see:

$$(-\Delta)^{1/2}v - T(v,u,u) = \Omega \cdot d_{1/2}v + f \qquad (2.47)$$

We notice that since each summand in (2.47) is integrable, we may include a summand with each, such that each summand has mean 0. This allows us to apply $(-\Delta)^{-1/2}$ and renders this operator injective in an appropriate sense.

The next step is to approximate the terms appearing in (2.47), similar to Rivière [70]. Therefore, let Ω_0 be a smooth map from $S^1 \times S^1$ into the anti-symmetric $n \times n$-matrices, such that:

$$\|\Omega_0 - \Omega\|_{L^2_{od}} < \varepsilon,$$

for $\varepsilon > 0$ to be determined. This can for example be obtained by cutting Ω off in a sufficiently small neighbourhood of the diagonal and then using convolutions to smoothen the function and thus approximate Ω by regular functions. This way, Ω_0 can also be assumed to be supported outside of the diagonal and thus to vanish in a neighbourhood of it. We may also assume that $\operatorname{div}_{1/2}\Omega_0 = 0$. This can be achieved by otherwise solving:

$$(-\Delta)^{1/2}h = \operatorname{div}_{1/2}\Omega,$$

in the weak sense and using $\Omega_0 - d_{1/2}h$ instead of Ω_0. One might argue by noticing that $\operatorname{div}_{1/2}\Omega_0$ is smooth by the function vanishing in a neighbourhood of the diagonal and then solving for h which immediately will be smooth as well. The right estimate can be obtained by the following train of thought:

$$\begin{aligned}
\|h\|_{\dot{H}^{1/2}}^2 &= \int_{S^1} (-\Delta)^{1/2} h \cdot h dz \\
&= \int_{S^1} \operatorname{div}_{1/2}(\Omega_0 - \Omega) h dz \\
&= \int_{S^1} \int_{S^1} (\Omega_0(z,w) - \Omega(z,w)) d_{1/2} h(z,w) \frac{dzdw}{|z-w|} \\
&\lesssim \|\Omega_0 - \Omega\|_{L^2_{od}} \|h\|_{\dot{H}^{1/2}}, \qquad (2.48)
\end{aligned}$$

providing an estimate for $d_{1/2}h$ that is required to ensure that $\Omega_0 - d_{1/2}h$ remains close to Ω, while becoming divergence-free. In addition, we may choose a smooth function $\tilde{u}$ to be arbitrarily close to u in $H^{1/2}(S^1)$, i.e. for any $\varepsilon > 0$ given, we can take $\tilde{u}$ in such a way that:

$$\|u - \tilde{u}\|_{H^{1/2}(S^1)} < \varepsilon$$

One proceeds now as in Rivière [68]: We may introduce the solution operator:

$$\begin{aligned}
\tau(v) := v + &(-\Delta)^{-1/2} \left((\Omega_0 - \Omega) \cdot d_{1/2} v + T(v, \tilde{u} - u, \tilde{u} - u) \right) \\
&= (-\Delta)^{-1/2} \Big((-\Delta)^{1/2} v + (\Omega_0 - \Omega) \cdot d_{1/2} v + T(v, \tilde{u} - u, \tilde{u} - u) \Big) \\
&= (-\Delta)^{-1/2} \left(\Omega_0 \cdot d_{1/2} v + T(v, u, \tilde{u}) + T(v, \tilde{u}, u - \tilde{u}) + f \right) \qquad (2.49)
\end{aligned}$$

We notice that the solution operator τ is well-defined, as we assumed that all summands have mean 0, thus we could also apply it to each summand individually. To deduce the desired regularity result, we now show that τ defines a bijective operator from $\dot{F}^{1/2}_{p,2}$ to itself for each $p \geq 2$. As in Rivière [68], let us split our considerations into two distinct cases:

The "easy" Case: $p > 2$ This case is an immediate consequence of the ellipticity of the fractional 1/2-Laplacian and the analogue of the fractional Sobolev embeddings. Fixing v to be the solution $u \in H^{1/2}(S^1)$ as in the Lemma on the RHS, we would like to solve:

$$\begin{aligned}
\tau(v) &= v + (-\Delta)^{-1/2} \left((\Omega_0 - \Omega) \cdot d_{1/2} v + T(v, \tilde{u} - u, \tilde{u} - u) \right) \\
&= (-\Delta)^{-1/2} \left(\Omega_0 \cdot d_{1/2} u + T(u, u, \tilde{u}) + T(u, \tilde{u}, u - \tilde{u}) + f \right), \qquad (2.50)
\end{aligned}$$

and we may conclude that the RHS on the last line of (2.50) lies in $\dot{F}^{1/2}_{p,2}$ thanks to Sobolev embeddings and the L^r-estimate for $1 \leq r < 2$ in (2.45), smoothness of $\Omega_0, \tilde{u}$ and the properties of $(-\Delta)^{1/2}$. Observe that the smallness of $\Omega_0 - \Omega$ in L^2_{od} and of $u - \tilde{u}$ in $H^{1/2}(S^1)$ is used in order to conclude that the solution operator is invertible by the usual perturbation argument. One invokes here the estimate proved above, see (2.42), (2.44) and Hölder's inequality applied to $(\Omega - \Omega_0) \cdot d_{1/2} u$ together with:

$$\begin{aligned}
\int_{S^1} (\Omega - \Omega_0) \cdot d_{1/2} u \varphi dx &\leq \|(\Omega - \Omega_0) \cdot d_{1/2} u\|_{L^{\frac{2p}{p+2}}} \|\varphi\|_{L^{\frac{2p'}{2-p'}}} \\
&\lesssim \|\Omega - \Omega_0\|_{L^2_{od}} \||d_{1/2} u|\|_{L^p} \|\varphi\|_{\dot{F}^{1/2}_{p',2}} \\
&\lesssim \|\Omega - \Omega_0\|_{L^2_{od}} \|u\|_{\dot{F}^{1/2}_{p,2}} \|\varphi\|_{\dot{F}^{1/2}_{p',2}}, \qquad (2.51)
\end{aligned}$$

where $1/p + 1/p' = 1$ and we used the Sobolev embeddings for Triebel-Lizorkin spaces, hence showing boundedness in $\dot{F}^{-1/2}_{p,2}$ of $(\Omega - \Omega_0) \cdot d_{1/2}u$. Similar estimates using Hölder's inequality as in (2.41) show that:

$$\|T(v, \tilde{u} - u, \tilde{u} - u)\|_{\dot{F}^{-1/2}_{p,2}} \lesssim \|v\|_{\dot{F}^{1/2}_{p,2}} \|u - \tilde{u}\|^2_{\dot{H}^{1/2}} \tag{2.52}$$

Using that $(-\Delta)^{1/2}$ defines an isomorphism between $\dot{F}^{1/2}_{p,2}$ and $\dot{F}^{-1/2}_{p,2}$, see Schmeisser-Triebel [80], the required estimate follows from (2.51) and (2.52):

$$\begin{aligned} \Big\|(-\Delta)^{-1/2}\big((\Omega_0 - \Omega) \cdot d_{1/2}v + &T(v, \tilde{u} - u, \tilde{u} - u)\big)\Big\|_{\dot{F}^{1/2}_{p,2}} \\ &\lesssim \Big(\|\Omega - \Omega_0\|_{L^2_{od}} + \|u - \tilde{u}\|^2_{\dot{H}^{1/2}}\Big) \|v\|_{\dot{F}^{1/2}_{p,2}} \end{aligned} \tag{2.53}$$

Hence, the perturbation is small, if $\Omega_0, \tilde{u}$ are sufficiently good approximations and thus τ invertible in this case. Notice that the RHS of (2.50) lies in L^q for all $q < 2$. Thus, using the same arguments as in (2.51) and also (2.41), we could deduce that $\Omega_0 \cdot d_{1/2}u + T(u, u, \tilde{u}) + T(u, \tilde{u}, u - \tilde{u}) + f \in \dot{F}^{-1/2}_{p,2}$, and therefore the RHS of (2.50) is in $\dot{F}^{1/2}_{p,2}$.

We emphasise that this step is crucially relying on the homogeneous Triebel-Lizorkin spaces $\dot{F}^{1/2}_{p,2}(S^1)$ for $p > 2$ which prove to be the right one and possess an equivalent norm description in terms of the L^p-norm of $|d_{1/2}f|$ for all f inside this space, see Theorem 2.1.2.1 as well as Appendix B. See also Schmeisser-Triebel [80], Triebel [97], Schikorra-Wang [79], Stein [85] and the references therein for further details.

The "hard" Case: $p = 2$ On the other hand, this case is more delicate and requires a version of the Wente-type result in Mazowiecka-Schikorra [57], see Lemma 2.1.2.1, for the circle. This can be obtained by a set of changes of variables under the stereographic projection and using a partition of unity, see (2.119) and the computations preceeding it. We postpone the details of this computation to Appendix B, as it is basically a technical analysis of a sequence of changes of variables. This immediately allows us to again proceed as in Rivière [68], as we may now estimate the perturbation of the solution operator by the Wente-type estimate below:

$$\|(\Omega_0 - \Omega) \cdot d_{1/2}v - c\|_{\dot{H}^{-1/2}} \leq C\|\Omega_0 - \Omega\|_{L^2_{od}} \|v\|_{\dot{H}^{1/2}} \leq C\varepsilon \|v\|_{\dot{H}^{1/2}},$$

where we take c as the constant rendering the term to have mean 0. To conclude the proof of Lemma 2.1.3.4, we now observe that for $\varepsilon > 0$ sufficiently small, the perturbation:

$$(-\Delta)^{-1/2}\left((\Omega_0 - \Omega) \cdot d_{1/2}v\right),$$

is small as well (when considered as an operator from $\dot{H}^{1/2}(S^1)$ to itself and hence the operator:

$$v \mapsto v + (-\Delta)^{-1/2}\left((\Omega_0 - \Omega) \cdot d_{1/2}v\right),$$

becomes an isomorphism, now for $\dot{H}^{1/2}(S^1)$ to itself as well as for different integrability exponents $p > 2$. If u is sufficiently small in $\dot{H}^{1/2}(S^1)$, the same remains true if we include the missing term under the smallness assumption on $\|u - \tilde{u}\|_{\dot{H}^{1/2}}$:

$$v \mapsto v + (-\Delta)^{-1/2}\left((\Omega_0 - \Omega) \cdot d_{1/2}v\right) + (-\Delta)^{-1/2}\left(T(v, \tilde{u} - u, \tilde{u} - u)\right)$$

This is clear, as we may use (2.44) to deduce that the second summand is again a small perturbation which does not affect invertibility.

As in Rivière [68], we may now deduce thanks to existence and uniqueness that $u \in \dot{F}^{1/2}_{4,2}(S^1)$ and, consequently, $|d_{1/2}u| \in L^4$, using the embedding:

$$\dot{F}^{1/2}_{4,2}(S^1) \subset \dot{F}^{1/2}_{2,2}(S^1) = H^{1/2}(S^1), \tag{2.54}$$

which shows that the unique solution v of (2.50) in $\dot{F}^{1/2}_{4,2}(S^1)$ (or, in fact, in any $\dot{F}^{1/2}_{p,2}(S^1)$ with $p \geq 2$ by replacing 4 by p in the argument) actually agrees with $u - \hat{u}(0)$, where u is as given in Lemma 2.1.3.4, due to the uniqueness in the case $p = 2$ and the embedding (2.54) of Triebel-Lizorkin spaces. So $(-\Delta)^{1/2}u \in L^2(S^1)$ by directly using (2.37), which immediately yields $\nabla u = \mathcal{R}(-\Delta)^{1/2}u \in L^2(S^1)$ and so $u \in H^1(S^1)$. Hence, we have established the desired regularity result for u. This concludes our proof. $\square$

2.1.3.3 Regularity

Next, we show that solutions to the fractional gradient flow (2.17) are smooth for all times $t > 0$. The main idea is to study the regularity of the RHS of (2.17) and bootstrap this information. In fact, a key step lies in studying the Fourier series of

$$|d_{1/2}u|^2(x),$$

and establishing sufficient H^s-estimates to bootstrap the regularity.

Some useful Results Let us assume that u, v are trigonometric polynomials. Thus, they are of the form:

$$u(x) = \sum_{n\in\mathbb{Z}} \hat{u}(n)e^{inx}, \quad v(x) = \sum_{n\in\mathbb{Z}} \hat{v}(n)e^{inx},$$

where $\hat{u}(n), \hat{v}(n) = 0$ for all but finitely many $n \in \mathbb{N}$. Let us consider $d_{1/2}u \cdot d_{1/2}v(x)$, or more precisely its Fourier coefficients:

$$\begin{aligned} \widehat{d_{1/2}u \cdot d_{1/2}}v(n) &= \frac{1}{2\pi}\langle d_{1/2}u \cdot d_{1/2}v, e^{-inx}\rangle \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d_{1/2}u \cdot d_{1/2}v(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{S^1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \frac{e^{ijh}-1}{|h|}\frac{e^{ikh}-1}{|h|}\hat{u}(j)\hat{v}(k)e^{i(j+k)x}dhe^{-inx}dx \\ &= \sum_{j\in\mathbb{Z}}\int_{S^1}\frac{e^{ijh}-1}{|h|}\frac{e^{i(n-j)h}-1}{|h|}dh \cdot \hat{u}(j)\hat{v}(n-j) \end{aligned} \tag{2.55}$$

where $n \in \mathbb{Z}$ is arbitrary and we used the formulas for u, v as trigonometric polynomials. Let us introduce:

$$C(j,k) := \int_{S^1}\frac{e^{ijh}-1}{|h|}\frac{e^{ikh}-1}{|h|}dh \tag{2.56}$$

Therefore, we may see using the previous computations:

$$\widehat{d_{1/2}u \cdot d_{1/2}v}(n) = \sum_{j \in \mathbb{Z}} C(j, n-j)\hat{u}(j)\hat{v}(n-j), \quad \forall n \in \mathbb{Z}$$

A first step to deduce the regularity of $d_{1/2}u \cdot d_{1/2}v$ lies in the study of $C(j,k)$. Namely, we observe that:

$$C(j,j) = \int_{S^1} \frac{|e^{ijh} - 1|^2}{|h|^2} dh = \int_{S^1} \frac{\sin\left(\frac{jh}{2}\right)^2}{\sin\left(\frac{h}{2}\right)^2} dh = |j| \int_{S^1} F_{|j|}(h) dh = |j|,$$

where F_n denotes the n-th Féjer kernel. Using Cauchy-Schwarz and Féjer kernels, we easily deduce:

$$\begin{aligned} |C(j,k)| &\leq \left(\int_{S^1} \frac{|e^{ijh} - 1|^2}{|h|^2} dh\right)^{1/2} \left(\int_{S^1} \frac{|e^{ikh} - 1|^2}{|h|^2} dh\right)^{1/2} \\ &\leq \sqrt{|j|}\sqrt{|k|} \cdot \left(\int_{S^1} F_{|j|}(h) dh\right)^{1/2} \left(\int_{S^1} F_{|j|}(h) dh\right)^{1/2} \\ &= \sqrt{|j|}\sqrt{|k|}, \end{aligned} \tag{2.57}$$

for all $j, k \in \mathbb{Z}$. The main goal is now to deduce regularity estimates leading to conclusions like $d_{1/2}u \cdot d_{1/2}v \in H^s(S^1)$ for some $s \in \mathbb{R}$ together with appropriate estimates in terms of u, v. Namely, we will prove the following:

Lemma 2.1.3.5. *Let u, v be trigonometric polynomials as above. Then we have for all $\varepsilon > 0$:*

$$\begin{aligned} \|d_{1/2}u \cdot d_{1/2}v\|_{\dot{H}^s(S^1)} &\lesssim \|(-\Delta)^{1/4+s/2+\varepsilon}u\|_{L^2}\|(-\Delta)^{1/2}v\|_{L^2} + \|(-\Delta)^{1/2}u\|_{L^2}\|(-\Delta)^{1/4+s/2+\varepsilon}v\|_{L^2} \\ &\lesssim \|u\|_{\dot{F}^{1/2+s+2\varepsilon}_{2,2}}\|v\|_{\dot{H}^1} + \|u\|_{\dot{H}^1}\|v\|_{\dot{F}^{1/2+s+2\varepsilon}_{2,2}}, \end{aligned} \tag{2.58}$$

as well as:

$$\begin{aligned} \|d_{1/2}u \cdot d_{1/2}v\|_{\dot{H}^s(S^1)} &\lesssim \|(-\Delta)^{1/4+s/2+2\varepsilon}u\|_{L^2}\|(-\Delta)^{1/2-\varepsilon}v\|_{L^2} + \|(-\Delta)^{1/2-\varepsilon}u\|_{L^2}\|(-\Delta)^{1/4+s/2+2\varepsilon}v\|_{L^2} \\ &\lesssim \|u\|_{\dot{F}^{1/2+s+4\varepsilon}_{2,2}}\|v\|_{\dot{F}^{1-2\varepsilon}_{2,2}} + \|u\|_{\dot{F}^{1-2\varepsilon}_{2,2}}\|v\|_{\dot{F}^{1/2+s+2\varepsilon}_{2,2}}, \end{aligned} \tag{2.59}$$

and by density, the same estimates continue to hold true for all u, v in the corresponding spaces. The constants depend on $s > 0$ and $\varepsilon > 0$.

Proof. By definition, we have:

$$\begin{aligned} \|d_{1/2}u \cdot d_{1/2}v\|^2_{\dot{H}^s} &= \sum_{n \in \mathbb{Z}} |n|^{2s} |\widehat{d_{1/2}u \cdot d_{1/2}v}(n)|^2 \\ &\lesssim \sum_{n \in \mathbb{Z}} |n|^{2s} \left(\sum_{j \in \mathbb{Z}} |\hat{u}(j)||\hat{v}(n-j)|\sqrt{|j||n-j|}\right)^2 \\ &\lesssim \sum_{n \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} |\widehat{(-\Delta)^{1/4}u}(j)||\widehat{(-\Delta)^{1/4}v}(n-j)\left(|j|^s + |n-j|^s\right)\right)^2 \end{aligned}$$

$$\lesssim \sum_{n\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} |\widehat{(-\Delta)^{1/4+s/2}u}(j)||\widehat{(-\Delta)^{1/4}v}(n-j)| \right)^2$$
$$+ \sum_{n\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} |\widehat{(-\Delta)^{1/4+s/2}v}(n-j)||\widehat{(-\Delta)^{1/4}u}(j)| \right)^2 \tag{2.60}$$

By symmetry, it suffices to restrict our attention to the first summand in (2.60). We observe:

$$|\widehat{(-\Delta)^{1/4+s/2}u}(j)||\widehat{(-\Delta)^{1/4}v}(n-j)| = |\widehat{(-\Delta)^{1/4+s/2+\varepsilon}u}(j)||\widehat{(-\Delta)^{1/2}v}(n-j)||n-j|^{-1/2}|j|^{-2\varepsilon},$$

which can be used to deduce by using Cauchy-Schwarz and Young's inequality:

$$\begin{aligned}\|d_{1/2}u \cdot d_{1/2}v\|^2_{\dot{H}^s} &\lesssim \sum_{n\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} |\widehat{(-\Delta)^{1/4+s/2+\varepsilon}u}(j)|^2 |\widehat{(-\Delta)^{1/2}v}(n-j)|^2 \right) \left(\sum_{j\in\mathbb{Z}} \frac{1}{|n-j||j|^{2\varepsilon}} \right) \\ &\lesssim \|u\|^2_{\dot{F}^{1/2+s+2\varepsilon}_{2,2}} \|v\|^2_{\dot{H}^1}, \end{aligned} \tag{2.61}$$

and completely analogous for the second summand in (2.60). Young's inequality is used to bound:

$$\sum_{j\in\mathbb{Z}} \frac{1}{|n-j||j|^{2\varepsilon}} \lesssim \sum_{j\in\mathbb{Z}} \left(\frac{1}{|n-j|^{p'}} + \frac{1}{|j|^{2\varepsilon p}} \right) < +\infty,$$

and choosing $p \in]1, +\infty[$ in such a way that $2\varepsilon p > 1$ and p' being the Hölder dual of p. The second estimate (2.61) follows analogously. This concludes therefore the proof of regularity. $\square$

Another useful result will be the following:

Lemma 2.1.3.6. *Assume that* $\alpha \in]0, 1[$ *and* $u \in C^{0,\alpha}(S^1)$*. Then:*

$$(-\Delta)^s u \in L^\infty(S^1),$$

if we know:

$$1 > \alpha > 2s$$

Proof. Up to constants, we know:

$$\begin{aligned}|(-\Delta)^s u(x)| &\lesssim \int_{S^1} \frac{|u(x)-u(y)|}{|x-y|^{1+2s}} dy \\ &\lesssim \int_{S^1} \frac{1}{|x-y|^{1+2s-\alpha}} dy \cdot \|u\|_{C^{0,\alpha}}, \end{aligned} \tag{2.62}$$

which yields the desired boundedness if:

$$1 + 2s - \alpha < 1 \Rightarrow 2s < \alpha,$$

by exploiting the integrability of $|x-y|^{1+2s-\alpha}$. This concludes our proof. $\square$

For convenience's sake, let us also state the version of Theorem 3.1 in Hieber-Prüss [46] that will be relevant in our discussion of the regularity of solutions to the fractional heat equation:

Lemma 2.1.3.7 (Theorem 3.1 in [46])**.** *Let $1 < p < +\infty$ and $I = [0, T]$ be any interval with $T < +\infty$. Then there exists for each $f \in L^p(I \times S^1)$ a unique solution $u \in W^{1,p}(I \times S^1)$ of the equation:*

$$u_t + (-\Delta)^{1/2} u = f,$$

and satisfying $u(0, \cdot) = 0$. Moreover, we have:

$$\|u\|_{W^{1,p}} \lesssim \|f\|_{L^p}. \tag{2.63}$$

The result follows from Theorem 3.1 in Hieber-Prüss [46] by observing that the 1/2-Laplacian is actually generating an analytic C^0-semigroup with the required properties (see for example [46, 3.2.E)]).

Local Regularity A key step in the study of regularity lies in the local regularity. Precisely, we will prove:

Proposition 2.1.3.2. *Let $u_0 \in C^\infty(S^1; S^{n-1})$ be any smooth map. Then there exists $T > 0$, possibly depending on u_0, and a smooth map $u \in C^\infty([0, T] \times S^1)$ which solves the half-harmonic gradient flow:*

$$u_t + (-\Delta)^{1/2} u = u|d_{1/2} u|^2, \tag{2.64}$$

and satisfies the initial condition $u(0, x) = u_0(x)$. Moreover, it holds for all $x \in S^1$ and $0 \leq t < T$:

$$u(t, x) \in S^{n-1}, \tag{2.65}$$

i.e. the solution u indeed assumes values in the desired target manifold.

A key observation is therefore, due to the previously proved uniqueness of the solution by Theorem 2.1.3.2, that any solution of the equation (2.17) is indeed regular at least for sufficiently small times t and provided the boundary data is smooth. If the 1/2-energy at $t = 0$ is small, the same holds for energy class solutions.

Proof of Proposition 2.1.3.2. We shall follow the presentation in Hamilton [43] and adapt the techniques to the non-local framework encountered here. Therefore, we want to study the following map for every $p > 2$:

$$H : W^{1,p}([0, T] \times S^1) \to L^p([0, T] \times S^1), \quad H(u) := u_t + (-\Delta)^{1/2} u - u|d_{1/2} u|^2 \tag{2.66}$$

We want to prove that we may apply the local Inversion Theorem for Banach spaces to H for sufficiently regular functions. This will then enable us to deduce the result in Proposition 2.1.3.2 by a slight modification, completely analogous to [43, p.122-124].

Observe that as $p > 2$, any $u \in W^{1,p}([0, T] \times S^1)$ will be continuous and bounded. Therefore, by using Sobolev-embeddings, we immediately deduce that the map is well-defined. In fact, the only critical part is dealt with by the following computation:

$$\begin{aligned} \|u|d_{1/2} u|^2\|_{L^p}^p &= \int_0^T \int_{S^1} |u|^p |d_{1/2} u|^{2p} dx dt \\ &\lesssim \|u\|_{L^\infty}^p \int_0^T \int_{S^1} |d_{1/2} u|^{2p} dx dt \end{aligned}$$

$$\lesssim \|u\|_{L^\infty}^{p}\|u\|^{2p}_{F^{1/2}_{2p,2}([0,T]\times S^1)}$$
$$\lesssim \|u\|^{3p}_{W^{1,p}([0,T]\times S^1)}, \tag{2.67}$$

where we used the Triebel-Lizorkin- as well as Morrey-embeddings $W^{1,p}([0,T]\times S^1) \hookrightarrow \dot{F}^{1/2}_{2p,2}([0,T]\times S^1)$ and $W^{1,p}([0,T]\times S^1) \hookrightarrow C^0([0,T]\times S^1) \subset L^\infty([0,T]\times S^1)$, see [96, Theorem 3.3.1] or [80, Theorem 3.5.5]. Furthermore, the map H is actually differentiable. Namely, we observe by computing directional derivatives with respect to $h \in W^{1,p}([0,T]\times S^1)$:

$$DH(u)h = h_t + (-\Delta)^{1/2}h - h|d_{1/2}u|^2 - 2ud_{1/2}u \cdot d_{1/2}h, \tag{2.68}$$

and by observing:

$$H(u+h) - H(u) - DH(u)h = u|d_{1/2}h|^2 + 2hd_{1/2}u \cdot d_{1/2}h + h|d_{1/2}h|^2,$$

one immediately sees, using similar estimates as above in (2.67), that H is actually a C^1-function.

In order to apply the local Inversion theorem, we would like to study the behaviour of the differential, in particular whether it is an isomorphism of Banach spaces. Assume for the moment that u is actually in $C^{0,\alpha}([0,T]\times S^1)$ for some $\alpha > 1/2$. Firstly, we observe that the map:

$$h \mapsto h|d_{1/2}u|^2 + 2ud_{1/2}u \cdot d_{1/2}h,$$

is a compact map from $h \in W^{1,p}([0,T]\times S^1)$ to $L^p([0,T]\times S^1)$. This is immediate due to the Hölder continuity of u which implies boundedness of $|d_{1/2}u|$ and compactness of Sobolev embeddings. Therefore, because $h \mapsto h_t + (-\Delta)^{1/2}h$ is invertible on the set $\tilde{W}^{1,p}_0([0,T]\times S^1)$ containing precisely all h with $h(0,\cdot) = 0$ (see [46, Theorem 3.1] which asserts existence and uniqueness), (2.68) defines an invertible linear operator $DH(u) : \tilde{W}^{1,p}_0([0,T]\times S^1) \to L^p([0,T]\times S^1)$ if and only if the kernel of $DH(u)$ is trivial. This is clear, as the operator is Fredholm with index 0, since it is a sum of an invertible (and thus Fredholm operator of index 0) and a compact operator. Therefore, we merely have to study the kernel of $DH(u)$. The result we will be proving is the following:

Lemma 2.1.3.8. *Assume that u is smooth. Then $DH(u)$ has trivial kernel and $DH(u)$ defines an invertible operator.*

To initiate the study of the kernel of (2.68) among all h with vanishing initial datum, we first prove regularity of h in the kernel of $DH(u)$. This will then allow us to employ maximum principles for fractional PDE similar to Hamilton [43]:

Lemma 2.1.3.9. *Let u be as in Lemma 2.1.3.8. Then, if $h \in \tilde{W}^{1,p}_0([0,T]\times S^1)$ lies in the kernel of $DH(u)$, then h is smooth on $[0,T]\times S^1$.*

Proof of Lemma 2.1.3.9. We first observe that given $h \in \tilde{W}^{1,p}_0([0,T]\times S^1)$, we have:

$$h|d_{1/2}u|^2 + 2ud_{1/2}u \cdot d_{1/2}h \in L^{\frac{4p}{4-p}},$$

if $p < 4$ and in any L^q with $q < \infty$, if $p \geq 4$. This follows again by Sobolev embeddings and using u smooth (a similar inclusion with $2p/(4-p)$ holds, if $u \in W^{1,p}$ only and a similar iteration applies in this case as well). Thus:

$$h_t + (-\Delta)^{1/2}h \in L^q,$$

for $q = 4p/(4-p)$ or $1 < q < \infty$, depending on p. Since:

$$(\partial_t - (-\Delta)^{1/2})(\partial_t + (-\Delta)^{1/2}) = \Delta_{t,x}, \tag{2.69}$$

i.e. the composition equals the Laplacian in 2D, we may invoke classical elliptic regularity theory to find:

$$\Delta_{t,x} h \in W^{-1,q} \Rightarrow h \in W^{1,q}$$

Observe that in case $p \geq 4$, this shows that $u \in W^{1,q}$ for all $1 < q < \infty$. If $p < 4$, then:

$$1/q = 1/p - 1/4 < 1/2 - 1/4 = 1/4 \Rightarrow q > 4,$$

so we may iterate the same argument to find ourselves in the case $p > 4$. In any case, we have that for h with:

$$DH(u)h = 0,$$

and vanishing initial datum that $h \in W^{1,q}$, for all $1 < q < \infty$. In particular, we have that for such h, the inclusion $h \in C^{0,\beta}$ holds for all $0 < \beta < 1$. This is immediate by Morrey's embedding. Observe that, for instance, this also means that $|d_{1/2}h|$ is bounded.

For the remainder of the argument, let us restrict our attention to u being smooth. By Lemma 2.1.3.6, we have:

$$(-\Delta)^{1/4+t} h \in L^\infty([0,T] \times S^1), \quad \forall t \in [0, 1/4[\tag{2.70}$$

Thus, combining this consideration with Lemma 2.1.3.5, we see that for all $0 < s < 1/2$ and $\varepsilon > 0$ sufficiently small:

$$\begin{aligned} \|d_{1/2}u \cdot d_{1/2}h\|_{\dot{H}^s(S^1)} &\lesssim \|(-\Delta)^{1/4+s/2+\varepsilon} h\|_{L^2} \|(-\Delta)^{1/2} u\|_{L^2} + \|(-\Delta)^{1/4+s/2+\varepsilon} u\|_{L^2} \|(-\Delta)^{1/2} h\|_{L^2} \\ &\lesssim \|h\|_{C^{0,\alpha}} \|u\|_{H^1} + \|u\|_{C^{0,\alpha}} \|h\|_{H^1}, \end{aligned} \tag{2.71}$$

and therefore, we know that:

$$(-\Delta)^{s/2} \left(d_{1/2}u \cdot d_{1/2}h \right) \in L^2([0,T] \times S^1)$$

The same then holds for $u d_{1/2}u \cdot d_{1/2}h$ as well as $h|d_{1/2}u|^2$ and thus leads to, by using elliptic regularity and [46, Theorem 3.1]:

$$(-\Delta)^{s/2} h \in H^1([0,T] \times S^1),$$

using (2.69). Bootstrapping using Lemma 2.1.3.5, we may deduce the same for every $s > 0$. For example, using (2.59) with $s = 3/4$ and $\varepsilon > 0$ sufficiently small, we find for almost every given time t, noting that $H^s \cap L^\infty$ is a Banach algebra for any $s > 0$:

$$\begin{aligned} &\|2u(t) d_{1/2}u(t) \cdot d_{1/2}h(t) + h(t)|d_{1/2}u(t)|^2\|_{\dot{H}^{3/4}(S^1)} \\ &\lesssim \|u(t)\|_{L^\infty} \left(\|h(t)\|_{\dot{H}^{5/4+4\varepsilon}} \|u(t)\|_{\dot{H}^{1-2\varepsilon}} + \|u(t)\|_{\dot{H}^{5/4+4\varepsilon}} \|h(t)\|_{\dot{H}^{1-2\varepsilon}} \right) + \|u(t)\|_{\dot{H}^{3/4}} \|u\|_{C^{0,\beta}} \|h\|_{C^{0,\beta}} \\ &+ \|h(t)\|_{L^\infty} \|u(t)\|_{\dot{H}^{5/4+4\varepsilon}} \|u(t)\|_{\dot{H}^{1-2\varepsilon}} + \|h\|_{\dot{H}^{3/4}} \|u\|^2_{C^{0,\beta}}, \end{aligned} \tag{2.72}$$

where $\beta > 1/2$ and the $\dot{H}^{1-2\varepsilon}$- and L^∞-norms can be uniformly bounded using Hölder continuity and Lemma 2.1.3.6. Therefore, we find by the previous step and integrating with respect to t:

$$(-\Delta)^{3/8} \left(2u d_{1/2}u \cdot d_{1/2}h + h|d_{1/2}u|^2 \right) \in L^2([0,T] \times S^1),$$

and so $D(-\Delta)^{3/8}h \in L^2([0,T]\times S^1)$, thus $(-\Delta)^{3/8}h \in H^1([0,T]\times S^1)$ similarily as before. This now enables us to apply the second part of Lemma 2.1.3.5 with $s = 3/2$ and $\varepsilon > 0$ sufficiently small to deduce, similar as in (2.72):

$$(-\Delta)^{3/4}\left(2ud_{1/2}u \cdot d_{1/2}h + h|d_{1/2}u|^2\right) \in L^2([0,T]\times S^1),$$

and thus using (2.69) and [46, Theorem 3.1] to find $(-\Delta)^{3/4}h \in H^1([0,T]\times S^1)$. This may now be iterated arbitrarily for an increasing sequence of s. Moreover, by inserting these expressions into the main equation $DH(u)h = 0$, we may deduce the same for higher derivatives in time direction, leading to:

$$h \in \bigcap_{s\in\mathbb{N}} H^s([0,T]\times S^1)$$

This shows that $h \in C^\infty([0,T]\times S^1)$ by Morrey-embeddings.

It should be noted, that due to using the 2D-Laplacian, we merely get regularity to times $t < T$, since we do not prescribe the boundary data at $t = T$. If we want regularity for all $t \leq T$, we have to use the result in [46, Theorem 3.1] regarding analytic operator semigroups and maximal L^p-regularity of heat flows (notice that the 1/2-Laplacian generates an analytic operator semigroup), which actually guarantee existence, uniqueness and estimates up to $t = T$. By uniqueness and the regularity for $t < T$, which we may deduce by using elliptic regularity, we may extend the estimates to $t = T$ for the solution h by [46, Theorem 3.1]. So the result is true as stated, but requires slightly more technical arguments at the endpoint. We emphasise that the treatment of $t < T$ is necessary, as the uniqueness result in Hieber-Prüss [46] requires some regularity to hold while $(-\Delta)^s h$ has a-priori not sufficient regularity for [46, Theorem 3.1] to be applied. □

Proof of Lemma 2.1.3.8. The smoothness of h satisfying $DH(u)h = 0$ with vanishing initial datum now enables us to prove that:

$$h = 0$$

Namely, let us compute the following:

$$\begin{aligned}\partial_t\left(|h|^2\right) &= 2h_t h \\ &= 2\left(-(-\Delta)^{1/2}h + 2ud_{1/2}u \cdot d_{1/2}h + h|d_{1/2}u|^2\right)h \\ &= -2h(-\Delta)^{1/2}h + 2uhd_{1/2}u \cdot d_{1/2}h + |h|^2|d_{1/2}u|^2,\end{aligned} \tag{2.73}$$

and observe that there exists a $C > 0$ (since u is smooth and thereforeHölder continuous), such that:

$$|h|^2|d_{1/2}u|^2 \leq C|h|^2$$

Moreover, we may easily find:

$$\begin{aligned}h(-\Delta)^{1/2}h &= P.V.\int_{S^1}\frac{h(x)-h(y)}{|x-y|^2}dyh(x) \\ &= \frac{1}{2}P.V.\int_{S^1}\frac{|h(x)|^2-|h(y)|^2}{|x-y|^2}dy + \frac{1}{2}P.V.\int_{S^1}\frac{|h(x)-h(y)|^2}{|x-y|^2}dy \\ &= \frac{1}{2}(-\Delta)^{1/2}\left(|h|^2\right) + \frac{1}{2}|d_{1/2}h|^2,\end{aligned} \tag{2.74}$$

where we used:

$$h(x) = \frac{h(x)+h(y)}{2} + \frac{h(x)-h(y)}{2}$$

as well as:

$$(h(x)+h(y))(h(x)-h(y)) = |h(x)|^2 - |h(y)|^2.$$

Therefore, we may estimate:

$$\begin{aligned}
\partial_t\left(|h|^2\right) + (-\Delta)^{1/2}\left(|h|^2\right) &\leq -|d_{1/2}h|^2 + 2uhd_{1/2}u \cdot d_{1/2}h + C|h|^2 \\
&\leq -|d_{1/2}h|^2 + |u|^2|h|^2|d_{1/2}u|^2 + |d_{1/2}h| + C|h|^2 \\
&\leq \hat{C}|h|^2
\end{aligned} \tag{2.75}$$

using the arithmetic-geometric mean to absorb $|d_{1/2}h|^2$ as well as the regularity of u. Here, $\hat{C} > 0$ is a constant not depending on h. Following the arguments in [43, p.101] for the maximum principle, we may here deduce:

$$h = 0,$$

due to the initial values vanishing. We emphasise that the argument merely relies on the fact that $(-\Delta)^{1/2}h(x) \geq 0$ at a global maximum and $h_t \geq 0$ and considering $e^{-(\hat{C}+1)t}h(t,x)$ instead of $h(t,x)$. □

Conclusion of the Proof of Proposition 2.1.3.2. The operator in (2.68) is invertible for smooth u between the spaces $\tilde{W}_0^{1,p}([0,T]\times S^1)$ and $L^p([0,T]\times S^1)$, as we have seen in Lemma 2.1.3.8. Thus, arguing as in [43, p.122] and invoking the Inverse Function Theorem for Banach spaces, we may deduce local existence of solutions to the fractional harmonic map equation in $W^{1,p}$, for $p > 2$. Observe that we use smooth boundary values u_0 at $t = 0$ to construct a smooth solution u to the fractional heat equation $u_t + (-\Delta)^{1/2}u = 0$ with $u(0) = u_0$. Indeed, such a solution exists and is smooth by using the explicit formula obtained from the Fourier coefficients of u_0:

$$u(t,x) = \sum_{n\in\mathbb{Z}} \hat{u}_0(n)e^{-|n|t}e^{inx}, \quad \forall t \in \mathbb{R}_+, \forall x \in S^1 \tag{2.76}$$

It can be directly verified that this is a smooth solution of the homogeneous fractional heat equation.

We then consider the operator $h \mapsto H(u+h)$ for h with vanishing initial datum, which is thus locally invertible. This is also the situation in Hamilton [43] and the key idea is to observe that if $f := H(u)$, then for $\tilde{f}_\delta$ being 0 for $[0,\delta]$ and agreeing with f for other times, then for $\delta > 0$ sufficently small, we know that $\tilde{f}_\delta$ lies in the image of $h \mapsto H(u+h)$, meaning that there is a $\tilde{h}_\delta$ such that $H(u+\tilde{h}_\delta) = \tilde{f}_\delta$. Then, $\tilde{u}_\delta := u + \tilde{h}_\delta$ is a local solution of the half-harmonic map equation up to some time $\delta > 0$ with the initial data u_0.

It should be observed that then the local solution, i.e. only on a subinterval of $[0,T]$, to the fractional harmonic gradient flow is also C^∞ up to some time. This can be proven analogous to the bootstrap for h above. Thanks to this smoothness property of the local solution, we may also deduce that u assumes values in S^{n-1} by following the arguments in Hamilton [43] and using similar tricks as above when we were proving $h = 0$ for solutions to $DH(u)h = 0$ with vanishing initial datum. We emphasise that is suffices to verify:

$$|u|^2 = 1 \text{ a.e. } \Rightarrow |u|^2 - 1 = 0 \text{ a.e.},$$

which can be seen again by using uniqueness of the solution to a specific flow. Namely, if u solves the half-harmonic gradient flow and is smooth, then we may deduce:

$$
\begin{aligned}
\partial_t\left(|u|^2-1\right) &= 2\partial_t u\cdot u \\
&= -2(-\Delta)^{1/2}u\cdot u + |u|^2|d_{1/2}u|^2 \\
&= -(\Delta)^{1/2}\left(|u|^2-1\right) - |d_{1/2}u|^2 + |u|^2|d_{1/2}u|^2 \\
&= -(\Delta)^{1/2}\left(|u|^2-1\right) + \left(|u|^2-1\right)|d_{1/2}u|^2,
\end{aligned}
\tag{2.77}
$$

using (2.74) and therefore, the function $v := |u|^2-1$ satisfies the flow equation:

$$v_t + (-\Delta)^{1/2}v = v|d_{1/2}u|^2$$

One should observe that by assumption, $u(0)\in S^{n-1}$ everywhere, so $v(0)=0$. Thus, arguing completely analogous to the proof of Theorem 2.1.3.2, we can easily deduce that $v=0$ everywhere and therefore that $u\in S^{n-1}$ for all t and x. This now concludes the proof of Proposition 2.1.3.2. □

By uniqueness of the solutions to the fractional harmonic gradient flow, this shows that the solutions to (2.17) are smooth, provided the initial value is smooth, at least for small times.

Approximation and Global Regularity It remains to check that regularity holds for all times and remove the restriction that the initial datum needs to be smooth. Both follow by arguing as in Struwe [89]. Firstly, we have the following result which will be crucial in reducing our considerations to the smooth case:

Lemma 2.1.3.10. *Let $u\in H^{1/2}(S^1;S^{n-1})$. Then there exists a sequence $u_k\in C^\infty(S^1)\cap H^{1/2}(S^1;S^{n-1})$ such that:*

$$\|u_k-u\|_{H^{1/2}(S^1)}\to 0,\quad n\to\infty.$$

This Lemma is a fractional version of an analogous result proven by Schoen-Uhlenbeck in [81], our proof follows the computations in Struwe [92].

Proof. Let ρ be a smooth, non-negative function on S^1 supported on a strict compact subset of S^1 with $\int_{S^1}\rho dx = 1$ and define ρ_ε as usual by:

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right),$$

for all $0<\varepsilon<1$. We shall assume that the support of ρ is $B_1(0)$, using the identification $S^1\simeq\mathbb{R}/2\pi\mathbb{Z}$. Then, as usual for approximations of the identity, we know:

$$\tilde{u}_\varepsilon := \rho_\varepsilon * u \to u \text{ in } H^{1/2}(S^1;\mathbb{R}^n),$$

and all convolutions are smooth. Moreover, we have:

$$
\begin{aligned}
d\left(\rho_\varepsilon * u(x), S^{n-1}\right) &\leq \inf_{z\in S^1}\left|\int_{S^1}\rho_\varepsilon(y)u(y)dy - u(z)\right| \\
&\leq \int_{S^1}\left|\int_{S^1}\rho_\varepsilon(x-y)u(y)dy - u(z)\right|\rho_\varepsilon(x-z)dz
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{S^1}\int_{S^1} \rho_\varepsilon(x-y)\rho_\varepsilon(x-z)\left|u(y)-u(z)\right| dydz \\
&\leq \left(\int_{S^1}\int_{S^1} \rho_\varepsilon(x-y)\rho_\varepsilon(x-z)\left|u(y)-u(z)\right|^2 dydz\right)^{1/2} \\
&\leq \frac{C}{\varepsilon}\left(\int_{B_\varepsilon(x)}\int_{B_\varepsilon(x)} \left|u(y)-u(z)\right|^2 dydz\right)^{1/2} \\
&\lesssim \frac{1}{\varepsilon}\left(\int_{B_\varepsilon(x)}\int_{B_\varepsilon(x)} \frac{\left|u(y)-u(z)\right|^2}{|x-y|^2}\varepsilon^2 dydz\right)^{1/2} \\
&\sim \left(\int_{B_\varepsilon(x)}\int_{B_\varepsilon(x)} \frac{\left|u(y)-u(z)\right|^2}{|x-y|^2} dydz\right)^{1/2} \\
&\lesssim \|u\|_{\dot{H}^{1/2}(S^1)},
\end{aligned}
\tag{2.78}
$$

where we used Hölder's inequality in the fourth line. Observe that we may thus use the absolute continuity of the integral in order to see that the distance between $\rho_\varepsilon * u$ and S^{n-1} becomes arbitrarily small, as $\varepsilon > 0$ goes to 0. Thus, for $\varepsilon > 0$ small enough, $\rho_\varepsilon * u$ is never 0 and thus we may use the projection $\pi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}, \pi(x) = x/|x|$ and apply it to the convolution. Hence, we may define:

$$
u_\varepsilon := \pi\left(\rho_\varepsilon * u\right)
$$

Clearly, these functions satisfy:

$$
u_\varepsilon \in C^\infty(S^1) \cap H^{1/2}(S^1; S^{n-1})
$$

Moreover, as π is Lipschitz on compact domains, we may also deduce, for $\varepsilon > 0$ sufficiently small, that:

$$
u_\varepsilon \text{ is bounded in } H^{1/2}(S^1)
$$

Therefore, an appropriate subsequence, which we shall now denote by u_k converges weakly in $H^{1/2}(S^1)$ to u and strongly in $L^2(S^1)$ as well as almost everywhere pointwise. Additionally, by weak lower semicontinuity of the seminorm:

$$
\|u\|_{H^{1/2}(S^1)} \leq \liminf_{n\to\infty} \|u_k\|_{H^{1/2}(S^1)}
$$

It suffices to check that we have:

$$
\limsup_{k\to\infty} \|u_k\|_{H^{1/2}(S^1)} \leq \|u\|_{H^{1/2}(S^1)},
\tag{2.79}
$$

since then, we also know:

$$
\lim_{k\to\infty} \|u_k\|_{H^{1/2}(S^1)} = \|u\|_{H^{1/2}(S^1)},
$$

which, combined with the weak convergence and the Hilbert space structure of $H^{1/2}(S^1;\mathbb{R}^n)$, shows that $u_k \to u$ strongly in $H^{1/2}(S^1;\mathbb{R}^n)$.

Instead of (2.79), it also suffices to verify:

$$
\limsup_{k\to\infty}\left(\int_{S^1}\int_{S^1} \frac{|u_k(x)-u_k(y)|^2}{|x-y|^2} dydx\right)^{1/2} = \limsup_{k\to\infty} \|u_k\|_{\dot{H}^{1/2}(S^1)} \leq \|u\|_{\dot{H}^{1/2}(S^1)},
\tag{2.80}
$$

and to deduce this, let us notice:

$$\begin{aligned}\left(\int_{S^1}\int_{S^1}\frac{|u_k(x)-u_k(y)|^2}{|x-y|^2}dydx\right)^{1/2} &= \left(\int_{S^1}\int_{S^1}\frac{|\pi(\tilde{u}_k(x))-\pi(\tilde{u}_k(y))|^2}{|x-y|^2}dydx\right)^{1/2}\\ &= Lip(\pi)\cdot\left(\int_{S^1}\int_{S^1}\frac{|\tilde{u}_k(x)-\tilde{u}_k(y)|^2}{|x-y|^2}dydx\right)^{1/2}\\ &= Lip(\pi)\cdot\|\tilde{u}_k\|_{\dot{H}^{1/2}(S^1)} \qquad (2.81)\end{aligned}$$

where $Lip(\pi) > 0$ denotes the Lipschitz constant associated with π. We observe that for k big, we may ensure that $\tilde{u}_k$ becomes arbitrarily close to S^{n-1}, see (2.78). We now just have to argue that for sufficently small neighbourhoods of S^{n-1}, the constant $Lip(\pi)$ can be chosen arbitrarily close to 1. If this was true, then for any $\delta > 0$ and n big enough, we would find:

$$\left(\int_{S^1}\int_{S^1}\frac{|u_k(x)-u_k(y)|^2}{|x-y|^2}dydx\right)^{1/2} \leq (1+\delta)|\tilde{u}_k\|_{\dot{H}^{1/2}(S^1)} \to (1+\delta)\|u\|_{\dot{H}^{1/2}(S^1)},$$

which would imply:

$$\limsup_{k\to\infty}\|u_k\|_{H^{1/2}(S^1)} \leq (1+\delta)\|u\|_{H^{1/2}(S^1)},$$

for all $\delta > 0$. The desired result then follows by letting $\delta \to 0$.

To deduce that the Lipschitz constant becomes arbitrarily small, we first observe that the function is Lipschitz in neighbourhoods of the $n-1$-sphere due to smoothness. Assume now that there is a $\delta > 0$, such that:

$$\sup_{k\in\mathbb{N}}\sup_{x,y\in B_{\frac{1}{k}}(S^{n-1})}\frac{|\pi(x)-\pi(y)|}{|x-y|} \geq 1+\delta,$$

where $B_{\frac{1}{k}}(S^{n-1})$ denotes $1/k$-neighbourhood of S^{n-1}. Choose then sequences $x_k, y_k \in B_{\frac{1}{k}}(S^{n-1}) \subset \mathbb{R}^n$ such that:

$$\frac{|\pi(x_k)-\pi(y_k)|}{|x_k-y_k|} \geq 1+\delta$$

Since the sequences are bounded, they have converging subsequences, still denoted by x_k, y_k, with limits $x_0, y_0 \in S^{n-1}$. If $x_0 \neq y_0$, then we see:

$$1+\delta \leq \frac{|\pi(x_k)-\pi(y_k)|}{|x_k-y_k|} \to \frac{|\pi(x_0)-\pi(y_0)|}{|x_0-y_0|} = 1,$$

which is a contradiction. So $x_0 = y_0$. But in this case, we know that for k sufficiently large, we also have that x_k and y_k remain in any given neighbourhood of $x_0 = y_0$. Choosing the neighbourhood small enough, we may assume that it is convex and that the differential of π has operator norm $< 1 + \delta/2$ for all points in the neighbourhood. The former is clear and the latter relies on π being smooth and $d\pi(x)$ being the orthogonal projection to the tangent plane at any given point $x \in S^{n-1}$, thus having operatornorm 1. Standard arguments then show that on such a neighbourhood of $x_0 = y_0$, π is Lipschitz with Lipschitz constant $\leq 1 + \delta/2$, again contradicting our choice of x_k, y_k. Therefore, we may conclude as previously outlined. $\square$

If we obtain uniform existence intervals and bounds depending merely on the energy, we may deduce regularity for general u_0 by the same result for smooth initial data using Lemma 2.1.3.10 and treat the general case analogous to Struwe [89] by approximation. So we may focus our attention on the smooth case.

The main idea is now to establish uniform bounds for solutions to the half-harmonic gradient flow that shall only depend on the energy and other harmless quantities and apply results like in Hieber-Prüss [46] to establish higher regularity and extensiability of solutions in a smooth way after any given time, similar to Struwe [89]. In order to do so, we shall first adapt Lemma 3.1 and Lemma 3.2 in Struwe [89] to our current situation:

Lemma 2.1.3.11. *There exist $C > 0$ not depending on R, u, T, such that for any smooth u on $[0,T] \times S^1$ and $0 < R < 1$, the following estimate holds for all $x_0 \in S^1$:*

$$\int_0^T \int_{B_{\frac{3R}{4}}(x_0)} |(-\Delta)^{1/4} u|^4 dx dt \leq C \sup_{0 \leq t \leq T} \int_{B_R(x_0)} |(-\Delta)^{1/4} u(t)|^2 dx \cdot \left(\int_0^T \int_{B_R(x_0)} |(-\Delta)^{1/2} u|^2 dx dt + \frac{1}{R^2} \int_0^T \int_{S^1} |(-\Delta)^{1/4} u|^2 dx dt \right), \tag{2.82}$$

by density the same result applies for all $u \in H^1([0,T] \times S^1)$ with bounded $1/2$-Dirichlet energy. Similarily, we have:

$$\int_0^T \int_{S^1} |(-\Delta)^{1/4} u|^4 dx dt \lesssim \sup_{0 \leq t \leq T, x \in S^1} \int_{B_R(x)} |(-\Delta)^{1/4} u(t)|^2 dx \cdot \left(\int_0^T \int_{S^1} |(-\Delta)^{1/2} u|^2 dx dt + \frac{1}{R^3} \int_0^T \int_{S^1} |(-\Delta)^{1/4} u|^2 dx dt \right). \tag{2.83}$$

The proof follows Struwe [89] and we refer to this reference for further details.

Proof. We only treat the case $x = 0$, again using $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, the general one follows by a simple rotation. Let φ be a smooth function supported on $B_1(0)$ and satisfying $0 \leq \varphi \leq 1$ as well as $\varphi = 1$ on $B_{3/4}(0)$. Then we define $\varphi_R(x) := \varphi(\frac{x}{R})$ for any $0 < R < 1$. For brevity, we shall suppress the subscript R in the following computations. We estimate using the Ladyzhenskaya-type inequality in Lemma 2.1.3.1 on lines with fixed time t:

$$\begin{aligned}
&\int_0^T \int_{B_{\frac{3R}{4}}(0)} |(-\Delta)^{1/4} u|^4 dx dt \\
&\leq \int_0^T \int_{S^1} |(-\Delta)^{1/4} u|^4 |\varphi|^4 dx dt \\
&\lesssim \int_0^T \int_{S^1} |(-\Delta)^{1/4} u \cdot \varphi - c|^4 dx dt + \int_0^T \int_{S^1} |c|^4 dx dt \\
&\lesssim \int_0^T \int_{S^1} |(-\Delta)^{1/4} u \cdot \varphi - c|^2 dx \cdot \int_{S^1} \left| (-\Delta)^{1/4} \left((-\Delta)^{1/4} u \cdot \varphi \right) \right|^2 dx dt + \int_{S^1} |c|^4 dx dt \\
&\lesssim \sup_{0 \leq t \leq T} \int_{S^1} |(-\Delta)^{1/4} u(t) \cdot \varphi|^2 dx \cdot \int_0^T \int_{S^1} \left| (-\Delta)^{1/4} \left((-\Delta)^{1/4} u \cdot \varphi \right) \right|^2 dx dt + \int_{S^1} |c|^4 dx dt
\end{aligned}$$

$$\lesssim \sup_{0\le t\le T}\int_{B_R(0)}|(-\Delta)^{1/4}u(t)|^2dx\cdot\int_0^T\int_{S^1}\left|(-\Delta)^{1/4}\left((-\Delta)^{1/4}u\cdot\varphi\right)\right|^2dxdt+\int_{S^1}|c|^4dxdt \tag{2.84}$$

where c is defined to be the average of $(-\Delta)^{1/4}u\cdot\varphi$ over S^1 and thus also the 0-th Fourier coefficient. Observe that the removal of the Fourier coefficient at 0 actually justifies the use of the seminorm above. Moreover, in the fourth inequality, we use that we can remove c due to minimality, cf. Struwe [89].

Let us now observe the following:

$$\begin{aligned}
\int_0^T\int_{S^1}|c|^4dxdt &= \int_0^T\int_{S^1}\left|\int_{S^1}(-\Delta)^{1/4}u(y)\varphi(y)dy\right|^4dxdt\\
&\lesssim \int_0^T\int_{S^1}\left|\int_{S^1}|(-\Delta)^{1/4}u(y)|^2\varphi(y)dy\right|^2\cdot R^2dxdt\\
&\lesssim \sup_{0\le t\le T}\int_{S^1}|(-\Delta)^{1/4}u(t)|^2\varphi dx\cdot\int_0^T\int_{S^1}|(-\Delta)^{1/4}u(y)|^2\varphi(y)dydt\\
&\le \sup_{0\le t\le T}\int_{B_R(0)}|(-\Delta)^{1/4}u(t)|^2dx\cdot\int_0^T\int_{B_R(0)}|(-\Delta)^{1/4}u(y)|^2dydt
\end{aligned}\tag{2.85}$$

as $R<1$. On the other hand, we may observe that:

$$\begin{aligned}
(-\Delta)^{1/4}\left((-\Delta)^{1/4}u\cdot\varphi\right)(x) &= P.V.\int_{S^1}\frac{(-\Delta)^{1/4}u(x)\varphi(x)-(-\Delta)^{1/4}u(y)\varphi(y)}{|x-y|^{3/2}}dy\\
&= (-\Delta)^{1/4}\left((-\Delta)^{1/4}u\right)(x)\varphi(x)+P.V.\int_{S^1}(-\Delta)^{1/4}u(y)\frac{\varphi(x)-\varphi(y)}{|x-y|^{3/2}}dy\\
&= (-\Delta)^{1/2}u(x)\varphi(x)+P.V.\int_{S^1}(-\Delta)^{1/4}u(y)\frac{\varphi(x)-\varphi(y)}{|x-y|^{3/2}}dy
\end{aligned}\tag{2.86}$$

In fact, by Gaia [36] and Roncal-Ral Stinga [74], there would need to be a slightly different kernel than $1/|x-y|^{3/2}$ involved. However, as there are good bounds in terms of $1/|x-y|^{3/2}$ (see Gaia [36]), the estimates we derive from here on continue to hold. The latter summand satisfies the following estimate:

$$\begin{aligned}
&\int_0^T\int_{S^1}\left|P.V.\int_{S^1}(-\Delta)^{1/4}u(y)\frac{\varphi(x)-\varphi(y)}{|x-y|^{3/2}}dy\right|^2dxdt\\
&\lesssim \int_0^T\int_{S^1}\int_{S^1}|(-\Delta)^{1/4}u(y)|^2\frac{1}{|x-y|^{1/2}}dy\cdot\int_{S^1}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{5/2}}dydxdt\\
&\lesssim \int_0^T\int_{S^1}\int_{S^1}|(-\Delta)^{1/4}u(y)|^2\frac{1}{|x-y|^{1/2}}dy\cdot\int_{S^1}\frac{\|\varphi\|_{L^\infty}^2}{|x-y|^{1/2}}dydxdt\\
&\lesssim \frac{1}{R^2}\int_0^T\int_{S^1}\int_{S^1}|(-\Delta)^{1/4}u(y)|^2\frac{1}{|x-y|^{1/2}}dydxdt\\
&\lesssim \frac{1}{R^2}\int_0^T\int_{S^1}|(-\Delta)^{1/4}u(y)|^2dydt
\end{aligned}\tag{2.87}$$

where we used that $\varphi=\varphi_R$ to obtain the uniform estimate in R. Combining (2.84), (2.85), (2.86) and (2.87), we therefore have:

$$\int_0^T\int_{B_{\frac{3R}{4}}(0)}|(-\Delta)^{1/4}u|^4dxdt\lesssim \sup_{0\le t\le T}\int_{B_R(0)}|(-\Delta)^{1/4}u(t)|^2dx$$

$$\cdot \left(\int_0^T \int_{B_R(0)} |(-\Delta)^{1/2} u|^2 dx dt + \frac{1}{R^2} \int_0^T \int_{S^1} |(-\Delta)^{1/4} u|^2 dx dt \right) \quad (2.88)$$

and the constant does not depend on R, u or T. As already noted at the beginning, the same inequality holds for all $x \in S^1$ instead of 0.

We may now cover S^1 by $\lceil \frac{2\pi}{\frac{3R}{4}} \rceil = \lceil \frac{8\pi}{3R} \rceil$ balls of radius $\frac{3R}{4}$ around points on S^1, such that each point is contained in at most 3 balls. Then, by adding the inequalities (2.88) in these points, we find:

$$\begin{aligned} \int_0^T \int_{S^1} |(-\Delta)^{1/4} u|^4 dx dt \lesssim & \sup_{0 \le t \le T, x \in S^1} \int_{B_R(x)} |(-\Delta)^{1/4} u(t)|^2 dx \\ & \cdot \left(\int_0^T \int_{S^1} |(-\Delta)^{1/2} u|^2 dx dt + \frac{1}{R^3} \int_0^T \int_{S^1} |(-\Delta)^{1/4} u|^2 dx dt \right) \end{aligned} \quad (2.89)$$

Observe that we have the power R^{-3} showing up due to including $\sim 1/R$ balls for any given R. □

As in Struwe [89], we shall use the following notation for $0 < R < 1$ and $t \in [0, T]$:

$$E_R(u; x, t) := \frac{1}{2} \int_{B_R(x)} |(-\Delta)^{1/4} u(t)|^2 dx, \quad (2.90)$$

for the local energy and introduce:

$$\varepsilon(R) = \varepsilon(R; u, T) := \sup_{x \in S^1, t \in [0,T]} E_R(u; x, t) \quad (2.91)$$

In analogy to Lemma 3.6 in Struwe [89], we have the following energy estimate:

Lemma 2.1.3.12. *There exists a constant $C > 0$ such that for every $u : [0,T] \times S^1 \to S^{n-1}$ in $H^1([0,T] \times S^1) \cap L^\infty([0,T]; \dot{H}^{1/2}(S^1))$ solving the half-harmonic flow equation* (2.17) *and satisfying the energy decrease property as in Lemma 2.1.3.3, any $0 < R < 1/2$ and $(t, x_0) \in [0,T] \times S^1$, the following estimate holds:*

$$\begin{aligned} E_R(u; x_0, t) &\le E_{2R}(u; x_0, 0) + C \left(\frac{t}{R^2} E(u_0) + \frac{\sqrt{t}}{R} \sqrt{\varepsilon(2R) E(u_0)} \right) \\ &\le E_{2R}(u; x_0, 0) + C \left(\frac{t}{R^2} + \frac{\sqrt{t}}{R} \right) E(u_0), \end{aligned} \quad (2.92)$$

where $E(u_0) = E_{1/2}(u_0)$. In the second inequality, we used the trivial estimate between the local energy and the global one under the energy decay.

The proof is as in Struwe [89].

Proof. Letting φ be any smooth, compactly supported, time-independent function on $B_{2R}(x_0)$, such that $\varphi = 1$ on $B_R(x_0)$ and $0 \le \varphi \le 1$, $|\nabla \varphi| \lesssim 1/R$ (see our choice in the proof of the previous Lemma). We now test (2.17) with $u_t \varphi^2$ and observe that $u_t \cdot u = 0$, as u maps to S^{n-1}. Therefore, we find:

$$0 = \int_0^t \int_{S^1} |u_t|^2 \varphi^2 dx ds + \int_0^t \int_{S^1} (-\Delta)^{1/2} u \cdot u_t \varphi^2 dx ds$$

$$= \int_0^t \int_{S^1} |u_t|^2 \varphi^2 dx ds + \int_0^t \int_{S^1} (-\Delta)^{1/4} u \cdot (-\Delta)^{1/4} \left(u_t \varphi^2 \right) dx ds \tag{2.93}$$

We observe that for smooth f:

$$(-\Delta)^{1/4} \left(f \varphi^2 \right)(x) = (-\Delta)^{1/4} f(x) \varphi(x)^2 + P.V. \int_{S^1} f(y) \frac{\varphi(x)^2 - \varphi(y)^2}{|x-y|^{3/2}} dy, \tag{2.94}$$

and therefore:

$$\begin{aligned} \int_{S^1} (-\Delta)^{1/4} u \cdot (-\Delta)^{1/4} \left(f \varphi^2 \right) dx &= \int_{S^1} (-\Delta)^{1/4} u \cdot (-\Delta)^{1/4} f \cdot \varphi^2 dx \\ &+ \int_{S^1} (-\Delta)^{1/4} u \cdot P.V. \int_{S^1} f(y) \frac{\varphi(x)^2 - \varphi(y)^2}{|x-y|^{3/2}} dy dx \end{aligned} \tag{2.95}$$

by approximation, the same holds true for L^2-functions like u_t, and thus:

$$\begin{aligned} &\int_0^t \int_{S^1} |u_t|^2 \varphi^2 dx ds + E_R(u; x_0, t) - E_{2R}(u; x_0, 0) \\ &\leq \int_0^t \int_{S^1} |u_t|^2 \varphi^2 dx ds + \int_0^t \int_{S^1} \frac{1}{2} \frac{d}{dt} \left(|(-\Delta)^{1/4} u|^2 \varphi^2 \right) dx ds \\ &\leq \left| \int_0^t \int_{S^1} (-\Delta)^{1/4} u \cdot P.V. \int_{S^1} u_t(y) \frac{\varphi(x)^2 - \varphi(y)^2}{|x-y|^{3/2}} dy dx ds \right| \\ &\lesssim \frac{1}{R} \left| \int_0^t \int_{S^1} (-\Delta)^{1/4} u \cdot P.V. \int_{S^1} u_t(y) \varphi(y) \frac{1}{|x-y|^{1/2}} dy dx ds \right| \\ &+ \frac{1}{R} \left| \int_0^t \int_{S^1} (-\Delta)^{1/4} u \cdot P.V. \int_{S^1} u_t(y) \varphi(x) \frac{1}{|x-y|^{1/2}} dy dx ds \right|, \end{aligned} \tag{2.96}$$

where we used the estimate for the gradient of φ in the last line and $\varphi(x)^2 - \varphi(y)^2 = (\varphi(x) + \varphi(y))(\varphi(x) - \varphi(y))$. Using Hölder's inequality, the RHS may be bounded by, up to a constant:

$$\frac{\sqrt{t}}{R} \sqrt{E_{1/2}(u_0)} \left(\int_0^t \int_{S^1} |u_t|^2 \varphi^2 dx dt \right)^{1/2} + \frac{\sqrt{t}}{R} \sqrt{\varepsilon(2R) E_{1/2}(u_0)},$$

the latter summand following from (the first summand may be estimated analogously):

$$\begin{aligned} &\frac{1}{R} \left| \int_0^t \int_{S^1} (-\Delta)^{1/4} u \cdot P.V. \int_{S^1} u_t(y) \varphi(x) \frac{1}{|x-y|^{1/2}} dy dx ds \right| \\ &\lesssim \frac{1}{R} \left| \int_0^t \int_{S^1} \int_{S^1} |(-\Delta)^{1/4} u(x)|^2 \varphi(x)^2 \frac{1}{|x-y|^{1/2}} dy dx ds \cdot \right|^{1/2} \left| \int_0^t \int_{S^1} \int_{S^1} |u_t(y)|^2 \frac{1}{|x-y|^{1/2}} dy dx ds \right|^{1/2} \\ &\lesssim \frac{1}{R} \left| \int_0^t \int_{B_{2R}(x_0)} |(-\Delta)^{1/4} u|^2 \varphi^2 dx ds \cdot \right|^{1/2} \left| \int_0^t \int_{S^1} |u_t|^2 dx ds \right|^{1/2} \\ &\lesssim \frac{\sqrt{t}}{R} \sqrt{\varepsilon(2R) E(u_0)}, \end{aligned} \tag{2.97}$$

where the second factor can be estimated as in Lemma 3.4 of Struwe [89], see also the monotone energy decay estimate for solutions of the half-harmonic flow. Therefore, the result follows after absorption in an obvious manner. $\square$

Having these tools available renders us able to establish the results (of course slightly adapted to our current situation) in Lemma 3.7, 3.8 and 3.10 of Struwe [89] and thus establish uniform L^p-estimates for the RHS of the fractional gradient flow (2.17) under restrictions on the local energy, global energy, R and T and independent of u. Let us state the appropriate adaptions to our current situation:

Lemma 2.1.3.13. *The following generalisations of the results in [89] hold true:*

1. *Lemma 3.7 in [89]: There exists $\epsilon_1 > 0$ such that for any $u \in H^1([0,T]\times S^1)\cap L^\infty([0,T];H^{1/2}(S^1))$ solving* (2.17) *and any $R < 1/2$, there holds:*
$$\int_0^T \int_{S^1} |\nabla u|^2 dxdt \leq CE(u_0)\left(1 + \frac{T}{R^3}\right), \tag{2.98}$$
with C independent of u, T, R, provided $\varepsilon(R) < \varepsilon_1$. Here, $u(0,\cdot) = u_0$ is the initial value.

2. *Lemma 3.8, Remark 3.9 in [89]: For any numbers $\varepsilon, \tau, E_0 > 0$, and if u_0 is smooth also $\tau = 0$, and $R_1 < 1/2$, there is a $\delta > 0$ such that for any u, satisfying the conditions as in 1., solving* (2.17) *and any $I \subset [\tau, T]$ with measure $|I| < \delta$, there holds:*
$$\int_I \int_{S^1} |(-\Delta)^{1/4} u|^2 dxdt < \varepsilon, \tag{2.99}$$
provided $\varepsilon(R_1) < \varepsilon_1, E(u_0) \leq E_0$.

3. *Lemma 3.10, Remark 3.11 in [89]: Let u be, in addition to the assumptions in 1., a $C^2([\tau,T]\times S^1)$-solution to* (2.17)*, then, for every $1 \leq p < +\infty$, there exists a $L^p([\tau,T]\times S^1)$-bound on $u_t + (-\Delta)^{1/2}u$ with a constant only depending on $E(u_0), \tau, T$ and R, provided $\varepsilon(R) < \varepsilon_1$. Here, $\tau > 0$ in general and $\tau \geq 0$ in case u_0 is smooth.*

For example, Lemma 3.7 in Struwe [89] follows by using $(-\Delta)^{1/2}u$ instead of Δu and applying the estimates in Lemma 2.1.3.11. Lemma 3.8 relies on choosing subsequences which can equally well be chosen for $(-\Delta)^{1/4}u$ and $(-\Delta)^{1/2}u$, compactness remains valid and the energy estimate in Lemma2.1.3.12 replaces local energy estimate used in Struwe [89]. Naturally, Remark 3.9 also carries over, as the uniform absolute continuity is guaranteed by Lemma 3.8. Lastly, arguing as in Struwe [89] Lemma 3.10, using twice differentiable solutions of the half-harmonic flow, we may differentiate with respect to t and test against u_t to deduce precisely the same estimates for the L^p-norm of the RHS independent of u, i.e. only depending on the analogous terms as in Lemma 3.10 of Struwe [89].

This also leads to higher order estimates following the bootstrap techniques above and using the result [46, Theorem 3.1], meaning that we may establish regularity up to time T. Extending as in Struwe [89] by restarting the flow at T and using approximating sequences as in Lemma 2.1.3.10 then show regularity of solutions with arbitrary initial datum by uniform convergence on sets with t strictly bounded from below (Remark 3.11 applies to the case of regular initial datum, so in this case smoothness is also given at $t = 0$). We emphasise that if we choose the initial energy sufficiently small, the localised energy E_R will satisfy the necessary inequalities for all times, meaning global smooth existence is justified.

We highlight at this point that the argument presented provides an alternative existence argument for the fractional harmonic gradient flow with values in S^{n-1}. Moreover, the techniques introduced can be used in order to study finite blow-up times and in the future investigate the types of blow-ups that can occur in finite time.

2.1.3.4 Convergence

Another important question is whether or not the solution u of the fractional harmonic gradient flow converges as $t \to +\infty$, or rather for specific subsequences $t_k \to +\infty$. The considerations are completely analogous to Struwe [91], [89].

Theorem 2.1.3.4. *Let $u \in L^2(\mathbb{R}_+; H^{1/2}(S^1))$ and $u_t \in L^2(\mathbb{R}_+; L^2(S^1))$ be a solution of the fractional harmonic gradient flow (2.17) with values in $S^{n-1} \subset \mathbb{R}^n$ and with initial data u_0. Assume that:*

$$\|(-\Delta)^{1/4}u(t)\|_{L^2} \leq \|(-\Delta)^{1/4}u_0\|_{L^2} \leq \varepsilon, \quad \forall t \in \mathbb{R}_+,$$

for $\varepsilon > 0$ sufficiently small. Then, for a suitably chosen subsequence $t_k \to +\infty$, the sequence of maps $(u(t_k, \cdot))_{k\in\mathbb{N}} \subset H^1(S^1; S^{n-1})$ converges weakly in $H^1(S^1)$ to a $1/2$-harmonic map in S^{n-1}.

The proof proceeds completely analogous to the one for Theorem 6.6 in Struwe [91].

Proof. By the considerations in the proof of Proposition 2.1.3.1, we know that for $\varepsilon > 0$ sufficiently small, we have for almost every t:

$$\|\nabla u(t)\|_{L^2(S^1)} \lesssim \|\partial_t u(t)\|_{L^2(S^1)} + 1 \tag{2.100}$$

As in [91], this implies the following:

$$\int_t^{t+1} \int_{S^1} |\nabla u|^2 dx dt \lesssim \int_t^{t+1} \int_{S^1} |u_t|^2 dx dt + 1 \lesssim \|u_t\|_{L^2(\mathbb{R}_+ \times S^1)} + 1, \tag{2.101}$$

for all $t \in [0, \infty[$. Observe that the right handside of the estimate is bounded independently of t. It is also clear that:

$$\lim_{t\to+\infty} \int_t^{t+1} \int_{S^1} |u_t|^2 dx dt = 0, \tag{2.102}$$

due to $u_t \in L^2(\mathbb{R}_+ \times S^1) = L^2(\mathbb{R}_+; L^2(S^1))$.

The observations in (2.101) and (2.102) show that we may choose a subsequence $t_k \to \infty$, such that:

$$u(t_k) \to u_\infty \quad \text{in } H^1(S^1) \text{ weakly,}$$

and $u_t(t_k) \to 0$ strongly in L^2. In fact, we may at first choose t_k such that L^2-convergence is satisfied and such that (2.100) holds for all elements in the sequence. Then extracting another subsequence, weak convergence in $H^1(S^1)$ is immediate due to the boundedness in (2.100). In addition, up to extracting another subsequence, the convergence also holds everywhere pointwise and thus:

$$u_\infty(x) \in S^{n-1} \quad \text{for almost every } x \in S^1$$

Let now $\varphi \in C^\infty(S^1)$ and test the equation (2.17) at the time t_k with φ which shows:

$$\begin{aligned}
&\int_{S^1} \big(u_t(t_k) + (-\Delta)^{1/2}u(t_k)\big)\varphi dx \\
&\qquad = \int_{S^1} u_t(t_k)\varphi dx + \int_{S^1}\int_{S^1} d_{1/2}\left(u(t_k)\right)(x,y) d_{1/2}\varphi(x,y) \frac{dydx}{|x-y|} \\
&\qquad = \int_{S^1}\int_{S^1} |d_{1/2}\left(u(t_k)\right)(x,y)|^2 u(x)\varphi(x) \frac{dydx}{|x-y|},
\end{aligned} \tag{2.103}$$

and since we know by $u_t(t_k) \to 0$ in $L^2(S^1)$ strongly and $u(t_k) \to u_\infty$ in $H^1(S^1)$ weakly, the left hand side converges for $t_k \to \infty$ to:

$$\int_{S^1} (-\Delta)^{1/2} u_\infty \varphi dx$$

On the other hand, the right handside does converge as well. Namely, observe that due to the compactness of $H^1(S^1) \hookrightarrow H^{1/2}(S^1)$ and Hölder's inequality, we have:

$$\int_{S^1}\int_{S^1} d_{1/2}\left(u(t_k) - u_\infty\right)(x,y) d_{1/2}u(t_k)(x,y) u(x)\varphi(x) \frac{dydx}{|x-y|} \to 0, \qquad \text{as } t_k \to \infty$$

Similarily, we may see:

$$\int_{S^1}\int_{S^1} d_{1/2}\left(u(t_k) - u_\infty\right)(x,y) d_{1/2}u_\infty(x,y) u(x)\varphi(x) \frac{dydx}{|x-y|} \to 0, \qquad \text{as } t_k \to \infty$$

So we merely have to consider:

$$\int_{S^1}\int_{S^1} |d_{1/2}u_\infty(x,y)|^2 \left(u(x) - u_\infty(x)\right) \varphi(x) \frac{dydx}{|x-y|},$$

which converges to 0 as well, which is an immediate consequence of dominated convergence and the boundedness of u, u_∞. Thus we have:

$$\begin{aligned} &\int_{S^1}\int_{S^1} |d_{1/2}u(x,y)|^2 u(x)\varphi(x) \frac{dydx}{|x-y|} \\ &\qquad \to \int_{S^1}\int_{S^1} |d_{1/2}u_\infty(x,y)|^2 u_\infty(x)\varphi(x) \frac{dydx}{|x-y|} \end{aligned} \tag{2.104}$$

So, we find the following by passing to the limit $t_k \to \infty$:

$$\int_{S^1} (-\Delta)^{1/2} u_\infty \varphi dx = \int_{S^1} u_\infty |d_{1/2}u_\infty|^2 \varphi dx, \tag{2.105}$$

which is equivalent to:

$$(-\Delta)^{1/2} u_\infty \perp T_{u_\infty} N$$

Therefore, u_∞ is actually 1/2-harmonic. □

One may even say more. By convergence, we may deduce:

$$\|(-\Delta)^{1/2} u_\infty\|_{L^2} \leq \|(-\Delta)^{1/2} u_0\|_{L^2} \leq \varepsilon,$$

meaning an energy bound for the limit function. If $\varepsilon > 0$ is sufficiently small, we may deduce:

$$u_\infty \text{ is a constant map}$$

This assertion follows by lower energy bounds for 1/2-harmonic maps, see for example Sire-Wei-Zheng [84] and the references presented therein.

2.1.4 Appendix A: Alternative Conclusion of Theorem 2.1.3.2: Estimate (2.27)

2.1.4.1 Preliminary Discussion

The goal of this first appendix is to provide an alternative proof of the final estimate (2.27) by using direct methods rather than Theorem 2.1.2.1 on S^1, see also Appendix B. We define the stereographic projection $\Pi : S^1 \setminus \{-i\} \to \mathbb{R}$ as follows:

$$\Pi(\cos(\alpha) + i\sin(\alpha)) := \frac{\cos(\alpha)}{1+\sin(\alpha)}, \quad \forall \alpha \in \mathbb{R}, \alpha \neq -\frac{\pi}{2} + 2\pi\mathbb{Z}$$

Let us state the following result found in Da Lio [15] as Proposition 1.1:

Proposition 2.1.4.1. *Let $u : \mathbb{R} \to \mathbb{R}^n$ and $v := u \circ \Pi : S^1 \setminus \{-i\} \to \mathbb{R}^n$. Then we have:*

$$(-\Delta)^{1/2}_{S^1} v(e^{i\theta}) = \frac{(-\Delta)^{1/2}_{\mathbb{R}} u(\Pi(e^{i\theta}))}{1+\sin(\theta)}, \tag{2.106}$$

and where we observe:

$$\Pi'(\theta) = \frac{1}{1+\sin(\theta)} \tag{2.107}$$

This hints at a connection between the 1/2-Laplacian on S^1 and the one on $\mathbb{R}$. We would like to exploit this relationship using the stereographic projection in order to apply the result in Schikorra-Wang [79], namely Theorem 2.1.2.1 on $\mathbb{R}$, directly as needed in our proof above. Our starting point is the following identity which was part of an earlier argument, where we now denote by $\Pi(x_0) = x$ and $v := u \circ \Pi^{-1}$.

Proposition 2.1.4.2. *We have the following identity for u, v, x, x_0 as previously introduced:*

$$\int_{\mathbb{R}} \frac{|v(x)-v(y)|^2}{|x-y|^2} dy = \int_{S^1} \frac{|u(x_0)-u(y)|^2}{|x_0-y|^2} dy \cdot (1+\sin(x_0)) \tag{2.108}$$

Proof. After a change of variables and obvious estimates, we arrive at:

$$\int_{\mathbb{R}} \frac{|v(x)-v(y)|^2}{|x-y|^2} dy = \int_{S^1} \frac{|v(\Pi(x_0))-v(\Pi(y))|^2}{4\sin\left(\frac{x_0-y}{2}\right)^2} \frac{4\sin\left(\frac{x_0-y}{2}\right)^2}{|\Pi(x_0)-\Pi(y)|^2} \frac{1}{1+\sin(y)} dy \tag{2.109}$$

Thus, the fractional gradient norm over $\mathbb{R}$ is bounded for v. Let us note:

$$\begin{aligned}
|\Pi(x_0)-\Pi(y)| &= \frac{|\cos(x_0)+\cos(x_0)\sin(y)-\cos(y)-\cos(y)\sin(x_0)|}{(1+\sin(x_0))(1+\sin(y))} \\
&= \frac{|\cos(x_0)-\cos(y)+\sin(y-x_0)|}{(1+\sin(x_0))(1+\sin(y))} \\
&= \frac{|-2\sin\left(\frac{y+x_0}{2}\right)\sin\left(\frac{x_0-y}{2}\right)+2\sin\left(\frac{y-x_0}{2}\right)\cos\left(\frac{y-x_0}{2}\right)|}{(1+\sin(x_0))(1+\sin(y))} \\
&= \frac{|2\sin\left(\frac{y+x_0}{2}\right)\sin\left(\frac{y-x_0}{2}\right)+2\sin\left(\frac{y-x_0}{2}\right)\cos\left(\frac{y-x_0}{2}\right)|}{(1+\sin(x_0))(1+\sin(y))} \\
&= 2\frac{|\sin\left(\frac{y-x_0}{2}\right)|}{(1+\sin(x_0))(1+\sin(y))}\left|\sin\left(\frac{y+x_0}{2}\right)+\cos\left(\frac{y-x_0}{2}\right)\right|
\end{aligned}$$

$$= 2\frac{|\sin\left(\frac{y-x_0}{2}\right)|}{(1+\sin(x_0))(1+\sin(y))}\cdot 2\left|\sin\left(\frac{y}{2}+\frac{\pi}{4}\right)\sin\left(\frac{x_0}{2}+\frac{\pi}{4}\right)\right| \tag{2.110}$$

Therefore:

$$\begin{aligned}\frac{4\sin\left(\frac{x_0-y}{2}\right)^2}{|\Pi(x_0)-\Pi(y)|^2}\frac{1}{1+\sin(y))} &= \frac{(1+\sin(y))(1+\sin(x_0))^2}{4\left|\sin\left(\frac{y}{2}+\frac{\pi}{4}\right)\sin\left(\frac{x_0}{2}+\frac{\pi}{4}\right)\right|^2}\\ &= \frac{1+\sin(y)}{2\left|\sin\left(\frac{y}{2}+\frac{\pi}{4}\right)\right|^2}\cdot\frac{(1+\sin(x_0))^2}{2\left|\sin\left(\frac{x_0}{2}+\frac{\pi}{4}\right)\right|^2}\end{aligned} \tag{2.111}$$

This is already sufficient to conclude the proof by combining (2.109), (2.110) and (2.111). Indeed, it can be obtained by observing that:

$$\frac{1+\sin(y)}{2\left|\sin\left(\frac{y}{2}+\frac{\pi}{4}\right)\right|^2} = 1,$$

by using the half-angle formula that:

$$\begin{aligned}\int_{\mathbb{R}}\frac{|v(x)-v(y)|^2}{|x-y|^2}dy &= \int_{S^1}\frac{|u(x_0)-u(y)|^2}{|x_0-y|^2}dy\cdot\frac{(1+\sin(x_0))^2}{2\left|\sin\left(\frac{x_0}{2}+\frac{\pi}{4}\right)\right|^2}\\ &= \int_{S^1}\frac{|u(x_0)-u(y)|^2}{|x_0-y|^2}dy\cdot(1+\sin(x_0))\end{aligned} \tag{2.112}$$

and thus providing the desired connecting identity between $\mathbb{R}$ and S^1. □

However, this immediately yields that changing domains by virtue of the stereographic projection is not sufficiently well-behaved for the fractional gradient to preserve arbitrary fractional norms, as the L^4-norm does not transform as required and thus obstructing an equivalence between L^4 on the circle and on $\mathbb{R}$. The obstruction is visible in the remaining factor $1+\sin(x_0))$ of Proposition 2.1.4.2. Therefore, further ideas are necessary.

A different approach involves periodically extending the function on S^1 to a function U with a cut-off after a finite number of periods, a technique explored afterwards. Let us present the main ideas informally first: This extension procedure allows us to have an immediate equivalence of the L^4-norms at the beginning of (2.27) and the corresponding one for U with the distance function changed suitably. The argument in (2.27) then carries on as specified on $\mathbb{R}$. The $(-\Delta)^{1/2}U$ norm can be easily estimated by using the Riesz transform to go over to the classical weak derivative which can be estimated by the corresponding quantity on S^1. It thus remains to connect the L^2-norm of $(-\Delta)^{1/4}U$ with the one of u. This is done in the same way as connecting the L^4-norms due to the immediate estimates for $|d_{1/2}u|^2$. This then finishes the proof of (2.27) and therefore also of Lemma 2.1.3.2.

As a final comment on the previous result, let us mention that naturally, by combining the results for periodic distributions in Schmeisser-Triebel [80] with the ideas in Schikorra-Wang [79], we could obtain the very same identifications as there, thus providing the first inequality in (2.27) immediately for free. This is the very same argument as we already mentioned in section 2 and explored in Appendix B. It then suffices to apply Lemma 2.1.3.1 to conclude.

2.1.4.2 Estimate for Fractional Gradients using Periodic Extension

Let us first take $u \in H^{1/2}(S^1)$ and extend it periodically to $\mathbb{R}$ and denote this extension by U. Next, we choose any $\varphi \in C_c^\infty([-3\pi, 3\pi])$ and define:

$$V := U \cdot \varphi$$

We may assume that $\varphi = 1$ on $[-2\pi, 2\pi]$ and $\varphi = 0$ for $x \in \mathbb{R}\backslash] -\frac{5}{2}\pi, \frac{5}{2}\pi[$. We notice that for every $x \in [-\pi, \pi]$:

$$\begin{aligned} |d_{1/2}V|^2(x) &= \int_{\mathbb{R}} \frac{|V(x) - V(y)|^2}{|x-y|^2} dy \\ &\geq \int_{[x-\pi, x+\pi]} \frac{|V(x) - V(y)|^2}{|x-y|^2} dy \\ &\geq C \int_{S^1} \frac{|u(x) - u(y)|^2}{|x-y|^2} dy \\ &= C|d_{1/2}u|^2(x) \end{aligned} \tag{2.113}$$

Observe that we exchanged the distance function on the real line for the one on the circle which are equivalent on the interval we consider with constants independent of x. Notice that the cut-off function φ has been chosen in such a way that the argument works. Therefore, we may deduce:

$$\int_{S^1} |d_{1/2}u|^4(x) dx \leq C \int_{\mathbb{R}} |d_{1/2}V|^4(x) dx$$

Assuming even that $u \in H^1(S^1)$, it is clear that $V \in H^1(\mathbb{R})$ and because of $H^1(S^1) \subset H^{1/2}(S^1)$, the estimate for the L^4-norm applies to this situation. We may therefore deduce from the Ladyzhenskaya-type estimate in Lemma 2.1.3.1 and the equivalent characterisation of the norm in Schikorra-Wang [79], see Theorem 2.1.2.1 for $\mathbb{R}$:

$$\int_{\mathbb{R}} |d_{1/2}V|^4(x) dx \sim \|(-\Delta)^{1/4}V\|_{L^4}^4 \leq C\|(-\Delta)^{1/4}V\|_{L^2}^2 \cdot \|(-\Delta)^{1/2}V\|_{L^2}^2 \leq C'\|(-\Delta)^{1/4}V\|_{L^2}^2 \cdot \|\nabla V\|_{L^2}^2$$

Notice that the equivalence at the beginning of the estimate is due to Schikorra-Wang [79]. We observe that:

$$\nabla V = \nabla U \cdot \varphi + U \cdot \nabla \varphi,$$

therefore the H^1-norm of V may be estimated by the H^1-norm of u:

$$\|\nabla V\|_{L^2}^2 \leq C\|u\|_{H^1(S^1)}^2$$

On the other hand, we may deduce that:

$$\|(-\Delta)^{1/4}V\|_{L^2}^2 \leq C \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|V(x) - V(y)|^2}{|x-y|^2} dx dy \leq C'\|u\|_{H^{1/2}(S^1)},$$

where the second inequality is easily established by direct means using the cut-off φ. Namely, if we write $I := [-3\pi, 3\pi]$:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|V(x) - V(y)|^2}{|x-y|^2} dx dy = \int_I \int_I \frac{|V(x) - V(y)|^2}{|x-y|^2} dx dy + 2 \int_I \int_{I^c} \frac{|V(x)|^2}{|x-y|^2} dy dx$$

$$\lesssim \int_I \int_I \frac{|U(x) - U(y)|^2}{|x-y|^2} dydx + \int_I \int_I |U(y)|^2 \|\nabla\varphi\|_\infty dydx + \int_I |V(x)|^2 dx$$
$$\lesssim \|U\|_{H^{1/2}(I)}, \tag{2.114}$$

where we used that the integral of $1/|x-y|^2$ in the second summand of the first line is taken over a domain $|x-y| > \delta > 0$ thanks to the cut-off ensuring that x lying in a strict subset of I is necessary for $|V(x)|/|x-y|^2 \neq 0$. We thus need to establish a connection between the norm of U and the one of u. For the L^2-norms, such a relationship is obvious. Regarding the $H^{1/2}$-seminorm, this follows rather easily as well by means of a direct comparison and using the decrease of $1/|x-y|^2$ and comparing it to the periodic distance on S^1. The claim is thus established.

Let us now observe that in the beginning of these calculations, we could have assumed that $⨍_{S^1} udx = 0$ or, alternatively, used $u - \hat{u}(0)$ instead of u, simply because of $d_{1/2}u$ annihilating constants. Notice that:

$$\|u - \hat{u}(0)\|_{H^{1/2}(S^1)} \leq C\|u - \hat{u}(0)\|_{\dot{H}^{1/2}(S^1)} = C\|u\|_{\dot{H}^{1/2}(S^1)}$$

So we arrive at the following estimate by combining all these considerations (using similar ones for H^1 versus $\dot{H}^1$) for $u - \hat{u}(0)$:

$$\int_{S^1} |d_{1/2}u|^4(x)dx = \int_{S^1} |d_{1/2}(u - \hat{u}(0))|^4(x)dx \leq \tilde{C}\|u\|^2_{\dot{H}^{1/2}(S^1)} \cdot \|u\|^2_{\dot{H}^1(S^1)} \tag{2.115}$$

This is precisely the inequality used in the proof of improved regularity in the proof of uniqueness and/or the proof of improved regularity.

2.1.5 Appendix B: Further useful Results

2.1.5.1 Wente-type result for Fractional Gradients on the Circle: Lemma 2.1.2.2

Let us assume that $F \in L^2_{od}(S^1 \times S^1)$ and $g \in H^{1/2}(S^1)$. Moreover, we assume that:

$$\operatorname{div}_{1/2} F = 0,$$

i.e. that:

$$\int_{S^1} \int_{S^1} F(x,y) d_{1/2}\varphi(x,y) \frac{dxdy}{|x-y|} = 0, \quad \forall \varphi \in C^\infty(S^1)$$

Our goal is to show that the following holds:

$$\forall \varphi \in C^\infty(S^1) : \left| \int_{S^1} F \cdot d_{1/2}g(x)\varphi(x)dx \right| \leq C\|\varphi\|_{\dot{H}^{1/2}},$$

with:

$$C \lesssim \|F\|_{L^2_{od}} \|g\|_{\dot{H}^{1/2}}$$

This implies that $F \cdot d_{1/2}g \in H^{-1/2}(S^1)$ which would in turn enable us to solve equations like:

$$(-\Delta)^{1/2}u = F \cdot d_{1/2}g - c,$$

where $c = ⨍_{S^1} F \cdot d_{1/2}g(x)dx$ for some $u \in H^{1/2}(S^1)$ with appropriate estimates. This is the kind of fractional Wente-type estimate we would like to use. To prove this, we observe that by using the stereographic projection Π as in Proposition 2.1.4.1, we then have for:

$$F' : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, F'(x,y) = F(\Pi^{-1}(x), \Pi^{-1}(y)), \quad g' : \mathbb{R} \to \mathbb{R}, g'(x) = g(\Pi^{-1}(x))$$

We observe the following for $\varphi \in C^\infty(S^1)$ compactly supported in $S^1\setminus\{-i\}$ and the previously studied factor $h(z) = \frac{1+\sin(z)}{2\left|\sin\left(\frac{z}{2}+\frac{\pi}{4}\right)\right|^2} = 1$ and thus also for:

$$\tilde{h}(x) := \frac{1}{h(\Pi^{-1}(x))} = 1,$$

which we may use to obtain the following chain of equations following the computations in the proof of Proposition 2.1.4.2, especially (2.111) to rewrite the quotient of the distance functions on $\mathbb{R}$ and S^1, and a change of variables:

$$\begin{aligned}
&\int_{S^1}\int_{S^1} F(z,w)\frac{g(z)-g(w)}{|z-w|^{1/2}}\varphi(z)\frac{dzdw}{|z-w|}\\
&\quad= \int_{\mathbb{R}}\int_{\mathbb{R}} F'(x,y)\frac{g'(x)-g'(y)}{|x-y|^{1/2}}\varphi(\pi^{-1}(x))\left(\frac{8\left|\sin\left(\frac{\Pi^{-1}(x)}{2}+\frac{\pi}{4}\right)\right|^3\cdot 8\left|\sin\left(\frac{\Pi^{-1}(y)}{2}+\frac{\pi}{4}\right)\right|^3}{(1+\sin(\Pi^{-1}(x)))(1+\sin(\Pi^{-1}(y)))}\right)^{1/2}\frac{dxdy}{|x-y|}\\
&\quad= \int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{F}(x,y)\frac{g'(x)-g'(y)}{|x-y|^{1/2}}\varphi(\Pi^{-1}(x))\frac{dxdy}{|x-y|},
\end{aligned} \tag{2.116}$$

(see below for the definition of $\tilde{F}$) and we observe that $\tilde{h} = 1$ on $\mathbb{R}$ and that:

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}\left(2\left|\sin\left(\frac{\Pi^{-1}(x)}{2}+\frac{\pi}{4}\right)\right|^{1/2}\left|\sin\left(\frac{\Pi^{-1}(y)}{2}+\frac{\pi}{4}\right)\right|^{1/2}\cdot F'(x,y)\right)^2\frac{dxdy}{|x-y|}\\
&\quad= \int_{\mathbb{R}}\int_{\mathbb{R}} 4\left|\sin\left(\frac{\Pi^{-1}(x)}{2}+\frac{\pi}{4}\right)\right|\left|\sin\left(\frac{\Pi^{-1}(y)}{2}+\frac{\pi}{4}\right)\right|\cdot|F'(x,y)|^2\frac{dxdy}{|x-y|}\\
&\quad= \int_{S^1}\int_{S^1}|F(z,w)|^2\frac{dzdw}{|z-w|},
\end{aligned} \tag{2.117}$$

so we observe that if $F \in L^2_{od}(S^1\times S^1)$, then the same holds true for

$$\tilde{F} := 2\left|\sin\left(\frac{\Pi^{-1}(x)}{2}+\frac{\pi}{4}\right)\right|^{1/2}\left|\sin\left(\frac{\Pi^{-1}(y)}{2}+\frac{\pi}{4}\right)\right|^{1/2}\cdot F'(x,y)$$

for the domain $\mathbb{R}$ instead of S^1. This is the ideal starting point for a generalisation of Theorem 2.1 in Mazowiecka-Schikorra [57], as we have now found the substitute for F on the real line. Next, we observe that we have for any constant $C \in \mathbb{R}$:

$$\begin{aligned}
\int_{S^1}\int_{S^1} F(z,w)\frac{g(z)-g(w)}{|z-w|^{1/2}}\varphi(z)\frac{dzdw}{|z-w|} &= \int_{S^1}\int_{S^1} F(z,w)(g(z)-C)\frac{\varphi(z)-\varphi(y)}{|z-w|^{1/2}}\frac{dzdw}{|z-w|}\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\tilde{F}(x,y)(g'(x)-C)d_{1/2}\varphi(\Pi^{-1}(x),\Pi^{-1}(y))\frac{dxdy}{|x-y|}
\end{aligned}$$

by using $\operatorname{div}_{1/2} F = 0$. If $\varphi = 1$, we may even notice (observe that the compact support is not relevant to the computations above) that then $\operatorname{div}_{1/2}(\tilde{F}(x,y)) = 0$. Therefore, the arguments in the proof of Theorem 2.1 become immediately applicable to $\tilde{F}$ and g'. Hence, this leads us to the realisation:

$$\tilde{F}\cdot d_{1/2}g' \in \mathcal{H}^1(\mathbb{R}),$$

with the Wente-type estimate found in the preliminary section as well as in Mazowiecka-Schikorra [57]. Observing that $\dot{H}^{1/2}(\mathbb{R})$ continuously embeds into $BMO(\mathbb{R})$, we therefore find that $\tilde{F} \cdot d_{1/2}g' \in H^{-1/2}(\mathbb{R})$. Pulling now back to S^1, we may obtain use for smooth compactly supported φ on $S^1 \backslash \{-i\}$:

$$\begin{aligned}\int_{S^1} \varphi(z) F \cdot d_{1/2}g(z) dz &= \int_{S^1}\int_{S^1} \varphi(z) F(z,w) \frac{g(z)-g(w)}{|z-w|^{1/2}} \frac{dzdw}{|z-w|} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{F}(x,y) \frac{g'(x)-g'(y)}{|x-y|^{1/2}} \varphi(\Pi^{-1}(x)) \frac{dxdy}{|x-y|}, \end{aligned} \tag{2.118}$$

The estimate on the circle may thus be obtained from the one on the real line, at least for smooth compactly supported functions on the complement of a point, since the very same argument works with respect to the stereographic projection with respect to any point on the circle.

To deduce the result on the entire circle, i.e. $F \cdot d_{1/2}g \in H^{-1/2}(S^1)$, we split any smooth function using a fixed partition of unity into two parts supported each on a compact subset of the complement of a point, the points for example being the north and south pole, and apply the estimate from the real line to each of these parts, using stereographic projections with respect to two different points. Observe that the Gagliardo seminorn and the L^2-norm of the parts are controlled by the original (semi-)norm of the smooth function. Therefore, we obtain the desired Wente-type estimate.

To close this argument, let us observe that for a suitable $c \in \mathbb{R}$ (given by the integral of $F \cdot d_{1/2}g$ over the circle), we can thus obtain the following estimate:

$$\left| \int_{S^1} (F \cdot d_{1/2}g(z) - c) \varphi(z) dz \right| \leq C \|F\|_{L^2_{od}} \|g\|_{H^{1/2}} \|\varphi\|_{\dot{H}^{1/2}}$$

This is clear by going over to Fourier coefficients on the circle. This can be rephrased as:

$$\|F \cdot d_{1/2}g - c\|_{H^{-1/2}} \leq C \|F\|_{L^2_{od}} \|g\|_{H^{1/2}} \tag{2.119}$$

2.1.5.2 Version of Theorem 2.1.2.1 on S^1

In this section, we shall prove the following:

Theorem 2.1.5.1. *Let $s \in (0,1)$, $p, q \in]1, \infty[$ and $f \in L^p(\mathbb{R})$. Then, if $p > \frac{nq}{n+sq}$, we also have the inclusion $\dot{F}^s_{p,q}(S^1) \subset \dot{W}^{s,(p,q)}(S^1)$ together with an estimate:*

$$\|f\|_{\dot{W}^{s,(p,q)}(S^1)} \lesssim \|f\|_{\dot{F}^s_{p,q}(S^1)}$$

The constant depends on s, p, q, n.

This is in fact the only part of Theorem 2.1.2.1 we use throughout the current paper. The proof proceeds as in Schikorra-Wang [79], see in particular the fourth section in this reference.

Proof. First, we notice that the following result, Lemma 4.4 in Schikorra-Wang [79], continues to hold true:

Lemma 2.1.5.1. *Let $k \in \mathbb{Z}, j \in \mathbb{N}$ and f_j be the j-th Littlewood-Paley projection of f a periodic distribution on $\mathbb{R}$ (or equivalently an distribution on S^1):*

$$f_j(x) := \sum_{k\in\mathbb{Z}} \varphi_j(k)\hat{f}(k)e^{ikx},$$

where φ_j are as in the definition of the Triebel-Lizorkin spaces in section 2. Assume that $x, y \in \mathbb{R}$ together with $|x-y| \sim 2^{-k}$. Then for every $r > 0$, we have:

$$|f_j(x) - f_j(y)| \lesssim 2^{j-k}(1+2^{j-k})^{n/r} \left(M|f_j|^r(x)\right)^{1/r} \tag{2.120}$$

$$|f_j(y)| \lesssim (1+2^{j-k})^{n/r} \left(M|f_j|^r(x)\right)^{1/r} \tag{2.121}$$

where Mg denotes the Littlewood-Paley maximal function and the constants only depend on r.

The proof is exactly the same as in Schikorra-Wang [79], only referring to Schmeisser-Triebel [80] Proposition 3.3.5 and Theorem 3.3.5 instead of the results for $\mathbb{R}^n$. Observe that only $j > 0$ need to be considered due to the discrete nature of Fourier coefficients.

Having Lemma 2.1.5.1 available, we can argue analogous to Schikorra-Wang [79]. Let us observe that:

$$\begin{aligned}\|f\|^p_{\dot{W}^{s,(p,q)}(S^1)} &= \int_{S^1}\left(\int_{S^1} \frac{\left|\sum_{j\in\mathbb{N}} f_j(x) - f_j(y)\right|^q}{|x-y|^{1+sq}} dy\right)^{p/q} dx \\ &\lesssim \int_{-\pi}^{\pi}\left(\sum_{k\in\mathbb{Z}} 2^{k(1+sq)} \int_{A_k(x)} \left|\sum_j f_j(x) - f_j(y)\right|^q dy\right)^{p/q} dx,\end{aligned} \tag{2.122}$$

where $A_k(x) := \{y | 2^{-k} \leq |x-y| < 2^{-k+1}\}$. Notice that we replaced the distance function on the circle S^1 by the one on $\mathbb{R}$ and chose the integration domain appropriately to still estimate the expression $\|f\|^p_{\dot{W}^{s,(p,q)}(S^1)}$.

As in Schikorra-Wang [79], let us introduce:

$$\int_{-\pi}^{\pi}\left(\sum_{k\in\mathbb{Z}} 2^{k(1+sq)} \int_{A_k(x)} \left|\sum_j f_j(x) - f_j(y)\right|^q dy\right)^{p/q} dx \lesssim R_1 + R_2 + R_3, \tag{2.123}$$

where:

$$R_1 := \int_{-\pi}^{\pi}\left(\sum_{k\in\mathbb{Z}} 2^{k(1+sq)} \int_{A_k(x)} \left(\sum_{j\leq k} \left|f_j(x) - f_j(y)\right|\right)^q dy\right)^{p/q} dx \tag{2.124}$$

$$R_2 := \int_{-\pi}^{\pi}\left(\sum_{k\in\mathbb{Z}} 2^{k(1+sq)} \int_{A_k(x)} \left(\sum_{j>k} \left|f_j(x)\right|\right)^q dy\right)^{p/q} dx \tag{2.125}$$

$$R_3 := \int_{-\pi}^{\pi}\left(\sum_{k\in\mathbb{Z}} 2^{k(1+sq)} \int_{A_k(x)} \left(\sum_{j>k} \left|f_j(y)\right|\right)^q dy\right)^{p/q} dx \tag{2.126}$$

The estimate for each contribution now proceeds as in Schikorra-Wang [79]: For example, R_1 can be dealt with by noticing that for some $s > \varepsilon > 0$

$$\begin{aligned}\left(\sum_{j\le k}\big|f_j(x)-f_j(y)\big|\right)^q &= \left(\sum_{j\le k}2^{j\varepsilon}2^{-j\varepsilon}\big|f_j(x)-f_j(y)\big|\right)^q\\ &\lesssim \left(\sum_{j\le k}2^{j\varepsilon}\right)^q \sup_{j\le k}2^{-j\varepsilon q}|f_j(x)-f_j(y)|^q\\ &\lesssim 2^{k\varepsilon q}\sum_{j\le k}2^{-j\varepsilon q}|f_j(x)-f_j(y)|^q \end{aligned}\tag{2.127}$$

Using now Lemma 2.1.5.1, we arrive at the following identity completely analogous to Schikorra-Wang [79]:

$$|f_j(x)-f_j(y)| \le C(r)2^{j-k}\left(M|f_j|^r(x)\right)^{1/r}, \quad \forall y\in A_k(x), \forall j\le k, \tag{2.128}$$

for some constant $C(r) > 0$ depending only on $r > 0$. Combining (2.127) and (2.128), we find:

$$\begin{aligned} R_1 &\lesssim \int_{-\pi}^{\pi}\left(\sum_{k\in\mathbb{Z}}2^{k(1+sq)}\int_{A_k(x)}2^{k\varepsilon q}\sum_{j\le k}2^{-j\varepsilon q}\big|f_j(x)-f_j(y)\big|^q dy\right)^{p/q}dx\\ &\lesssim \int_{-\pi}^{\pi}\left(\sum_{k\in\mathbb{Z}}2^{k(1+sq)}\int_{A_k(x)}2^{k\varepsilon q}\sum_{j\le k}2^{-j\varepsilon q}C(r)^q 2^{(j-k)q}\left(M|f_j|^r(x)\right)^{q/r}dy\right)^{p/q}dx\\ &\lesssim \int_{-\pi}^{\pi}\left(\sum_{k\in\mathbb{Z}}2^{k(1+sq)}2^{-k}2^{k\varepsilon q}\sum_{j\le k}2^{-j\varepsilon q}2^{(j-k)q}\left(M|f_j|^r(x)\right)^{q/r}\right)^{p/q}dx\\ &\lesssim \int_{-\pi}^{\pi}\left(\sum_{j>0}2^{-j\varepsilon q}\left(M|f_j|^r(x)\right)^{q/r}2^{jq}\sum_{k\ge j}2^{k(s-1+\varepsilon)q}\right)^{p/q}dx\\ &\lesssim \int_{-\pi}^{\pi}\left(2^{jsq}\left(M|f_j|^r(x)\right)^{q/r}\right)^{p/q}dx, \end{aligned}\tag{2.129}$$

where we use $\varepsilon > 0$ suffciently small, such that $s+\varepsilon < 1$. Applying Proposition 3.2.4 in Schmeisser-Triebel [80], i.e. the maximal function estimate for $L^p l^q$-functions on S^1, we thus deduce by the very definition of f_j and the Triebel-Lizorkin norm:

$$R_1 \lesssim \|f\|_{\dot{F}^s_{p,q}}$$

The other contributions R_2 and R_3 may also be deduced completely analogous to Schikorra-Wang [79], but using the corresponding results for S^1 as found in Schmeisser-Triebel [80]. We thus may conclude. □

2.2 The General Case: Arbitrary Closed Target Manifolds [103]

Now, we would like to extend the results from the previous section to arbitrary target manifolds. To achieve this, we have to overcome various technicalities which we were able to avoid in the spherical case. For instance, there is no longer a simple characterisation in terms of the distance function or an arbitrary global smooth function, which imposes difficulties in the choice of operators one considers when solving the problem for short times. The arguments remain mostly similar to the ones in the section before, but we have to be more careful at times.

2.2.1 Introduction

The goal of this paper is to etend the results obtained in the author's previous work [102], where the half-harmonic gradient flow with values in S^{n-1} was studied. More precisely, the following result was proven:

Theorem 2.2.1.1. *Let $u_0 \in H^{1/2}(S^1; S^{n-1})$ be any initial data. There exists $\varepsilon > 0$, such that if:*

$$\|(-\Delta)^{1/4}u_0\|_{L^2(S^1)} \leq \varepsilon,$$

then there exists a unique energy class solution $u : \mathbb{R}_+ \times S^1 \to S^{n-1} \subset \mathbb{R}^n$ of the weak fractional harmonic gradient flow:

$$u_t + (-\Delta)^{1/2}u = u|d_{1/2}u|^2,$$

satisfying $u(0,\cdot) = u_0$ in the sense $u(t,\cdot) \to u_0$ in L^2, as $t \to 0$. Moreover, the solution fulfills the energy decay estimate:

$$\|(-\Delta)^{1/4}u(t)\|_{L^2(S^1)} \leq \|(-\Delta)^{1/4}u_0\|_{L^2(S^1)}.$$

In fact, $u \in C^\infty(]0,\infty[\times S^1)$ and for an appropriate subsequence $t_k \to \infty$, the sequence $u(t_k)$ converges weakly in $H^1(S^1)$ to a point.

Let us briefly recall the definition of harmonic and fractional harmonic maps. Harmonic maps are the critical points of the following Dirichlet energy which is given for all maps $u : M \to N \subset \mathbb{R}^n$ in $H^1(M;N)$ by:

$$E(u) := \frac{1}{2}\int_M g^{\alpha\beta}(x)\gamma_{ij}(u(x))\frac{\partial u^i}{\partial x_\alpha}(x)\frac{\partial u^j}{\partial x_\beta}(x)dx,$$

where $(M,g), (N,\gamma)$ smooth Riemannian manifolds, $u = (u^1,\dots,u^n)$ and employing Einstein's summation convention. In case $M = \Omega \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ are isometrically embedded in $\mathbb{R}^m$ and $\mathbb{R}^n$ and equipped with the Riemannian metrics induced by the standard scalar product, this reduces to:

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx$$

One naturally is lead to the extension of the definition above to fractional harmonic maps. Namely, we say that a map $u : S^1 \to N \subset \mathbb{R}^n$ is weakly 1/2-harmonic, if it is a critical point of the following energy:

$$E_{1/2}(u) := \frac{1}{2}\int_{S^1} |(-\Delta)^{1/4}u|^2 dx, \tag{2.130}$$

with respect to variations in the following set:

$$H^{1/2}(S^1;N) := \left\{v \in H^{1/2}(S^1;\mathbb{R}^n) \;\middle|\; u(x) \in N, \text{ for a.e. } x \in S^1\right\}$$

Namely, the criticality condition means that for every $\Phi \in \dot{H}^{1/2}(S^1;\mathbb{R}^n) \cap L^\infty(S^1)$, in particular all smooth $\Phi \in C^\infty(S^1;\mathbb{R}^n)$, we have:

$$\frac{d}{dt}E_{1/2}\left(\pi(u + t\Phi)\right)\Big|_{t=0} = 0, \tag{2.131}$$

where π is the orthogonal closest-point projection to N, which is defined in a sufficiently small neighbourhood of N and smooth due to N being smooth. As we shall see, this condition is equivalent to:

$$d\pi(u)(-\Delta)^{1/2}u = 0 \quad \text{ in } \mathcal{D}'(S^1), \tag{2.132}$$

which is sometimes also stated informally in the following form, observing that $d\pi(x)$ is the orthogonal projection to T_xN for every $x \in N$:

$$(-\Delta)^{1/2}u \perp T_uN$$

It is clear that, in order to study the regularity of 1/2-harmonic maps, the first step lies in the reformulation of (2.132). Naturally, corresponding definitions for $\mathbb{R}$ instead of S^1 are possible and due to the conformal invariance, the theory of half-harmonic maps on S^1 and $\mathbb{R}$ are equivalent by virtue of composing with the (conformal) stereographic projection. Such equations were first studied in Da Lio-Rivière [21] and questions regarding regularity, bubbling and general properties of such maps have been adressed in the literature, see Da Lio [16]; Schikorra [76]; Da Lio-Schikorra [28], [27]; Da Lio-Pigati [20].

In this paper, we will study the associated evolution problem with the energy (2.130) for arbitrary closed manifolds $N \subset \mathbb{R}^n$. We shall see that this equation could be phrased as:

$$u_t + (-\Delta)^{1/2}u = (Id - d\pi(u))(-\Delta)^{1/2}u, \tag{2.133}$$

or:

$$u_t + (-\Delta)^{1/2}u = d_{1/2}u \cdot d_{1/2}\left(d\pi^{\perp}(u)\right) + \operatorname{div}_{1/2}\left(\frac{A^i_u(du,du)(x,y)}{|x-y|^{1/2}} d\pi^{\perp}(u(y))_{ij}\right), \tag{2.134}$$

for u being a function assuming values a.e. in N with appropriate initial condition $u(0) = u_0$ with values in N. Other types of reformulations are possible and will appear later on. Similar problems in the local setup have been studied in Struwe [89], [90] for the evolution problems associated with the harmonic map equation and found existence in an appropriate sense. However, the case of weak solutions was resolved by Rivière [68] in the case of small energy with values in spheres by means of uniqueness and Freire [35] later on in general.

As in [102], we will focus here on the case of small energy for solutions in the weakest sense and study a class of general solutions for arbitrary energy much like in Struwe [89]. Let us mention here that a weak solution (see also Schikorra-Sire-Wang [77]) of the half-harmonic gradient flow is a function $u \in H^1([0,+\infty[; L^2(S^1;N)) \cap L^\infty([0,+\infty[; H^{1/2}(S^1;N))$ such that

$$\int_0^\infty \int_{S^1} \partial_t u(t,x) \cdot \varphi(t,x) dx dt + \int_0^\infty \int_{S^1} d_{1/2}u(t,x) \cdot d_{1/2}\varphi(t,x) dx dt = 0, \tag{2.135}$$

for all φ sufficiently nice functions with $\varphi(0,\cdot) = 0$ and $\varphi(t,x) \in T_{u(t,x)}N$ in all points $(t,x) \in [0,+\infty[\times S^1$. The restriction on the values of φ can be alleviated, but is omitted in order not to obscure the definition (see (2.170) and the following computations for details on the precise right hand side in case of general φ). We shall now prove:

Theorem 2.2.1.2. *Let $u_0 \in H^{1/2}(S^1;N)$ be any initial datum and N be any closed manifold. There exists $\varepsilon > 0$, such that if:*

$$\|(-\Delta)^{1/4}u_0\|_{L^2(S^1)} \leq \varepsilon, \tag{2.136}$$

then there exists a unique energy class solution $u : \mathbb{R}_+ \times S^1 \to N \subset \mathbb{R}^n$ *of the weak fractional harmonic gradient flow* (2.133), (2.134) *satisfying* $u(0,\cdot) = u_0$ *in the sense* $u(t,\cdot) \to u_0$ *in* L^2, *as* $t \to 0$. *Moreover, the solution fulfills the energy decay estimate:*

$$\|(-\Delta)^{1/4}u(t)\|_{L^2(S^1)} \leq \|(-\Delta)^{1/4}u_0\|_{L^2(S^1)}. \tag{2.137}$$

In fact, $u \in C^\infty(]0,\infty[\times S^1)$ *and for an appropriate subsequence* $t_k \to \infty$, *the sequence* $u(t_k)$ *converges weakly in* $H^1(S^1)$ *to a point. Without the small energy assumption, a unique solution* $u \in C^\infty(]0,T[\times S^1) \cap H^1([0,T] \times S^1; N)$ *with non-increasing energy exists up to some time* T *that can be bounded from below by the initial energy* $\|(-\Delta)^{1/4}u_0\|_{L^2(S^1)}$.

We would like to emphasise here that this result is new to our knowledge. It should especially be noticed that while some existence results for weak solutions were known before (see Schikorra-Sire-Wang [77]), the question remained open for non-homogeneous closed target manifolds. In our discussion, there is no need to restrict to particularly symmetric target manifolds, the arguments are completely general.

It would be interesting to study the behaviour of solutions to the half-harmonic gradient flow with initial datum with high energy and see what happens. In particular, it would be worth investigating blow-ups of the solution in finite time. If no blow-ups exist, then one may argue as in Struwe [89] to extend solutions to arbitrary times, i.e. global smooth existence would be proven for all initial data, with uniqueness of the solution among all that have non-increasing energy. Another intriguing issue pertains to study the bubbling behaviour, as was initiated in Sire-Wei-Zheng [84] for bubbling at $t = +\infty$ for the target space S^1 using a parabolic inner-outer gluing scheme well-known from other parabolic PDEs. In this paper, the authors also conjecture that the half-harmonic gradient flow does not blow-up in finite time, an intriguing conjecture that, to the autor's knowledge, has yet to be verified or disproven.

The key techniques employed in the current paper are similar to those encountered in [102]. Nonetheless, let us briefly sketch the general procedure in the proof of uniqueness and existence for Theorem 2.2.1.2. Firstly, existence is obtained following an argument involving the inverse function theorem in Banach spaces. We use the Fredholm properties of the linearisation of the non-linear operator associated with the half-harmonic gradient flow to prove local existence for sufficiently nice boundary data and then show general local existence by the techniques found in Struwe [89], establishing uniform estimates and using suitable approximating sequences. The most interesting point in the argument involves a bootstrap argument based on commutator estimates from Da Lio-Pigati [20] and regularity results for the fractional heat flow as seen, for example, in Hieber-Prüss [46]. A crucial step is the investigation of the kind of fractional heat equations solved by the difference between a candidate for a solution of the half harmonic gradient flow and its projection onto N, which ultimately allows us to prove that the candidate u indeed assumes values in N, which is a-priori unclear due to the general nature of the arguments involved.

Secondly, uniqueness follows similar to Struwe [89], using ideas and reformulations from Mazowiecka-Schikorra [57] based on (2.134) and arguments based on Rivière [68] to treat the energy class case with small energy by some compensation phenomenon. In fact, the slight gain in integrability that is obtained heavily relies on the emergence of an anti-symmetric potential after suitably rewriting the the system and may be quantified as, for instance, in Da Lio-Pigati [20]. We refer to Proposition 2.2.4.1 for some details and the Appendix. The desirable properties of anti-symmetric properties have first been observed in Rivière [70] and since been used also in the context of fractional harmonic maps

(Da Lio-Rivière [21]) and the outlined argument is completely in line with the results in there. Lastly, the convergence result is an immediate adaption of Struwe [89], as has previously been seen in [102] for the case $N = S^{n-1}$.

The paper is organised as follows: In Section 2, we discuss and introduce some of the key notions for our later arguments. Section 3 starts our investigation of the fractional harmonic gradient flow in the case of N being a closed, orientable hypersurface. The formula we find is reminiscent of the one in Mazowiecka-Schikorra [57] and [102] and emphasises the increased technical difficulty of dealing with general N. In Section 4, we turn to arbitrary closed target manifolds N and first investigate different formulations of the fractional harmonic gradient flow in Section 4.1. These turn out to be useful under different circumstances and the equivalence of all of these forms is ensured for smooth solutions. Then, we prove uniqueness of solutions under varying assumptions in Sections 4.2 by following Struwe [89] and Rivière [68]. Uniqueness among strong solutions, i.e. solutions which are in $H^1_{loc}(\mathbb{R}_+ \times S^1)$, of the half-harmonic gradient flow is a consequence of estimating the L^2-norm of the difference between two solutions with the same initial value, while uniqueness in the class of weak solutions relies on slightly better integrability as established in the Appendix. Next, in Section 4.3, we deal with local existence for smooth boundary data using ideas similar to Hamilton [43] and use estimates as in Struwe [89] to deduce local existence and global existence for small initial energy as in [102]. Indeed, the differences in the proofs in [102] and the current paper are rather technical, as the technique relies on general properties of the non-linearity (quadratic growth in an appropriate sense, orthogonality to the tangent space of N, etc.), as would be expected from a reasonable approach to the problem at hand. Convergence results as $t \to \infty$ are discussed in Section 4.4. The appendices complement the presentation.

2.2.2 Preliminaries

Before we enter our discussion of the main result of this paper, we recall some notions from non-local analysis. In particular, we present the definition of the Triebel-Lizorkin spaces on the unit circle, give an equivalent characterisation under some technical assumptions as in Prats-Saksman [65] and define the fractional gradient and fractional divergence that will appear later on, together with some useful identities.

2.2.2.1 Fractional Laplacian and Triebel-Lizorkin Spaces

In this section, we recall the definition of the Triebel-Lizorkin spaces on the unit circle $S^1 \subset \mathbb{R}^2$ as well as some of the most relevant properties of the fractional Laplacian, at least for our purposes. The current presentation follows the one in Prats-Saksman [65], Prats [64] and Schmeisser-Triebel [80].

We may define a natural metric on S^1 stemming from the identification $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, providing a useful formula for the metric on the universal covering of S^1. The natural distance function given by:

$$|x-y|^2 = |e^{ix} - e^{iy}|^2 = |e^{i(x-y)} - 1|^2,$$

which can be rewritten as:

$$|x-y| = 2\left|\sin\left(\frac{x-y}{2}\right)\right|. \tag{2.138}$$

We shall implicitly use this metric, whenever we are working over S^1, without emphasizing this fact further. Next, we define for any $f : S^1 \to \mathbb{R}$:

$$\mathcal{D}_{s,q}(f)(x) := \left(\int_{S^1} \frac{|f(x) - f(y)|^q}{|x-y|^{sq}} \frac{dy}{|x-y|} \right)^{1/q},$$

for all $1 \leq q < \infty$ and $0 < s < 1$. This results in the following definition as seen previously in Prats-Saksman [65]:

$$\|f\|_{\dot{W}^{s,(p,q)}(S^1)} := \|\mathcal{D}_{s,q}(f)(x)\|_{L^p(S^1)},$$

for every $1 \leq p \leq \infty$. If $p = q$, these spaces correspond to the usual homogeneous Gagliardo-Sobolev spaces $\dot{W}^{s,p}(S^1)$. The operator $\mathcal{D}_{s,q}$ and its main properties are studied in Prats-Saksman [65] and the references therein.

As per usual, one denotes by $\mathcal{D}'(S^1)$ the collection of distributions on S^1 and sometimes denote, for notational convenience, by $\mathcal{D}(S^1)$ the space $C^\infty(S^1)$ of smooth functions (the collection of test functions). $\hat{f}(k)$ will always denote the k-th Fourier coefficient of f, for all $f \in \mathcal{D}'(S^1)$ and $k \in \mathbb{Z}$:

$$\hat{f}(k) := \frac{1}{2\pi} \langle f, e^{-ikx} \rangle = \frac{1}{2\pi} f\left(e^{-ikx}\right), \quad \forall k \in \mathbb{Z}$$

Completely analogous to the situation on $\mathbb{R}^n$, the Triebel-Lizorkin spaces for S^1, denoted by $F^s_{p,q}(S^1)$, are defined for all $s \in \mathbb{R}$, $p, q \in [1, \infty[$ by the following identity:

$$F^s_{p,q}(S^1) := \left\{ f \in \mathcal{D}'(S^1) \mid \|f\|_{F^s_{p,q}} < +\infty \right\}$$

Here we employ the norm defined below, analogous to the construction of function spaces on $\mathbb{R}^n$:

$$\|f\|_{F^s_{p,q}} := \left\| \left\| \left(\sum_{k \in \mathbb{Z}} 2^{js} \varphi_j(k) \hat{f}(k) e^{ikx} \right)_{j \in \mathbb{N}} \right\|_{l^q} \right\|_{L^p(S^1)}, \tag{2.139}$$

for an appropriate partition of unity $(\varphi_j)_{j \in \mathbb{N}}$ consisting of smooth, compactly supported functions on $\mathbb{R}$ satisfying:

$$\operatorname{supp} \varphi_0 \subset B_2(0), \quad \operatorname{supp} \varphi_j \subset \{x \in \mathbb{R} \mid 2^{j-1} \leq |x| \leq 2^{j+1}\}, \forall j \geq 1$$

as well as the boundedness property:

$$\forall k \in \mathbb{N} : \sup_{j \in \mathbb{N}} 2^{jk} \|D^k \varphi_j\|_{L^\infty} \lesssim 1$$

Such a family of functions can be easily constructed by the usual methods for Littlewood-Paley decompositions involving scalings. The Triebel-Lizorkin spaces on S^1, and more generally on the n-torus, possess a theory analogous to the classical case of function spaces on $\mathbb{R}^n$, see Schmeisser-Triebel [80], Chapter 3. In particular, Sobolev embeddings continue to hold ([80, Section 3.5.5]), identifications with classical spaces such as $L^p(S^1)$ ([80, Section 3.5.4]) and duality results ([80, Section 3.5.6]). We shall use the properties of these spaces throughout this paper and shall refer to the given reference for details. The homogeneous spaces is now defined by omitting the Fourier coefficient of 0th-order and adapting the notions accordingly.

In our later considerations, it will be most convenient to be able to work with norms different from, but equivalent to (2.139). The reason lies in the technical nature of the norm (2.139) which we shall not see explicitely emerge from the structure of the fractional gradient flow, but rather a different incarnation. More precisely, in Prats-Saksman [65], the authors prove the following result:

Theorem 2.2.2.1. *Let $s \in (0,1)$, $p, q \in]1,\infty[$ and $f \in L^p(\mathbb{R}^n)$. Then:*

(i) We know $\dot{W}^{s,(p,q)}(\mathbb{R}^n) \subset \dot{F}^s_{p,q}(\mathbb{R}^n)$ together with:

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{W}^{s,(p,q)}(\mathbb{R}^n)} \tag{2.140}$$

(ii) If $p > \frac{nq}{n+sq}$, then we also have the converse inclusion together with:

$$\|f\|_{\dot{W}^{s,(p,q)}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \tag{2.141}$$

The constants depend on s, p, q, n.

We mention that the spaces introduced before easily generalize to $\mathbb{R}^n$ with the obvious modifications. Thus, the result makes sense over this domain.

As seen in Prats-Saksman [65] and by using the properties in Schmeisser-Triebel [80], Triebel [97] for periodic functions, we can similarily discover the following equivalence with Triebel-Lizorkin spaces for all $1 < q < \infty$ and $1 < p < \infty$:

$$\dot{W}^{s,(p,q)}(S^1) = \dot{F}^s_{p,q}(S^1), \tag{2.142}$$

with equivalence of the corresponding seminorms, provided $p > \frac{q}{1+sq}$. For a proof of a part of the result above in the case S^1, we refer to the Appendix in [102]. If $s = 1/2$ and $q = 2$, then $p > 1$ is the requirement in Theorem 2.2.2.1 for the equality of $\dot{F}^{1/2}_{p,2}$ and $\dot{W}^{1/2,(p,2)}$ to hold. Moreover, if $q = 2$, an ubiquitous situation throughout this paper, the result surely applies for all $p \geq 2$. It should be observed that while $\dot{F}^s_{p,2}(S^1) \subset \dot{W}^{s,p}(S^1) = \dot{W}^{s,(p,p)}(S^1)$ for $p \geq 2$, there does not hold equality except for $p = 2$. The reader is reminded of the difference between the Bessel potential spaces and the Gagliardo-Sobolev spaces, which is more or less the underlying statement of this inclusion. We will generally omit mentioning the domain, if it is clear from the context.

On S^1, the fractional s-Laplacian is defined as a Fourier multiplier operating on Fourier series:

$$\widehat{(-\Delta)^s f}(k) = |k|^{2s}\hat{f}(k),$$

for every $k \in \mathbb{Z}$ and all $0 < s$. On the other hand, for $0 < s \leq 1$, this operator can be defined by a singular integral as well:

$$(-\Delta)^s f(x) = C(s) \cdot P.V. \int_{S^1} \frac{f(x) - f(y)}{|x-y|^{1+2s}} dy,$$

where $C(s) > 0$ denotes some constant depending on s. By the Fourier multiplier properties, fractional Laplacians interact in a natural way with Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(S^1)$, as is usual for this type of function spaces. This means that it induces an isomorphism:

$$(-\Delta)^s : \dot{F}^{t+2s}_{p,q} \to \dot{F}^t_{p,q},$$

for all $p, q \in (1,\infty)$ and $t, t+2s \in \mathbb{R}$, see [80, Section 3.6.3] and the proof of the analogous statement in the case $\mathbb{R}^n$.

In analogy, the s-Laplacian can be defined on $\mathbb{R}$ as a Fourier multiplier using the Fourier transform rather than the Fourier series and leads again to an object which can also be characterised by a similar principal value. We omit the details, as the formulas are virtually the same as for the circle, see for example Garofalo [37] for an overview of different aspects of the fractional Laplacian.

2.2.2.2 Fractional Gradients and Divergences

For our later use, we summarise and collect some of the ideas in Mazowiecka-Schikorra [57]. Namely, we are most interested in the fractional gradient and fractional divergence and we recapitulate some of the notions, as was already done in [102].

We denote by $\mathcal{M}_{od}(\mathbb{R}\times\mathbb{R})$ the set of all measurable functions $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ with respect to the measure $\frac{dxdy}{|x-y|}$. One can make this definition equally well on S^1 by exchanging the domain $\mathbb{R}$ for the S^1 and using the metric previously mentioned. Whenever a definition/property applies for both domains, we shall sometimes denote this space by $\mathcal{M}_{od}$.

For a measurable function $f:\mathbb{R}\to\mathbb{R}$ or $f:S^1\to\mathbb{R}$, we define the *fractional s-gradient* as follows:

$$d_s f(x,y) = \frac{f(x)-f(y)}{|x-y|^s} \in \mathcal{M}_{od},$$

for all $0\leq s<1$. The corresponding s-divergence is then introduced by means of duality. It should be clear, but is often useful to know that:

$$d_s f(y,x) = -d_s f(x,y)$$

As stated above, by duality, for $F\in\mathcal{M}_{od}(\mathbb{R}\times\mathbb{R})$ or $F\in\mathcal{M}_{od}(S^1\times S^1)$, we are consequently able to define for every φ smooth and compactly supported on $\mathbb{R}$ or just smooth on S^1 in the latter case:

$$\operatorname{div}_s F(\varphi) := \int\int F(x,y) d_s\varphi(x,y)\frac{dxdy}{|x-y|}$$

This expression is hence defined merely in a distributional sense, i.e. by its duality relation with d_s. For later use, we generally denote for $F,G\in\mathcal{M}_{od}$ over $\mathbb{R}$ or S^1:

$$F\cdot G(x) := \int F(x,y)G(x,y)\frac{dy}{|x-y|}$$

As an obvious special case, if $F=G$ we also write:

$$F\cdot F(x) = |F|^2(x) \Rightarrow |F|(x) := \sqrt{F\cdot F(x)}$$

One should immediately notice the relationship between the previously defined norms on $W^{s,(p,q)}(S^1)$. Indeed, we have:

$$\||d_s f|\|_{L^p(S^1)} = \|f\|_{\dot{W}^{s,(p,2)}(S^1)}.$$

This provides a powerful characterisation of Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(S^1)$ in terms of the fractional gradients d_s, under certain special technical conditions on s,p,q.

It is also possible to prove up to constants which we shall ignore, as they have no effect on the results:

$$(-\Delta)^s f = \operatorname{div}_s d_s f,$$

which is particularily useful for the weak formulation of PDEs involving non-local operators. This equation is to be understood in the following sense:

$$C_s\int d_s f\cdot d_s g(x)dx = \int(-\Delta)^s f\cdot g dx = \int(-\Delta)^{s/2}f\cdot(-\Delta)^{s/2}g dx,$$

for the domains S^1 and $\mathbb{R}$. Lastly, the following identity, sometimes referred to as fractional Leibniz' rule, is often useful:

$$d_s\,(fg)\,(x,y) = d_s f(x,y) g(x) + f(y) d_s g(x,y). \tag{2.143}$$

This identity can be verified by directly inserting the definition. Another type of Leibniz rule is summarised in the following formula:

$$(-\Delta)^{1/2}(fg) = (-\Delta)^{1/2} f \cdot g + f(-\Delta)^{1/2} g - d_{1/2} f \cdot d_{1/2} g, \tag{2.144}$$

which again can be verified by directly inserting definitions. This formula also accounts for the commutator behaviour. Therefore, the fractional gradient may be used to account for the error in the Leibniz rule for the fractional Laplacian and specifies the order of the error.

In general, one defines $L^p_{od}(S^1 \times S^1)$ or $L^p_{od}(\mathbb{R} \times \mathbb{R})$ as the set of all measurable functions, such that the following norm is finite:

$$\|F\|_{L^p_{od}} := \left(\int \int |F(x,y)|^p \frac{dydx}{|x-y|} \right)^{1/p},$$

for $1 \leq p < \infty$. Obviously, $L^\infty_{od}(S^1 \times S^1)$ and $L^\infty_{od}(\mathbb{R} \times \mathbb{R})$ are then to be introduced in the usual manner. These spaces are, in some sense, related to the spaces $W^{s,(p,q)}$.

2.2.3 The Fractional Harmonic Gradient Flow with Values in an orientable Hypersurface

Before we turn our attention to the case of a general target manifold, we dedicate some time to the uniqueness under improved regularity for the special case of an embedded hypersurface which is orientable and closed. This case exhibits similar properties as in the case of the $n-1$-sphere while essentially containing all features encountered in the general case. Moreover, the harmonic map equation possesses a slightly simpler form than in the general case, rendering this special case more tractable. However, the main reason to consider this special case lies in the emergence of all phenomena which we shall encounter in the general case, in particular the inclusion of a fractional divergence term, and thus providing a toy example which will simplify our treatment of the case of a general target manifold.

Indeed, one of the main differences between the sphere S^{n-1} and N a hypersurface will be that the latter is described by a non-local PDE for the fractional harmonic flow which involves a fractional divergence. The techniques used here can then be rather easily adapted to the more general framework, as all estimates used are in some sense independent of the restrictions on N. The remaining properties contained in Theorem 2.2.1.2, i.e. (local) existence as well as the convergence as $t \to +\infty$, shall be proven in the next section for all N at the same time.

2.2.3.1 The Euler-Lagrange Equation of the Half-Harmonic Map

Let us consider $N \subset \mathbb{R}^n$ a closed hypersurface, i.e. an orientable, compact submanifold of dimension $n-1$ without boundary. An important example is of course $N = S^{n-1}$. Under these circumstances, there exists a smooth unit normal field ν over N which, using the tubular neighbourhood theorem and some cut-off-function, can be extended to a smooth vector field $\tilde{\nu}$ on all of $\mathbb{R}^n$, such that $\nu = \tilde{\nu}$

on N and that ν is a unit vectorfield in a neighbourhood of N.

Our goal is now to rewrite the 1/2-harmonic map equation for maps with values in N. Following the computations in [102], one may find along the same lines a formulation for the 1/2-harmonic gradient flow. First, we recall from the introduction that a map $u: S^1 \to N \subset \mathbb{R}^n$ is called 1/2-harmonic, if it is a critical point of the fractional 1/2-Dirichlet energy:

$$E(u) := \frac{1}{2}\int_{S^1} |(-\Delta)^{1/4}u|^2 dx,$$

with respect to variations in $H^{1/2}(S^1; N)$. By compactness of N, we know that any element in this function space is almost everywhere bounded:

$$H^{1/2}(S^1; N) \subset L^\infty(S^1)$$

Let now $u(t)$ be a variation in the set of functions introduced in the introduction, such that $u(0) = u$ is a critical point of the fractional energy E. We may use the tubular neighbourhood theorem to construct $u(t) = \pi(u + t\varphi)$ for some $\varphi \in C^\infty(S^1)$. Here, we used π to denote the projection onto N which is well-defined and smooth on a sufficently small neighbourhood and thus for t small enough. This means:

$$u'(0) := \frac{d}{dt}u(t)\big|_{t=0} = d\pi(u)\varphi$$

Then, we have for a critical point u of E:

$$\begin{aligned}
0 &= \frac{d}{dt}E(u(t))\big|_{t=0} \\
&= \frac{d}{dt}\left(\frac{1}{2}\int_{S^1} |(-\Delta)^{1/4}u(t)|^2 dx\right)\Big|_{t=0} \\
&= \int_{S^1} (-\Delta)^{1/4}u \cdot (-\Delta)^{1/4}u'(0) dx \\
&= \int_{S^1} (-\Delta)^{1/2}u \cdot d\pi(u)\varphi dx, \qquad (2.145)
\end{aligned}$$

which, thanks to $d\pi$ being an orthogonal projection onto the tangent space T_uN, can be rephrased as:

$$(-\Delta)^{1/2}u \perp T_uN \qquad (2.146)$$

This computation holds even for N which are merely closed, there is no need to assume for example codimension 1. This condition becomes however useful, if we would like to find an explicit formula for the harmonic map equation like in the local case, see Struwe [89], or as we have seen for $N = S^{n-1}$ in Mazowiecka-Schikorra [57] or [102]. Namely, we know that:

$$(-\Delta)^{1/2}u = \lambda \cdot \nu(u),$$

where λ is a scalar and depends on the point on S^1 inserted into u. Equivalently:

$$\lambda = (-\Delta)^{1/2}u \cdot \nu(u)$$

Take ψ to be any smooth, scalar-valued function on S^1. We may then compute, by using the fractional Leibniz rule (2.143):

$$\int_{S^1} \lambda \cdot \psi dx = \int_{S^1} (-\Delta)^{1/2}u \cdot \psi\nu(u) dx$$

$$
\begin{aligned}
&= \int_{S^1} d_{1/2}u(x,y)\cdot d_{1/2}\left(\psi\nu(u)\right)(x,y)\frac{dxdy}{|x-y|}\\
&= \int_{S^1} d_{1/2}u(x,y)\cdot \left(d_{1/2}\left(\nu\circ u\right)(x,y)\psi(x)+d_{1/2}\psi(x,y)\nu(u(y))\right)\frac{dxdy}{|x-y|}\\
&= \int_{S^1} d_{1/2}u(x,y)\cdot \left(d_{1/2}\left(\nu\circ u\right)(x,y)\psi(x)+d_{1/2}\psi(x,y)\frac{\nu(u(x))+\nu(u(y))}{2}\right)\frac{dxdy}{|x-y|},
\end{aligned}
\tag{2.147}
$$

where we used a change of variables (i.e. exchanging x with y and vice versa) to justify the last equation. Let us observe that we therefore have:

$$
\lambda = d_{1/2}u\cdot d_{1/2}\left(\nu\circ u\right) + \operatorname{div}_{1/2}\left(d_{1/2}u(x,y)\frac{\nu(u(x))+\nu(u(y))}{2}\right)
\tag{2.148}
$$

We emphasise that the operator $\operatorname{div}_{1/2}$ is defined precisely as the dual of the fractional gradient $d_{1/2}$, so that the identity in fact holds true. Let us observe that the first summand is actually hiding a quadratic structure similar to the one in the case $N=S^{n-1}$ (see [102]) or the local case. Namely, we observe that by the fundamental theorem of calculus:

$$
\begin{aligned}
d_{1/2}\left(\nu\circ u\right)(x,y) &= \frac{\nu(u(x))-\nu(u(y))}{|x-y|^{1/2}}\\
&= \frac{1}{|x-y|^{1/2}}\int_0^1 d\tilde{\nu}\left(u(y)+s\left(u(x)-u(y)\right)\right)\left(u(x)-u(y)\right)ds\\
&= \int_0^1 d\tilde{\nu}\left(u(y)+s\left(u(x)-u(y)\right)\right)ds\cdot d_{1/2}u(x,y)\\
&=: \tilde{A}_u(x,y)d_{1/2}u(x,y),
\end{aligned}
\tag{2.149}
$$

where we notice the similarity of $\tilde{A}_u$ in a certain sense with the term appearing in the local case. We notice that $\tilde{A}_u$ is bounded, therefore giving the estimate:

$$
\left|d_{1/2}u\cdot\tilde{A}_u d_{1/2}u(x,y)\right| \leq \|\tilde{A}_u\|_{L^\infty}|d_{1/2}u|^2
$$

2.2.3.2 Toy Example: Uniqueness under Improved Regularity

We now turn to the gradient flow associated with the fractional harmonic map with values in $N\subset\mathbb{R}^n$. Therefore, as in the case $N=S^{n-1}$ treated in [102], let us assume that u,v are two solutions to the fractional gradient flow taking a.e. values in the closed, orientable hypersurface $N\subset\mathbb{R}^n$ and we suppose the following regularity conditions hold:

$$
u,v\in L^\infty(\mathbb{R}_+;H^{1/2}(S^1))\cap L^2_{loc}(\mathbb{R}_+;H^1(S^1));\quad u_t,v_t\in L^2(\mathbb{R}_+;L^2(S^1))),
\tag{2.150}
$$

In addition, they satisfy the gradient flow associated with the 1/2-harmonic map as described below (see the discussion in [102] and the previous subsection for a justification of this equation):

$$
w_t+(-\Delta)^{1/2}w = d_{1/2}w\cdot\tilde{A}_w d_{1/2}w\cdot\nu(w)+\operatorname{div}_{1/2}\left(d_{1/2}w(x,y)\frac{\nu(w(x))+\nu(w(y))}{2}\right)\cdot\nu(w),
\tag{2.151}
$$

for both $w=u$ and $w=v$, together with the boundary condition $u(0,\cdot)=v(0,\cdot)=u_0\in H^{1/2}(S^1;N)$. It is intuitively clear that the same arguments as in the proof of Theorem 3.2 in [102] should be applicable to the current situation to deduce an analogous uniqueness result, as long as we assume the same kind of regularity for the solution as there. Indeed, we shall prove:

Theorem 2.2.3.1. *If u, v both solve (2.151) with the same initial datum $u_0 \in H^{1/2}(S^1; N)$ and we assume that:*

$$\|(-\Delta)^{1/4}u(t)\|_{L^2(S^1)}, \|(-\Delta)^{1/4}v(t)\|_{L^2(S^1)} \leq \|(-\Delta)^{1/4}u_0\|_{L^2(S^1)}, \quad \forall t \in \mathbb{R}_+,$$

then we have:

$$u = v \quad \mu\text{-a.e.}$$

The strategy of the proof is analogous to the one for $N = S^{n-1}$, cf. [102]. The main changes mostly consist of finding suitable decompositions of the different contributions for the situation at hand, in particular the divergence term. We therefore focus on providing the key estimates needed for the proof and refer to our previous work for the remaining details:

Proof. The main idea is to study the non-local PDE solved by the difference between u and v. Therefore, we are led to define:

$$w := u - v,$$

and observe that:

$$w(0, \cdot) = u(0, \cdot) - v(0, \cdot) = 0.$$

We find that w solves the following PDE by linearity of derivatives and the fractional Laplacians:

$$\begin{aligned} w_t + (-\Delta)^{1/2}w &= u_t + (-\Delta)^{1/2}u - v_t - (-\Delta)^{1/2}v \\ &= R_1 + R_2, \end{aligned} \tag{2.152}$$

where:

$$R_1 := d_{1/2}u \cdot \tilde{A}_u d_{1/2}u \cdot \nu(u) - d_{1/2}v \cdot \tilde{A}_v d_{1/2}v \cdot \nu(v) \tag{2.153}$$

$$R_2 := \operatorname{div}_{1/2}\left(d_{1/2}u \frac{\nu(u(x)) + \nu(u(y))}{2}\right)\nu(u) - \operatorname{div}_{1/2}\left(d_{1/2}v \frac{\nu(v(x)) + \nu(v(y))}{2}\right)\nu(v) \tag{2.154}$$

Naturally, we would like to estimate R_1 and R_2 in a similar way as we have done for the $n-1$-sphere. We treat both contributions individually:

Claim: We have the following estimate:

$$|R_1|(x) \lesssim \left(|d_{1/2}u|(x) + |d_{1/2}v|(x)\right)|d_{1/2}w|(x) + \int_{S^1} \left(|w(x)| + |w(y)|\right) \frac{|d_{1/2}v(x,y)|^2}{|x-y|} dy$$

Proof of Claim: We observe that by using the fundamental theorem of calculus, we can deal with the fractional gradients of $\nu(u), \nu(v)$, i.e. the additional term $\tilde{A}_u, \tilde{A}_v$. Namely, we have:

$$\begin{aligned} R_1 &= d_{1/2}u \cdot \tilde{A}_u d_{1/2}u - d_{1/2}v \cdot \tilde{A}_v d_{1/2}v \\ &= \left(d_{1/2}u \cdot \tilde{A}_u d_{1/2}u - d_{1/2}v \cdot \tilde{A}_u d_{1/2}v\right) + d_{1/2}v \cdot \left(\tilde{A}_u - \tilde{A}_v\right) d_{1/2}v \\ &=: R_{1,1} + R_{1,2} \end{aligned} \tag{2.155}$$

For $R_{1,1}$, we proceed by using the fundamental theorem of calculus:

$$R_{1,1} = \int_0^1 \frac{d}{ds}\left(d_{1/2}\left((1-s)v + su\right) \cdot \tilde{A}_u d_{1/2}\left((1-s)v + su\right)\right) ds$$

$$
\begin{aligned}
&= \int_0^1 \frac{d}{ds}\left(\int_{S^1} d_{1/2}\left((1-s)v+su\right)(x,y)\cdot \tilde{A}_u(x,y) d_{1/2}\left((1-s)v+su\right)(x,y)\frac{dy}{|x-y|}\right) ds \\
&\lesssim \int_0^1 \int_{S^1} |d_{1/2}\left((1-s)v+su\right)(x,y)| \cdot |\tilde{A}_u(x,y)| \cdot |d_{1/2}\left(u-v\right)(x,y)| \frac{dy}{|x-y|} ds \\
&\lesssim \int_0^1 \left|d_{1/2}\left(v+s(u-v)\right)\right|(x)\left|d_{1/2}w\right|(x) ds \\
&\lesssim \left(|d_{1/2}u|(x)+|d_{1/2}v|(x)\right)\cdot |d_{1/2}w|(x)
\end{aligned}
\tag{2.156}
$$

This contribution can be dealt with just as in the case $N = S^{n-1}$ after we test against w itself, see [102]. Concerning $R_{1,2}$, we may use:

$$
\begin{aligned}
&\tilde{A}_u(x,y) - \tilde{A}_v(x,y) \\
&= \int_0^1 d\tilde{\nu}\left(u(y)+s\left(u(x)-u(y)\right)\right) ds - \int_0^1 d\tilde{\nu}\left(v(y)+s\left(v(x)-v(y)\right)\right) ds \\
&= \int_0^1 \frac{d}{dt}\left(\int_0^1 d\tilde{\nu}\left(v(y)+tw(y)+s\left((v(x)+tw(x)-v(y)-tw(y)\right)\right) ds\right) dt \\
&= \int_0^1\int_0^1 d(d\tilde{\nu})\left((1-s)(v(y)+tw(y))+s(v(x)+tw(x))\right)\cdot\left((1-s)w(y)+sw(x)\right) dsdt
\end{aligned}
\tag{2.157}
$$

This concludes the **proof of the claim**. $\square$

Now we may obtain an estimate as in the case $N = S^{n-1}$ by testing against w and:

$$
w_i(y)w_j(x) \leq \frac{w_i(y)^2+w_j(x)^2}{2} \leq \frac{|w|^2(y)+|w|^2(x)}{2},
$$

which enables us, together with the symmetry of the emerging integrals in x, y and thus exchanging the order of integrals in x and y, to arrive at an estimate reminiscent of the first term on the right hand side of equation (20) in [102].

It remains to consider R_2. This term does not possess an immediate analogue in the case $N = S^{n-1}$, thus it seems to require some additional work. Fortunately, the ideas that were involved in the estimation of R_1 (together with using the duality definition of $\mathrm{div}_{1/2}$ and Leibniz' rule for fractional gradients as in Section 2) may be expanded upon to reach a suitable estimates. Let us first notice that testing against w yields for $\nu_u(x,y) := (\nu(u(x))+\nu(u(y)))/2$:

$$
\int_{S^1} R_2 w dx = \int_{S^1}\int_{S^1} d_{1/2}u d_{1/2}\left(\nu(u)w\right)\nu_u(x,y) - d_{1/2}v d_{1/2}\left(\nu(v)w\right)\nu_v(x,y)\frac{dxdy}{|x-y|}
\tag{2.158}
$$

To conclude, we have to estimate this expression appropriately. We see:

$$
\begin{aligned}
d_{1/2}u d_{1/2}&\left(\nu(u)w\right)\nu_u(x,y) - d_{1/2}v d_{1/2}\left(\nu(v)w\right)\nu_v(x,y) \\
&= d_{1/2}u d_{1/2}\left(\nu(u)w\right)\nu_u(x,y) - d_{1/2}u d_{1/2}\left(\nu(v)w\right)\nu_u(x,y) \\
&+ d_{1/2}u d_{1/2}\left(\nu(v)w\right)\nu_u(x,y) - d_{1/2}v d_{1/2}\left(\nu(v)w\right)\nu_u(x,y) \\
&+ d_{1/2}v d_{1/2}\left(\nu(v)w\right)\nu_u(x,y) - d_{1/2}v d_{1/2}\left(\nu(v)w\right)\nu_v(x,y) \\
&=: R_{2,1}+R_{2,2}+R_{2,3}
\end{aligned}
\tag{2.159}
$$

More precisely, we have that:

$$R_{2,1} = d_{1/2} u d_{1/2} \left((\nu(u) - \nu(v)) w \right) \nu_u(x,y),$$

and:

$$R_{2,2} = d_{1/2} w d_{1/2} \left(\nu(v) w \right) \nu_u(x,y),$$

and finally:

$$R_{2,3} = d_{1/2} v d_{1/2} \left(\nu(v) w \right) \left(\nu_u(x,y) - \nu_v(x,y) \right)$$

We shall use the fractional Leibniz' rule as already seen in Section 2, see (2.143):

$$d_{1/2} \left(f w \right) (x,y) = f(x) \frac{w(x) - w(y)}{|x-y|^{1/2}} + w(y) \frac{f(x) - f(y)}{|x-y|^{1/2}},$$

which leads us to:

$$d_{1/2} \left((\nu(u) - \nu(v)) w \right) (x,y) = \left(\nu(u(x)) - \nu(v(x)) \right) d_{1/2} w(x,y) + d_{1/2} \left(\nu(u) - \nu(v) \right) w(y)$$

So we may estimate $R_{2,1}$ for example as follows:

$$\begin{aligned}
\left| \int_{S^1} R_{2,1} dx \right| &\leq \int_{S^1} |d_{1/2} u|(x) |d_{1/2} w|(x) |\nu(u(x)) - \nu(v(x))| dx \\
&+ \int_{S^1} |d_{1/2} u|(y) \left| d_{1/2} \left(\nu(u) - \nu(v) \right) \right| (y) |w| dy \\
&\lesssim \int_{S^1} |d_{1/2} u|(x) |d_{1/2} w|(x) |w| dx, \qquad (2.160)
\end{aligned}$$

by using again the smoothness of $\tilde{\nu}$ and renaming the variable of integration. This is a term that can be treated as before.

The remaining summands $R_{2,2}$ and $R_{2,3}$ are a bit more delicate to deal with, mainly because we have to use another kind of estimate in our proceedings. Let us start with $R_{2,2}$ and observe the following: We shall denote by π the smooth closest point projection defined in a neighbourhood of N and extended to all of $\mathbb{R}^n$ by using a cut-off function (compare with the construction in the next section for some more details). We have for all $x \in S^1$, using Einstein's summation convention and $u = (u_1, \dots, u_n)$ and similarily for v and w:

$$\begin{aligned}
\nu(v(x))_i w_i(x) &= \nu(v(x))_i (u_i(x) - v_i(x)) \\
&= \nu(v(x))_i \left(\partial_j \pi(v(x))_i (u_j(x) - v_j(x)) \right) \\
&+ \nu(v(x))_i \left(\int_0^1 \int_0^1 t \partial_{kj} \pi(v(x) + tsw(x))_i (u_j(x) - v_j(x))(u_k(x) - v_k(x)) ds dt \right) \\
&= \nu(v(x))_i \left(\int_0^1 \int_0^1 t \partial_{kj} \pi(v(x) + tsw(x))_i (u_j(x) - v_j(x))(u_k(x) - v_k(x)) ds dt \right) \\
&=: D^{u,v}_{jk}(x) (u_j(x) - v_j(x))(u_k(x) - v_k(x)) = D^{u,v}_{j,k}(x) w_j(x) w_k(x) \qquad (2.161)
\end{aligned}$$

by using Taylor expansion around the point $v(x)$. Observe that we exploited the fact that $d\pi(v(x))$ maps to the tangent space $T_{v(x)}N$ of N at $v(x)$, which immediately shows:

$$\nu(v(x)) d\pi(v(x)) w(x) = 0,$$

as $\nu(v(x))$ is orthogonal to the projected vector $d\pi(v(x))w(x)$. If we insert this into $R_{2,2}$ instead of $\nu(v)w$, we see by using $d_0 w = d_0 u - d_0 v$, boundedness of u, v and therefore also of w, and estimating D^{uv}_{jk} and its fractional gradient using the smoothness of π:

$$\left| \int_{S^1} R_{2,2} dx \right| \lesssim \int_{S^1} |d_{1/2}w|(x)|w| \left(|d_{1/2}u|(x) + |d_{1/2}v|(x)\right) dx \tag{2.162}$$

In fact, we used implicitly a decomposition of the following form:

$$\begin{aligned} d_{1/2}\left(\nu(v)w\right)(x,y) &= d_{1/2}\left(D^{u,v}_{j,k} w^j w^k\right)(x,y) \\ &= d_{1/2} D^{u,v}_{j,k}(x,y) \cdot w^j(x) w^k(x) \\ &+ D^{u,v}_{j,k}(y) \cdot d_{1/2} w^j(x,y) \cdot w^k(x) \\ &+ D^{u,v}_{j,k}(y) w^j(y) \cdot d_{1/2} w^k(x,y), \end{aligned} \tag{2.163}$$

which can then be dealt with similar to the term R_1 and $R_{1,2}$. Note that $|d_{1/2}w|(x)$ and $|d_{1/2}D^{u,v}_{j,k}|(x)$ can both be bounded by $|d_{1/2}u|(x) + |d_{1/2}v|(x)$ and that $\|w\|_{L^\infty} < +\infty$, as N is compact and u, v both assume values a.e. in N. Similarily, it is clear that $D^{u,v}_{j,k}$ is bounded, due to the regularity of the extended closest-point-projection π.

To arrive at a similar estimate for $R_{2,3}$, we notice the following by using the formula above for $\nu(v)w$ and exchanging the labels x, y at some point:

$$\left| \int_{S^1} R_{2,3} dx \right| \lesssim \int_{S^1} (|d_{1/2}u|(x) + |d_{1/2}v|(x))|d_{1/2}w|(x)|w| + (|d_{1/2}u|(x) + |d_{1/2}v|(x))^2(x)|w|^2 dx \tag{2.164}$$

Comparing this with the estimates in the case $N = S^{n-1}$, we see that we may now proceed as in the proof there, since the main estimate in equation (23) of [102] can now be generalized and the remainder of the proof is of general nature and does not rely on any particular structure of S^{n-1}. To be more precise, the main point now is to apply Cauchy-Schwarz in order to deduce estimates for the L^2-norm of w at fixed times and absorbing all these terms, after maximizing over a small enough time interval, in the left side. The absorption relies on a fractional Ladyzenskaya-type estimate and the characterisation of Triebel-Lizorkin norms by fractional gradients. As a result, this concludes the proof of Theorem 2.2.3.1. □

Naturally, the proof of uniqueness also continues to work for a variety of non-linearities which are in some sense quadratic in the fractional gradient and sufficiently smooth by precisely the same arguments. Moreover, as we have seen in the case of general hypersurfaces, certain perturbation terms are allowed to appear without obstructing the argument, namely some kinds of divergence terms. Therefore, it is expected that analogous results hold for fractional harmonic flows with values in arbitrary closed smooth manifolds by means virtue of a suitable quadratic structure similar to the local case, where it is intimately connected to the curvature of the manifold. In fact, there is an explicit formula for the half-harmonic map given in Mazowiecka-Schikorra [57] which we shall exploit in the next section. There, we shall also present the missing argument for the improvement in regularity needed to conclude the proof of uniqueness in the energy class for solutions with small 1/2-energy.

To conclude this section, let us, for completeness' sake, mention the following: The choice of equation (2.151) is natural for orientable hypersurfaces, due to the existence of a unit normal vector field ν. Nevertheless, one could omit the use of such a vector field by using that the non-linearity in the 1/2-harmonic map equation could be phrased as:

$$
\begin{aligned}
&(-\Delta)^{1/2}u(x)\\
&= (Id - d\pi(u)(x))\,(-\Delta)^{1/2}u(x)\\
&= P.V.\int_{S^1}\frac{u(x)-u(y)}{|x-y|^2}dy - d\pi(u)(x)P.V.\int_{S^1}\frac{u(x)-u(y)}{|x-y|^2}dy\\
&= P.V.\int_{S^1}\frac{u(x)-u(y)-d\pi(u)(x)\,(u(x)-u(y))}{|x-y|^2}dy\\
&= P.V.\int_{S^1}\int_0^1\int_0^1(s-1)d(d\pi)(u(x)+(st-t)(u(x)-u(y)))dsdtd_{1/2}u(x,y)d_{1/2}u(x,y)\frac{dy}{|x-y|}\\
&=: \int_{S^1}P(x,y)d_{1/2}u(x,y)d_{1/2}u(x,y)\frac{dy}{|x-y|}, \qquad (2.165)
\end{aligned}
$$

where we denote again by π also an extension by cut-off of the closest point projection and use $\pi(u(x)) = u(x)$ for u being a solution to the 1/2-harmonic map equation in N. The map P is defined appropriately:

$$P(x,y) := \int_0^1\int_0^1(s-1)d(d\pi)(u(x)+(st-t)(u(x)-u(y)))dsdt,$$

and one sees that it is clearly bounded. Observe that (2.165) includes an implicit quadratic form summing over the components of $d_{1/2}u$. The equation follows by using Taylor-type expansions. It is clear that we could immeidately estimate (2.165) directly in the same way as R_1 in the proof above and obtain uniqueness this way. Moreover, it is easily seen that the gradient flow equation with the non-linearity in (2.165) and (2.151) actually are equivalent for sufficiently regular solutions. Lastly, this formulation also gives a closer connection with the case $N = S^{n-1}$ as we have previously seen in [102].

2.2.4 Fractional Harmonic Gradient Flow with Values in a General Manifold

After having treated the case of orientable, closed hypersurfaces in some detail, we perform analogous investigations to resolve the general case and finally also discuss existence and convergence of solutions to the fractional harmonic gradient flow. In our computations and simplifications of the half-harmonic map equation, we follow Mazowiecka-Schikorra [57], where the half-harmonic map equation was already stated as well as treated and some of its features were already highlighted. We enhance the exposition given there by analysing certain estimates in more detail, leading to the uniqueness result under improved regularity following the process outlined in the introduction and seen in the previous section. To prove that energy class solutions actually possess slightly better regularity properties than assumed under some smallness condition on the energy which is uniform in time, one proceeds similar to the one for S^{n-1}. The major difference lies in the use of Morrey estimates in the case $p = 2$ following Da Lio-Pigati [20], as we would like to deduce slightly better integrability and then apply the techniques in Rivière [68] for arbitrary integrability of the fractional gradient. This change of method should however not obscure the fact that the regularity gain once more stems from the hidden structure of an anti-symmetric potential within the forumlation of the half-harmonic gradient flow. The discussion of local existence and convergence on the other hand is very similar to [102], once we fix the right formulation for the 1/2-harmonic gradient flow.

2.2.4.1 Half-Harmonic Map Equation for general Target Manifolds

Preliminaries Before we begin our analysis of the fractional harmonic gradient flow, we want to study the 1/2-harmonic map equation and its features. Throughout this chapter, we assume that $N \subset \mathbb{R}^n$ is a closed and smooth manifold, in particular it is compact and without a boundary. Let us denote by $B_\delta(N)$ the δ-neighbourhood of N, i.e. the collection of all points in $\mathbb{R}^n$ at a distance $< \delta$ from N. This definition obviously makes sense for any $\delta > 0$. By standard theory of smooth manifolds, i.e. the tubular neighbourhood theorem, we know that the closest point projection is a well-defined and smooth map:

$$\pi : B_\delta(N) \to N,$$

for sufficiently small $\delta > 0$, such that:

$$\|\pi(x) - x\| = \inf_{y \in N} \|x - y\|, \quad \forall x \in B_\delta(N)$$

One can show that the differential of π at any point $x \in N$ is actually the orthogonal projection onto the tangent space of N at x. The orthogonality of the projection means that the following holds:

$$\forall x \in N : \quad d\pi(x)^2 = d\pi(x) = d\pi(x)^T$$

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a smooth, compactly supported function, such that $\operatorname{supp} \varphi \subset B_\delta(N)$ and assume that:

$$\varphi(x) = 1, \quad \forall x \in B_{\delta/2}(N)$$

Then we may extend π to a map $\tilde{\pi} : \mathbb{R}^n \to \mathbb{R}^n$ by the following formula:

$$\tilde{\pi}(x) = \varphi(x) \cdot \pi(x), \quad \forall x \in B_\delta(N),$$

as well as $\tilde{\pi}(x) = 0$ for every $x \notin B_\delta(N)$. This map is clearly well-defined and smooth. It agrees with π on the set $B_{\delta/2}(N)$, but has the advantage of being defined globally, rendering certain definitions and computations easier later on. By abuse of notation, we shall from now on refer to $\tilde{\pi}$ as π to keep the notation as simple as possible. It should be noticed at this point that the set of fixed points of $\tilde{\pi}$ consists of N and a set $\tilde{N}$ with positive distance from N due to the definition of the extension of the closest point projection.

We recall that a function $u \in H^{1/2}(S^1; N)$ is called weakly 1/2-harmonic, if and only if:

$$(-\Delta)^{1/2} u \perp T_u N \Leftrightarrow d\pi(u)(-\Delta)^{1/2} u = 0, \tag{2.166}$$

see Da Lio-Rivière [21], Mazowiecka-Schikorra [57] and the references therein. This can be easily verified, as 1/2-harmonic maps are precisely the critical points in $H^{1/2}(S^1; N)$ of the energy:

$$E(u) := \frac{1}{2} \int_{S^1} |(-\Delta)^{1/4} u|^2 dx$$

Arguing as for hypersurfaces shows that the orthogonality relation above is another way to phrase this relation. Computing the Euler-Lagrange equation using a variation of the form:

$$u(t) := \pi(u + t\varphi),$$

for $\varphi \in C^\infty(S^1)$ and t small enough, such that $u + t\varphi \in B_{\delta/2}(N)$, i.e. such that π maps $u + t\varphi$ to a point on N a.e. and thus to conclude $u(t) \in H^{1/2}(S^1; N)$. By criticality, the Euler-Langrange equation can be computed using differentiation of $E(u(t))$ with respect to t at $t = 0$:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\int_{S^1} |(-\Delta)^{1/4}u(t)|^2\right)\Big|_{t=0} &= \int_{S^1} (-\Delta)^{1/4}u \cdot (-\Delta)^{1/4}\left(\frac{d}{dt}u(t)\Big|_{t=0}\right) dx \\
&= \int_{S^1} (-\Delta)^{1/2}u \cdot d\pi(u)\varphi dx \\
&= \int_{S^1}\int_{S^1} d_{1/2}u(x,y) d_{1/2}\left(d\pi(u)\varphi\right)(x,y)\frac{dydx}{|x-y|},
\end{aligned} \tag{2.167}$$

which is precisely how we understand (2.166). The product of the vectors is naturally understood as a scalar product in the usual sense.

For later use, we shall sometimes write $d\pi^\perp(u)$ for the following:

$$d\pi^\perp(u) := Id - d\pi(u) \tag{2.168}$$

One should observe that for every $x \in N$, the differential $d\pi^\perp(p)$ is actually another orthogonal projection, meaning:

$$d\pi^\perp(x)^T = d\pi^\perp(x) = d\pi^\perp(x)^2$$

This can be easily deduced from the corresponding identities for $d\pi(x)$.

Rewriting the Half-Harmonic Map Equation (2.166) Our first goal lies in the simplification of the half-harmonic map equation (2.166), similar considerations then apply to the 1/2-harmonic gradient flow which is defined by the equation:

$$u_t + (-\Delta)^{1/2}u \perp T_uN, \quad \forall (t,x) \in \mathbb{R}_+ \times S^1,$$

see also [102] for the case $N = S^{n-1}$. Therefore, working merely with the 1/2-harmonic map equation instead of the associated flow is for brevity's sake.

In fact, we would like to write the fractional harmonic map equation in a similar form as in the case of $N = S^{n-1}$ or even the case of closed, orientable hypersurfaces, since this form seems appropriate for the proof of a Regularity Lemma similar to Lemma 3.4 in [102] and Rivière [68]. In addition, such considerations are useful when studying uniqueness under improved regularity assumptions, as we have seen both for $N = S^{n-1}$ and N being a closed, orientable hypersurface. Indeed, we rewrote the half-harmonic map equation in both cases in such a way that enabled us to estimate certain summands more easily and reveal a quadratic structure in the non-local PDE. The computations we make are precisely the ones found Mazowiecka-Schikorra [57], we merely rewrite certain terms in a manner more appropriate for our purposes and provide more general estimates than the ones found in Mazowiecka-Schikorra [57].

Our goal is to establish the following slightly informal proposition:

Proposition 2.2.4.1. *Let $u : \mathbb{R}_{\geq 0} \times S^1 \to N$ be a solution to the half-harmonic gradient flow. Then the half-harmonic gradient flow can be rewritten as:*

$$u_t + (-\Delta)^{1/2} u = \Omega_u \cdot d_{1/2} u + R_1^{u,u} + R_2^{u,u} + R_3^{u,u}, \tag{2.169}$$

where Ω_u is anti-symmetric and all terms and operators on the righthand side depend on u. Moreover, the terms $R_1^{u,u}, R_2^{u,u}$ and $R_3^{u,u}$ satisfy estimates like:

$$|R_1^{v,w}(\varphi)| \lesssim \|\varphi\|_{\dot{F}^{1/2}_{p',2}} \|v\|_{\dot{F}^{1/2}_{2,2}} \|w\|_{\dot{F}^{1/2}_{p,2}},$$

for all appropriate v, w, φ and $p \geq 2$.

The terms involved in the result above will be introduced step-by-step below.

First of, let $\varphi \in C^\infty(S^1)$. Then we know by using the fractional version of Leibniz' rule as seen for example in (2.143) with $s = 1/2$:

$$\begin{aligned}
\int_{S^1} d_{1/2} u \cdot d_{1/2} \varphi dx &= \int_{S^1} d_{1/2} u \cdot d_{1/2} \left(d\pi(u)\varphi\right) dx + \int_{S^1} d_{1/2} u \cdot d_{1/2} \left(d\pi^\perp(u)\varphi\right) dx \\
&= \int_{S^1} d_{1/2} u \cdot d_{1/2} \left(d\pi^\perp(u)\varphi\right) dx \\
&= \int_{S^1} \int_{S^1} d_{1/2} u(x,y) d_{1/2}(d\pi^\perp(u))(x,y) \frac{dy}{|x-y|} \varphi(x) dx \\
&+ \int_{S^1} \int_{S^1} d_{1/2} u(x,y) d\pi^\perp(u(y)) d_{1/2}\varphi(x,y) \frac{dydx}{|x-y|}
\end{aligned} \tag{2.170}$$

This is the computation for the half-harmonic map, for the associated gradient flow, it would be analogous since:

$$\begin{aligned}
&\int_{S^1} u_t \cdot \varphi dx + \int_{S^1} d_{1/2} u \cdot d_{1/2} \varphi dx \\
&= \int_{S^1} u_t \cdot d\pi(u) \varphi dx + \int_{S^1} d_{1/2} u \cdot d_{1/2} \left(d\pi(u)\varphi\right) dx + \int_{S^1} d_{1/2} u \cdot d_{1/2} \left(d\pi^\perp(u)\varphi\right) dx \\
&= \int_{S^1} d_{1/2} u \cdot d_{1/2} \left(d\pi^\perp(u)\varphi\right) dx,
\end{aligned} \tag{2.171}$$

and the remaining steps being now analogous, by noticing $u_t \in T_u N$. Observe also that we used (2.166) in the second step of (2.170) for u half-harmonic. We shall refer to the second summand as $R_1(\varphi)$:

$$R_1(\varphi) := \int_{S^1} \int_{S^1} d_{1/2} u(x,y) d\pi^\perp(u(y)) d_{1/2}\varphi(x,y) \frac{dydx}{|x-y|}$$

The goal is to treat R_1 as a perturbation of a "main order term", which we consider to be the remaining one in (2.170). To achieve this, we need the following Lemma that was already stated in Mazowiecka-Schikorra [57]:

Lemma 2.2.4.1. *Let $x, y \in S^1$ and $u \in H^{1/2}(S^1; N)$. Then the following holds for all $x, y \in S^1$ such that $u(x), u(y) \in N$:*

$$u(x) - u(y) = d\pi(u(y))\,(u(x) - u(y)) + \int_0^1 \int_0^1 t d\,(d\pi)\,((1-ts)u(y) + tsu(x))\,(u(x) - u(y))^2 dsdt$$

More precisely, for $i = 1, \dots, n$, we have for the i-th component of $u = (u_1, \dots, u_n)$:

$$\begin{aligned} u_i(x) - u_i(y) &= d\pi(u(y))_{ij}(u_j(x) - u_j(y)) \\ &\quad + \int_0^1 \int_0^1 t\partial_{kj}\pi_i\,((1-ts)u(y) + tsu(x))\,(u_j(x) - u_j(y))(u_k(x) - u_k(y))dsdt, \end{aligned}$$

adopting Einstein's summation convention.

Similar identities and/or estimates already appeared in Da Lio-Pigati [20], Mazowiecka-Schikorra [57]. We shall define the following quantity based on Lemma 2.2.4.1:

$$A_u(dv, dw)(x, y) := \int_0^1 \int_0^1 t d\,(d\pi)\,((1-ts)u(y) + tsu(x))\,(v(x) - v(y))(w(x) - w(y))dsdt \quad (2.172)$$

This simplifies the result above considerably. In fact, we could restate Lemma 2.2.4.1 as:

$$u_i(x) - u_i(y) = d\pi(u(y))_{ij}(u_j(x) - u_j(y)) + A^i_u(du, du)(x, y),$$

We emphasise that this is one of the main points why we would like to consider π as a map defined on all of $\mathbb{R}^n$. This way, we may insert any value into the function, meaning that even if $(1-ts)u(y)+tsu(x)$ is not necessarily in N or even close enough for the closest point projection to be well-defined, the extension of the projection and thus the formula above remain defined everywhere. The proof is an immediate and standard application of Taylor's formula and/or repeated applications of the fundamental theorem of calculus and therefore omitted. We refer to Mazowiecka-Schikorra [57] for details.

Using the orthogonality of the projections $d\pi$ and $d\pi^\perp$, we may deduce (again using Einstein's summation convention):

$$\begin{aligned} R_1(\varphi) &= \int_{S^1}\int_{S^1} \frac{u_i(x) - u_i(y)}{|x-y|^{1/2}} d\pi^\perp(u(y))_{ij} d_{1/2}\varphi_j(x, y) \frac{dydx}{|x-y|} \\ &= \int_{S^1}\int_{S^1} \frac{d\pi(u(y))_{ik}(u_k(x) - u_k(y))}{|x-y|^{1/2}} d\pi^\perp(u(y))_{ij} d_{1/2}\varphi_j(x, y) \frac{dydx}{|x-y|} \\ &\quad + \int_{S^1}\int_{S^1} \frac{A^i_u(du, du)(x, y)}{|x-y|^{1/2}} d\pi^\perp(u(y))_{ij} d_{1/2}\varphi_j(x, y) \frac{dydx}{|x-y|} \\ &= \int_{S^1}\int_{S^1} \frac{A^i_u(du, du)(x, y)}{|x-y|^{1/2}} d\pi^\perp(u(y))_{ij} d_{1/2}\varphi_j(x, y) \frac{dydx}{|x-y|}, \end{aligned} \quad (2.173)$$

since:

$$d\pi(u(y))_{ik} d\pi^\perp(u(y))_{ij} = 0, \quad \forall k, j,$$

due to orthogonality of the projection and $d\pi^\perp = Id - d\pi$. We observe that the formula above for R_1 could also be stated as:

$$R_1 = \operatorname{div}_{1/2}\left(\frac{A^i_u(du, du)(x, y)}{|x-y|^{1/2}} d\pi^\perp(u(y))_{ij}\right)$$

This combined with the computations in (2.170) show us that the fractional harmonic map equation can actually be rephrased as:

Lemma 2.2.4.2. *Assume that $u \in H^{1/2}(S^1; N)$ is a half-harmonic map. Then u solves the following non-local PDE:*

$$(-\Delta)^{1/2} u = d_{1/2} u \cdot d_{1/2} \left(d\pi^{\perp}(u) \right) + \operatorname{div}_{1/2} \left(\frac{A_u^i(du, du)(x, y)}{|x - y|^{1/2}} d\pi^{\perp}(u(y))_{ij} \right) \tag{2.174}$$

The proof is an immediate consequence of our computations above.

Comparing this with the previously investigated case of orientable, closed hypersurfaces and keeping (2.148) in mind, we notice the immediate similarities between the two equations. Indeed, this will be a crucial point in reducing our computations to establish uniqueness under improved regularity and obtaining the result basically for free from what we have already done.

We may prove the following estimates:

Lemma 2.2.4.3. *Let $\varphi \in \dot{F}^{1/2}_{p',2}(S^1)$ for $1/p + 1/p' = 1$ and $p \geq 2$ finite. Then, for all $v \in \dot{F}^{1/2}_{2,2}(S^1), w \in \dot{F}^{1/2}_{p,2}(S^1)$, we have:*

$$|R_1^{v,w}(\varphi)| := \left| \int_{S^1} \int_{S^1} \frac{A_u^i(dv, dw)(x, y)}{|x - y|^{1/2}} d\pi^{\perp}(u(y))_{ij} d_{1/2}\varphi_j(x, y) \frac{dydx}{|x - y|} \right| \lesssim \|\varphi\|_{\dot{F}^{1/2}_{p',2}} \|v\|_{\dot{F}^{1/2}_{2,2}} \|w\|_{\dot{F}^{1/2}_{p,2}} \tag{2.175}$$

In addition, for $v, w \in \dot{F}^{1/2}_{4,2}(S^1)$, we have:

$$\begin{aligned} &\int_{S^1} \int_{S^1} \frac{A_u^i(dv, dw)(x, y)}{|x - y|^{1/2}} d\pi^{\perp}(u(y))_{ij} d_{1/2}\varphi_j(x, y) \frac{dydx}{|x - y|} \\ &\quad = \int_{S^1} \int_{S^1} A_u^i(dv, dw)(x, y) d\pi^{\perp}(u(y))_{ij} - A_u^i(dv, dw)(y, x) d\pi^{\perp}(u(x))_{ij} \frac{dy}{|x - y|^2} \varphi_j(x) dx, \end{aligned}$$

and as a result $R_1^{v,w} \in L^2(S^1)$ with:

$$\|R_1\|_{L^2} \lesssim \|v\|_{\dot{F}^{1/2}_{4,2}} \|w\|_{\dot{F}^{1/2}_{4,2}} \tag{2.176}$$

Proof. The first estimate follows by letting $0 < s < 1$ and observing:

$$\begin{aligned} &\Big| \int_{S^1} \int_{S^1} \frac{A_u^i(dv, dw)(x, y)}{|x - y|^{1/2}} d\pi^{\perp}(u(y))_{ij} d_{1/2}\varphi(x, y) \frac{dydx}{|x - y|} \Big| \\ &\quad \lesssim \left| \int_{S^1} \int_{S^1} |d_s v(x, y)| |d_{1/2-s} w(x, y)| |d_{1/2}\varphi(x, y)| \frac{dydx}{|x - y|} \right| \\ &\quad \lesssim \|v\|_{\dot{W}^{s,(1/s,4)}} \|w\|_{\dot{W}^{1/2-s,(p/(1-sp),4)}} \|\varphi\|_{\dot{W}^{1/2,(p',2)}}, \end{aligned} \tag{2.177}$$

where we used the triangle inequality and Hölder's inequality. If we can choose s in a way, such that:

$$\frac{p}{1 - sp} > \frac{4}{3 - 4s}, \frac{1}{s} > \frac{4}{1 + 4s},$$

we may use the identification mentioned in the preliminary section. One notices that the latter inequality is trivially true, while the first one reduces to:

$$p \geq 2 > \frac{4}{3},$$

which again holds trivially. So the choice of $s \in (0,1/2)$ is immaterial here (we just need to ensure that we do not divide by 0 or have negative Hölder exponents). Indeed these conditions lead to $0 < s < 1/p$, which clearly allows for a choice of s. Consequently, we may identify:

$$\dot{W}^{s,(1/s,4)} = \dot{F}^{s}_{1/s,4}, \quad \dot{W}^{1/2-s,(p/(1-sp),4)} = \dot{F}^{1/2-s}_{p/(1-sp),4}, \quad \dot{W}^{1/2,(p',2)} = \dot{F}^{1/2}_{p',2},$$

and the norms are equivalent, as mentioned in the preliminary section, see Prats-Saksman [65]. By using Sobolev embeddings for Triebel-Lizorkin spaces, this shows:

$$\|v\|_{\dot{F}^{s}_{1/s,4}} \lesssim \|v\|_{\dot{F}^{s}_{1/s,2}} \lesssim \|v\|_{\dot{F}^{1/2}_{2,2}},$$

as well as:

$$\|w\|_{\dot{F}^{1/2-s}_{p/(1-sp),4}} \lesssim \|w\|_{\dot{F}^{1/2-s}_{p/(1-sp),2}} \lesssim \|w\|_{\dot{F}^{1/2}_{p,2}}.$$

Combining these inequalities with the estimate in (2.177), we find the desired result:

$$\Big| \int_{S^1}\int_{S^1} \frac{A^i_u(dv,dw)(x,y)}{|x-y|^{1/2}} d\pi^{\perp}(u(y))_{ij} d_{1/2}\varphi(x,y) \frac{dydx}{|x-y|} \Big| \lesssim \|\varphi\|_{\dot{F}^{1/2}_{p',2}} \|v\|_{\dot{F}^{1/2}_{2,2}} \|w\|_{\dot{F}^{1/2}_{p,2}}$$

For the second identity mentioned above, one observes that the equality mentioned holds by a simple change of role between x and y. The estimate is then obtained by noticing:

$$\begin{aligned}
&\Big| \int_{S^1}\int_{S^1} A^i_u(dv,dw)(x,y) d\pi^{\perp}(u(y))_{ij} - A^i_u(dv,dw)(y,x) d\pi^{\perp}(u(x))_{ij} \frac{dy}{|x-y|^2} \varphi_j(x) dx \Big| \\
&\quad \lesssim \int_{S^1}\int_{S^1} |d_{1/2}v(x,y)||d_{1/2}w(x,y)| \frac{dy}{|x-y|} |\varphi_j(x)| dx \\
&\quad \lesssim \int_{S^1} |d_{1/2}v|(x)|d_{1/2}w|(x) \frac{dy}{|x-y|} |\varphi_j(x)| dx \\
&\quad \lesssim \||d_{1/2}v|(x)\|_{L^4} \||d_{1/2}w|(x)\|_{L^4} \|\varphi\|_{L^2} \\
&\quad \lesssim \|v\|_{\dot{F}^{1/2}_{4,2}} \|w\|_{\dot{F}^{1/2}_{4,2}} \|\varphi\|_{L^2},
\end{aligned} \tag{2.178}$$

by applying Hölder's inequality twice as well as boundedness of $d\pi$ and consequently for $d\pi^{\perp}$. Observe that we used once more the equivalent characterisation of the Triebel-Lizorkin norm in the case $q = 2, s = 1/2$. □

The formulation in (2.174) of the fractional harmonic map equation will already suffice for proving uniqueness under improved regularity assumptions.

We shall continue similar to Mazowiecka-Schikorra [57] and further explore simplifications of the leading term, i.e. the term:

$$\int_{S^1}\int_{S^1} d_{1/2}u_i(x,y) d_{1/2}(d\pi^{\perp}(u))_{ij}(x,y) \frac{dy}{|x-y|} \varphi_j(x) dx = \int_{S^1} d_{1/2}u_i \cdot d_{1/2}(d\pi^{\perp}(u))_{ij}(x) \varphi_j(x) dx$$

Our main goal is to unveil an anti-symmetric potential, i.e. show that the half-harmonic map equation could be rephrased as:

Claim: There exists a remainder $\tilde{R}_u$ depending on u with good estimates to be established later, such that the half-harmonic map equation can be restated as:

$$(-\Delta)^{1/2}u = \Omega \cdot d_{1/2}u + \tilde{R}_u, \tag{2.179}$$

where $\Omega \in L^2_{od}(S^1 \times S^1; so(n))$ is an anti-symmetric potential.

Proof of Claim: Applying our previous computations and the formulation in (2.174), we have made the first steps and already are aware of estimates for R_1. It remains to further simplify the other term in (2.174) to find an anti-symmetric potential. Using Lemma 2.2.4.1, we may replace $d_{1/2}u(x,y)$ by the following expression:

$$\begin{aligned}
&\int_{S^1} d_{1/2}u_i(x,y) d_{1/2}(d\pi^\perp(u))_{ij}(x,y)\frac{dy}{|x-y|} \\
&\quad = \int_{S^1} d_{1/2}\left(d\pi^\perp(u)\right)_{ij}(x,y) d\pi_{ik}(u(y)) d_{1/2}u_k(x,y)\frac{dy}{|x-y|} \\
&\quad + \int_{S^1} d_{1/2}\left(d\pi^\perp(u)\right)_{ij}(x,y) A^i_u(du,du)(x,y)\frac{dy}{|x-y|^{3/2}}
\end{aligned}$$

The second summand is defined to be:

$$R_2(x) := \int_{S^1} d_{1/2}\left(\pi^\perp(u)\right)_{ij}(x,y) A^i_u(du,du)(x,y)\frac{dy}{|x-y|^{3/2}} \tag{2.180}$$

The first summand may be rewritten one last time to obtain:

$$\int_{S^1} d_{1/2}\left(\pi^\perp(u)\right)_{ij}(x,y) d\pi_{ik}(u(y)) d_{1/2}u_k(x,y)\frac{dy}{|x-y|} = \Omega_{jk} \cdot d_{1/2}u_k + R_3, \tag{2.181}$$

where:

$$\Omega_{jk}(x,y) := d\pi(u(y))_{ik} d_{1/2}\left(d\pi^\perp(u)\right)_{ij}(x,y) - d\pi(u(y))_{ij} d_{1/2}\left(d\pi^\perp(u)\right)_{ik}(x,y), \quad \forall j,k, \tag{2.182}$$

as well as:

$$\begin{aligned}
R_3(x) &:= \int_{S^1} d\pi(u(y))_{ij} d_{1/2}\left(d\pi^\perp(u)\right)_{ik}(x,y) d_{1/2}u_k(x,y)\frac{dy}{|x-y|} \\
&= \int_{S^1} d\pi(u(y))_{ij} d\pi^\perp(u(x))_{ik} d_{1/2}u_k(x,y)\frac{dy}{|x-y|^{3/2}} \\
&= -\int_{S^1} d_{1/2}\left(d\pi(u)\right)_{ij}(x,y) d\pi^\perp(u(x))_{ik} d_{1/2}u_k(x,y)\frac{dy}{|x-y|},
\end{aligned} \tag{2.183}$$

simply because of orthogonality of the projections. We highlight that $\Omega = -\Omega^T$, i.e. we have found an anti-symmetric potential similar to the case $N = S^{n-1}$. We will sometimes write Ω_u to emphasise the dependence on u. This also suggests that an increase in integrability should be obtainable in the critical case of the non-local PDE, i.e. for $u \in H^{1/2}(S^1)$. This ends the **proof of the Claim**. □

We close this preliminary examination of the half-harmonic map equation by summarising these computations in the following equivalent equation for fractional harmonic maps:

$$(-\Delta)^{1/2}u = \Omega \cdot d_{1/2}u + R_1 + R_2 + R_3, \tag{2.184}$$

where we use the definitions (2.173), (2.180), (2.183) as well as (2.182). This is exactly the form we have previously mentioned in the Claim above. Let us remark that the following holds:

Lemma 2.2.4.4. *Let $v \in \dot{F}^{1/2}_{p,2}(S^1), w \in \dot{F}^{1/2}_{2,2}(S^1)$ with $p > 2$. Then:*

$$\|R_2^{v,w}\|_{L^{\frac{2p}{2+p}}} \lesssim \|u\|_{\dot{F}^{1/2}_{2,2}} \|v\|_{\dot{F}^{1/2}_{p,2}} \|w\|_{\dot{F}^{1/2}_{2,2}},$$

and if $v, w \in \dot{F}^{1/2}_{4,2}$

$$\|R_2^{v,w}\|_{L^2} \lesssim \|u\|_{\dot{F}^{1/2}_{2,2}} \|v\|_{\dot{F}^{1/2}_{4,2}} \|w\|_{\dot{F}^{1/2}_{4,2}},$$

where:

$$R_2^{v,w} := \int_{S^1} d_{1/2}\left(\pi^{\perp}(u)\right)_{ij}(x,y) A^i_u(dv, dw)(x,y) \frac{dy}{|x-y|^{3/2}}$$

Similar estimates can be obtained for R_3 by using a variant of Lemma 2.2.4.1.

The proof is similar to the one in Lemma 2.2.4.3. One notices that by Sobolev embeddings for Triebel-Lizorkin spaces $\dot{F}^{1/2}_{p',2}(S^1) \subset L^{\frac{2p}{p-2}}(S^1)$, we may deduce that:

$$R_2^{v,w}, R_3^{v,w} \in \dot{F}^{-1/2}_{p,2}(S^1),$$

with estimates analogous to the ones in Lemma 2.2.4.4.

Proof. We see:

$$|R_2^{v,w}(x)| \lesssim \int_{S^1} |d_s u(x,y)||d_s w(x,y)||d_{1-2s} v(x,y)| \frac{dy}{|x-y|},$$

and therefore, by using Hölder's inequality:

$$\begin{aligned} \|R_2^{v,w}\|_{L^{\frac{2p}{p+2}}} &\lesssim \|u\|_{\dot{W}^{s,(1/s,3)}} \|w\|_{\dot{W}^{s,(1/s,3)}} \|v\|_{\dot{W}^{1-2s,(2p/(p+2-4sp),3)}} \\ &\lesssim \|u\|_{\dot{F}^{1/2}_{2,2}} \|w\|_{\dot{F}^{1/2}_{2,2}} \|v\|_{\dot{F}^{1/2}_{p,2}}, \end{aligned} \tag{2.185}$$

where we used Sobolev embeddings for Triebel-Lizorkin spaces in the last step. We emphasise that changing between the spaces $\dot{W}^{s,(p,q)}$ and $\dot{F}^s_{p,q}$ is possible by Prats-Saksman [65], see Theorem 2.2.2.1 and the comment afterwards in section 2, due to:

$$\frac{1}{s} > \frac{3}{1+3s}, \quad \frac{2p}{p+2-4ps} > \frac{3}{4-6s},$$

of which the first inequality is trivially true and the latter reduces to:

$$8p - 12sp > 3p + 6 - 12ps \Rightarrow p > \frac{6}{5},$$

which is trivially true for all s because $p \geq 2$. So one merely has to take care that the Hölder exponents remain in $(1, +\infty)$, which is easily ensured as in Lemma 2.2.4.3. The second estimate is obtained along the same lines, merely changing Triebel-Lizorkin spaces. The estimates for R_3 follow from similar considerations by using Lemma 2.2.4.1, but this time by expanding around $u(x)$ instead of $u(y)$ which only requires minor modifications. Thus we are done. □

As in [102], we may also find the corresponding fractional harmonic gradient flow to be therefore:

$$u_t + (-\Delta)^{1/2} u = d_{1/2} u \cdot d_{1/2}\left(d\pi^{\perp}(u)\right) + \operatorname{div}_{1/2}\left(\frac{A^i_u(du, du)(x,y)}{|x-y|^{1/2}} d\pi^{\perp}(u(y))_{ij}\right) \tag{2.186}$$

$$= \Omega_u \cdot d_{1/2} u + R_1 + R_2 + R_3, \tag{2.187}$$

where the latter equation uses the expressions for Ω_u, R_1, R_2, R_3 introduced above in (2.182), (2.175), (2.180), (2.183).

Other useful Formulations for the Fractional Harmonic Gradient Flow In the proof of existence of local solutions to the fractional harmonic gradient flow, we shall make use of a slightly different formulation for the 1/2-harmonic gradient flow. Therefore, we list already here several equivalent (at least for sufficiently regular u) formulations of the main equation (2.186):

The most basic formulation of the gradient flow we are studying is:

$$u_t + (-\Delta)^{1/2}u \perp T_uN,$$

which characterises solutions of the gradient flow with values in N a.e.. This can be rewritten as:

$$u_t + (-\Delta)^{1/2}u = (-\Delta)^{1/2}\pi(u) - d\pi(u)(-\Delta)^{1/2}u, \tag{2.188}$$

where we use for $u \in N$ almost everywhere:

$$\begin{aligned}(-\Delta)^{1/2}\pi(u) - d\pi(u)(-\Delta)^{1/2}u &= (-\Delta)^{1/2}u - d\pi(u)(-\Delta)^{1/2}u \\ &= \left(Id - d\pi(u)\right)(-\Delta)^{1/2}u \\ &= d\pi^{\perp}(u)(-\Delta)^{1/2}u \perp T_uN, \end{aligned}\tag{2.189}$$

which is in fact the same as $d\pi^{\perp}(u)(u_t + (\Delta)^{1/2}u)$, since $u_t \in T_uN$ a.e.. A key advantage of this formulation is that we may encode the condition $u \in N$ a.e. directly inside the equation, see the proof of local existence of solutions below. A major drawback on the other hand is that this formulation obscures the compensation phenomena at hand. Lastly, there also exists a formulation analogous to (2.165) by using arguments analogous to the ones needed to prove Lemma 2.2.4.1. This leads to:

$$u_t + (-\Delta)^{1/2}u = \sum_{k,l=1}^{n} P.V. \int_{S^1} P^{kl}(u(x),u(y)) d_{1/2}u_k(x,y) d_{1/2}u_l(x,y)\frac{dy}{|x-y|}, \tag{2.190}$$

where $P^{kl} = (P^{kl}_1, \dots, P^{kl}_n)$ with:

$$\begin{aligned}&P^{kl}_j(u(x),u(y))(u_k(x)-u_k(y))(u_l(x)-u_l(y)) \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n}\int_0^1\int_0^1 (t-1)\partial_{kl}\pi_j((s-st)u(y)+(1+st-s)u(x))(u_k(x)-u_k(y))(u_l(x)-u_l(y))dsdt.\end{aligned}\tag{2.191}$$

One observes the immediate similarity with (2.165) as well as the quadratic structure of the RHS of this formulation. This formulation is again useful in proving uniqueness and regularity, but the connection to the formulation in [102] for $N = S^{n-1}$ is less apparent than with (2.186).

2.2.4.2 Uniqueness of Solutions to the $1/2$-Harmonic Gradient Flow

We discuss now the uniqueness of solutions to the 1/2-harmonic gradient flow. Our approach is similar to the one in [102]. Thus, we first discuss uniqueness in a class of functions which have more regularity than strictly required to make sense of solutions and follow the approach in Struwe [91] to establish uniqueness for such "strong solutions". Then, we expand this result to arbitrary energy class solution, i.e. the class of functions with minimal regularity for the fractional gradient flow to make sense, by using techniques similar to Rivière [68] to squeeze some more integrability out of solutions with small energy. In fact, we will rely on the techniques in Da Lio-Pigati [20] which will turn out to be important to establish a slight gain in regularity by means of compensation phenomena tied to the emergence of an antisymmetric potential (notice that the appearance of such a potential from a slightly different point of view is already hinted at in Section 4.1 and in Mazowiecka-Schikorra [57]).

Uniqueness under Improved Regularity Assumptions We are now able to turn to the study of the gradient flow associated with the fractional harmonic map with values in $N \subset \mathbb{R}^n$ being a general closed manifold. As previously in the case of a closed orientable hypersurface, let us assume that u, v are two solutions to the fractional harmonic gradient flow taking a.e. values in a general closed $N \subset \mathbb{R}^n$. Clearly, this implies boundedness of u, v due to the compactness of N. As before in Section 3, we assume that the following regularity conditions hold:

$$u, v \in L^\infty(\mathbb{R}_+; H^{1/2}(S^1)); \quad u_t, v_t \in L^2(\mathbb{R}_+; L^2(S^1)); \quad u, v \in L^2_{loc}(\mathbb{R}_+; H^1(S^1)), \tag{2.192}$$

In addition, they satisfy the gradient flow associated with the 1/2-harmonic map as described below:

$$w_t + (-\Delta)^{1/2} w = d_{1/2} w \cdot d_{1/2} \left(d\pi^\perp(w) \right) + \operatorname{div}_{1/2} \left(\frac{A^i_w(dw, dw)(x,y)}{|x-y|^{1/2}} d\pi^\perp(w(y))_{ij} \right), \tag{2.193}$$

for both $w = u$ and $w = v$, together with the boundary condition $u(0,\cdot) = v(0,\cdot) = u_0 \in H^{1/2}(S^1)$. We mention that, as we have seen in the previous subsection, the right hand side of (2.193) is precisely the non-linearity associated with the fractional harmonic map equation for functions taking values in N. The equation (2.193) could be derived along the same lines, cf. [102].
Let us present the main uniqueness statement in analogy to Theorem 2.2.3.1:

Theorem 2.2.4.1. *If u, v both solve* (2.193) *with the same initial datum $u_0 \in H^{1/2}(S^1; N)$ and we assume that:*

$$\|(-\Delta)^{1/4} u(t)\|_{L^2(S^1)}, \|(-\Delta)^{1/4} v(t)\|_{L^2(S^1)} \leq \|(-\Delta)^{1/4} u_0\|_{L^2(S^1)}, \quad \forall t \in \mathbb{R}_+,$$

then we have:

$$u = v$$

The proof is actually going to proceed analogous to the case of closed orientable hypersurfaces. We will provide some of the details below and the proof naturally extends to the case of solutions u, v defined only on a subinterval $[0, T] \subset \mathbb{R}_+$.

Proof. First, we observe that since $d\pi^\perp = Id - d\pi$, we know:

$$d_{1/2} \left(d\pi^\perp(u) \right)(x,y) = -d_{1/2} \left(d\pi(u) \right)(x,y),$$

and we therefore would like to estimate the following:

$$d\pi(u(x)) - d\pi(u(y)) = \int_0^1 d\left(d\pi\right)\left((1-s)u(y) + su(x)\right) ds \cdot (u(x) - u(y)), \tag{2.194}$$

using Taylor expansion and understanding the differential $d(d\pi)$ as previously in Lemma 2.2.4.1. If we define:

$$B_u(x,y) := \int_0^1 d\left(d\pi\right)\left((1-s)u(y) + su(x)\right) ds,$$

which is clearly bounded thanks to the smoothness of π and its definition as an extension, we may rewrite:

$$d_{1/2}u \cdot d_{1/2}\left(d\pi^{\perp}(u)\right) = -d_{1/2}u \cdot B_u(x,y)d_{1/2}u,$$

which renders the fractional harmonic flow equation virtually the same as previously in the case of N being an orientable closed hypersurface:

$$u_t + (-\Delta)^{1/2}u = -d_{1/2}u \cdot B_u(x,y)d_{1/2}u + \operatorname{div}_{1/2}\left(\frac{A_u^i(du,du)(x,y)}{|x-y|^{1/2}}d\pi^{\perp}(u(y))_{ij}\right),$$

and completely analogous for v.

We may now use a decomposition as in (2.152). The first remainder involving $B_u(x,y)$ can be estimated analogous to (2.155) by obvious modifications of the estimates provided there. The second remainder, i.e. the fractional divergence, has a similar form to (2.158) and may be decomposed as in (2.159):

$$\begin{aligned}
&\frac{A_u^i(du,du)(x,y)}{|x-y|^{1/2}}d\pi^{\perp}(u(y))_{ij}d_{1/2}w - \frac{A_v^i(dv,dv)(x,y)}{|x-y|^{1/2}}d\pi^{\perp}(v(y))_{ij}d_{1/2}w \\
&= \frac{A_u^i(du,du)(x,y)}{|x-y|^{1/2}}d\pi^{\perp}(u(y))_{ij}d_{1/2}w - \frac{A_v^i(du,du)(x,y)}{|x-y|^{1/2}}d\pi^{\perp}(u(y))_{ij}d_{1/2}w \\
&+ \frac{A_v^i(du,du)(x,y)}{|x-y|^{1/2}}d\pi^{\perp}(u(y))_{ij}d_{1/2}w - \frac{A_v^i(du,du)(x,y)}{|x-y|^{1/2}}d\pi^{\perp}(v(y))_{ij}d_{1/2}w \\
&+ \frac{A_v^i(du,du)(x,y)}{|x-y|^{1/2}}d\pi^{\perp}(v(y))_{ij}d_{1/2}w - \frac{A_v^i(dv,du)(x,y)}{|x-y|^{1/2}}d\pi^{\perp}(v(y))_{ij}d_{1/2}w \\
&+ \frac{A_v^i(dv,du)(x,y)}{|x-y|^{1/2}}d\pi^{\perp}(v(y))_{ij}d_{1/2}w - \frac{A_v^i(dv,dv)(x,y)}{|x-y|^{1/2}}d\pi^{\perp}(v(y))_{ij}d_{1/2}w
\end{aligned} \tag{2.195}$$

One may now estimate these terms as in the case N a hypersurface summand by summand. For example, in the first summand we may estimate the difference between A_u and A_v by $|w(x)|+|w(y)|$ and estimate one of the d_0u by its L^∞-bound to arrive at an estimate of the form:

$$\begin{aligned}
&\Big|\int_{S^1}\int_{S^1}\frac{A_u^i(du,du)(x,y)}{|x-y|^{1/2}}d\pi^{\perp}(u(y))_{ij}d_{1/2}w(x,y) - \frac{A_v^i(du,du)(x,y)}{|x-y|^{1/2}}d\pi^{\perp}(u(y))_{ij}d_{1/2}w(x,y)dxdy\Big| \\
&\qquad\qquad \lesssim \int_{S^1}|d_{1/2}u|(x)|d_{1/2}w|(x)|w(x)|dx
\end{aligned}$$

Observe that we have to exchange labels x, y at some point. The other terms can be estimated in a similar manner.

The estimates are therefore obtained completely analogous to the case N a hypersurface, applying Young's and Hölder's inequality, leading to an estimate for $w=u-v$ of the following form:

$$\begin{aligned}
&\frac{1}{2}\|w(T)\|_{L^2(S^1)} + \frac{1}{2}\int_0^T \|(-\Delta)^{1/4}w(t)\|_{L^2(S^1)}dt \\
&\qquad \le \tilde{C}\left(\int_0^T\int_{S^1}|w|^4 dxdt\right)^{1/2}\cdot\left(\int_0^T\int_{S^1}\left(|d_{1/2}u|+|d_{1/2}v|\right)^4 dxdt\right)^{1/2}
\end{aligned} \tag{2.196}$$

which can then be treated as in $N = S^{n-1}$ in order to conclude uniqueness by an iteration argument, we refer to [102] for the details. The estimates invoked are independent of S^{n-1} and rely on general properties of u provided by (2.192), therefore generalising to our current situation. $\square$

Naturally, as in the case of N being a hypersurface, one may also use the formulation in (2.190) to deduce uniqueness, see also Section 3 for some more details. The proof proceeds analogously and is omitted.

Small Initial Energy: Regularity Lemma in the Case $p > 2$ To deduce uniqueness of fractional gradient flow in energy class under some additional assumption, we have to establish some higher regularity for energy class solutions of the fractional gradient flow. In the case of the $n-1$-sphere, we managed to achieve slightly better regularity properties of the solution by using Wente-type estimates from Mazowiecka-Schikorra [57] and the invertibility of certain operators as in Rivière [68]. In the general case, we will also prove an existence and uniqueness result for a modified operator under higher integrability, which will then be useful to establish the higher regularity needed. In a second step, we shall show that higher integrability actually holds for solutions in $H^{1/2}(S^1)$ by means of compensation-compactness as in Da Lio-Pigati [20], completing the proof.

A key Lemma is the following:

Lemma 2.2.4.5. *Let $u \in H^{1/2}(S^1)$ and $f \in L^2(S^1)$ and assume that u solves:*

$$(-\Delta)^{1/2}u = d_{1/2}u \cdot d_{1/2}\left(d\pi^{\perp}(u)\right) + \mathrm{div}_{1/2}\left(\frac{A_u^i(du,du)(x,y)}{|x-y|^{1/2}} d\pi^{\perp}(u(y))_{ij}\right) + f, \tag{2.197}$$

Then, if there exists a $p > 2$, such that (up to adding a constant):

$$u \in \dot{F}^{1/2}_{p,2},$$

then we immediately conclude:

$$u \in \dot{F}^{1/2}_{q,2}(S^1), \quad \forall q \geq 2.$$

The proof relies on the techniques in Rivière [68] and using the remainders and their associated estimates established earlier, see Lemma 2.2.4.3 and 2.2.4.4. One should observe that (2.193)is naturally of the form (2.197) for almost every fixed time $t \in \mathbb{R}_+$. This is apparent by chosing $f = -\partial_t u$.

Proof. First, we write:

$$(-\Delta)^{1/2}u = \Omega \cdot d_{1/2}u + R_1 + R_2 + R_3 + f,$$

where Ω, R_1, R_2, R_3 are all as previously introduced in this chapter, see (2.182) as well as (2.173), (2.180), (2.183). We may approximate Ω by a smooth $\tilde{\Omega}$ vanishing in a neighbourhood of the diagonal and lying in $L^2_{od}(S^1 \times S^1)$ and u by a smooth function $\tilde{u}$ in $H^{1/2}(S^1)$, the norms of the differences being arbitrarily small in the respective spaces:

$$\|\Omega - \tilde{\Omega}\|_{L^2_{od}}, \|u - \tilde{u}\|_{H^{1/2}} < \varepsilon,$$

for $\varepsilon > 0$ to be determined later. Then, we may rewrite this equation as:

$$(-\Delta)^{1/2}u + (\tilde{\Omega} - \Omega) \cdot d_{1/2}u + R_1^{\tilde{u}-u,u} + R_2^{\tilde{u}-u,u} + R_2^{\tilde{u}-u,u}$$

$$= \tilde{\Omega} \cdot d_{1/2} u + R_1^{\tilde{u},u} + R_2^{\tilde{u},u} + R_3^{\tilde{u},u} + f \tag{2.198}$$

As for $N = S^{n-1}$, we may restrict our attention to the case of vanishing Fourier coefficients by removing the averages (i.e. 0-th order Fourier coefficients) associated with each summand individually in (2.197). This means, in order to render the fractional Laplacian invertible, we consider $v = u - \hat{u}(0)$ instead of $v = u$ below and remove averages for the contributions R_1, R_2, R_3 and $\Omega \cdot d_{1/2} v$. Let us observe that:

$$\begin{aligned} v \mapsto & (-\Delta)^{-1/2} \left((-\Delta)^{1/2} v + (\tilde{\Omega} - \Omega) \cdot d_{1/2} v + R_1^{\tilde{u}-u,v} + R_2^{\tilde{u}-u,v} + R_3^{\tilde{u}-u,v} \right) \\ &= v + (-\Delta)^{-1/2} \left((\tilde{\Omega} - \Omega) \cdot d_{1/2} v \right) + (-\Delta)^{-1/2} \left(R_1^{\tilde{u}-u,v} + R_2^{\tilde{u}-u,v} + R_3^{\tilde{u}-u,v} \right) \end{aligned} \tag{2.199}$$

defines an invertible mapping from $\dot{F}^{1/2}_{p,2}$ to itself, if we assume that $\tilde{\Omega}$ and $\tilde{u}$ are sufficiently good approximations. Use the estimates in Lemma 2.2.4.3 and 2.2.4.4 as well as the estimate:

$$\|(\tilde{\Omega} - \Omega) \cdot d_{1/2} v\|_{L^{\frac{2p}{p+2}}} \lesssim \|\Omega - \tilde{\Omega}\|_{L^2_{od}} \|v\|_{\dot{F}^{1/2}_{p,2}},$$

and the continuity of the embedding $L^{\frac{2p}{p+2}}(S^1) \hookrightarrow \dot{F}^{-1/2}_{p,2}(S^1)$ to deduce that the maps are sufficiently small in operatornorm, rendering (2.199) a small perturbation of the identity map and therefore invertible itself. One may also observe that the RHS of (2.198) lies in $L^q(S^1)$ for any $q < 2$ (using smoothness of the approximating terms), which is in the dual of $\dot{F}^{-1/2}_{p,2}$ thanks to Sobolev embeddings. This implies that there exists a unique solution $v \in \dot{F}^{1/2}_{p,2}$:

$$\begin{aligned} v + (-\Delta)^{-1/2} & \left((\tilde{\Omega} - \Omega) \cdot d_{1/2} v + R_1^{\tilde{u}-u,v} + R_2^{\tilde{u}-u,v} + R_2^{\tilde{u}-u,v} \right) \\ &= (-\Delta)^{-1/2} \left(\tilde{\Omega} \cdot d_{1/2} u + R_1^{\tilde{u},u} + R_2^{\tilde{u},u} + R_3^{\tilde{u},u} + f \right) \end{aligned} \tag{2.200}$$

This shows that if $u \in \dot{F}^{1/2}_{p,2}(S^1)$ for some $p > 2$, it also lies in this space for any $q \geq 2$ by existence and uniqueness of solutions to the equation above due to the invertibility of the map and the natural inclusions $\dot{F}^{1/2}_{p,2} \hookrightarrow \dot{F}^{1/2}_{q,2}$ for every $p \geq q$, keeping in mind that the RHS of (2.200) is independent of v. This concludes the proof of Lemma 2.2.4.5. □

Small Initial Energy: Regularity Lemma in the Case $p = 2$ The missing step in order to be able to apply Lemma 2.2.4.5 to our energy class solution of the fractional harmonic gradient flow at a fixed time is provided by the following:

Lemma 2.2.4.6. *Let $u \in H^{1/2}(S^1) \cap L^\infty(S^1)$ and $f \in L^2(S^1)$ and assume that u solves the non-local PDE:*

$$(-\Delta)^{1/2} u = d_{1/2} u \cdot d_{1/2} \left(d\pi^\perp(u) \right) + \mathrm{div}_{1/2} \left(\frac{A^i_u(du, du)(x,y)}{|x-y|^{1/2}} d\pi^\perp(u(y))_{ij} \right) + f,$$

Then there exists $p > 2$, such that:

$$u \in \dot{F}^{1/2}_{p,2}(S^1).$$

Combining Lemma 2.2.4.5 and 2.2.4.6 yields the following corollary:

Corollary 2.2.4.1. *Let $u \in H^{1/2}(S^1) \cap L^\infty(S^1)$ and $f \in L^2(S^1)$ and assume that u solves:*

$$(-\Delta)^{1/2} u = d_{1/2} u \cdot d_{1/2} \left(d\pi^\perp(u) \right) + \operatorname{div}_{1/2} \left(\frac{A_u^i(du, du)(x, y)}{|x-y|^{1/2}} d\pi^\perp(u(y))_{ij} \right) + f,$$

Then we know:

$$u \in \dot{F}^{1/2}_{p,2}(S^1), \quad \forall p \geq 2.$$

We observe that this corollary is essentially the analogue of [103, Lemma 3.4] in the case $N = S^{n-1}$ and thus should suffice in order to conclude our investigation into the uniqueness of the fractional harmonic gradient flow with small initial energy. Indeed, we shall show that the same arguments, like proving a uniform bound for the H^1-norm of $u(t)$ at fixed times, carry over in the next subsection. For the proof of this result, we refer to the Appendix of the current section, as it is an adaption of the work in Da Lio-Pigati [20] which is not of particular interest on its own. It relies on commutator estimates that essentially quantify the gain of integrability precisely and allow for a Morrey bootstrap argument in order to arrive at increased integrability and, as a result, show that $u \in \dot{F}^{1/2}_{p,2}(S^1)$ for some $p > 2$. Thus, we shall continue with the uniqueness result for weak solutions with small initial energy.

Small Initial Energy: Application of the Regularity Lemmas We are finally able to apply the results from the previous subsections to the case of an energy class solution u of the $1/2$-harmonic gradient flow with values in $N \subset \mathbb{R}^n$ a closed manifold. The steps are completely analogous to the case $N = S^{n-1}$ and Rivière [68], cf. [102]. To be precise, we shall:

1. Conclude that for fixed times t, we have $u(t) \in H^1(S^1)$ by Corollary 2.2.4.1
2. Apply a fractional Ladyzhenskaya-type estimate, i.e.:
$$\|v\|_{L^4} \leq C \|v\|_{L^2}^{1/2} \|v\|_{H^{1/2}}^{1/2}, \quad \forall v \in H^{1/2}(S^1), \tag{2.201}$$
analogous to [102] and smallness of the energy (together with non-increasing energy) to obtain a uniform estimate for $\|u(t)\|_{H^1}$ for almost all times $t \in [0, +\infty[$
3. Conclude that u is actually a strong solution and therefore, the previous uniqueness result applies

We will be proving the following as the main result, which we have already mentioned in the introductionas part of Theorem 2.2.1.2:

Theorem 2.2.4.2. *Let $u : \mathbb{R}_+ \times S^1 \to N \subset \mathbb{R}^n$, N a closed and smooth manifold, be a solution of the weak fractional harmonic gradient flow* (2.193) *with initial datum $u_0 \in H^{1/2}(S^1; N)$ and satisfying the following regularity assumptions:*

$$u \in L^\infty(\mathbb{R}_+; H^{1/2}(S^1)); \quad u_t \in L^2(\mathbb{R}_+; L^2(S^1))$$

Then there exists $\varepsilon > 0$, such that among all such u satisfying the smallness condition:

$$\|(-\Delta)^{1/4} u(t)\|_{L^2(S^1)} \leq \varepsilon, \quad \forall t \in \mathbb{R}_+,$$

the solution to the fractional harmonic gradient flow (2.193) *with initial datum u_0 is unique.*

In particular, if we assume that the 1/2-energy:

$$E_{1/2}(u(t)) := \frac{1}{2}\|(-\Delta)^{1/4}u(t)\|^2_{L^2(S^1)},$$

is non-increasing in time, as motivated in the case $N = S^{n-1}$ by [103, Lemma 3.3], the smallness condition could be rephrased as:

$$\|(-\Delta)^{1/4}u_0\|_{L^2(S^1)} \leq \varepsilon$$

An important first step in the proof of Theorem 2.2.4.2 is the following:

Proposition 2.2.4.2. *Let $u : \mathbb{R}_+ \times S^1 \to N \subset \mathbb{R}^n$ satisfy the following regularity assumptions:*

$$u \in L^\infty(\mathbb{R}_+; H^{1/2}(S^1)); \quad u_t \in L^2(\mathbb{R}_+; L^2(S^1))$$

Moreover, assume u solves the half-harmonic gradient flow equation (2.193)*. Then for almost every time $t > 0$, we have:*

$$u(t) \in H^1(S^1)$$

The proof is more or less an immediate application of Corollary 2.2.4.1, once we observe that $u \in N$ for almost every $(t, x) \in \mathbb{R}_+ \times S^1$ implies $u(t) \in L^\infty(S^1)$ for almost every time t.

Proof. First, by noticing that we may apply Corollary 2.2.4.1 with $p = 4$, we notice that $u(t) \in H^1(S^1)$ for almost every $t \in \mathbb{R}$, since the RHS of the rephrasing of the fractional harmonic flow (2.202) below is in L^2 and the Riesz potential preserves the L^2-norm. Indeed, we see for a fixed time t:

$$(-\Delta)^{1/2}u(t) = d_{1/2}u \cdot d_{1/2}\left(d\pi^\perp(u)\right) + \mathrm{div}_{1/2}\left(\frac{A^i_u(du, du)(x, y)}{|x - y|^{1/2}} d\pi^\perp(u(y))_{ij}\right) - \partial_t u(t) \tag{2.202}$$

Use the remainder estimates in Lemma 2.2.4.3 and the bounds on $B_u(x, y)$ to find that the first two summands lie in $L^2(S^1)$, see (2.176) and the estimate in Theorem 2.2.2.1 for S^1 with $p = 4, q = 2, s = 1/2$ for the first summand in (2.202). Hence, by standard elliptic estimates for the fractional Laplacian or simply observing that with $\mathcal{R}$ being the Riesz transform, we have:

$$\nabla u(t) = \mathcal{R}\left(d_{1/2}u \cdot d_{1/2}\left(d\pi^\perp(u)\right) + \mathrm{div}_{1/2}\left(\frac{A^i_u(du, du)(x, y)}{|x - y|^{1/2}} d\pi^\perp(u(y))_{ij}\right) - \partial_t u(t)\right), \tag{2.203}$$

and $\mathcal{R}$ being a continuous operator on $L^2(S^1)$, we are led to the following estimate:

$$\begin{aligned}\|u(t)\|^2_{H^1(S^1)} &\leq C\left(\|u(t)\|^2_{L^2} + \||d_{1/2}u(t)|^2\|^2_{L^2} + \|\partial_t u(t)\|^2_{L^2}\right) \\ &\leq C\left(1 + \||d_{1/2}u(t)|^2\|^2_{L^2} + \|\partial_t u(t)\|^2_{L^2}\right),\end{aligned} \tag{2.204}$$

where we used $u(t) \in N$ almost everywhere for almost every time t to bound the L^2-norm of $u(t)$ by an L^∞-bound depending only on N. To provide some more details, the first two summands on the RHS of the equation (2.202) may be estimated by the L^2-norm of $|d_{1/2}u|^2(x)$, see Lemma 2.2.4.3 in the second case with $u = v = w$ and the treatment of uniqueness for a rephrasing of $d_{1/2}u \cdot d_{1/2}\left(d\pi^\perp(u)\right)$ in terms of $B_u(x, y)$ which is bounded. We highlight that:

$$\||d_{1/2}u(t)|^2\|^2_{L^2} \sim \|u(t)\|^2_{\dot{F}^{1/2}_{4,2}},$$

by Theorem 2.2.2.1, which together with Corollary 2.2.4.1 in the case $p = 4$ allows us to conclude that $u(t) \in H^1(S^1)$. $\square$

The remainder of the proof of Theorem 2.2.4.2 is now contained in the following Lemma:

Lemma 2.2.4.7. *Let u be as in Theorem 2.2.4.2. If $\varepsilon > 0$ is sufficiently small, then for almost all $t \in [0, +\infty[$:*

$$\|u(t)\|_{H^1(S^1)} \lesssim 1 + \|\partial_t u(t)\|_{L^2(S^1)},$$

with a constant independent of u and t. This implies $u \in L^2_{loc}(\mathbb{R}_+; H^1(S^1))$ and thus proves Theorem 2.2.4.2.

Proof. It is clear that thanks to the (local) L^2-integrability with respect to time, it thus remains to study the following contribution:

$$\||d_{1/2}u(t)|^2\|_{L^2}^2 = \|u(t)\|_{\dot{W}^{1/2,(4,2)}}^4 \sim \|u(t)\|_{\dot{F}^{1/2}_{4,2}}^4 \sim \|(-\Delta)^{1/4}u(t)\|_{L^4}^4 \tag{2.205}$$

Using the same ideas as in the proof of the uniqueness statement for $N = S^{n-1}$ (see [102]), we may estimate this using fractional Ladyzhenskaya-type estimate (2.201) (obtained by some cut-off function applied to the periodic extension of u or using [103, Lemma 3.1] and Theorem 2.2.2.1) for $v = (-\Delta)^{1/4}u$ by:

$$\||d_{1/2}u(t)|^2\|_{L^2}^2 \leq C'\|u(t)\|_{\dot{H}^{1/2}}^2 \|u(t)\|_{\dot{H}^1}^2$$

We recall that the homogeneous norm may be used by means of a perturbation-argument using constants as in the case $N = S^{n-1}$. Therefore, we have the energy term appearing as for $N = S^{n-1}$ and provided it is smaller than some $\varepsilon > 0$, we find:

$$\||d_{1/2}u(t)|^2\|_{L^2}^2 \leq C'\|(-\Delta)^{1/4}u(t)\|_{L^2}^2 \|u(t)\|_{H^1}^2 \leq C'\varepsilon \cdot \|u(t)\|_{H^1}^2,$$

where $\varepsilon > 0$ is an a priori energy estimate as in Rivière [68]. If $\varepsilon > 0$ is sufficiently small, we may absorb this term in the left hand side of (2.204) to arrive at:

$$(1 - CC'\varepsilon) \cdot \|u(t)\|_{H^1(S^1)}^2 \leq \tilde{C}\left(1 + \|\partial_t u(t)\|_{L^2}^2\right) \Rightarrow \|u(t)\|_{H^1(S^1)} \leq \frac{\tilde{C}}{1 - C'C\varepsilon}\left(1 + \|\partial_t u(t)\|_{L^2}^2\right),$$

which thus yields an estimate for the H^1-norm, if, for example, $0 < \varepsilon \leq 1/(2C'C)$. We observe that hence, by the integrability properties of $\partial_t u$ and the constant function:

$$u \in L^2_{loc}(\mathbb{R}_+; H^1(S^1)) \tag{2.206}$$

This allows us to employ Theorem 2.2.4.1, as the local regularity of the derivative is now apparent. This proves precisely the result in Theorem 2.2.4.2 which is also stated in the introduction of the paper and thus concludes our investigation of the fractional harmonic gradient flow with small initial energy. □

2.2.4.3 Existence and Regularity of Solutions

It remains to establish that solutions of the fractional harmonic gradient flow exist and are smooth, at least locally, and even globally smooth, provided the initial energy is sufficiently small. The ideas behind the proof are mostly the same as in [102], once we have rewritten the fractional harmonic gradient flow in a slightly different way. It should be noted that the existence result we prove is extending the one in Schikorra-Sire-Wang [77] which only deals with certain closed manifolds N that possess nice symmetry properties. The result in its generality presented here is, to the author's knowledge, new.

Before entering the proof, let us quickly recap the steps that we shall take in analogy to [102]:

1. Define the half-harmonic gradient flow operator and compute its linearisation which is a Fredholm operator
2. Show that the kernel of the linearisation is trivial by first showing that any element of the kernel is smooth and then applying a maximum principle for the fractional heat equation
3. Establish local existence for smooth boundary data using the injectivity of the linearisation and the Fredholm properties by using the Inverse Function Theorem
4. Deduce general local existence and global existence in the case of small initial energy by providing uniform estimates for solutions that allow us to approximate the boundary values and show that the corresponding solutions converge then to a solution of the half-harmonic gradient flow

An Equivalent Reformulation of the Main Equation A key step in [102] in order to derive local existence lies in the application of the Inverse Function Theorem in Banach spaces to argue along the lines of Hamilton [43]. As one can see in the author's previous work [102], the property that the solution assumes values only in S^{n-1} is merely proven after establishing existence and therefore crucially relies on the fact that $u \in N$ is ensured by the 1/2-harmonic gradient flow, provided the initial datum takes values in N. It is thus reasonable to expect that we shall treat the target space after establishing local existence. Nevertheless, the choice of formulation of the fractional harmonic gradient flow we study will be of great importance when it comes to verifying $u \in N$ a.e.. As a result, we first would like to think about the right kind of equation to study.

First, in [102] we used the following sequence of equivalent characterisations:

$$u(t,x) \in S^{n-1} \Leftrightarrow |u(t,x)|^2 = 1 \Leftrightarrow |u(t,x)|^2 - 1 = 0$$

Unfortunately, quite such a simple characterisation is not available for general N. However, if we let $\pi : \mathbb{R}^n \to \mathbb{R}^n$ be the extended version of the closest point projection, see Section 4.1, we may see for u at least continuous and $u(0) = u_0$ smooth with values in N:

$$u(t,x) \in N \Leftrightarrow \pi(u(t,x)) = u(t,x) \Leftrightarrow |u(t,x) - \pi(u(t,x))|^2 = 0,$$

for all $(t,x) \in [0,\infty[\times S^1$. The key observation is that due to the continuity and the fact that $\pi(x) = x$ only on N and possibly on a subset of the complement of a sufficiently small neighbourhood of N, the identity $\pi(u(t,x)) = u(t,x)$ for all (t,x) actually necessitates $u(t,x) \in N$, as $u(0,x) \in N$ for all $x \in S^1$. The minimal regularity imposed by assuming u is continuous actually prevents u from ever leaving N, since the set of fixed points of π is a disconnected union of N and a second set disjoint from N. To summarise, if we start with $u(0,\cdot) \in N$ and we know that u remains in the set of fixed points of π and is continuous, then due to the disconnectedness of the fixed point set, we auutomatically get $\pi(u(t,x)) = u(t,x)$ for all t,x. Thus, even though the extension π is not canonical, the condition:

$$|u(t,x) - \pi(u(t,x))|^2 = 0, \quad \forall (t,x) \in [0,\infty[\times S^1,$$

is the analogue we are looking for to the function $|u|^2 - 1$ in the case $N = S^{n-1}$.

Now, we would like to think about the fractional heat-type equation solved by $|u - \pi(u)|^2$. This will provide us crucial information about the "correct" choice of non-linearity to study in connection with the 1/2-harmonic gradient flow. So we are interested in computing:

$$\partial_t \left(|u - \pi(u)|^2 \right) + (-\Delta)^{1/2} \left(|u - \pi(u)|^2 \right)$$

For now, we assume that u is actually smooth to justify our calculations. Then:

$$\partial_t\left(|u-\pi(u)|^2\right) = 2\left(u_t - d\pi(u)u_t\right)\cdot(u-\pi(u))$$

Completely analogous to the computations for local existence and regularity in [102], we have:

$$(-\Delta)^{1/2}\left(|u-\pi(u)|^2\right) = 2(-\Delta)^{1/2}\left(u-\pi(u)\right)\cdot(u-\pi(u)) - |d_{1/2}\left(u-\pi(u)\right)|^2$$

Therefore:

$$\begin{aligned}
&\partial_t\left(|u-\pi(u)|^2\right) + (-\Delta)^{1/2}\left(|u-\pi(u)|^2\right)\\
&= 2\left(u_t - d\pi(u)u_t\right)\cdot(u-\pi(u)) + 2(-\Delta)^{1/2}\left(u-\pi(u)\right)\cdot(u-\pi(u)) - |d_{1/2}\left(u-\pi(u)\right)|^2\\
&= 2\left(Id - d\pi(u)\right)\left(u_t + (-\Delta)^{1/2}u\right)\cdot(u-\pi(u))\\
&+ 2\left(d\pi(u)(-\Delta)^{1/2}u - (-\Delta)^{1/2}(\pi(u))\right)\cdot(u-\pi(u)) - |d_{1/2}\left(u-\pi(u)\right)|^2\\
&= 2(Id - d\pi(u))r(u)\cdot(u-\pi(u)) + 2\left(d\pi(u)(-\Delta)^{1/2}u - (-\Delta)^{1/2}(\pi(u))\right)\cdot(u-\pi(u))\\
&- |d_{1/2}\left(u-\pi(u)\right)|^2,
\end{aligned}\tag{2.207}$$

where we write:

$$u_t + (-\Delta)^{1/2}u = r(u),\tag{2.208}$$

with r the non-linearity depending on u we are trying to find. We emphasise here that so far, we have only used simple manipulations and $r(u)$ is a quantity we still need to choose in such a way that (2.208) is a reformulation of the half-harmonic gradient flow.

Thinking about (2.207), one might be led to consider the following non-linearity:

$$r(u) := C(d\pi(u), u),\tag{2.209}$$

where we define:

$$C(a,b) := \mathcal{R}(a\nabla b) - a(-\Delta)^{1/2}b\tag{2.210}$$

In particular, for the expression relevant to our further computations this implies:

$$C(d\pi(u),u) = \mathcal{R}\left(d\pi(u)\nabla u\right) - d\pi(u)\mathcal{R}(\nabla u) = (-\Delta)^{1/2}(\pi(u)) - d\pi(u)(-\Delta)^{1/2}u\tag{2.211}$$

Here, we denote by $\mathcal{R}$ the Riesz-Hilbert transform on S^1. Naturally, (2.210) extends to vector-valued or matrix-valued maps in the natural way. Additionally the operator is already studied in Da Lio-Pigati [20] and one of the results there, which easily translates to S^1 by the same arguments, is the following:

Proposition 2.2.4.3 (Lemma E.2, [20]). *Assume that $a \in F^s_{p,2}(S^1), b \in F^1_{q,2}(S^1)$ with $s > 1/p, 1 < p, q < \infty$. Then, for any $\gamma > 1/p$, we have the following estimate:*

$$\|C(a,b)\|_{F^{s-\gamma}_{q,2}(S^1)} \lesssim \|a\|_{F^s_{p,2}(S^1)}\|b\|_{F^1_{q,2}(S^1)}$$

The proof only relies on the characterisations of the Bessel-Sobolev spaces and the use of Littlewood-Paley decompositions, both of which continue to hold on S^1 by Schmeisser-Triebel [80] and our discussion in Section 2. We thus refer to Da Lio-Pigati [20] for the proof.

Observe that Proposition 2.2.4.3 actually also hints at nice bootstrapping estimates available for $C(a,b)$ and therefore, solutions of (2.208) can be expected to be smooth, provided the initial datum is smooth as well, compare this with the ideas in [102]. However, for the moment, we would like to explore how this choice of $r(u)$ affects the computation in (2.207) by using the form (2.211):

$$
\begin{aligned}
&\partial_t \left(|u-\pi(u)|^2\right) + (-\Delta)^{1/2}\left(|u-\pi(u)|^2\right) \\
&= 2(Id - d\pi(u))r(u)\cdot(u-\pi(u)) + 2\left(d\pi(u)(-\Delta)^{1/2}u - (-\Delta)^{1/2}(\pi(u))\right)\cdot(u-\pi(u)) - |d_{1/2}\left(u-\pi(u)\right)|^2 \\
&= 2(Id - d\pi(u))\left((-\Delta)^{1/2}(\pi(u)) - d\pi(u)(-\Delta)^{1/2}u\right)\cdot(u-\pi(u)) \\
&+ 2\left(d\pi(u)(-\Delta)^{1/2}u - (-\Delta)^{1/2}\left(\pi(u)\right)\right)\cdot(u-\pi(u)) - |d_{1/2}\left(u-\pi(u)\right)|^2 \\
&= -2d\pi(u)\left((-\Delta)^{1/2}(\pi(u)) - d\pi(u)(-\Delta)^{1/2}u\right)\cdot(u-\pi(u)) - |d_{1/2}(u-\pi(u))|^2 \\
&= 2\left(d\pi(\pi(u)) - d\pi(u)\right)\left((-\Delta)^{1/2}(\pi(u)) - d\pi(u)(-\Delta)^{1/2}u\right)\cdot(u-\pi(u)) - |d_{1/2}(u-\pi(u))|^2, \quad (2.212)
\end{aligned}
$$

where in the last line, we used that $u-\pi(u)$ is orthogonal to $T_{\pi(u)}N$ and thus:

$$d\pi(\pi(u))(u-\pi(u)) = 0.$$

Applying the fact that $d\pi(\pi(u))$ is an orthogonal projection and therefore symmetric, we may deduce the equality above in (2.212). Using now Lipschitz-continuity of π and its derivatives as well as smoothness, we may therefore show:

$$\partial_t \left(|u-\pi(u)|^2\right) + (-\Delta)^{1/2}\left(|u-\pi(u)|^2\right) \leq C\cdot|u-\pi(u)|^2, \tag{2.213}$$

where C is a constant depending on u. We expand a bit on this step in the next subsection when determining the kernel of the linearisation of the operator induced by the fractional harmonic gradient flow. Since $u-\pi(u)=0$ at time $t=0$, we may therefore invoke a maximum principle inspired by the one in Hamilton [43] just like in [102] to deduce:

$$|u-\pi(u)|^2 = 0, \quad \forall (t,x) \in [0,\infty[\times S^1,$$

and consequently:

$$u(t,x) \in N, \quad \forall (t,x) \in [0,\infty[\times S^1$$

We notice that this precisely proves the required condition on the values of u. Additionally, the argument works equally well on $[0,T]\times S^1$, thus it applies also to local solutions of (2.208) in time. This observation will be crucial, as it will allow us to forget about the condition on the values at first and focus on the analytic aspects of the PDE.

Local Regularity Similar to [102], we shall prove the following result dealing with local regularity of solutions:

Proposition 2.2.4.4. *Let $u_0 \in C^\infty(S^1; N)$. Then there exists a $T > 0$, possibly depending on u_0, and a smooth map $u \in C^\infty([0,T] \times S^1)$ solving the following non-local PDE:*

$$u_t + (-\Delta)^{1/2} u = (-\Delta)^{1/2} \pi(u) - d\pi(u)(-\Delta)^{1/2} u, \tag{2.214}$$

and satisfying $u(0, \cdot) = u_0$. Additionally, by the result in the previous subsection:

$$u(t,x) \in N, \quad \forall (t,x) \in [0,T] \times S^1,$$

and, as a result, (2.214) *becomes the half-harmonic gradient flow equation* (2.193).

The last observation is due to the fact that if $u \in N$, then $d\pi(u)$ is the projection onto the tangent space $T_u N$ and $\pi(u) = u$, i.e.:

$$(-\Delta)^{1/2} \pi(u) - d\pi(u)(-\Delta)^{1/2} u = (-\Delta)^{1/2} u - d\pi(u)(-\Delta)^{1/2} u = (Id - d\pi(u))(-\Delta)^{1/2} u,$$

which implies that the RHS of (2.214) is actually orthogonal to $T_u N$. This is pecisely the meaning of (2.193), see also the computation in Section 3.1, Section 4.1 and [103, Section 3.1].

Proof of Proposition 2.2.4.4. The proof actually goes along the very same lines as the proof of Proposition 3.2 in [102], using the local Inversion Theorem for Banach spaces and slightly better integrability combined with a bootstrap procedure which now uses Proposition 2.2.4.3 instead of the Lemma proven in [102].

Step 1: Setup
As seen in [102] by Fourier representation, we may find $\tilde{u}$ solving the fractional heat equation:

$$\tilde{u}_t + (-\Delta)^{1/2} \tilde{u} = 0, \quad \tilde{u}(0, \cdot) = u_0$$

By the same argument using an explicit formula for $\tilde{u}$, we know that $\tilde{u}$ is actually smooth, as u_0 is smooth. Let us now define the following map for $1 < p < \infty$:

$$\begin{aligned} &H : W_0^{1,p}([0,T] \times S^1) \to L^p([0,T] \times S^1) \\ &H(v) := (\tilde{u} + v)_t + (-\Delta)^{1/2} (\tilde{u} + v) - C(d\pi(\tilde{u} + v), \tilde{u} + v) \end{aligned} \tag{2.215}$$

Here, $u \in W_0^{1,p}([0,T] \times S^1) \subset W^{1,p}([0,T] \times S^1)$ denotes the subspace of functions in $W^{1,p}([0,T] \times S^1)$ with $u(0, \cdot) = 0$. We observe that if $H(\tilde{u} + v)$ is vanishing on some subinterval $[0, T_0] \subset [0,T]$, then $\tilde{u} + v$ is actually a local solution to the half-harmonic gradient flow (2.214). Therefore, it suffices to establish the existence of v with this property. To achieve this, as in [102], we will show that H maps a sufficiently small neighbourhood of the zero function to an open neighbourhood of $H(\tilde{u})$ and then choose $f \in L^p([0,T] \times S^1)$ such that f equals 0 on an interval $[0, T_0]$ and agreeing with $H(\tilde{u})$ on the remainder of $[0,T]$. Choosing T_0 sufficiently small, f then lies in the image of H and thus a function v with the desired properties exists.

Step 2: Fredholm Property of the Linearisation
To follow the program outlined above, we would like to invoke the inverse function theorem for Banach spaces. Let us observe that H is Frèchet-differentiable and:

$$dH(0)h = h_t + (-\Delta)^{1/2} h - C\left(d\left(d\pi\right)(\tilde{u})h, \tilde{u}\right) - C\left(d\pi(\tilde{u}), h\right)$$

If we are able to show that $dH(0)$ is invertible, then the inverse function theorem would apply and we may argue as previously stated. Using Theorem 3.1 in Hieber-Prüss [46], we deduce that:

$$h \mapsto h_t + (-\Delta)^{1/2} h,$$

on the function space above is actually invertible. Therefore, if we can establish that the remaining summand:

$$h \mapsto C\left(d\left(d\pi\right)(\tilde{u})h, \tilde{u}\right) + C\left(d\pi(\tilde{u}), h\right), \tag{2.216}$$

is compact, then $dH(0)$ would be Fredholm and thus:

$$dH(0) \text{ is invertible } \Leftrightarrow dH(0) \text{ is injective}$$

Indeed, one may observe that (by using $\nabla(ab) = \nabla a \cdot b + a \cdot \nabla b$ and the definition of $\mathcal{R}$):

$$C(a,b) = (-\Delta)^{1/2} a \cdot b - \mathcal{R}\left(\nabla a \cdot b\right) - 2 d_{1/2} a \cdot d_{1/2} b,$$

which shows that the second summand in (2.216) is a compact map. Indeed, we have:

$$\begin{aligned}
C(a,b) &= \mathcal{R}(a \nabla b) - a(-\Delta)^{1/2} b \\
&= \mathcal{R}(\nabla(ab)) - \mathcal{R}(\nabla a \cdot b) - a(-\Delta)^{1/2} b \\
&= (-\Delta)^{1/2}(ab) - (-\Delta)^{1/2} a \cdot b - a(-\Delta)^{1/2} b - \mathcal{R}(\nabla a \cdot b) + (-\Delta)^{1/2} a \cdot b \\
&= -2 d_{1/2} a \cdot d_{1/2} b - \mathcal{R}(\nabla a \cdot b) + (-\Delta)^{1/2} a \cdot b,
\end{aligned} \tag{2.217}$$

where we used in the last equality the singular integral formulations of the fractional Laplacian. Reordering now provides the desired formula. For the first summand in (2.216), we may just use Proposition 2.2.4.3 and compactness of Sobolev embeddings. In both cases, we may easily deal with all terms involving $\tilde{u}$ by using the smoothness of this function. Thus, the operator $dH(0)$ is Fredholm. Since $dH(0)$ consists of an invertible operator and a compact one and adding a compact operator does not affect the Fredholm index, which means that $dH(0)$ has index 0. As a result, invertibility of $dH(0)$ becomes equivalent to injectivity.

Step 3: Regularity of Elements in the Kernel

Next, we want to investigate the kernel of $dH(0)$ to ultimately show that the linearised operator $dH(0)$ is injective. We argue by contradiction, i.e., assume that h is such that:

$$dH(0)h = 0.$$

Using Proposition 2.2.4.3, we now may deduce that $h \in C^{\infty}([0,T] \times S^1)$. Namely, we observe that the summand $C\left(d\pi(\tilde{u}), h\right)$ is arbitrarily regular with respect to x by using Proposition 2.2.4.3. Integrability in L^p follows, as we may uniformly estimate all terms involving $\tilde{u}$ due to smoothness. For the second summand, we may argue analogous to [102]: By Proposition 2.2.4.3, we see:

$$\begin{aligned}
\|(-\Delta)^{s/2-1/4} C(d(d\pi)(\tilde{u})h, \tilde{u})\|_{L^p(S^1)} &\lesssim \|d(d\pi(\tilde{u}))h\|_{F^s_{p,2}(S^1)} \|\nabla \tilde{u}\|_{L^p(S^1)} \\
&\lesssim \|h\|_{L^p(S^1)} + \|(-\Delta)^{s/2} h\|_{L^p(S^1)},
\end{aligned} \tag{2.218}$$

where the inequality involves constants depending on $\tilde{u}$, and therefore also:

$$\|(-\Delta)^{s/2-1/4} C(d(d\pi)(\tilde{u})h, \tilde{u})\|_{L^p([0,T]\times S^1)} \lesssim \|h\|_{L^p([0,T]\times S^1)} + \|(-\Delta)^{s/2} h\|_{L^p([0,T]\times S^1)}$$

Let us now see the following for any $\varphi \in C^\infty([0,T]\times S^1)$ with compact support strictly contained in $[0,T]\times S^1$. Then we have:

$$\begin{aligned}
&\int_0^T \int_{S^1} (-\Delta)^{s/2} h \cdot (-\partial_t + (-\Delta)^{1/2}) \varphi dx dt \\
&= \int_0^T \int_{S^1} h \cdot (-\partial_t + (-\Delta)^{1/2}) (-\Delta)^{s/2} \varphi dx dt \\
&= \int_0^T \int_{S^1} \left(C(d(d\pi)(\tilde{u})h, \tilde{u}) + C(d\pi(\tilde{u}), h) \right) (-\Delta)^{s/2} \varphi dx dt + \int_{S^1} h(0,x) (-\Delta)^{s/2} \varphi(0,x) \\
&= \int_0^T \int_{S^1} (-\Delta)^{s/2} \left(C(d(d\pi)(\tilde{u})h, \tilde{u}) + C(d\pi(\tilde{u}), h) \right) \varphi dx dt
\end{aligned} \tag{2.219}$$

where we observed that $(-\Delta)^{s/2}\varphi$ is still compactly supported and smooth as well as the equation solved by h and the initial condition $h(0,\cdot) = 0$. Therefore, $(-\Delta)^{s/2}h$ solves an inhomogeneous fractional heat equation with RHS in L^p. Arguing by using the connection of $\partial_t + (-\Delta)^{1/2}$ with the Laplacian and using an extension:

$$\tilde{h}(t,x) := \begin{cases} h(t,x), & \text{if } t \geq 0 \\ -h(-t,x), & \text{if } t \leq 0 \end{cases}$$

by symmetrising of h to times $t \in [-T, 0]$ (the resulting map $(-\Delta)^{s/2}\tilde{h}$ solves a Laplace equation with RHS in $W^{-1,p}$ in the distributional sense and we may thus argue by elliptic regularity), we find that $(-\Delta)^{s/2}h \in W^{1,p}_{loc}([0,T]\times S^1)$. Notice that as $h(0) = 0$, the extension $\tilde{h}$ is well-behaved. Arguing as in [102] using Hieber-Prüss [46], where existence and uniqueness of solutions to the half-harmonic gradient flow in appropriate Sobolev spaces is implicitly treated, we may thus deduce:

$$(-\Delta)^{s/2} h \in W^{1,p}([0,T]\times S^1), \quad \forall s \in [0, 3/4], \tag{2.220}$$

by using the estimate in Proposition 2.2.4.3 as specified before. The next step is to actually find an equation solved by $\partial_x h =: h'$. This is achieved by using a slightly modified version of the equation $dH(0)h = 0$ using (2.217). Namely, we use:

$$h_t + (-\Delta)^{1/2} h = C(d(d\pi(\tilde{u})h, \tilde{u}) - 2 d_{1/2}\left(d\pi(\tilde{u})\right) \cdot d_{1/2} h - \mathcal{R}\left(\nabla d\pi(\tilde{u}) \cdot h\right) + (-\Delta)^{1/2}\left(d\pi(\tilde{u})\right) \cdot h \tag{2.221}$$

If we differentiate both sides with respect to x, this leads to:

$$\begin{aligned}
\left(\partial_t + (-\Delta)^{1/2}\right) h' &= C((d(d\pi(\tilde{u}))' h + d(d\pi(\tilde{u}))h', \tilde{u}) \\
&\quad - 2 d_{1/2}\left(d\pi(\tilde{u})\right)' \cdot d_{1/2} h - 2 d_{1/2}\left(d\pi(\tilde{u})\right) \cdot d_{1/2} h' \\
&\quad - \mathcal{R}\left(\nabla \left(d\pi(\tilde{u})\right)' \cdot h\right) - \mathcal{R}\left(\nabla d\pi(\tilde{u}) \cdot h'\right) \\
&\quad + (-\Delta)^{1/2}\left(d\pi(\tilde{u})\right)' \cdot h + (-\Delta)^{1/2}\left(d\pi(\tilde{u})\right) \cdot h'
\end{aligned} \tag{2.222}$$

A direct computation using (2.220), we thus know that the RHS of the equation lies in $L^p([0,T]\times S^1)$. Arguing as before using distributional solutions for the symmetrisation and the connection to the Laplacian as well as Hieber-Prüss [46], we deduce:

$$h' \in W^{1,p}([0,T]\times S^1])$$

Inserting this into the main equation $dH(0)h = 0$, we thus find by differentiating with respect to t:

$$h \in W^{2,p}([0,T] \times S^1)$$

By iterating similar to [102] and our computations starting from (2.222) as above and repeating the same steps for higher and higher derivatives, using the connection to the Laplacian and Theorem 3.1 in Hieber-Prüss [46] repeatedly, this shows:

$$\forall s \in \mathbb{R}_{\geq 0} : (-\Delta)^s h \in W^{1,p}([0,T] \times S^1)$$

Using the equation $dH(0)h = 0$, we may also discover estimates for higher order derivatives in t-direction. Hence:

$$h \in \bigcap_{k \in \mathbb{N}} W^{k,p}([0,T] \times S^1) \subset C^\infty([0,T] \times S^1)$$

Therefore, any h in the kernel of $dH(0)$ is actually smooth.

Step 4: Kernel is trivial and thus $dH(0)$ is invertible

It remains to establish that actually $h = 0$. The trick is as in [102] and the argument presented actually provides the argument for the missing step to prove (2.213) in Section 4.3.1: We study the fractional heat-type equation satisfied by $|h|^2$. One finds:

$$\partial_t \left(|h|^2\right) = 2h_t \cdot h, \tag{2.223}$$

as well as:

$$(-\Delta)^{1/2} \left(|h|^2\right) = 2(-\Delta)^{1/2} h \cdot h - |d_{1/2} h|^2. \tag{2.224}$$

Combining (2.223) and (2.224), we find:

$$\begin{aligned}
&\partial_t \left(|h|^2\right) + (-\Delta)^{1/2} \left(|h|^2\right) \\
&= 2h_t \cdot h + 2(-\Delta)^{1/2} h \cdot h - |d_{1/2} h|^2 \\
&= 2 \left(h_t + (-\Delta)^{1/2} h\right) \cdot h - |d_{1/2} h|^2 \\
&= 2 \left(C\left(d\left(d\pi\right)(\tilde{u})h, \tilde{u}\right) + C\left(d\pi(\tilde{u}), h\right)\right) \cdot h - |d_{1/2} h|^2
\end{aligned} \tag{2.225}$$

Notice the similarity with (2.213). Our goal is now to estimate the terms involving C in an appropriate manner. For example, we have:

$$\begin{aligned}
|C(d(d\pi)(\tilde{u})h, \tilde{u})| &= \left| \mathcal{R}\left(d(d\pi)(\tilde{u})h \cdot \nabla \tilde{u}\right) - d(d\pi)(\tilde{u})h \cdot (-\Delta)^{1/2} \tilde{u} \right| \\
&\leq \left| \mathcal{R}\left(d(d\pi)(\tilde{u})h \cdot \nabla \tilde{u}\right)\right| + |h| \, \|(-\Delta)^{1/2} \tilde{u}\|_{L^\infty([0,T] \times S^1)},
\end{aligned} \tag{2.226}$$

which shows that we merely have to estimate the contribution of the Riesz operator. We know that up to a constant, we have for any $x \in S^1$:

$$\begin{aligned}
&\mathcal{R}\left(d(d\pi)(\tilde{u})h \cdot \nabla \tilde{u}\right)(x) \\
&\sim -P.V. \int_{S^1} \left(d(d\pi)(\tilde{u}(x))h(x)\nabla \tilde{u}(x) - d(d\pi)(\tilde{u}(y))h(y)\nabla \tilde{u}(y)\right) \cot\left(\frac{x-y}{2}\right) dy \\
&\sim -P.V. \int_{S^1} \frac{d(d\pi)(\tilde{u}(x))h(x)\nabla \tilde{u}(x) - d(d\pi)(\tilde{u}(y))h(y)\nabla \tilde{u}(y)}{|x-y|} \cos\left(\frac{x-y}{2}\right) dy
\end{aligned}$$

$$\lesssim |h(x)| \|u\|_{C^2} + |d_{1/2}h|(x) \|d(d\pi)(\tilde{u})\|_{L^\infty} \|\nabla u\|_{L^\infty}, \tag{2.227}$$

where we used the formula for the distance on the circle $|x-y| = 2\sin((x-y)/2)$. The formula follows by using the fractional Leibniz rule adapted appropriately here. By analogous computations using (2.217), we find:

$$|C(d\pi(\tilde{u}), h)(x)| \lesssim |h(x)| + |d_{1/2}h|(x) \tag{2.228}$$

The constants in (2.227) and (2.228) may depend on $\tilde{u}$ and on the target manifold N (via the projection π and its derivatives), but they are independent of h. To summarise, we have found the following:

$$\partial_t \left(|h|^2\right) + (-\Delta)^{1/2} \left(|h|^2\right) \leq \tilde{C}_{\tilde{u},N} |h| \left(|h(x)| + |d_{1/2}h|(x)\right) - |d_{1/2}h|^2,$$

where $\tilde{C}_{\tilde{u},N} > 0$ is a constant depending on $\tilde{u}$ and N. By using the arithmetic geometric mean inequality, we may deduce:

$$\tilde{C}_{\tilde{u},N} |h| \left(|h(x)| + |d_{1/2}h|(x)\right) - |d_{1/2}h|^2 \leq \tilde{C}_{\tilde{u},N} |h|^2 + \frac{\tilde{C}_{\tilde{u},N}}{4\delta} |h|^2 + \delta |d_{1/2}h|^2 - |d_{1/2}h|^2,$$

and choosing $\delta = 1$, we find:

$$\partial_t \left(|h|^2\right) + (-\Delta)^{1/2} \left(|h|^2\right) \leq \hat{C}_{\tilde{u},N} |h|^2.$$

Invoking the maximum principle for the fractional heat flow yields just as in [102] that h must assume its global extremum on $[0,T] \times S^1$ at time $t = 0$. However, as $h(0) = 0$ and $|h|^2 \geq 0$, this implies:

$$|h|^2 = 0 \Rightarrow h = 0,$$

which finally implies injectivity of $dH(0)$. Therefore, $dH(0)$ is an injective Fredholm operator of index 0, which shows that it is surjective, thus invertible. Local existence of $W^{1,p}$-solutions to the equation (2.214) exist.

It remains to verify smoothness of such a solution u. This follows by a bootstrap argument similar to the one for h, but taking a bit more care. The key observation is that in each step of the bootstrap of h, Hölder regularity with sufficiently close α to 1 is sufficient to obtain the desired estimates, i.e. $\alpha > 1/2$ and thus $p > 4$ suffice. Let us for now take $p > 8$ to make the arguments easier, as we shall see below any p is possible anyways. Namely, we have at every fixed time:

$$\begin{aligned} \|(-\Delta)^{s/2-1/16} C(d\pi(u), u)\|_{L^p(S^1)} &\lesssim \|d\pi(u)\|_{F^s_{p,2}(S^1)} \|\nabla u\|_{L^p(S^1)} \\ &\lesssim \|u\|_{C^{0,\alpha}(S^1)} \|u\|_{W^{1,p}(S^1)}, \end{aligned} \tag{2.229}$$

if $1 - 1/8 = 7/8 = \alpha > s$ and by integrating in time-direction:

$$\|(-\Delta)^{s/2-1/16} C(d\pi(u), u)\|_{L^p([0,T]\times S^1)} \lesssim \|u\|_{C^{0,\alpha}([0,T]\times S^1)} \|u\|_{W^{1,p}([0,T]\times S^1)} \tag{2.230}$$

Arguing as for h before by symmetrisation and Hieber-Prüss [46], this shows:

$$(-\Delta)^{t/2} u \in W^{1,p}([0,T] \times S^1),$$

for all $0 \leq t < 3/4$. Next, we verify that $u \in W^{2,p}$ for all $1 < p < \infty$. This follows from the formulation (2.214) by rewriting:

$$(-\Delta)^{1/2} \left(\pi(u)\right)(x) - d\pi(u(x))(-\Delta)^{1/2} u(x) = P.V. \int_{S^1} \frac{\pi(u(x)) - \pi(u(y)) - d\pi(u(x))(u(x) - u(y))}{|x-y|^2} dy$$

By using Taylorapproximation, we see for every $j \in \{1, \dots n\}$:

$$
\begin{aligned}
&\pi_j(u(x)) - \pi_j(u(y)) - d\pi_j(u(x))(u(x) - u(y)) \\
&= \int_0^1 d\pi_j((1-t)u(y) + u(x))(u(x) - u(y)) - d\pi_j(u(x))(u(x) - u(y))dt \\
&= \sum_{k=1}^n \int_0^1 \left(\partial_k \pi_j((1-t)u(y) + tu(x)) - \partial_k \pi_j(u(x))\right)(u_k(x) - u_k(y))dy \\
&= \sum_{k=1}^n \sum_{l=1}^n \int_0^1 \int_0^1 (t-1)\partial_{kl}\pi_j((s-st)u(y) + (1+st-s)u(x))(u_k(x) - u_k(y))(u_l(x) - u_l(y))dsdt \\
&=: P_j^{kl}(u(x), u(y))(u_k(x) - u_k(y))(u_l(x) - u_l(y))
\end{aligned}
\tag{2.231}
$$

This is precisely the form we alluded to in Section 4.1.3. Therefore:

$$
\begin{aligned}
&(-\Delta)^{1/2}\left(\pi(u)\right)(x) - d\pi(u(x))(-\Delta)^{1/2}u(x) \\
&= \sum_{k,l=1}^n P.V. \int_{S^1} P^{kl}(u(x), u(y)) d_{1/2}u_k(x,y) d_{1/2}u_l(x,y) \frac{dy}{|x-y|}
\end{aligned}
\tag{2.232}
$$

Notice that P^{jk} are bounded and thus the RHS of the flow is actually bounded, since we know $u \in C^{0,\alpha}$ for $\alpha > 1/2$ and:

$$
\begin{aligned}
\left|(-\Delta)^{1/2}\left(\pi(u)\right)(x) - d\pi(u(x))(-\Delta)^{1/2}u(x)\right| &= \left|\sum_{k,l=1}^n P.V. \int_{S^1} P^{kl}(u(x), u(y)) d_{1/2}u_k(x,y) d_{1/2}u_l(x,y) \frac{dy}{|x-y|}\right| \\
&\lesssim \int_{S^1} \frac{|u(x) - u(y)|^2}{|x-y|^2} dy = |d_{1/2}u|(x)^2 \\
&\lesssim \|u\|_{C^{0,\alpha}},
\end{aligned}
\tag{2.233}
$$

where $\alpha > 1/2$. This implies that $u \in W^{1,p}([0,T] \times S^1)$ for all $1 < p < +\infty$, since the RHS of the fractional harmonic gradient flow for u is thus bounded and therefore in all L^p-spaces, see Hieber-Prüss [46].

Our goal is now to establish higher integrability: We may now differentiate this expression with respect to x to find:

$$
\begin{aligned}
&\frac{d}{dx}\left((-\Delta)^{1/2}\left(\pi(u)\right)(t,x) - d\pi(u(x))(-\Delta)^{1/2}u(t,x)\right) \\
&= \sum_{k,l=1}^n P.V. \int_{S^1} P^{kl}(u(x), u(y)) d_{1/2}u_k'(x,y) d_{1/2}u_l(x,y) \frac{dy}{|x-y|} \\
&+ \sum_{k,l=1}^n P.V. \int_{S^1} P^{kl}(u(x), u(y)) d_{1/2}u_k(x,y) d_{1/2}u_l'(x,y) \frac{dy}{|x-y|} \\
&+ \sum_{k,l=1}^n P.V. \int_{S^1} \left(dP^{kl}(u(x), u(y)) \begin{pmatrix} u'(x) \\ u'(y) \end{pmatrix}\right) d_{1/2}u_k(x,y) d_{1/2}u_l(x,y) \frac{dy}{|x-y|}
\end{aligned}
\tag{2.234}
$$

It can now he seen, using again Hölder continuity and the previously proven regularity as well as Sobolev embeddings:

$$\frac{d}{dx}\left((-\Delta)^{1/2}\left(\pi(u)\right)(t,x) - d\pi(u(x))(-\Delta)^{1/2}u(t,x)\right) \in L^p([0,T]\times S^1).$$

This now shows:

$$u \in W^{2,p}([0,T]\times S^1), \tag{2.235}$$

by inserting the regularity $u' \in W^{1,p}$ into the main equation to establish higher regularity in time-direction. Higher order regularity can now be proven by iteration. The result of Proposition 2.2.4.4 therefore follows. □

Let us observe that Proposition 2.2.4.4 actually proves existence of solutions to the fractional harmonic gradient flow for sufficiently small times for all closed target manifolds N, provided the initial datum is smooth. This is due to the openness of $H(0)$ being an interior point of the image and therefore, if we modify $H(0)$ for times $t < \delta$ to be 0 and else as $H(0)$, then the resulting map is in the image and thus for some v, it is the same as $H(v)$. However, this already implies that $\tilde{u} + v$ solve the half-harmonic gradient flow for times up to $t = \delta$ and by the computations in the introduction to this section, $\tilde{u} + v$ takes values in N.

In the next section shall remove the regularity assumption on the boundary data by following Struwe [89] as in [102].

Global Regularity by Approximation, Existence as a Byproduct To prove existence and regularity of solutions in the case of general initial data, we first have to be able to approximate the boundary data sufficiently well by smooth functions, The following result follows precisely as in [102] and the proof is therefore omitted:

Lemma 2.2.4.8. *Assume N is an arbitrary closed manifold. Let $u \in H^{1/2}(S^1;N)$. Then there exists a sequence $u_k \in C^\infty(S^1) \cap H^{1/2}(S^1;N)$ such that:*

$$\|u_k - u\|_{H^{1/2}(S^1)} \to 0, \quad k \to \infty.$$

The next lemma proven in [102] continues to hold, as its proof relies on general properties of the Triebel-Lizorkin spaces on the unit circle, while the target manifold is irrelevant:

Lemma 2.2.4.9. *There exist $C > 0$ not depending on R, u, T, such that for any smooth u on $[0,T]\times S^1$ and $0 < R < 1$, the following estimate holds for all $x_0 \in S^1$:*

$$\int_0^T \int_{B_{\frac{3R}{4}}(x_0)} |(-\Delta)^{1/4}u|^4 dxdt \leq C \sup_{0\leq t\leq T} \int_{B_R(x_0)} |(-\Delta)^{1/4}u(t)|^2 dx \cdot \left(\int_0^T \int_{B_R(x_0)} |(-\Delta)^{1/2}u|^2 dxdt + \frac{1}{R^2}\int_0^T \int_{S^1} |(-\Delta)^{1/4}u|^2 dxdt\right), \tag{2.236}$$

by density the same result applies for all $u \in H^1([0,T] \times S^1)$, and all boundary terms $u_0 = u(0,\cdot) \in H^{1/2}(S^1)$, with bounded $1/2$-Dirichlet energy. Similarily, we have:

$$\int_0^T \int_{S^1} |(-\Delta)^{1/4} u|^4 dxdt \lesssim \sup_{0 \le t \le T, x \in S^1} \int_{B_R(x)} |(-\Delta)^{1/4} u(t)|^2 dx \\ \cdot \left(\int_0^T \int_{S^1} |(-\Delta)^{1/2} u|^2 dxdt + \frac{1}{R^3} \int_0^T \int_{S^1} |(-\Delta)^{1/4} u|^2 dxdt \right). \tag{2.237}$$

Furthermore, due to the orthogonality of the RHS of (2.193) with respect to the tangent space of N, we also may generalise the following lemmas found in [102], as the orthogonality is the only property used:

Lemma 2.2.4.10. *Let u be a sufficiently regular solution of the $1/2$-harmonic gradient flow in N as previously defined with $u(0,\cdot) = u_0$ taking values in N. Then the following holds for all $T \ge 0$:*

$$\frac{1}{2} \|(-\Delta)^{1/4} u(T)\|^2_{L^2(S^1)} \le \frac{1}{2} \|(-\Delta)^{1/4} u_0\|^2_{L^2(S^1)}$$

In fact, the energy $T \mapsto \|(-\Delta)^{1/4} u(T)\|_{L^2(S^1)}$ monotonically decreases in T.

As in Struwe [89], we may introduce for $0 < R < 1$ and $t \in [0,T]$:

$$E_R(u; x, t) := \frac{1}{2} \int_{B_R(x)} |(-\Delta)^{1/4} u(t)|^2 dx, \tag{2.238}$$

for the local energy and also:

$$\varepsilon(R) = \varepsilon(R; u, T) := \sup_{x \in S^1, t \in [0,T]} E_R(u; x, t) \tag{2.239}$$

The local energy estimate from Struwe [89] and [102] continues to hold by the same proof:

Lemma 2.2.4.11. *There exists a constant $C > 0$ such that for every $u : [0,T] \times S^1 \to N$ in $H^1([0,T] \times S^1) \cap L^\infty([0,T]; \dot{H}^{1/2}(S^1))$, $u_0 = u(0,\cdot) \in H^{1/2}(S^1; N)$ solving the half-harmonic flow equation (2.193) and satisfying the energy decrease property as in Lemma 2.2.4.10, any $0 < R < 1/2$ and $(t, x_0) \in [0,T] \times S^1$, the following estimate holds:*

$$E_R(u; x_0, t) \le E_{2R}(u; x_0, 0) + C \left(\frac{t}{R^2} E(u_0) + \frac{\sqrt{t}}{R} \sqrt{\varepsilon(2R) E(u_0)} \right) \\ \le E_{2R}(u; x_0, 0) + C \left(\frac{t}{R^2} + \frac{\sqrt{t}}{R} \right) E(u_0), \tag{2.240}$$

where $E(u_0) = E_{1/2}(u_0)$. In the second inequality, we used the trivial estimate between the local energy and the global one under the energy decay.

Again, the proof is referred to Lemma 3.17 in [102], there are no real differences as the orthogonality of the RHS in (2.193) to the tangent space of N removes the non-linearity in the computations.

It therefore remains to verify the following results as in Struwe [89]:

Lemma 2.2.4.12. *The following generalisations of the results in [89] hold true:*

1. *Lemma 3.7 in [89]: There exists* $\varepsilon_1 > 0$ *such that for any* $u \in H^1([0,T]\times S^1)\cap L^\infty([0,T];H^{1/2}(S^1))$ *solving* (2.193) *with values in* N *and any* $R < 1/2$, *there holds:*
$$\int_0^T \int_{S^1} |\nabla u|^2 dxdt \leq CE(u_0)\left(1+\frac{T}{R^3}\right), \tag{2.241}$$
with C *independent of* u, T, R, *provided* $\varepsilon(R) < \varepsilon_1$. *Here,* $u(0,\cdot) = u_0 \in H^{1/2}(S^1;N)$ *is the initial value.*

2. *Lemma 3.8, Remark 3.9 in [89]: For any numbers* $\varepsilon, \tau, E_0 > 0$ *and* $R_1 < 1/2$, *there is a* $\delta > 0$ *such that for any* u, *satisfying the conditions as in 1., solving* (2.193) *with values in* N *and any* $I \subset [\tau, T]$ *with measure* $|I| < \delta$, *there holds:*
$$\int_I \int_{S^1} |(-\Delta)^{1/4}u|^2 dxdt < \varepsilon, \tag{2.242}$$
provided $\varepsilon(R_1) < \varepsilon_1, E(u_0) \leq E_0$. *The same holds with* $\tau = 0$, *if we consider a sequence* u_n *associated with converging initial data* $u_n(0)$ *in* $H^{1/2}(S^1)$.

3. *Lemma 3.10, Remark 3.11 in [89]: Let* u *be, in addition to the assumptions in 1., a* $C^2([\tau,T]\times S^1)$*-solution to* (2.193), *then, for every* $1 \leq p < +\infty$, *there exists a* $L^p([\tau,T]\times S^1)$*-bound on* $u_t + (-\Delta)^{1/2}u$ *with a constant only depending on* $E(u_0), \tau, T$ *and* R, *provided* $\varepsilon(R) < \varepsilon_1$. *Here,* $\tau > 0$ *in general and* $\tau \geq 0$ *in case* u_0 *is smooth.*

The proof is analogous to [89], we merely rely on the quadratic estimate for the non-linearity given by:
$$\left|\sum_{k,l=1}^n P.V. \int_{S^1} a^{kl}(u(x),u(y)) d_{1/2}u_k(x,y) d_{1/2}u_l(x,y) \frac{dy}{|x-y|}\right| \lesssim |d_{1/2}u|^2(x)$$
Therefore, as in [102], we refer to [89], as the proofs are obvious modifications of Struwe's techniques and the previously presented bootstrap procedure for solutions to fractional heat-type equations. Arguing as in Theorem 4.1 in [89], we may also deduce that for sufficiently small energy at time $t = 0$, global existence is ensured. Otherwise, blow-ups may occur.

2.2.4.4 Convergence of Solutions as $t \to +\infty$

If we look at the proof of [103, Theorem 3.4], it is clear that the arguments immediately generalises to the following Theorem by the same proof:

Theorem 2.2.4.3. *Let* $u \in L^2(\mathbb{R}_+;H^{1/2}(S^1))$ *and* $u_t \in L^2(\mathbb{R}_+;L^2(S^1))$ *be a solution of the fractional harmonic gradient flow* (2.193) *with values in a closed manifold* $N \subset \mathbb{R}^n$ *and with initial data* $u_0 \in H^{1/2}(S^1;N)$. *Assume that:*
$$\|(-\Delta)^{1/4}u(t)\|_{L^2} \leq \|(-\Delta)^{1/4}u_0\|_{L^2} \leq \varepsilon, \quad \forall t \in \mathbb{R}_+,$$
for $\varepsilon > 0$ *sufficiently small. Then, for a suitably chosen subsequence* $t_k \to +\infty$, *the sequence of maps* $(u(t_k,\cdot))_{k\in\mathbb{N}} \subset H^1(S^1;N)$ *converges weakly in* $H^1(S^1)$ *to a* $1/2$*-harmonic map in* N.

We refer to the proof in [102] for details. Again, for sufficiently small $\varepsilon > 0$, we may even deduce that the limit function is a constant map to some point in N.

2.2.5 Appendix: Morrey Regularity and Increased Integrability as in [20]

In this appendix, we briefly go into some more details of the proof of Lemma 2.2.4.6. We recall that in the paper, we referred to Da Lio-Pigati [20], in particular Theorem D.7 and Corollary D.8. Let us expand upon this:

We assume that u solves the following equation:

$$(-\Delta)^{1/2}u = d_{1/2}u \cdot d_{1/2}\left(d\pi^{\perp}(u)\right) + \operatorname{div}_{1/2}\left(\frac{A^i_u(du,du)(x,y)}{|x-y|^{1/2}} d\pi^{\perp}(u(y))_{ij}\right) + f, \tag{2.243}$$

then we notice that:

$$\int_{S^1} d_{1/2}u \cdot d_{1/2}\left(d\pi(u)\varphi\right) dx = \int_{S^1} f\varphi dx,$$

by arguing as in Section 3.1.2. Therefore, this shows:

$$d\pi(u)(-\Delta)^{1/2}u = d\pi(u)f$$

We shall sometimes write $d\pi$ instead of $d\pi(u)$ and $d\pi^{\perp}$ instead of $d\pi^{\perp}(u)$, implying the appropriate functions to be inserted.

If we define $w = u \circ \Pi^{-1}$ using the stereographic projection as in Da Lio-Pigati [20], this becomes:

$$d\pi(w)(-\Delta)^{1/2}w = d\pi(w)\tilde{f},$$

and therefore:

$$(-\Delta)^{1/2}w = d\pi(w)\tilde{f} + d\pi^{\perp}(w)(-\Delta)^{1/2}w, \tag{2.244}$$

where:

$$\tilde{f} = \frac{2}{1+x^2} f \circ \Pi^{-1}.$$

We let:

$$v := (-\Delta)^{1/4}w, \tag{2.245}$$

and by following precisely the arguments as in Da Lio-Pigati [20] on p.31-32, we find:

$$(-\Delta)^{1/4}v = \Omega_0 v + \Omega_1 v + (-\Delta)^{1/4}\left(d\pi^{\perp}v\right) + 2(-\Delta)^{1/4}d\pi^{\perp} \cdot d\pi^{\perp}v - T(d\pi^{\perp}, v) + \tilde{f} \tag{2.246}$$

Here, $\Omega_0 := d\pi^{\perp}(-\Delta)^{1/4}d\pi^{\perp} - (-\Delta)^{1/4}d\pi^{\perp}d\pi^{\perp}, \Omega_1 := T^*(d\pi^{\perp}, d\pi)$ and T are the same objects as defined in Da Lio-Pigati [20], in particular T and T^* are the following commutators:

$$T(Q,v) := (-\Delta)^{1/4}(Qv) + (-\Delta)^{1/4}Q \cdot v - Q(-\Delta)^{1/4}v \tag{2.247}$$

$$T^*(P,Q) := (-\Delta)^{1/4}(PQ) - (-\Delta)^{1/4}P \cdot Q - P(-\Delta)^{1/4}Q, \tag{2.248}$$

satisfying the estimates due to compensation properties of T, T^*:

$$\|T(Q,v)\|_{\mathcal{H}^1(\mathbb{R};\mathbb{R}^m)} \lesssim \|Q\|_{\dot{H}^{1/2}(\mathbb{R};\mathbb{R}^{m\times m})}\|v\|_{L^2(\mathbb{R};\mathbb{R}^m)} \tag{2.249}$$

$$\|T^*(P,Q)\|_{L^{2,1}(\mathbb{R},\mathbb{R}^{m\times m})} \lesssim \|P\|_{\dot{H}^{1/2}(\mathbb{R},\mathbb{R}^{m\times m})}\|Q\|_{\dot{H}^{1/2}(\mathbb{R},\mathbb{R}^{m\times m})}, \tag{2.250}$$

for $P, Q \in \dot{H}^{1/2}(\mathbb{R};\mathbb{R}^{m\times m}) \cap L^{\infty}(\mathbb{R})$ and $v \in L^2(\mathbb{R};\mathbb{R}^m)$. See Appendix C in Da Lio-Pigati [20] for further details. Due to the similar structure of the equation, it is not surprising that the following holds:

Theorem 2.2.5.1 (Theorem D.7, [20]). *The map* $v = (-\Delta)^{1/4} w$ *has* $(-\Delta)^{1/4}(d\pi v), \mathcal{R}(-\Delta)^{1/4}(d\pi^{\perp} v) \in L^1(\mathbb{R})$ *and there exists* $\alpha > 0$*, such that:*

$$\|(-\Delta)^{1/4}(d\pi v)\|_{L^1(B_r(x_0))} + \|\mathcal{R}(-\Delta)^{1/4}(d\pi^{\perp} v)\|_{L^1(B_r(x_0))} \lesssim r^{\alpha},$$

for all $r > 0$ *and uniformly in* $x_0 \in \mathbb{R}$*.*

Proof. The change of gauge argument and localisation estimates work equally well in the case of our new equation (2.246). Thus, Step 1 carries over word by word. In Step 2, we just need to change the estimate slightly to account for $\tilde{f}$. Namely, we replace $(-\Delta)^{1/4}(Qh)$ in Da Lio-Pigati [20] immediately by $(-\Delta)^{1/4}\tilde{w}_0$ with $\tilde{w}_0$ being the pullback under the stereographic projection of w_0 which solves:

$$(-\Delta)^{1/2} w_0 = Q \circ \Pi \cdot f,$$

with Q the gauge from Da Lio-Pigati [20]. The argument proceeds as outlined in section 5 and shows that $(-\Delta)^{1/4}\tilde{w}_0$ lies in all $L^q(\mathbb{R})$, for $2 \leq q < \infty$, by Hardy-Littlewood-Sobolev inequality. Therefore, we insert $(-\Delta)^{1/4}\tilde{w}_0$, obtaining the same expression as on the bottom of p.32 of Da Lio-Pigati [20] and by using Hölder's inequality to estimate:

$$\|(-\Delta)^{1/4}\tilde{w}_0\|_{L^2(B_r(x_0))} \lesssim r^{\beta},$$

for any $\beta \in]0, 1/2[$ and the estimates on p.33-34 in Da Lio-Pigati [20], one may deduce completely analogous for some $0 < \gamma < 1/4$:

$$\|v\|_{L^{2,\infty}(B_r(x_0))} \lesssim r^{\gamma},$$

for all $r > 0$ and x_0. Thus, Step 2 of the proof of Theorem D.7 still applies.

The remainder of the proof of Theorem D.7 in Da Lio-Pigati [20] can now be generalized as well. The application of Adams' embedding is immediate, the L^2-Morrey decay of v can be obtained by the same trick and due to this, Step 3 holds. Finally, Step 4 and thus the conclusion of the proof of Theorem 2.2.5.1 follow completely analogous by commutator estimates. □

Looking at Corollary D.8 in Da Lio-Pigati [20] and its proof reveals that the local integrability $(-\Delta)^{1/4} w \in L^p_{loc}(\mathbb{R})$ follows immediately by the same arguments as given there. Therefore, the remainder of the argument in section 5 can be applied and provides the desired gain in integrability.

2.3 Bubbling and Global Weak Existence Theory [104]

To conclude our investigations of the half-harmonic gradient flow, we shall now focus on the behaviour at points where energy concentrates in finite time. We see that when energy concentrates, a bubble forms, i.e. after suitable rescalings the maps converge to a half-harmonic map with values in a sphere. Thus, one also obtains natural bounds on the energy required to accumulate at such points and we may strengthen the existence results in the previous subsections by gluing together solutions of the flow. This is the second part of the current section and treats global existence of weak solutions in two ways: Directly by a variational and limiting argument as well as by a gluing procedure. The former may even provide an example of a solution with potentially increasing energy, a phenomena known in the case of the harmonic gradient flow.

2.3.1 Introduction

Among the most prominent partial differential equations is the harmonic map equation. Its relevance derives from the way they emerge naturally[2] as well as the fact that the associated PDE, especially in the critical realm (i.e. if the domain is two-dimensional), is related to the creation of powerful techniques in the context of regularity theory for PDEs such as Hélein's moving frames method (Hélein [45]), an adaptation by Rivière [70] of Uhlenbeck's Coulomb gauge construction (Uhlenbeck [98], Wehrheim [100]) and bubbling techniques (Sacks-Uhlenbeck [75]). The harmonic map equation has later on also inspired the introduction of fractional harmonic maps by Da Lio-Rivière [21], [22]. The corresponding regularity theory for fractional harmonic maps, based on contributions by a variety of authors including Da Lio-Rivière [22]; Schikorra [76]; Da Lio [16]; Da Lio-Schikorra [27], [28]; Mazowiecka-Schikorra [57], and bubbling analysis (Da Lio [17], Da Lio-Laurain-Rivière [18]) have led to generalisations of various ideas from the local world, such as Wente/Coifman-Lions-Meyer-Semmes-type estimates, gauge techniques, Pohozaev identities and many others to the fractional world.

Let us take a step back and recall the main definitions that we shall be using. For the moment, let us take (M, g), (N, γ) to be arbitrary closed Riemannian manifolds. We mention here that we shall usually assume N to be isometrically embedded in $\mathbb{R}^n$, a property guaranteed by Nash's embedding theorem for sufficiently big n. Using such manifolds, one is naturally led to define the following Dirichlet energy for maps $u : M \to N$:

$$E(u) := \frac{1}{2} \int_M g^{\alpha\beta}(x)\gamma_{ij}(u(x)) \frac{\partial u^i}{\partial x_\alpha}(x) \frac{\partial u^j}{\partial x_\beta}(x) dx, \tag{2.251}$$

where we assume for convenience that M, N are embedded submanifolds of the Euclidean space and use Einstein's summation convention. Naturality of this definition becomes apparent if one realises that given $M = \mathbb{T}^m$ the m-dimensional torus:

$$\mathbb{T}^m = \underbrace{S^1 \times \ldots \times S^1}_{m \text{ times}}$$

as a domain as well as the target space $N = \mathbb{R}^n$ with the corresponding natural, flat Riemannian metrics, the energy simplifies to the usual Dirichlet energy:

$$E(u) = \frac{1}{2} \int_{\mathbb{T}^m} |\nabla u|^2 dx \tag{2.252}$$

Critical points of (2.252) satisfy the corresponding Euler-Lagrange equation:

$$-\Delta u = 0,$$

which immediately, by means of elliptic regularity, yields $u \in C^\infty(\mathbb{T}^m)$. As a result, it is natural to wonder what one is able to say about the regularity of critical points of the general energy (2.251). In some sense (see also Evans [34]), the condition that u takes values in N may be interpreted as a Lagrange-multiplier, introducing non-linearity into the Euler-Lagrange equation. This immediately leads us to the first key definition:

[2] The action induced by the Dirichlet energy in Quantum Field Theory leads to the so-called sigma model, connecting harmonic maps with instatons. See also Chapter 2.4 in Jost [51].

Definition 2.3.1.1. *A map $u \in H^1(M;N)$ is called (weakly) harmonic, if and only if it is a critical point of the energy function E defined in* (2.251) *among competitors in $H^1(M;N)$.*

Here, we define $H^1(M;N)$ to consist of all functions $u \in H^1(M;\mathbb{R}^n)$ such that $u(x) \in N$ for almost all $x \in M$. Consequently, the choice of space is appropriate for both the energy functional E as well as the condition of u assuming values in N. It should be emphasised here that critical points do not only include (global/local) extrema, but also saddle points.

As usual in variational problems, computing the Euler-Lagrange equation is the first step towards regularity results and in the case of harmonic maps, one obtains the following sequence of equivalences:

$$\begin{aligned} u \text{ is (weakly) harmonic } &\Leftrightarrow (-\Delta)_M u \perp T_u N \text{ in } \mathcal{D}'(M) \\ &\Leftrightarrow (-\Delta)_M u = A(u)(\nabla u, \nabla u), \end{aligned} \tag{2.253}$$

where A denotes the second fundamental form of N and Δ_M is the Laplace-Beltrami operator of M. This already highlights the intimate connection of harmonic maps to curvature and geometry of the manifolds and indeed, existence of harmonic maps may be tied to conditions on the curvature, see for example Schoen-Yau [82]. A particularily striking feature of the PDE (2.253) is the quadratic structure inherent to the last equivalence above:

$$|A(u)(\nabla u, \nabla u)| \leq C|\nabla u|^2,$$

for some constant $C > 0$, which immediately singles out the case of M being two-dimensional as being critical. Unsurprisingly, if $u \in H^1$, then the RHS of (2.253) is in L^1. In this case, as Calderon-Zygmund theory does not apply to L^1-functions, one merely deduces $\nabla u \in L^{2,\infty}$, so we have regularity properties of the same homogeneity as at the start and therefore, standard bootstrapping techniques are unsuccessful in obtaining higher regularity. What is even worse, there is no general regularity theory for solutions of similar equations with smooth quadratic non-linearity. This may be seen from the following scalar PDE for $u \in H^1(B_1(0);\mathbb{R})$, $B_1(0)$ denoting here the unit ball in $\mathbb{R}^2$:

$$-\Delta u = |\nabla u|^2.$$

In this case, by using $v := e^u$, one may immediately construct counterexamples by using the observation that the above PDE is equivalent to $-\Delta v = 0$ and then considering suitable versions of the fundamental solution, see Rivière [73, p.32-34] for details.

Fortunately, in the case of the harmonic map equation (2.253), the non-linearity behaves much better due to its special geometric structure. Namely, by the combined efforts of various authors including (but not limited to) Béthuel [4], Grüter [41], Hélein [44], Morrey [62], Rivière [70], Shatah [83] and others over decades, we know nowadays that weakly harmonic maps are always regular. A common feature among proofs of regularity is the use of compensation results based on properties of 2D-jacobians as summarised by Wente's estimate, see Wente [101], and later extended to arbitrary div-curl quantities and determinants by Coifman-Lions-Meyer-Semmes [14] which we state here for the reader's convenience:

Proposition 2.3.1.1 ([101])**.** *Let $r > 0$ be arbitrary, $1 \leq p < 2$ and $B_r(0) \subset \mathbb{R}^2$ be the ball of radius r around 0. If $u \in W^{1,p}(B_r(0))$ and $a, b \in W^{1,2}(B_r(0))$ are such that:*

$$-\Delta u = \partial_x a \partial_y b - \partial_x b \partial_y a, \tag{2.254}$$

and u has vanishing trace on $\partial B_r(0)$, then u is actually continuous and the following estimate holds:

$$\|u\|_{L^\infty} + \|\nabla u\|_{L^{2,1}} + \|\nabla^2 u\|_{L^1} \lesssim \|\nabla a\|_{L^2} \|\nabla b\|_{L^2} \tag{2.255}$$

The reader is reminded that both $\partial_x a \partial_y b$ and $\partial_x b \partial_y a$ individually only lie in $L^1(B_r(0))$ (at best guaranteeing $\nabla u \in L^{2,\infty}$), but their linear combination $\partial_x a \partial_y b - \partial_x b \partial_y a$ still behaves better and thus leads to $\nabla u \in L^{2,1}$.

A similar result continues to hold for the RHS of (2.254), provided it consists of a product of a divergence-free and a rotation-free vector field for higher dimensional domains (see Coifman-Lions-Meyer-Semmes [14]). In fact, the underlying key result is a Hardy-regularity estimate for the RHS[3]. One may wonder how this estimate assists us in establishing regularity for equations such as (2.253), as at first glance the non-linearity in (2.253) appears to have a completely different character. The idea is that, despite the initial lack of 2D-jacobians in (2.253), by either employing Hélein's moving frames [45] or Rivière's change of gauge approach [70], one may reveal the hidden structure of jacobians in the harmonic map equation. This is best illustrated in the case $M = \mathbb{T}^2, N = S^{n-1}$, where the harmonic map equation reads:

$$-\Delta u = u|\nabla u|^2. \tag{2.256}$$

Also in this very special case, there is no 2D-jacobian or div-curl quantity involved in the formulation at this point. Nevertheless, Shatah ([83]) observed that the equation (2.256) is actually equivalent to the following conservation laws:

$$\forall i, j \in \{1, \dots n\} : \operatorname{div}(u_i \nabla u_j - u_j \nabla u_i) = 0. \tag{2.257}$$

Thus, we have (see Section 2.6 in Hélein [45] for regularity also in the more general case of N being arbitrary):

$$\begin{aligned}
-\Delta u_i &= \sum_j u_i \nabla u_j \cdot \nabla u_j \\
&= \sum_j (u_i \nabla u_j - u_j \nabla u_i) \cdot \nabla u_j + u_j \nabla u_i \cdot \nabla u_j \\
&= \sum_j (u_i \nabla u_j - u_j \nabla u_i) \cdot \nabla u_j,
\end{aligned} \tag{2.258}$$

where in the last line we used the observation that $\sum_j u_j \nabla u_j = 0$, as u takes values in S^{n-1} and thus this may be interpreted as a vector of scalar products between u, which belongs to the normal space of S^{n-1}, and partial derivatives of u, which lie in the tangent space of S^{n-1} by virtue of u assuming values in the $n-1$-sphere. Consequently, it is thus completely clear that $\sum_j u_j \nabla u_j = 0$ holds. Keeping Shatah's conservation laws (2.257) in mind, the structure of Wente's/Coifman-Lions-Meyer-Semmes' estimate is now uncovered by applying Hodge decompositions to find suitable vector fields for a formulation involving 2D-jacobians. By localising, splitting u into harmonic and zero-boundary value parts, one may establish a suitable Morrey decrease and this ultimately allows, by virtue of Adams embedding, to conclude that $\nabla u \in L^p_{loc}$ for some $p > 2$, the details may be found in Hélein [45]. The remaining part of the regularity proof is then a bootstrap argument.

A generalisation of the notion of harmonic maps was developed years later by Da Lio and Rivière in [21] following the spirit above. Let us denote by $H^s(S^1; \mathbb{R}^n)$ the space of functions, such that:

$$H^s(S^1; \mathbb{R}^n) := \Big\{ u : S^1 \to \mathbb{R}^n \text{ measurable } \Big| \sum_{n \in \mathbb{Z}} (1 + |n|)^{2s} |\hat{u}(n)|^2 < +\infty \Big\}, \tag{2.259}$$

[3]In contrast to L^1, the Hardy space $\mathcal{H}^1$ is well-behaved with respect to Caldéron-Zygmund operators and thus elliptic regularity results apply in this case, see Coifman-Lions-Meyer-Semmes [14] and Grafakos [40].

where $\hat{u}(n)$ denotes the n-th Fourier coefficient of u:

$$\hat{u}(n) := \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-inx}, \quad \forall n \in \mathbb{Z}.$$

The space $H^1(S^1; N)$ is then defined as the subspace of $H^1(S^1; N)$, again using Nash's embedding theorem, such that $u(x) \in N$ for almost every $x \in S^1$.

Taking N to be an embedded submanifold of $\mathbb{R}^n$ with the induced Riemannian structure, we are able to introduce the *s-Dirichlet energy*, $s > 0$ any positive real number, given by:

$$E_s(u) := \int_{S^1} |(-\Delta)^{s/2} u|^2 dx, \quad \forall u \in H^s(S^1, N),$$

and in the same step generalise the notion of (weakly) harmonic maps as well:

Definition 2.3.1.2. *A map $u \in H^s(S^1; N)$ is called (weakly) s-harmonic, if and only if it is a critical point of the energy function E_s for variations in the space $H^s(S^1; N)$.*

The fractional Laplacians $(-\Delta)^{s/2}$[4] may be defined by Fourier multipliers or using principal value integrals, we refer to Nezza-Palatucci-Valdinocci [63] for some exposition or the next subsection where both kinds of definitions are introduced. These maps are related to (branched) free-boundary minimal discs (Da Lio-Rivière [21], Da Lio-Pigati [20]) and singular limits of Ginzburg-Landau approximations (Millot-Sire [60]), so there are again interesting connections to geometry.

There is no particular reason to use S^1 instead of $\mathbb{R}$ as the domain and most results are available in both cases. If $s = 1/2$, which will interest us particularily, $E_{1/2}$ is conformally invariant (under the trace of Möbius transformations) and, as a result, the stereographic projection between $\mathbb{R}$ and S^1 enables us to switch between $\mathbb{R}$ and S^1 seemlessly. For the remainder of this paper, we shall restrict our attention to the case $s = 1/2$.

As a very simple first case, if $N = \mathbb{R}$, the Euler-Lagrange equation reads:

$$(-\Delta)^{1/2} u = 0,$$

which, similar to the harmonic case, immediately proves regularity[5]. In general for an arbitrary closed target manifold N, the Euler-Lagrange equation has a similar formulation as well as structure to (2.253), namely it becomes:

$$u \text{ is } \frac{1}{2}\text{-harmonic } \Leftrightarrow (-\Delta)^{1/2} u \perp T_u N \text{ in } \mathcal{D}'(S^1) \tag{2.260}$$

In Da Lio-Rivière [21], the equation above has been rewritten using Three-commutators to reveal compensation structures inherent to (2.260). In addition, in the case $N = S^{n-1}$, we may push the similarity with (2.256) further by noting the equivalence of (2.260) with:

$$(-\Delta)^{1/2} u = u |d_{1/2} u|^2, \tag{2.261}$$

[4] For instance, in terms of Fourier coefficients:

$$\widehat{(-\Delta)^s} u(n) = |n|^{2s} \hat{u}(n), \quad \forall n \in \mathbb{Z}.$$

[5] This is clear in the case S^1 as the domain, as it requires $\hat{u}(n) = 0, \forall n \neq 0$, hence u must be constant.

where we use the framework of fractional gradients as introduced in Mazowiecka-Schikorra [57]:

$$d_{1/2}u(x,y) = \frac{u(x)-u(y)}{|x-y|^{1/2}}, \quad |d_{1/2}u|^2(x) := \int_{S^1} |d_{1/2}u(x,y)|^2 \frac{dy}{|x-y|}.$$

In [103], similar formulations with a quadratic non-linearity were obtained. For completeness' sake, let us emphasise here that we use the natural distance $|x-y| = |e^{ix} - e^{iy}| = 2|\sin(\frac{x-y}{2})|$ on $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$. It should also be stated here that the fractional gradients provide a very natural framework for non-local operators and relate to the Gagliardo-Sobolev spaces and Bessel-Sobolev potential spaces (see Chapter 6.1.2 in [40]), see also Prats [64], Prats-Saksman [65].

It is interesting to observe that the key features of (2.253) are also present in (2.260): Both possess a quadratic RHS (see (2.261) as well as [103] for general N) and are critical equations not allowing for simple regularity by bootstrap techniques. Indeed, in [103], we even establish a "curvature-like" formulation similar to (2.261) in general, but the structure is most easily recognizable in the case $N = S^{n-1}$.

Regularity properties of half-harmonic maps have been studied extensively and we know that every critical point of the half-Dirichlet energy is smooth, provided N is smooth as well. The investigation was started by Da Lio-Rivière [21], [22] and later expanded by other authors, for example in Schikorra [76] and Mazowiecka-Schikorra [57], the proofs becoming increasingly similar to approach in the case of the harmonic map equation. In the literature, there are essentially two established approaches which are in a certain sense two sides of the same medal: Three-term commutators based on improved regularity of certain linear combinations of terms and non-local Wente/Coifman-Lions-Meyer-Semmes estimates. For illustration, Three commutator estimates are in some sense various incarnations of estimates of operators such as:

$$\mathcal{T} : L^2(\mathbb{R};\mathbb{R}^m) \times \dot{H}^{1/2}(\mathbb{R};\mathbb{R}^{m\times m}) \to \dot{H}^{-1/2}(\mathbb{R};\mathbb{R}^m),$$

defined by:

$$\mathcal{T}(v,Q) := (-\Delta)^{1/4}(Qv) - Q(-\Delta)^{1/4}v + (-\Delta)^{1/4}Q \cdot v$$

It is immediately clear that this operator quantifies the defect of Leibniz' rule for the 1/4-Laplacian. For instance, it is proven in Da Lio-Rivière [21] that:

$$\|\mathcal{T}(v,Q)\|_{\dot{H}^{-1/2}} \lesssim \|Q\|_{\dot{H}^{1/2}} \|v\|_{L^2}$$

One should keep in mind that a-priori, each summand in $\mathcal{T}(v,Q)$ individually does not belong to $\dot{H}^{-1/2}$.

To draw even more similarities to the harmonic map equation, we focus for now on the non-local Wente/Coifman-Lions-Meyer-Semmes estimate found in Mazowiecka-Schikorra [57]:

Proposition 2.3.1.2 ([57]). *Let $s \in (0,1)$ and $p \in (1,\infty)$. For $F \in L^p_{od}(\mathbb{R}\times\mathbb{R})$ and $g \in \dot{W}^{s,p'}(\mathbb{R})$, where p' denotes the Hölder dual of p, we assume that $\operatorname{div}_s F = 0$. Then $F \cdot d_s g$ lies in the Hardy* *space $\mathcal{H}^1(\mathbb{R})$*[6] *and we have the estimate:*

$$\|F \cdot d_s g\|_{\mathcal{H}^1(\mathbb{R})} \lesssim \|F\|_{L^p_{od}(\mathbb{R}\times\mathbb{R})} \|g\|_{\dot{W}^{s,p'}(\mathbb{R})}.$$

The general s-gradient is introduced in analogy to the 1/2-gradient and we say that $\operatorname{div}_s F = 0$, if for all $\varphi \in C_c^\infty(\mathbb{R})$:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} F(x,y) d_s\varphi(x,y) \frac{dydx}{|x-y|} = 0.$$

Lastly, we define:

$$F \cdot d_s g(x) := \int_{\mathbb{R}} F(x,y) d_s g(x,y) \frac{dy}{|x-y|}.$$

These notions and results also apply for S^1 as a domain and we refer to Mazowiecka-Schikorra [57] for details. In particular, we have used the following spaces for $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ in the formulation of Proposition 2.3.1.2:

$$F \in L^p_{od}(\mathbb{R}\times\mathbb{R}) \Leftrightarrow \|F\|_{L^p_{od}(\mathbb{R}\times\mathbb{R})} := \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |F(x,y)|^p \frac{dydx}{|x-y|} \right)^{1/p} < +\infty, \tag{2.262}$$

and:

$$g \in \dot{W}^{s,p'}(\mathbb{R}) \Leftrightarrow \|g\|_{\dot{W}^{s,p}(\mathbb{R})} := \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{g(x)-g(y)}{|x-y|^s} \right|^{p'} \frac{dydx}{|x-y|} \right)^{1/p'} < +\infty, \tag{2.263}$$

equipped both with the natural (semi-)norms associated with these spaces. We refer to the paper Mazowiecka-Schikorra [57] for further details.

Regularity theory for half-harmonic maps is now mostly analogous to the harmonic map case, up to taking care of tail estimates (i.e. estimates involving an infinite sum of localisations to larger and larger balls with a smaller and smaller weight, see p.15-19 in Mazowiecka-Schikorra [57] as an example) emerging due to the non-locality of the equations. In the current paper, we shall just give an outline in the case $N = S^{n-1}$ following the observations and work in Mazowiecka-Schikorra [57]: Shatah-like fractional conservation laws hold

$$\forall i,j \in \{1,\dots,n\} : \operatorname{div}_{1/2}\left(u_i d_{1/2} u_j - u_j d_{1/2} u_i\right) = 0,$$

leading to a reformulation of (2.261) as:

$$(-\Delta)^{1/2} u_i = \sum_j \left(u_i d_{1/2} u_j - u_j d_{1/2} u_i\right) \cdot d_{1/2} u_j + \sum_j u_j d_{1/2} u_i \cdot d_{1/2} u_j,$$

where, in contrast to the harmonic map equation, the last term is now not vanishing as the fractional gradient is not necessarily tangential to S^{n-1}. Nevertheless, using π the closest point projection to the target manifold S^{n-1}, or in general N, extended suitably and Taylor expansion, we see:

$$u(x) - u(y) = \pi(u(x)) - \pi(u(y)) = d\pi(u(x))\left(u(x)-u(y)\right) + R(u(x),u(y)),$$

[6] The Hardy space $\mathcal{H}^1(\mathbb{R})$ is the subspace of $L^1(\mathbb{R})$-functions such that:

$$M_\Phi(f)(x) := \sup_{t>0} |\Phi_t * f|(x) \in L^1(\mathbb{R}),$$

where Φ is a Schwartz function on $\mathbb{R}$ with $\int \Phi dx = 1$ and $\Phi_t(x) = 1/t \cdot \Phi(x/t)$. Various other, sometimes simpler characterisations (for example $\mathcal{H}^1(\mathbb{R}) \simeq F^0_{1,2}(\mathbb{R})$) exist and the relevance of Hardy spaces stem from their "good" behaviour with respect to Calderón-Zygmund operators, especially when compared to $L^1(\mathbb{R})$. We refer to [40], Chapter 6.4, for details on the theory of Hardy spaces and their relation to Triebel-Lizorkin spaces.

with $|R(u(x), u(y))| \lesssim |u(x) - u(y)|^2$. Hence, the defect of $d_{1/2}u$ being non-tangential is controlled. Noting that $d\pi(u(x))$, written as a differential to highlight the emphasis on its interpretation as linear approximation, is the tangent projection at $u(x)$, we may deduce, as is done in detail in [57, p.15-19], that the remainder may be considered to be of lower order and thus does not obstruct the argument. Then, localising and applying Wente/Coifman-Lions-Meyer-Semmes-type estimates, we arrive again at a Morrey decrease and, by Adams embedding (Adams [1]), at higher integrability and thus Hölder regularity of half-harmonic maps. Following for instance Da Lio-Pigati [20], one may bootstrap this information to deduce smoothness of solutions.

The considerations above hopefully convinced the reader of the close connection between 1/2-harmonic and harmonic maps and their Euler-Lagrange equations, at least in the structure of their proof and, in some sense, their relation to the geometry of the target manifold. This relation is something that we exploited in the papers [102], [103] to establish a theory for the half-harmonic gradient flow in analogy to Struwe [89] and obtain even a weak uniqueness statement in the small energy realm by extending the techniques introduced in Rivière [68]. Thus, we shall embark on a short survey of some results pertaining to the harmonic gradient flow before turning to the discussion of the half-harmonic gradient flow.

To start, let us present the main equation: We consider functions $u : [0, T[\times M \to N$, M, N being isometrically embedded submanifolds of some $\mathbb{R}^K$ and $T \in \mathbb{R} \cup \{+\infty\}$ and would like to solve the following equation:

$$\partial_t u - \Delta_M u = A(u)(\nabla u, \nabla u), \tag{2.264}$$

for a given initial datum $u(0, \cdot) = u_0(\cdot) \in H^1(M; N)$. The relevance of this PDE derives from approximations of stationary points (i.e. harmonic maps) and questions pertaining to the homotopy of maps like whether any given map in $H^1(M; N)$ is homotopic to a harmonic one (see also Eells-Sampson [33]). The latter question not being true in general, one may wonder what kind of convergence and regularity properties are to be expected in finite time as well as when $t \to +\infty$. We shall see, also in the case of the half-harmonic gradient flow, that bubbling may potentially occur and especially in the harmonic flow, a variety of different types of bubbling (finite time, infinite time, reverse bubbling) may manifest, see especially the extraordinarily well-suited approach to bubbling using the inner-outer gluing scheme (Davila-Del Pino-Wei [29]). We recall that, rather informally, a bubble is a harmonic map which is created by energy concentration in smaller and smaller neighbourhoods of a point after blow up using rescalings. Outside of the blow up point, the flow will behave nicely and be smooth, but at the bubbling point itself, energy accumulates and results in the formation of a so-called bubble. We refer to Struwe [89] for more details.

The harmonic gradient flow was first studied by Eells and Sampson [33] culminating in an existence result for minimizers of the Dirichlet energy homotopic to any given initial datum, provided that the target manifold has non-positive sectional curvature. However, the first general result that applies independent of any geometric properties (such as sectional curvature) was given by Struwe in [89] for two-dimensional domains and extended to arbitrary domains in Struwe [90]. We state the result here in a special case, as it may be found for example in Rivière [68]:

Theorem 2.3.1.1 ([89]). *Let $u_0 \in H^1(\mathbb{T}^2; S^{n-1})$. Then there exists a solution $u \in H^1(]0, +\infty[; L^2(\mathbb{T}^2))$ of the harmonic gradient flow:*

$$\partial_t u - \Delta u = u|\nabla u|^2 \quad \textit{in } \mathcal{D}'(]0, T[\times \mathbb{T}^2), \quad \forall T > 0, \tag{2.265}$$

together with the boundary conditions:

$$u(0,x) = u_0(x), \quad \text{for all } x \in \mathbb{T}^2, \tag{2.266}$$

and satisfying $E(u(t,\cdot)) \leq E(u_0)$ for all times $t \geq 0$. The solution u is regular on $]0,+\infty[\times\mathbb{T}^2$, except in a finite number of points (t_k, x_k), $k = 1, \ldots, K$, for some $K \in \mathbb{N}$. Additionally, u is unique in the class $\mathcal{E} \subset H^1_{loc}([0,+\infty[\times\mathbb{T}^2)$ which consists precisely of the $u \in H^1_{loc}([0,+\infty[\times\mathbb{T}^2)$, such that:

$$\exists m \in \mathbb{N}, \exists T_0 = 0 < T_1 < \ldots < T_m < \infty: \quad u \in L^2([T_i, T_{i+1}[; W^{2,2}(\mathbb{T}^2)), \forall i \leq m-1 \tag{2.267}$$

Finally, there exists a constant $C > 0$ independent of u_0, such that:

$$K \leq C \cdot E(u_0) \tag{2.268}$$

Theorem 2.3.1.1 immediately addresses existence, regularity and bubbling questions as well as providing a first uniqueness result (at least for strong solutions, i.e. solutions with sufficient regularity to make sense of the equation in the L^2-sense). Interesting features include the fact that bubbling may occur (Chang-Ding-Ye [12]), nevertheless no energy is lost in the process. So the energy as a function of time is continuous and monotone up to the creation of bubbles which account for the jump down in energy completely. This also guarantees that the amount of bubbles that form is finite, due to the quantisation of energy associared with bubbling and harmonic maps.

The proof of Theorem 2.3.1.1 relies on testing the equation (2.265) against itself or appropriate derivatives (testing here referring to integrating the PDE multiplied by the test function). As a first simple example, let us test (2.265) against $\partial_t u$, i.e. we integrate the equation against $\partial_t u$, in order to find:

$$\int_0^T \int_{\mathbb{T}^2} |\partial_t u|^2 dxdt + E(u(T)) - E(u_0) = \int_0^T \int_{\mathbb{T}^2} |\partial_t u|^2 dxdt + \int_0^T \int_{\mathbb{T}^2} \partial_t \left(\frac{1}{2} |\nabla u|^2 \right) dxdt = 0, \tag{2.269}$$

since $\partial_t u$ is tangential to S^{n-1}, while u is perpendicular to the tangent space, it spans the normal space to $S^{n-1} \subset \mathbb{R}^n$. This proves for instance that the energy is non-increasing along the harmonic gradient flow, a property expected as we follow the steepest desecent of the Dirichlet energy. Testing against $\varphi \partial_t u$ instead of $\partial_t u$, where φ is some cutoff-function leads in a similar way to control of the localised energy and hence of energy concentration, a crucial tool in establishing estimates in the proof in Struwe [89]. Testing against $-\Delta u$ yields control for the second order derivatives:

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^2} \langle \partial_t u, -\Delta u \rangle dxdt + \int_0^T \int_{\mathbb{T}^2} |-\Delta u|^2 dxdt \\
&= \int_0^T \int_{\mathbb{T}^2} \langle u |\nabla u|^2, -\Delta u \rangle dxdt \\
&\leq \frac{1}{2} \int_0^T \int_{\mathbb{T}^2} |\nabla u|^4 dxdt + \frac{1}{2} \int_0^T \int_{\mathbb{T}^2} |-\Delta u|^2 dxdt,
\end{aligned} \tag{2.270}$$

which after absorption may be rewritten as:

$$\begin{aligned}
\frac{1}{2} \int_0^T \int_{\mathbb{T}^2} |-\Delta u|^2 dxdt &\leq \int_0^T \int_{\mathbb{T}^2} |\nabla u|^4 dxdt + \int_0^T \frac{d}{dt} \left(\frac{1}{2} \int_{\mathbb{T}^2} |\nabla u|^2 \right) dt \\
&\leq \int_0^T \int_{\mathbb{T}^2} |\nabla u|^4 dxdt + E(u_0).
\end{aligned} \tag{2.271}$$

The L^4-norm may be estimated as in Struwe [89] using Ladyzhenskaya's estimate (Ladyzhenskaya [52]). As a result, allows for uniform estimates in terms of the initial energy and the energy concentration. The latter, i.e. concentration of energy, is controlled by the inequalities obtained by testing against $\varphi\partial_t u$ as we mentioned above, which allows us to reduce this dependence to an estimate based on the energy distribution at time 0. So ultimately, by an approximation process as well as the uniform estimates established just as in Struwe [89] along the lines sketched above (as well as testing against further suitably chosen functions), one obtains existence of solutions for arbitrary initial values in $H^1(\mathbb{T}^2; S^{n-1})$. The rigorous treatment of the estimates is referred to Struwe [89].

There are several natural questions to ask from here: Firstly, one may wonder if bubbling in finite time is actually possible. This question is answered by Chang, Ding, Ye [12] by using explicit solutions in the corotational setting. They construct subsolutions blowing up in finite time and prove that appropriate boundary conditions exist to transfer the blow up to a solution of the harmonic map flow.

Another question pertains to whether the energy decay is necessary as an assumption. Indeed it is, without this no uniqueness statement is possible. For instance, in Topping [94], other kinds of blow ups violating the monotone decay of energy are constructed by using so-called reverse bubbling and prove existence of "non-physical" solutions. Furthermore, various types of blow ups may be considered using the inner-outer gluing scheme, studied by Davila, del Pino and Wei [29] and various other authors. It should be stated that a kind of non-uniqueness phenomena can already be observed for the linear heat equation in $\mathbb{R}^n$ (for example in John [49]) where we need some decay at ∞ to ensure uniqueness, so these kinds of issues are not unexpected.

Lastly, it is natural to inquire whether uniqueness also holds among weak solutions/energy class solutions. One may show that this is true, at least for solutions with non-increasing energy and has been shown by Rivière [68] and Freire [35]. Rivière's argument works for the small energy regime and for the target manifold S^{n-1} and employs an ingenious absorption argument that allows to deduce that the solution is actually a solution in the strong sense. The key result in such an approach may be stated as follows:

Lemma 2.3.1.1 ([68])**.** *Let $u \in H^1(\mathbb{T}^2; S^{n-1})$ and $f \in L^2(\mathbb{T}^2; \mathbb{R}^n)$ and assume that u solves the following non-linear quadratic PDE:*

$$-\Delta u = u|\nabla u|^2 + f \tag{2.272}$$

Then $u \in H^2(\mathbb{T}^2; S^{n-1})$.

It should be noted that Lemma 2.3.1.1 gives the maximal amount of regularity one may expect from a general u solving such an equation. In the context of the harmonic gradient flow, it may be applied (for almost every fixed time $t > 0$) to:

$$-\Delta u(t) = u(t)|\nabla u(t)|^2 - \partial_t u(t).$$

This equation actually holds for almost every time, provided $u \in H^1(\mathbb{T}^2; S^{n-1})$ is a weak solution, as can be seen from standard arguments. Thus, we deduce that for all such times $u(t) \in H^2(\mathbb{T}^2; S^{n-1})$. However, this does not suffice to prove uniqueness, as we need an L^2-bound on the H^2-norms when integrated over time. This follows however immediately by using a version of Ladyzhenskaya's estimate (Ladyzhenskaya [52]):

$$\|\nabla u(t)\|_{L^4} \lesssim \|\nabla u(t)\|_{L^2} \|\nabla u(t)\|_{H^1} \lesssim E(u_0)\|u(t)\|_{H^2},$$

so we arrive at, by using elliptic regularity as well as $|u(t)| = 1$ almost everywhere:

$$\|u(t)\|_{H^2} \lesssim \|u(t)\|_{L^2} + \left\| u(t)|\nabla u(t)|^2 - \partial_t u(t) \right\|_{L^2} \lesssim 1 + E(u_0)\|u(t)\|_{H^2} + \|\partial_t u(t)\|_{L^2}, \tag{2.273}$$

so if $E(u_0)$ is sufficiently small, the H^2-norm on the RHS of (2.273) may be absorbed in the LHS of (2.273) to deduce:

$$\|u(t)\|_{H^2} \lesssim 1 + \|\partial_t u(t)\|_{L^2},$$

and by integrating over time intervals, the desired local integrability follows and allows for the application of Theorem 2.3.1.1 to conclude.

The proof of Lemma 2.3.1.1 may be found in Rivière [68] or [102] for the half-harmonic gradient flow where a natural analogue holds. Indeed, the techniques discussed so far naturally generalise to the framework of the half-harmonic gradient flow. The motivation to study this gradient flow stems once more from approximation of solutions to the half-harmonic map equation as well as the interest in expanding ideas from the local world to the fractional one. This is what the author has done in [102] for the case of the target manifold being a sphere and in [103] for the target manifold being any closed, isometrically embedded N. Since we shall restrict our considerations later on to the case $N = S^{n-1}$ anyways, let us focus on this special case, where the equation takes the form:

$$\partial_t u + (-\Delta)^{1/2} u = u|d_{1/2} u|^2$$

The most natural formulation of the half-harmonic gradient flow equation in N is phrased as:

$$\partial_t u + (-\Delta)^{1/2} u \perp T_u N \quad \text{in } \mathcal{D}'([0, T[\times S^1)$$

First, one may wonder what was known about the half-harmonic gradient flow before [102]. In Schikorra-Sire-Wang [77], the authors had already constructed solutions to the gradient flow of various non-local energies of similar type as the 1/2-Dirichlet energy by a discretisation procedure. Unfortunately, the result was limited to target spaces with inherent symmetry such as the sphere due to the limits taken. In a different paper Sire-Wei-Zheng [84], the bubbling as $t \to +\infty$ was investigated by adapting the inner-outer gluing scheme well-known in the case of the harmonic map flow to the non-local framework, establishing that bubbling is possible for $N = S^1$ at time $t = +\infty$. The authors of Sire-Wei-Zheng [84] further conjectured that bubbling may actually only occur asymptotically, so no finite time bubbling is possible due to dimensional peculiarities of $\mathbb{R}$ and S^1. The conjecture is, according to our best knowledge, still open and under investigation.

Let us now turn to the main result as found in [102] that answered some of the questions about the half-harmonic gradient flow:

Theorem 2.3.1.2 ([102])**.** *Let $u_0 \in H^{1/2}(S^1; S^{n-1})$ be any initial data. There exists $\varepsilon > 0$, such that if:*

$$E_{1/2}(u_0) \leq \varepsilon, \tag{2.274}$$

then there exists a unique energy class solution $u : \mathbb{R}_+ \times S^1 \to S^{n-1} \subset \mathbb{R}^n$ of the weak fractional harmonic gradient flow:

$$\partial_t u + (-\Delta)^{1/2} u = u|d_{1/2} u|^2, \tag{2.275}$$

satisfying $u(0, \cdot) = u_0$ in the sense $u(t, \cdot) \to u_0$ in L^2, as $t \to 0$. Moreover, the solution fulfills the energy decay estimate:

$$E_{1/2}(u(t)) \leq E_{1/2}(u(s)) \leq E_{1/2}(u_0), \quad \forall t \geq s \in [0, +\infty[.$$

In fact, $u \in C^\infty(]0,\infty[\times S^1)$ *and for an appropriate subsequence* $t_k \to \infty$*, the sequence* $u(t_k)$ *converges weakly in* $H^1(S^1)$ *to a point.*

If u_0 *has arbitrary energy, then we still get the existence of a solution to* (2.275) *on some time interval* $[0,T[$*, where* $T = T(u_0) > 0$ *depends on the initial datum. There exists a similar characterisation of* $T(u_0)$ *as in Struwe [89], namely saying that* $T(u_0)$ *is the first time such that:*

$$\limsup_{t\to T} \varepsilon(R;u,t) \geq \varepsilon_1, \quad \forall R \in]0,\frac{1}{2}[, \tag{2.276}$$

where:

$$\varepsilon(R;u,T) := \sup_{x\in S^1, t\in[0,T]} E_R(u;x,t) = \sup_{x\in S^1,t\in[0,T]} \frac{1}{2}\int_{B_R(x)} |(-\Delta)^{1/4}u(t)|^2 dx, \tag{2.277}$$

measures concentration of energy and $\varepsilon_1 > 0$ *is a quantity appearing in the proof of the result and is independent of* u_0, R, T.

The latter half of Theorem 2.3.1.2 is implicit in [102] due to the nature of the local existence proof. As already hinted at earlier, the entire result continues to hold true if we use an arbitrary N instead of S^{n-1}. As the proof of this result is similar in spirit, but different in various technical and computational aspects from Struwe [89], we do not go into the details of the proof, but we just present some of the basic steps as in the case of the harmonic gradient flow to indicate key similarities and differences: The main idea is, as for the harmonic gradient flow, that testing (2.275) against u and its derivatives again yields the desired control of energy concentration and higher regularity estimates. As a result, one may deduce existence and regularity similar to Struwe [89], once we have a sufficient local existence theory for smooth u_0. Once more, as a first example of the testing-technique used, testing (2.275) against $\partial_t u$ enables us to deduce that:

$$\begin{aligned}\int_0^T \int_{S^1} &|\partial_t u|^2 dxdt + E_{1/2}(u(T)) - E_{1/2}(u_0)\\ &= \int_0^T \int_{S^1} |\partial_t u|^2 dxdt + \int_0^T \int_{S^1} \partial_t \left(\frac{1}{2}|(-\Delta)^{1/4}u|^2\right) dxdt = 0,\end{aligned} \tag{2.278}$$

again using the fact that $\partial_t u \in T_u S^{n-1} \perp u$, leading to monotonicity for smooth solutions of (2.275). If we were to test against $\varphi\partial_t u$ instead of $\partial_t u$, local energy control and thus estimates on the concentration of energy in points of S^1 is obtainable (some technicalities due to the non-locality of $(-\Delta)^{1/4}$ require attention in this step). To have uniform estimates, we remember that in the case of the harmonic gradient flow we tested against $-\Delta u$, so it is natural to expect (and this of course holds) that a similar control can be derived by testing against $(-\Delta)^{1/2}u$ for the half-harmonic gradient flow:

$$\begin{aligned}&\int_0^T \int_{S^1} \langle \partial_t u, (-\Delta)^{1/2}u\rangle dxdt + \int_0^T \int_{S^1} |(-\Delta)^{1/2}u|^2 dxdt\\ &= \int_0^T \int_{S^1} \langle u|d_{1/2}u|^2, (-\Delta)^{1/2}u\rangle dxdt\\ &\leq \frac{1}{2}\int_0^T \int_{S^1} |d_{1/2}u|^4 dxdt + \frac{1}{2}\int_0^T \int_{S^1} |(-\Delta)^{1/2}u|^2 dxdt,\end{aligned} \tag{2.279}$$

and from here, one argues by employing a fractional version of Ladyzhenskaya's inequality as well as absorbing suitable terms on the RHS into the LHS of the inequality above. The uniform estimates in terms of initial energy and the concentration of energy are obtained by testing with further maps derived from u in much the same way, see [102], [103]. Thus, once we are able to establish existence of solutions to the half-harmonic gradient flow for small times and regular boundary data, we may deduce the general existence and regularity result by approximation completely analogous to Struwe [89] building on the uniform estimates that are obtained in this way.

This leads us to consider the half-harmonic gradient flow for smooth boundary data u_0. Local existence for the half-harmonic gradient flow in this case can be obtained by studying the following operator:

$$H : W^{1,p}([0,T] \times S^1) \to L^p([0,T] \times S^1), \quad H(u) := u_t + (-\Delta)^{1/2}u - u|d_{1/2}u|^2, \tag{2.280}$$

for $p > 2$. Notice that for solutions u of the half-harmonic gradient flow, we of course have $H(u) = 0$. Applying the inverse function theorem to H, therefore investigating invertibility of the linearisation $DH(v)$ for some suitable extension v of u_0, allows us to deduce that solutions with arbitrary prescribed smooth boundary data do exist (at least for times $t \in [0, \delta_0]$ with $\delta_0 > 0$ sufficiently small and depending on u_0). Later on in the current paper, we shall provide an alternative proof of the local existence result for the half-harmonic gradient flow in [103, Prop. 3.2] based on a fixed-point argument involving Banach's fixed point theorem that again allows us to show existence for a short period of time by different means (a previously unpublished, yet unsurprising approach).

Lastly, we shall turn to the uniqueness of solutions to the half-harmonic gradient flow in the energy class: The argument in Rivière [68] generalises thanks to the following Lemma which is a fractional version of Lemma 2.3.1.1:

Lemma 2.3.1.2 ([102]). *Let $f \in L^2(S^1; \mathbb{R}^n)$ and assume that $u \in H^{1/2}(S^1; S^{n-1})$ solves the following equation:*

$$(-\Delta)^{1/2}u = u|d_{1/2}u|^2 + f. \tag{2.281}$$

Then, we have the following improved regularity property:

$$u \in H^1(S^1; S^{n-1}).$$

The proof of uniqueness for weak solutions with sufficiently small energy now follows as before in the local case by absorption and employing a fractional Ladyzhenskaya-type estimate that may be proven by means of Fourier coefficients and Sobolev embeddings. Namely, one obtains as in the outline of the proof of Lemma 2.3.1.1:

$$(-\Delta)^{1/2}u(t) = u(t)|d_{1/2}u(t)|^2 - \partial_t u(t),$$

for almost every fixed time $t > 0$, so that by applying Lemma 2.3.1.2:

$$u(t) \in H^1(S^1), \tag{2.282}$$

for almost every t. As a result, we have by using the definition of the H^1-norm:

$$\begin{aligned}
\|u(t)\|_{H^1}^2 &\lesssim \|u(t)\|_{L^2}^2 + \|(-\Delta)^{1/2}u(t)\|_{L^2}^2 \\
&\lesssim 1 + \||d_{1/2}u(t)|^2\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2 \\
&\lesssim 1 + \|(-\Delta)^{1/4}u(t)\|_{L^2}\| \Big(\|(-\Delta)^{1/4}u(t)\|_{L^2} + \|(-\Delta)^{1/2}u(t)\|_{L^2}\Big) + \|\partial_t u(t)\|_{L^2}^2
\end{aligned}$$

$$\lesssim 1 + \|(-\Delta)^{1/4}u(t)\|_{L^2}^2 + \|(-\Delta)^{1/4}u(t)\|_{L^2}\|u(t)\|_{H^1}^2 + \|\partial_t u(t)\|_{L^2}$$
$$\lesssim 1 + E_{1/2}(u_0)^2 + E_{1/2}(u_0)\|u(t)\|_{L^2} + \|\partial_t u(t)\|_{L^2}. \tag{2.283}$$

Details on the individual estimates may be found in [102], in particular the fractional Ladyzhenskaya-type estimate in line 3 of (2.283). From here on, the considerations become the same as in the proof of Lemma 2.3.1.1, by absorbing the H^1-norm on the RHS of the equation (2.283) into the LHS, provided $E_{1/2}(u_0)$ is sufficiently small. Integrating both sides of (2.283) in time yields the desired regularity properties to apply the uniqueness result in Theorem 2.3.1.2.

For completeness' sake, we would like to mention that in a later work Hyder-Segatti-Sire-Wang [47], the authors consider an alternative version of the half-harmonic heat flow not based on the L^2-gradient flow, but rather on ideas similar to the connection noticed by Caffarelli and Silvestre [11] between fractional Laplacians and Dirichlet-to-Neumann and reading:

$$(\partial_t - \Delta)^{1/2}u \perp T_u N,$$

which allows for a monotonicity formula. Such a formula is unknown in the case of the flow (2.275). It is obvious that both flows allow for the same stationary solutions, namely half-harmonic maps, but the approaches are independent. Additionally, the authors of [47] establish regularity outside of a lower dimensional set, but not answering questions relating to the uniqueness of such solutions.

In our current work, we will use the characterisation (2.276) above in order to state and prove results regarding bubbling. In fact, we shall show that at most finitely many points exist where bubbles form. By observing that the bubbles are non-constant half-harmonic maps, we may further show that there must localise a quantum of the 1/2-Dirichlet energy which gets removed from the flow. To summarise, we shall obtain the following new result:

Theorem 2.3.1.3. *Let u be a solution as in Theorem 2.3.1.2 and let $x_0 \in S^1$ be a point, such that:*

$$\limsup_{t\to T} \int_{B_R(x_0)} |(-\Delta)^{1/4}u|^2 dx \geq \varepsilon_1, \quad \forall R > 0, \tag{2.284}$$

where $\varepsilon_1 > 0$ is as in [103, Lemma 4.10]. Then there exists a half-harmonic map $v : \mathbb{R} \to S^{n-1}$, such that:

$$u_m \to v \quad \text{weakly in } H^1(\mathbb{R}) \text{ and strongly in } H^{1/2}(\mathbb{R}), \tag{2.285}$$

where u_m is a suitable rescaling and translation of u.

Therefore, by using such considerations, we may conclude that the solution constructed in [103] may be extended by L^2-continuity and will be smooth except for finitely many times $0 < T_1 < \ldots < T_l < +\infty$, which may be characterised by concentration identities as in Theorem 2.3.1.2:

Theorem 2.3.1.4. *Let $u_0 \in H^{1/2}(S^1; S^{n-1})$, then there exists a weak solution of (2.275) with non-increasing 1/2-Dirichlet energy:*

$$u : [0, +\infty[\times S^1 \to N,$$

with $u \in L^\infty([0, +\infty[; H^{1/2}(S^1; S^{n-1})) \cap H^1([0, +\infty[; L^2(S^1; S^{n-1}))$ such that, except for finitely many times $0 < T_1 < \ldots < T_l < T_{l+1} := +\infty$, the function u is smooth:

$$u \in C^\infty(]T_k, T_{k+1}[; S^{n-1}), \quad \forall k = 1, \ldots l.$$

Moreover, we may bound the number $l = l(u_0)$ as follows:

$$l(u_0) \leq \frac{E(u_0)}{\varepsilon_0},$$

where $\varepsilon_0 > 0$ is the minimum amount of $1/2$-energy a non-constant, half-harmonic map with values in S^{n-1} must possess.

Of course, both of the results for S^{n-1} carry over to the general manifold case N without major changes.

In a future paper, we will investigate the smoothness at the critical times outside of bubbling points. This issue is quite delicate due to the non-local nature of the equation at hand and thus requires more care than in Struwe [89] where suitable localisations are immediately available. Furthermore, the question whether bubbling may even occur remains open and under investigation as well, see also p.3 in Sire-Wei-Zheng [84] for a conjecture in this direction. Additionally, the global existence result in Theorem 2.3.1.4 is still new in the case of arbitrary target manifolds, as previous papers such as Schikorra-Sire-Wang [77] have either only dealt with special target manifolds or with solutions for a possibly short amount of time as in the author's previous work. Finally, the author is aware of current research Struwe [93] concerning the half-harmonic gradient flow based on techniques involving harmonic extensions and, once more, arguments similar to Struwe [89]. Thus, alternative approaches to the problem at hand may be possible.

The structure of the paper is as follows: In Section 2.3.2, we will quickly recall the necessary notions used throughout the paper, in particular fractional gradients and divergences, Triebel-Lizorkin spaces on S^1 and the fractional Laplacian. In the following Section 2.3.3, we provide proofs and statements for properties of the half-harmonic gradient flow. In particular, in Section 2.3.3.1, we provide an alternative proof of local existence of solutions based on a fixed point argument. Afterwards, we investigate bubbling in finite time in Section 2.3.3.3, studying the concentration of energy, resulting in a proof of Theorem 2.3.1.3. Lastly, Section 2.3.3.4 deals with extensions and other ideas to find global solutions to our main PDE, proving Theorem 2.3.1.4.

2.3.2 Preliminaries

In this brief preliminary section, we shall introduce some of the most important notions used throughout. In particular, we discuss Triebel-Lizorkin spaces on S^1, provide a short summary of fractional gradient and fractional divergences based on Mazowiecka-Schikorra [57] and finally recall some of the main results associated with the fractional heat flow. Most of the results are discussed in more detail in [102] and the references provided therein.

2.3.2.1 Triebel-Lizorkin Spaces on the Unit Circle and Fractional Laplacians

Firstly, we shall discuss Triebel-Lizorkin spaces on the unit circle $S^1 \subset \mathbb{R}^2$ and recall some of the most important properties of and formulas for the fractional Laplacian. Much of the current presentation is due to Prats [64], Prats-Saksman [65] and Schmeisser-Triebel [80]. Throughout, we shall use the distance:

$$|x-y| = 2|\sin\left(\frac{x-y}{2}\right),$$

for all $x, y \in S^1 \simeq \mathbb{R} \mod 2\pi\mathbb{Z}$.

We define for any $f : S^1 \to \mathbb{R}$ the following quantity based on the fractional gradients $d_s f(x,y) = \frac{f(x)-f(y)}{|x-y|^s}$:

$$\mathcal{D}_{s,q}(f)(x) := \left(\int_{S^1} |d_s f(x,y)|^q \frac{dy}{|x-y|} \right)^{1/q}, \tag{2.286}$$

for all $1 \leq q < \infty$ and $0 < s < 1$. We refer to the next subsection for some details on the fractional gradient $d_s f$. Then:

$$\|f\|_{\dot{W}^{s,(p,q)}(S^1)} := \|\mathcal{D}_{s,q}(f)(x)\|_{L^p(S^1)}, \tag{2.287}$$

for every $1 \leq p \leq \infty$. If $p = q$, these spaces correspond to the usual homogeneous Gagliardo-Sobolev spaces $\dot{W}^{s,p}(S^1)$, see Prats [64], Prats-Saksman [65].

Furthermore, we shall denote as per usual by $\mathcal{D}'(S^1)$ the set of all distributions on S^1 and occasionally use $\mathcal{D}(S^1)$ as an alternative notation for the space $C^\infty(S^1)$. Finally, $\hat{f}(k)$ shall always be the k-th Fourier coefficient of f, for all $f \in \mathcal{D}'(S^1)$. It is formally defined by:

$$\hat{f}(k) := \frac{1}{2\pi} \langle f, e^{-ikx} \rangle = \frac{1}{2\pi} f\left(e^{-ikx}\right), \quad \forall k \in \mathbb{Z} \tag{2.288}$$

In Schmeisser-Triebel [80], it is shown that one may define Triebel-Lizorkin spaces for S^1, denoted by $F^s_{p,q}(S^1)$, completely analogous to the usual space $F^s_{p,q}(\mathbb{R}^n)$ for any parameters $s \in \mathbb{R}$ and $p, q \in [1, \infty[$:

$$F^s_{p,q}(S^1) := \left\{ f \in \mathcal{D}'(S^1) \mid \|f\|_{F^s_{p,q}} < +\infty \right\} \tag{2.289}$$

The norm is defined by:

$$\|f\|_{F^s_{p,q}} := \left\| \left\| \left(\sum_{k \in \mathbb{Z}} 2^{js} \varphi_j(k) \hat{f}(k) e^{ikx} \right)_{j \in \mathbb{N}} \right\|_{l^q} \right\|_{L^p(S^1)}, \tag{2.290}$$

for a suitable partition of unity $(\varphi_j)_{j \in \mathbb{N}}$ consisting of smooth, compactly supported functions on $\mathbb{R}$ with the properties:

$$\operatorname{supp} \varphi_0 \subset B_2(0), \quad \operatorname{supp} \varphi_j \subset \{x \in \mathbb{R} \mid 2^{j-1} \leq |x| \leq 2^{j+1}\}, \forall j \geq 1$$

as well as:

$$\forall k \in \mathbb{N} : \sup_{j \in \mathbb{N}} 2^{jk} \|D^k \varphi_j\|_{L^\infty} \lesssim 1$$

One may develop, as for example seen in [80, Chapter 3], a complete theory of Triebel-Lizorkin spaces on S^1 and more generally on the n-torus $\mathbb{T}^n$ by following the techniques of these spaces on $\mathbb{R}^n$. We refer for further details to Schmeisser-Triebel [80], but for now it suffices to be aware that all tools and results for Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ are also available for $F^s_{p,q}(\mathbb{T}^n)$.

It turns out that the fractional gradients are exceptionally useful in studying non-local problems. As an example, the following result found in Prats-Saksman [65] is key to many of our arguments, allowing us to restrict our considerations to fractional gradients rather than fractional Laplacians:

Theorem 2.3.2.1 (Theorem 1.2, [65])**.** *Let $s \in (0,1)$, $p, q \in]1, \infty[$ and $f \in L^p(\mathbb{R})$. Then:*

(i) We know $\dot{W}^{s,(p,q)}(\mathbb{R}^n) \subset \dot{F}^s_{p,q}(\mathbb{R}^n)$ *together with:*

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{W}^{s,(p,q)}(\mathbb{R}^n)}. \tag{2.291}$$

(ii) If $p > \frac{nq}{n+sq}$, *then we also have the converse inclusion together with:*

$$\|f\|_{\dot{W}^{s,(p,q)}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \tag{2.292}$$

The constants depend on s, p, q, n.

By using the properties in Schmeisser-Triebel [80] for periodic functions and employing Theorem 2.3.2.1, we can arrive at the following equivalence with Triebel-Lizorkin spaces for all $1 < q < \infty$ and $1 < p < \infty$:

$$\dot{W}^{s,(p,q)}(S^1) = \dot{F}^s_{p,q}(S^1), \tag{2.293}$$

with equivalence of the corresponding seminorms, provided $p > \frac{q}{1+sq}$. As a simple, but important special case, let us observe that if $s = 1/2$ and $q = 2$, then $p > 1$ is the requirement in Theorem 2.3.2.1 for the equality of $\dot{F}^{1/2}_{p,2}$ and $\dot{W}^{1/2,(p,2)}$ to hold. Some more details and a proof of one direction of Theorem 2.3.2.1 can be found in the appendix of [102].

Finally, we would like to briefly address the fractional Laplacian. The simplest definition is based on the Fourier multiplier properties of the Laplacian itself, leading ultimately to the following definition for the fractional s-Laplacian on Fourier series on S^1:

$$\widehat{(-\Delta)^s} f(k) = |k|^{2s} \hat{f}(k), \tag{2.294}$$

for every $k \in \mathbb{Z}$ and all $0 < s < 1$. There is an alternative formulation as a Cauchy principal value, which actually leads to the same operator and is often useful:

$$(-\Delta)^{1/2} f(x) = C \cdot P.V. \int_{S^1} \frac{f(x) - f(y)}{|x-y|^2} dy, \tag{2.295}$$

where $C > 0$ denotes a suitable constant. Similar formulas with less explicit kernels exist for $0 < s < 1$, these are omitted for accessibility of the presentation. Additionally, it is of course possible to define the fractional Laplacians also on $\mathbb{R}^n$, leading again to two different characterisations (as a Fourier multiplier and Cauchy principal value) with the same type of formulas. The details are thus omitted.

An essential property of function spaces is their behaviour under Fourier multipliers, for example extending results such as Mikhlin's multiplier theorem for L^p-spaces. As the fractional Laplacian is obviously a Fourier multiplier operator, one expects characterisations of the spaces $F^s_{p,q}(S^1)$ based on these operators, compare with the Bessel potentials. Indeed, one easily sees (Schmeisser-Triebel [80]):

$$(-\Delta)^s : \dot{F}^{t+2s}_{p,q} \to \dot{F}^t_{p,q},$$

for all $p, q \in (1, \infty)$ and $t, t+2s \in \mathbb{R}$. This should not be surprising and follows along the same lines as in the case of Triebel-Lizorkin spaces on $\mathbb{R}^n$. Observe the use of $\dot{F}^s_{p,q}(S^1)$ rather than $F^s_{p,q}(S^1)$ indicating the use of homogeneous Triebel-Lizorkin spaces, which are again defined as usual, see also Schmeisser-Triebel [80].

2.3.2.2 Fractional Gradients and Divergences

Next, we would like to discuss in some depth the notion of fractional gradient and its derivatives, like the fractional divergence and certain weighted L^p-spaces. The presentation greatly draws from Mazowiecka-Schikorra [57] and is a shortened version of [102]:

One may introduce $\mathcal{M}_{od}(\mathbb{R}\times\mathbb{R})$ as the set of all measurable functions $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ with respect to the weighted Lebesgue measure $\frac{dxdy}{|x-y|}$. In complete analogy, we do the same for S^1 instead of $\mathbb{R}$ as the domain, denoting this space by $\mathcal{M}_{od}$ if both $\mathbb{R}$ or S^1 are possible as domains. Naturally, the associated L^p-spaces, denoted L^p_{od} are of interest and the defining (semi-)norms are given by:

$$\|F\|_{L^p_{od}} := \left(\int\int |F(x,y)|^p \frac{dydx}{|x-y|}\right)^{1/p}, \tag{2.296}$$

for $1\leq p<\infty$. The space $L^\infty_{od}(S^1\times S^1)$ and $L^\infty_{od}(\mathbb{R}\times\mathbb{R})$ as the sets of essentially bounded functions with the essential supremum as the (semi-)norm. Later on, the following quantity, defined in terms of $F,G\in\mathcal{M}_{od}$, will be useful:

$$F\cdot G(x) := \int F(x,y)G(x,y)\frac{dy}{|x-y|} \tag{2.297}$$

In the special case $F=G$, this becomes:

$$F\cdot F(x) = |F|^2(x), \quad |F|(x) := \sqrt{F\cdot F(x)} \tag{2.298}$$

Of course, this shows:

$$\|F\|^2_{L^2_{od}} = \int |F|^2(x)dx$$

Let us finally turn to the definition of fractional gradients: For a measurable function $f:\mathbb{R}\to\mathbb{R}$ or $f:S^1\to\mathbb{R}$, we define for $0\leq s<1$ the fractional s-gradient by:

$$d_s f(x,y) = \frac{f(x)-f(y)}{|x-y|^s}\in\mathcal{M}_{od},$$

and the corresponding s-divergence by means of duality, i.e. for $F\in\mathcal{M}_{od}$:

$$\langle \operatorname{div}_s F,\varphi\rangle = \int\int F(x,y)d_s\varphi(x,y)\frac{dydx}{|x-y|}, \quad \forall\varphi\in C^\infty_c(\mathbb{R}) \text{ or } C^\infty(S^1) \tag{2.299}$$

It is obvious that:

$$d_s f(y,x) = -d_s f(x,y)$$

Also, a version of Leibniz' rule holds true:

$$d_s\left(fg\right)(x,y) = d_s f(x,y)g(x) + f(y)d_s g(x,y)$$

Naturally, $\operatorname{div}_s F$ is only well-defined in a distributional sense.

Using the notions introduced for functions $F\in L^p_{od}$ and as we have already stated in the subsection before, we do now have:

$$\||d_s f|(\cdot)\|_{L^p(S^1)} = \|f\|_{\dot{W}^{s,(p,2)}(S^1)}, \tag{2.300}$$

We refer to Theorem 2.3.2.1 for the significance of this. Finally, the fractional Laplacian also has a place in the setting of fractional gradients and divergences, behaving much as expected from $\Delta = \operatorname{div} \circ \nabla$:

$$(-\Delta)^s f = C_s \operatorname{div}_s d_s f, \tag{2.301}$$

for some constant $C_s > 0$ depending on s. Equation (2.301) has to be read as follows for g a smooth test function:

$$C_s \int d_s f \cdot d_s g(x) dx = \int (-\Delta)^s f \cdot g dx = \int (-\Delta)^{s/2} f \cdot (-\Delta)^{s/2} g dx$$

A key result to establish, for instance, regularity of fractional harmonic maps or the uniqueness of weak solutions to the half-harmonic gradient flow with small initial energy is the following Wente-type estimate:

Lemma 2.3.2.1 (Theorem 2.1, [57])**.** *Let $s \in (0,1)$ and $p \in (1,\infty)$. For $F \in L^p_{od}(\mathbb{R} \times \mathbb{R})$ and $g \in \dot{W}^{s,p'}(\mathbb{R})$, where p' denotes the Hölder dual of p, we assume that $\operatorname{div}_s F = 0$. Then $F \cdot d_s g$ lies in the Hardy space $\mathcal{H}^1(\mathbb{R})$ and we have the estimate:*

$$\|F \cdot d_s g\|_{\mathcal{H}^1(\mathbb{R})} \lesssim \|F\|_{L^p_{od}(\mathbb{R}\times\mathbb{R})} \cdot \|g\|_{\dot{W}^{s,p'}(\mathbb{R})}. \tag{2.302}$$

In the case where $s = 1/2$ and $p = p' = 2$, we may immediately deduce $F \cdot d_s g \in H^{-1/2}(\mathbb{R})$ following the Sobolev embedding $\dot{H}^{1/2}(\mathbb{R}) \hookrightarrow BMO(\mathbb{R})$ with analogous estimates. Naturally, the result also remains valid in the case of the domain being S^1:

Lemma 2.3.2.2. *For $F \in L^2_{od}(S^1 \times S^1)$ and $g \in \dot{H}^{1/2}(S^1)$, we assume that $\operatorname{div}_{1/2} F = 0$. Then $F \cdot d_{1/2} g$ lies in the space $H^{-1/2}(S^1)$ and we have the estimate:*

$$\|F \cdot d_{1/2} g\|_{H^{-1/2}(S^1)} \lesssim \|F\|_{L^2_{od}(S^1\times S^1)} \cdot \|g\|_{\dot{H}^{1/2}(S^1)}. \tag{2.303}$$

We refer to [102] for some details of the proof.

2.3.2.3 Fractional Heat Flow

The presentation of this subsection follows Garofalo [37] and we refer to it and the references mentioned therein for details.

A natural PDE to consider is the fractional heat flow equation derived from the operator $\partial_t + (-\Delta)^s$. One may be motivated by the ubiquity of the heat equation in general mathematics or by the interest in the fractional harmonic gradient flow, whose linearisation is closely connected to this equation. Of course, semi-group theory provides a suitable theoretical framework to discuss questions of existence, regularity and uniqueness of such solutions. For our purpose, it will be sufficient to introduce the heat kernel (at least in the special case $s = 1/2$) and discuss some of its basic properties.

A natural approach to solve the equation for the fundamental solution of the homogeneous equation:

$$\partial_t u + (-\Delta)^s u = 0, \quad u(0,x) = \delta_0(x),$$

on $[0,\infty[\times\mathbb{R}$ would be to apply a spatial Fourier transform, leading to the following equation for the Fourier transform:

$$\partial_t \hat{u}(t,\xi) + |\xi|^{2s} \hat{u}(t,\xi) = 0$$

Solving this ODE for fixed ξ leads us to:

$$\hat{u}(t,\xi) = e^{-|\xi|^{2s}t} \cdot \hat{\delta_0}(\xi) = e^{-|\xi|^{2s}t} \tag{2.304}$$

The fundamental solution is thus the Fourier inverse of this expression and in the case $s = 1/2$, the following explicit formula exists:

$$u(t,x) = C \cdot \frac{t}{t^2 + x^2}, \tag{2.305}$$

C being a suitable constant. A fundamental solution on S^1 may be constructed by periodic extension, so we discover an analogous kind of heat kernel. It should be noted that outside of $(t,x) = (0,0)$, the heat kernel is smooth, thus implying the smoothing property already well-known from the standard heat flow.

The fractional heat semigroup may be used for various things, such as a formula for the fractional Laplacians by subordination, see Garofalo [37]. We are more interested in the immediate regularity properties. By using Duhamel's principle, one may indeed solve the problem:

$$\partial_t v(t,x) + (-\Delta)^s v(t,x) = f(t,x), \qquad \forall (t,x) \in]0,\infty[\times S^1 \tag{2.306}$$

$$v(0,x) = g(x), \qquad \forall x \in S^1 \tag{2.307}$$

Regularity may be obtained either by semigroup theory or, if $s = 1/2$, using the ellipticity of $\partial_t + (-\Delta)^{1/2}$ which is contained in:

$$\left(\partial_t - (-\Delta)^{1/2}\right)\left(\partial_t + (-\Delta)^{1/2}\right) = \left(\partial_t + (-\Delta)^{1/2}\right)\left(\partial_t - (-\Delta)^{1/2}\right) = \partial_t^2 + \partial_x^2$$

Therefore, if $s = 1/2$, an L^p-theory with estimates as expected does exist, see also Hieber-Prüss [46] and the discussion in [102] on the regularity of local solutions.

2.3.3 Half-Harmonic Gradient Flow

In this section, we go into some depth regarding some specific aspects of the proof of Theorem 2.3.1.2. To be precise, we shall supply the reader with an alternative proof to the local existence result for smooth u_0 given in [102], [103] for some brief interval of time based on Banach's fixed point theorem, present a detailed account of the proof of Lemma 2.3.1.2, since this argument is beautiful and provides potential insight into the way Hodge decomposition may be substituted in the non-local case. The way to conclude from this uniqueness for weak solutions follows by using similar arguments as in the introduction and we refer to [102] for the details. Following this, we shall then discuss bubbling processes based on concentration estimates in localised Gagliardo seminorms and rescaled versions of the solution. The approach is quite similar to Struwe [89], however the non-locality renders quite a few steps more difficult and requires us to refine some estimates we have previously established in [102, Lemma 3.11-3.12]. Only after having this estimate available are we in a position to address boundedness of suitable rescalings of the solution to the half-harmonic gradient flow. To conclude this section, we discuss global existence of solutions by using two distinct approaches, one producing a solution based on Theorem 2.3.1.2 with non-increasing energy, while the other proves existence based on variational arguments, but does not immediately exhibit monotonicity of energy.

2.3.3.1 A Local Existence and Regularity Result

In this first subsection, our goal is to prove the following:

Proposition 2.3.3.1. *Let $u_0 \in C^\infty(S^1; S^{n-1})$. Then there exists a solution $u : [0,T] \times S^1 \to S^{n-1}$ with $u(0,\cdot) = u_0$ of the equation (2.275) which is smooth on some time interval $[0,T]$, where $T = T(u_0)$.*

This result was already proven in [102] by introducing an appropriate solution operator H (see (2.280)) and applying the inverse function theorem. The key observations were that firstly, the linearisation of H at any smooth function is indeed Fredholm with index 0 and thus injectivity and invertibility become equivalent. Secondly, an argument based on maximum principles shows that smooth elements in the kernel of the linearisation are always trivial. Bootstrapping to deduce sufficient regularity then bridges the gap between the two observations and amounts to the existence result stated as Proposition 2.3.3.1.

Here, we will take a slightly different approach and substitute the use of Fredholm theory by employing a standard argument based on Banach's fixed point theorem. For the remainder of this section, we shall denote by:

$$W^{1,p}_{u_0}([0,T] \times S^1) := \big\{ u \in W^{1,p}([0,T] \times S^1) \in \big| u(0,\cdot) = u_0 \big\},$$

where $u_0 \in C^\infty(S^1; S^{n-1})$ is a given boundary value. Indeed, we shall prove:

Lemma 2.3.3.1. *Let $u_0 \in C^\infty(S^1; S^{n-1})$ and $p > 4$. Then the map:*

$$S : W^{1,p}_{u_0}([0,T] \times S^1) \to W^{1,p}_{u_0}([0,T] \times S^1), \tag{2.308}$$

mapping $u \in W^{1,p}_{u_0}([0,T] \times S^1)$ to the unique solution $S(u) \in W^{1,p}_{u_0}([0,T] \times S^1)$ of the following system:

$$\partial_t S(u) + (-\Delta)^{1/2} S(u) = u|d_{1/2}u|^2, \qquad \forall (t,x) \in [0,T] \times S^1 \tag{2.309}$$

$$S(u)(0,x) = u_0(x), \qquad \forall x \in S^1 \tag{2.310}$$

Given $R > 0$ sufficiently big and $T > 0$ sufficiently small, then S is a contraction of the closed ball of radius R around u_0, denoted $B_R(u_0)$, onto itself and thus possesses a fixed point.

We remark at this point that by then employing the same kind of bootstrap procedure as in [103], we immediately deduce that the fixed point is smooth, thus Proposition 2.3.3.1 holds, once we have established Lemma 2.3.3.1. The reader should notice that we tacitly omit any assumption ensuring $u(t,x) \in S^{n-1}$ for $(t,x) \in [0,T] \times S^1$. This is no oversight, but relates to the fact that by employing the maximum principle for parabolic equations immediately proves this from the equation (2.275), see the proof of Proposition 3.12 in [102].

Proof. First, one should observe that $u|d_{1/2}u|^2 \in L^\infty([0,T] \times S^1) \subset L^p([0,T] \times S^1)$. This follows by Sobolev embeddings into Hölder spaces and the compactness of the domain. Therefore, the operator S is actually well-defined.

By abuse of notation, we denote by u_0 also its extension to $[0,T] \times S^1$ which is independent of time. Let us consider the following for arbitrary $u, v \in W^{1,p}_{u_0}([0,T] \times S^1)$:

$$\|S(u) - S(v)\|_{W^{1,p}} \lesssim \|u|d_{1/2}u|^2 - v|d_{1/2}v|^2\|_{L^p}$$

$$
\begin{aligned}
&\lesssim \|u-v\|_{L^p}\|u\|_{C^{1/2}}^2 + \|v\|_{L^\infty}\||d_{1/2}(u-v)|\|_{L^p}\left(\|u\|_{C^{1/2}} + \|v\|_{C^{1/2}}\right) \\
&\lesssim T^{1/p}\|u-v\|_{L^\infty}\|u\|_{W^{1,p}} + \|v\|_{W^{1,p}} \cdot T^{1/p}\|u-v\|_{C^{1/2}} \cdot \left(\|u\|_{W^{1,p}} + \|v\|_{W^{1,p}}\right) \\
&\lesssim (R+\|u_0\|_{W^{1,p}})^2 \cdot T^{1/p} \cdot \|u-v\|_{W^{1,p}}, \qquad (2.311)
\end{aligned}
$$

where we emphasise that all estimates have no further dependence on T. This may be seen by mirror-extensions and applying the Sobolev embeddings on potentially larger sets. Thus, we may conclude, provided R is given:

$$S \text{ is a contraction, if } T \text{ is sufficently small.}$$

Thus, it remains to be seen that provided $R > 0$ is sufficiently large, then for every $u \in B_R(u_0)$, we also have:

$$S(u) \in B_R(u_0)$$

To see this, we have to consider the difference:

$$d := \|u_0 - S(u_0)\|_{W^{1,p}}$$

We define for now $R = 2d$ and then choose $T > 0$ so small, that the Lipschitz constant in (2.311) is $1/2$. Let us notice that for any $u \in B_R(0)$, we have:

$$
\begin{aligned}
\|u_0 - S(u)\|_{W^{1,p}} &\leq \|u_0 - S(u_0)\|_{W^{1,p}} + \|S(u_0) - S(u)\|_{W^{1,p}} \\
&\leq \frac{R}{2} + \frac{1}{2}\|u_0 - u\|_{W^{1,p}} \\
&\leq \frac{R}{2} + \frac{R}{2} = R, \qquad (2.312)
\end{aligned}
$$

thus:

$$S(u) \in B_R(u_0).$$

This now concludes the proof of Lemma 2.3.3.1, as $W^{1,p}_{u_0}([0,T]\times S^1)$ is a complete metric space due to the continuity of the trace operator. $\square$

2.3.3.2 Uniqueness: Lemma 2.3.1.2

In this short section, we shall explain the proof of Lemma 2.3.1.2 which can be seen as Rivière's Lemma ([68]). Recall that we are interested in solutions $u \in H^{1/2}(S^1; S^{n-1})$ of an equation of the form:

$$(-\Delta)^{1/2} u = u|d_{1/2}u|^2 + f, \qquad (2.313)$$

where $f \in L^2(S^1; \mathbb{R}^n)$. We observe the following (using Einstein's summation convention):

$$
\begin{aligned}
&\langle \mathrm{div}_{1/2}\left(u_i d_{1/2}u_j - u_j d_{1/2}u_i\right), \varphi\rangle_{\mathcal{D}'(S^1)} \\
&= \langle u_i d_{1/2}u_j - u_j d_{1/2}u_i, d_{1/2}\varphi\rangle_{L^2_{od}(S^1\times S^1)} \\
&= \int_{S^1}\int_{S^1} u_i(x)\frac{(u_j(x)-u_j(y))(\varphi(x)-\varphi(y)}{|x-y|^2} - u_j(x)\frac{(u_i(x)-u_i(y))(\varphi(x)-\varphi(y)}{|x-y|^2} dydx \\
&= \int_{S^1}\int_{S^1} d_{1/2}u_j(x,y)\left(d_{1/2}(u_i\varphi)(x,y) - d_{1/2}u_i(x,y)\varphi(y)\right)\frac{dydx}{|x-y|} \\
&- \int_{S^1}\int_{S^1} d_{1/2}u_i(x,y)\left(d_{1/2}(u_j\varphi)(x,y) - d_{1/2}u_j(x,y)\varphi(y)\right)\frac{dydx}{|x-y|}
\end{aligned}
$$

$$= \int_{S^1}\int_{S^1} d_{1/2}u_i(x,y)d_{1/2}(u_j\varphi)(x,y) - d_{1/2}u_j(x,y)d_{1/2}(u_i\varphi)(x,y)\frac{dydx}{|x-y|}$$
$$= \int_{S^1} u_i(x)|d_{1/2}u|^2(x)\cdot u_j(x)\varphi(x) - u_j(x)|d_{1/2}u|^2(x)\cdot u_i(x)\varphi(x) - f_i(x)u_j(x)\varphi(x) + f_j(x)u_i(x)\varphi(x)dx$$
$$= \int_{S^1} (u_i(x)f_j(x) - u_j(x)f_i(x))\,\varphi(x)dx, \tag{2.314}$$

which reveals, in analogy to Rivière [68]:

$$\forall i,j\in\{1,\dots,n\} : \mathrm{div}_{1/2}\left(u_i d_{1/2}u_j - u_j d_{1/2}u_i\right) = u_i f_j - u_j f_i \tag{2.315}$$

One may solve now for i,j as above the equation:

$$(-\Delta)^{1/2}\psi_{ij} = u_i f_j - u_j f_i,$$

for $\psi_{ij}\in H^1(S^1)$. Observe that we may choose these in such a way that $\psi_{ij} = -\psi_{ji}$. Then it becomes clear:

$$\mathrm{div}_{1/2}\ \left(u_i d_{1/2}u_j - u_j d_{1/2}u_i - d_{1/2}\psi_{ij}\right) = 0$$

Thus, defining $\Omega_{ij} := u_i d_{1/2}u_j - u_j d_{1/2}u_i - d_{1/2}\psi_{ij}$, we find:

$$(-\Delta)^{1/2}u = \Omega\cdot d_{1/2}u + T(u) + d_{1/2}\psi\cdot d_{1/2}u + f \tag{2.316}$$

Here, $T(u)$ is the remainder as already found in Mazowiecka-Schikorra [57] and [102]: It is given by $T(u) = (T^1(u),\dots,T^n(u))$ and

$$\forall i\in\{1,\dots,n\} : T^i(u) := \frac{1}{2}\sum_{k=1}^{n}\int_{S^1} d_{1/2}u_i(x,y)|d_{1/4}u_k(x,y)|^2\frac{dy}{|x-y|}$$

One may generalise this remainder as follows:

$$T^i(u,v,w) := \frac{1}{2}\sum_{k=1}^{n}\int_{S^1} d_{1/2}u_i(x,y)d_{1/4}v_k(x,y)d_{1/4}w_k(x,y)\frac{dy}{|x-y|},$$

such that $T(u) = T(u,u,u)$. This term has good integrability properties, see the proof of Lemma 3.8 in [102]. To simplify, let us notice that $\psi\in H^1(S^1)\hookrightarrow W^{1/2,p}(S^1)$ by Sobolev embeddings for every $p<+\infty$ and thus, using Prats [64]:

$$d_{1/2}\psi\cdot d_{1/2}u\in L^q(S^1),\quad \forall 1\le q<2,$$

since $|d_{1/2}u|\in L^2(S^1)$ by $u\in H^{1/2}(S^1)$. Thus, (2.316) can be rephrased as:

$$(-\Delta)^{1/2}u = \Omega\cdot d_{1/2}u + T(u,u,u) + \tilde{f}, \tag{2.317}$$

where $\tilde{f} := f + d_{1/2}\psi\cdot d_{1/2}u\in L^q(S^1)$ for all $1\le q<2$.

The key idea in Rivière [68] is now the following: We try to approximate Ω by a smooth $\tilde{\Omega}$ with vanishing 1/2-divergence, such that:

$$\|\Omega - \tilde{\Omega}\|_{L^2_{od}}\le\varepsilon,$$

for $\varepsilon > 0$ small. Similarily, we approximate u by a smooth $\tilde{u}$ in $H^{1/2}(S^1)$. Then (2.317) leads us to:

$$(-\Delta)^{1/2}u - \left(\Omega - \tilde{\Omega}\right)\cdot d_{1/2}u - T(u, u-\tilde{u}, u-\tilde{u}) = \tilde{\Omega}\cdot d_{1/2}u + T(u,u,\tilde{u}) + T(u,\tilde{u},u-\tilde{u}) + \tilde{f} =: \hat{f} \quad (2.318)$$

Since $\operatorname{div}_{1/2}(\Omega - \tilde{\Omega}) = 0$, we notice that we are in the realm of the fractional Wente-type estimate in Proposition 2.3.1.2. Namely, if $v \in \dot{F}^{1/2}_{p,2}(S^1)$ for some $p > 2$, then we have by Hölder's inequality:

$$\left\| \left(\Omega - \tilde{\Omega}\right)\cdot d_{1/2}v \right\|_{L^{\frac{2p}{p+2}}} \lesssim \|\Omega - \tilde{\Omega}\|_{L^2_{od}} \|v\|_{\dot{F}^{1/2}_{p,2}} \leq \varepsilon \cdot \|v\|_{\dot{F}^{1/2}_{p,2}}$$

Since $F^{1/2}_{p',2}(S^1) \hookrightarrow L^{\frac{2p'}{2-p'}}$ for p' the Hölder conjugate of p by Sobolev embedding, the inequality above immediately yields:

$$\left\| \left(\Omega - \tilde{\Omega}\right)\cdot d_{1/2}v \right\|_{F^{-1/2}_{p,2}} \lesssim \varepsilon \cdot \|v\|_{\dot{F}^{1/2}_{p,2}}$$

In the case $p = 2$, i.e. $v \in \dot{F}^{1/2}_{2,2}(S^1) = \dot{H}^{1/2}(S^1)$, then by Proposition 2.3.1.2 we get immediately:

$$\left\| \left(\Omega - \tilde{\Omega}\right)\cdot d_{1/2}u \right\| \lesssim \|\Omega - \tilde{\Omega}\|_{L^2_{od}} \|v\|_{\dot{H}^{1/2}} \leq \varepsilon \cdot \|v\|_{\dot{H}^{1/2}}$$

In an analogous manner, the estimates in the preliminary section show us:

$$\|T(v, u-\tilde{u}, u-\tilde{u})\|_{\dot{F}^{-1/2}_{p,2}} \lesssim \|u-\tilde{u}\|^2_{H^{1/2}} \|v\|_{\dot{F}^{1/2}_{p,2}} \leq \varepsilon^2 \cdot \|v\|_{\dot{F}^{1/2}_{p,2}},$$

for all $p \geq 2$. This shows us that the operator:

$$\tau : v \mapsto v - (-\Delta)^{-1/2}\left(\left(\Omega - \tilde{\Omega}\right)\cdot d_{1/2}v + T(v, u-\tilde{u}, u-\tilde{u})\right),$$

actually defines an invertible operator (one has to be slightly careful at this point and restrict to v having vanishing mean), for any $p \geq 2$, from $\dot{F}^{1/2}_{p,2}(S^1)$ to itself, provided $\varepsilon > 0$ is sufficiently small.

Keeping the RHS of (2.318) in mind, it is immediate that it lies in L^q for all $1 \leq q < 2$ by estimates from the previous subsection. Thus:

$$(-\Delta)^{-1/2}\left(\tilde{\Omega}\cdot d_{1/2}u + T(u,u,\tilde{u}) + T(u,\tilde{u},u-\tilde{u}) + \tilde{f}\right) \in \dot{F}^1_{q,2}(S^1) \hookrightarrow \dot{F}^{1/2}_{\frac{2q}{2-q},2}(S^1),$$

for again all $q \in [1,2[$.

The conclusion of Lemma 2.3.1.2 follows now by noticing that $\tau(v) = (-\Delta)^{-1/2}\hat{f}$ does possess a solution $v \in \dot{F}^{1/2}_{p,2}(S^1)$ by invertibility for each fixed $p \geq 2$, provided $\varepsilon > 0$ is sufficiently small. Observing that due to compactness of S^1, we have:

$$\dot{F}^{1/2}_{p,2}(S^1) \subset \dot{F}^{1/2}_{2,2}(S^1) = H^{1/2}(S^1),$$

by choosing $\varepsilon > 0$ so small, that invertibility holds for $p = 2$ and some $p > 2$, we conclude that the solution $v \in \dot{F}^{1/2}_{p,2}(S^1)$ must also lie in $H^{1/2}(S^1)$. Since $u \in H^{1/2}(S^1)$ is already a solution and by invertibility actually the unique one, we deduce:

$$v = u \Rightarrow u \in \dot{F}^{1/2}_{p,2}(S^1)$$

As $p > 2$ was arbitrary up to possibly choosing better approximations for even smaller $\varepsilon > 0$, we find:

$$u \in \dot{F}^{1/2}_{p,2}(S^1), \quad \forall p \in [1, +\infty[$$

Taking $p = 4$, we deduce:

$$|d_{1/2}u|^2 \in L^2(S^1), \tag{2.319}$$

which combined with $|u| = 1$ almost everywhere and (2.313), we thus conclude:

$$(-\Delta)^{1/2}u = u|d_{1/2}u|^2 + f \in L^2(S^1) \Rightarrow u \in H^{1/2}(S^1; S^{n-1}),$$

which is the required conclusion.

All that remains to do is to justify the approximation of Ω and u. Since the latter is standard and does not require any further interesting considerations, it is omitted here. The former, however, requires some care. Thus, let $\varepsilon > 0$ be arbitrary and we shall consider the following approximation:

$$\Omega_\delta := \Omega \cdot 1_{D_\delta},$$

where:

$$\forall \delta > 0 : D_\delta := \{(x, y) \in S^1 \times S^1 | |x - y| \geq \delta\}$$

So D_δ omits a neighbourhood of the diagonal. It is clear by Lebesgue's dominated convergence, that:

$$\Omega_\delta \to \Omega, \text{ as } \delta \to 0, \tag{2.320}$$

in the space $L^2_{od}(S^1 \times S^1)$. Thus, take δ so small that:

$$\|\Omega - \Omega_\delta\|_{L^2_{od}} < \frac{\varepsilon}{2} \tag{2.321}$$

Now, we may argue by convolution by a suitable smooth kernel to replace Ω_δ by a smooth function, again denoted Ω_δ, which vanishes close to the diagonal $\{x = y\}$. This is again standard and thus omitted.

The final obstacle to overcome is to adapt $\Omega_\delta \in C^\infty(S^1 \times S^1)$ in such a way that:

$$\operatorname{div}_{1/2} \Omega_\delta = 0$$

To achieve this, we shall solve the following problem:

$$(-\Delta)^{1/2}h_\delta = \operatorname{div}_{1/2} \Omega_\delta,$$

i.e. solving the weak equation:

$$\langle (-\Delta)^{1/2}h_\delta, \varphi \rangle = \int_{S^1} \int_{S^1} d_{1/2}h_\delta(x, y) d_{1/2}\varphi(x, y) \frac{dydx}{|x - y|} = \int_{S^1} \int_{S^1} \Omega_\delta(x, y) d_{1/2}\varphi(x, y) \frac{dydx}{|x - y|}, \quad \forall \varphi \in C^\infty(S^1).$$

Existence of such a solution is immediate, as in the case of Ω_δ, one may define the divergence directly as a smooth function. One may immediately notice that since $\Omega_\delta \in L^2_{od}$, we have:

$$(-\Delta)^{1/2}h_\delta \in H^{-1/2}(S^1) \Rightarrow h_\delta \in H^{1/2}(S^1),$$

together with the estimate:

$$
\begin{aligned}
\|h_\delta\|_{\dot{H}^{1/2}}^2 &= \int_{S^1} |d_{1/2}h_\delta|^2 dx \\
&= \int_{S^1} \left(\Omega_\delta(x,y) - \Omega(x,y)\right) d_{1/2}h_\delta(x,y) \frac{dydx}{|x-y|} \\
&\lesssim \|\Omega_\delta - \Omega\|_{L^2_{od}} \|h_\delta\|_{\dot{H}^{1/2}},
\end{aligned}
\tag{2.322}
$$

ultimately proving:

$$
\|h_\delta\|_{\dot{H}^{1/2}} \lesssim \|\Omega_\delta - \Omega\|_{L^2_{od}} \leq \frac{\varepsilon}{2},
$$

where we used in the computation above that $\operatorname{div}_{1/2}\Omega = 0$. Therefore, by choosing $\delta > 0$ sufficiently small, we have:

$$
\|\Omega_\delta - d_{1/2}h_\delta - \Omega\|_{L^2_{od}} \lesssim \varepsilon, \tag{2.323}
$$

as well as:

$$
\operatorname{div}_{1/2}\left(\Omega_\delta - d_{1/2}h_\delta\right) = \operatorname{div}_{1/2}\Omega_\delta - (-\Delta)^{1/2}h_\delta = 0. \tag{2.324}
$$

It should be emphasised that $(-\Delta)^{1/2} = \operatorname{div}_{1/2} \circ d_{1/2}$ in complete analogy to $\Delta = \operatorname{div} \circ \nabla$. This is precisely the desired approximation and thus concludes the proof of Lemma 2.3.1.2. □

2.3.3.3 Bubbling-Analysis and Concentration of Energy

In the current section, we will first study the concentration of energy in greater detail and with more precise estimates. Two main results shall be obtained: Firstly, we shall improve the following Lemma 3.16 in [102]:

Lemma 2.3.3.2 (Lemma 3.16 in [102]). *There exist $C > 0$ not depending on R, u, T, such that for any smooth u on $[0,T] \times S^1$ and $0 < R < 1$, the following estimate holds for all $x_0 \in S^1$:*

$$
\begin{aligned}
\int_0^T \int_{B_{\frac{3R}{4}}(x_0)} |(-\Delta)^{1/4}u|^4 dxdt \leq C \sup_{0\leq t\leq T} \int_{B_R(x_0)} |(-\Delta)^{1/4}u(t)|^2 dx \\
\cdot \left(\int_0^T \int_{B_R(x_0)} |(-\Delta)^{1/2}u|^2 dxdt + \frac{1}{R^2} \int_0^T \int_{S^1} |(-\Delta)^{1/4}u|^2 dxdt \right),
\end{aligned}
\tag{2.325}
$$

by density the same result applies for all $u \in H^1([0,T] \times S^1)$, and all boundary terms $u_0 = u(0,\cdot) \in H^{1/2}(S^1)$, with bounded $1/2$-Dirichlet energy. Similarily, we have:

$$
\begin{aligned}
\int_0^T \int_{S^1} |(-\Delta)^{1/4}u|^4 dxdt \lesssim \sup_{0\leq t\leq T, x\in S^1} \int_{B_R(x)} |(-\Delta)^{1/4}u(t)|^2 dx \\
\cdot \left(\int_0^T \int_{S^1} |(-\Delta)^{1/2}u|^2 dxdt + \frac{1}{R^3} \int_0^T \int_{S^1} |(-\Delta)^{1/4}u|^2 dxdt \right).
\end{aligned}
\tag{2.326}
$$

The improvement will be in the order of power of R that occurs and this is indeed crucial for a non-local rescaling argument to work, for instance giving an improved exponent in Lemma 2.3.3.4. Namely, we shall show that R^{-3} may be replaced by R^{-2} which allows for suitable rescaling and a blow-up procedure. Secondly, we will connect the condition (2.276) to an analogous condition for the localised energy in balls, sacrificing potentially focus by allowing for "larger" balls in which the localised Gagliardo-seminorms are bounded from below. Observe that due to the non-local nature of the 1/4-Laplacian, $\varepsilon(R; u, t)$ takes into account not only value of $u(t,x)$ in a ball, but on the entire S^1. However, contributions "far away" are not as important (these are dealt with by enlarging the balls under consideration) and thus we may restrict our attention to the local Gagliardo seminorms on balls.

An Improved Version of Lemma 3.16 in [102] In this brief subsection, we shall argue why the following refinement of Lemma 3.16 in [102] holds true:

Lemma 2.3.3.3. *There exist $C > 0$ not depending on R, u, T, such that for any smooth u on $[0,T]\times S^1$ and $0 < R < 1/2$, the following estimate holds for all $x_0 \in S^1$:*

$$\int_0^T \int_{B_{\frac{3R}{4}}(x_0)} |(-\Delta)^{1/4}u|^4 dxdt \leq C \sup_{0\leq t\leq T} \int_{B_R(x_0)} |(-\Delta)^{1/4}u(t)|^2 dx \cdot \left(\int_0^T \int_{B_R(x_0)} |(-\Delta)^{1/2}u|^2 dxdt + \frac{1}{R^2} \int_0^T \int_{S^1} |(-\Delta)^{1/4}u|^2 dxdt \right), \tag{2.327}$$

by density the same result applies for all $u \in H^1([0,T]\times S^1)$, and all boundary terms $u_0 = u(0,\cdot) \in H^{1/2}(S^1)$, with bounded 1/2-Dirichlet energy. Similarily, we have:

$$\int_0^T \int_{S^1} |(-\Delta)^{1/4}u|^4 dxdt \lesssim \sup_{0\leq t\leq T, x\in S^1} \int_{B_R(x)} |(-\Delta)^{1/4}u(t)|^2 dx \cdot \left(\int_0^T \int_{S^1} |(-\Delta)^{1/2}u|^2 dxdt + \frac{1}{R^2} \int_0^T \int_{S^1} |(-\Delta)^{1/4}u|^2 dxdt \right). \tag{2.328}$$

Proof. The key observation lies in the following estimate: In [102] and [103], we used the rather crude estimate:

$$\begin{aligned}
&\int_0^T \int_{S^1} \left| P.V. \int_{S^1} (-\Delta)^{1/4}u(y) \frac{\varphi(x)-\varphi(y)}{|x-y|^{3/2}} dy \right|^2 dxdt \\
&\lesssim \int_0^T \int_{S^1} |(-\Delta)^{1/4}u(y)|^2 \frac{1}{|x-y|^{1/2}} dy \cdot \int_{S^1} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{5/2}} dydxdt && (2.329) \\
&\lesssim \frac{1}{R^2} \int_0^T \int_{S^1} |(-\Delta)^{1/4}u(y)|^2 dydt, && (2.330)
\end{aligned}$$

where φ is a cut-off function on some subset $B_R(x_0)$, $x_0 \in S^1$. In [102] and [103], we then obtained (2.326) by summing for a suitable covering by balls with finite-intersection property the terms (2.330). Instead of using (2.330), we will now use (2.329) and obtain a more precise estimate. For each fixed value $x \in S^1$ (observe that if we are able to obtain a bound independent of x, one may argue as in (2.330) to finish), we have then a sum:

$$\sum_{j\in I} \frac{|\varphi_j(x)-\varphi_j(y)|^2}{|x-y|^{5/2}},$$

which we want to estimate in order to establish (2.328) using (2.329) and summation over a suitable covering. Here, φ_j are the corresponding cut-offs to a suitable covering, i.e. they are supported in balls of radius R (φ_j being equal to 1 on the ball with same center and radius $3/4R$) with the property that every point is contained in at most 3 of these balls. In fact, the covering should be as in [102], taking balls of radius R around $\frac{8\pi}{3R}$ points which are evenly distributed along the circle and observe that such a covering immediately has the finite intersection property (indeed, any given point lies in at most 3 such balls). Now, if $\varphi_j(x) \neq 0$, we use the estimate:

$$\frac{|\varphi_j(x)-\varphi_j(y)|^2}{|x-y|^{5/2}} \leq \|\nabla\varphi\|_{L^\infty}^2 \frac{1}{|x-y|^{1/2}} \lesssim \frac{1}{R^2}\cdot\frac{1}{|x-y|^{1/2}}$$

Notice that $\varphi(x) \neq 0$ only holds true for finitely many j (due to the choice of balls above, recalling that $\varphi_j(x) \neq 0$ only if x lies inside the associated ball), this number being independent of R, so by integrating over S^1 and exploiting the integrability of $1/|x-y|^{1/2}$ on S^1 in y, we deduce that the contribution of these terms may be bounded by $1/R^2$.

Next, we have to consider all terms with $\varphi_j(x) = 0$. By choice of the covering by balls with equal radius and equally spaced along the circle, see [102], it is clear that then:

$$|x-y| \geq \delta R,$$

for some $\delta > 0$ independent of R. Indeed, the cover may be chosen in such a way that for $\delta > 0$ small and independent of R, we have that for $x \in S^1$, $B_{\delta R}(x)$ lies in one of the balls of the covering. Then only finitely many have non-empty intersection with this ball around x and thus all others must satisfy

$$|x-y| \geq \delta R,$$

for y in the remaining balls. Taking next the ball which gets closest to x among all with empty intersection with $B_{\delta R}(x)$, we have that again only finitely many have non-empty intersection with this one, namely a maximum of 3 again, all others satisfy

$$|x-y| \geq \left(\delta + \frac{3}{4}\right) R,$$

for y these balls. The reason for this lies in the fact that provided there is no intersection with a ball B_j, then the midpoint M_j of B_j must have at least one midpoint of a different ball between x and M_j. So the distance must be larger by $3R/4$ than the distance for the closest point of the ball with midpoint in between. Iterating such an argument, see also Figure 2.3.1[7], and observing that the number of intersecting balls may be controlled independent of R, we arrive at the estimate ultimately required. Thus, by integration of these summands, we obtain a sum of the form:

$$\sum_{j\in\mathbb{N}_0} \frac{1}{R^{3/2}} \cdot \frac{2}{\left(\frac{3j}{4}+\delta\right)^{3/2}} \lesssim \frac{1}{R^{3/2}} \lesssim \frac{1}{R^2}, \quad \forall R \in]0, 1/2[$$

[7]The points labelled are the midpoints, the coloured arrows give the size of balls and x is marked purple. It is clear that x lies in the balls around M_0, M_1, M_2. These are treated as in the case $\varphi_0(x) \neq 0$. Every point y in the ball around M_3 has now a fixed distance in terms of slightly less than R from x at least, say δR as in the proof. For the ball around M_4, the minimum distance is increased by a step between midpoints, i.e. $3R/4$. The same holds for M_5, etc.. Naturally, the same argument may be done for going over to $M_{-1}, M_{-2}, \ldots$, ultimately giving the result in the proof. Reducing the overlap of balls leads to better estimates. Moreover, the estimates may be made uniform in x which is done in the proof (the argument just requires one to treat all balls that contain one point of the ball around M_0 with the first case, leading to 5 balls being covered by this case, while the others are still treated precisely the same way as outlined.

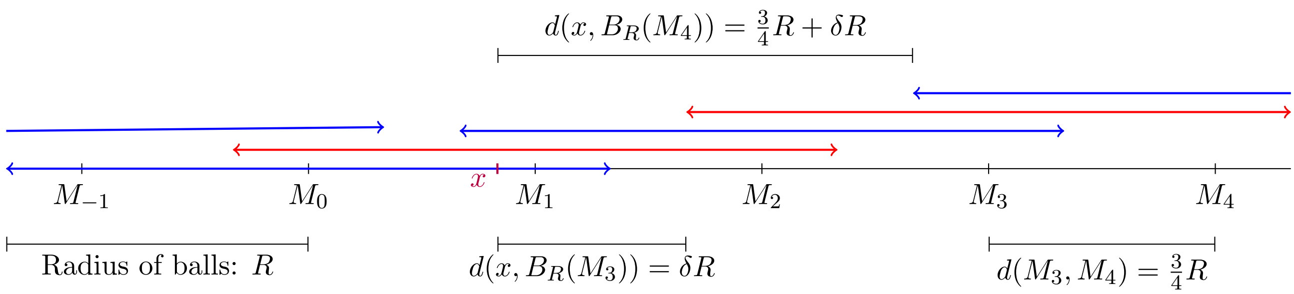

Figure 2.3.1: Sketch of the splitting of points with varying distances as in the Proof of Lemma 2.3.3.3.

Indeed, observe that the covering may be chosen in such a way that at each point, at most 3 of the balls intersect. Noting that we may select balls and describe the distance between x and the corresponding balls in terms of $(j+\delta)R$, the statement becomes apparent. Then by integrating $1/|x-y|^{5/2}$ explicitly, we obtain the sum above. Combining both contributions, we get the improved estimate (2.328) by arguing as in [103]. □

Such a result also allows for a slightly more refined version of Lemma 3.19 in [102]:

Lemma 2.3.3.4. *There exists $\varepsilon_1 > 0$ such that for any $u \in H^1([0,T] \times S^1) \cap L^\infty([0,T]; H^{1/2}(S^1))$ solving:*

$$\partial_t u + (-\Delta)^{1/2} u \perp T_u N \quad \text{in } \mathcal{D}'([0,T] \times S^1)$$

with values in N and any $R < 1/2$, there holds:

$$\int_0^T \int_{S^1} |\nabla u|^2 dx dt \leq C E(u_0) \left(1 + \frac{T}{R^2}\right), \tag{2.331}$$

with C independent of u, T, R, provided $\varepsilon(R) < \varepsilon_1$. Here, $u(0,\cdot) = u_0 \in H^{1/2}(S^1; N)$ is the initial value.

The proof is as in [103] or Struwe [89], the only change lies in the application of Lemma 2.3.3.3 instead of Lemma 3.16 in [102]. This improved version will be crucial in the blow-up procedure, as it will enable us to deduce that the H^1-energy is bounded and thus leads to a good solution after extracting a weakly convergent subsequence, since we have now an appropriate scaling-behaviour of time and space variable.

Lower Bound for Local Gagliardo Seminorms Next, we would like to establish a connection between the concentration condition (2.276) at blow-up points and the Gagliardo-seminorms at the same points. The intuition behind the estimate is that whenever 1/2-Dirichlet energy concentrates close to a point, then also the localised Gagliardo seminorm around the same point should concentrate, just as it is the case for the harmonic gradient flow in some sense (the statement is however tautological in this case, as the energy is already local). Due to the non-local nature, however, contributions from further away may still be significant, forcing us to include a bigger domain in the estimate of the seminorm than in the 1/2-energy to avoid concentration in "neck regions" that we would otherwise not account for. The key connection is the following:

Proposition 2.3.3.2. *Let $\varepsilon > 0$ be given and N sufficiently large depending on ε. Assume that $u \in H^{1/2}(S^1)$ with $|u| \leq 1$ and such that:*

$$\int_{B_R(x_0)} |(-\Delta)^{1/4} u|^2 dx \geq \varepsilon,$$

for some $R < 2^{-N-1}$. Then there is a $\delta > 0$ depending only on n and ε, such that:

$$\int_{B_{2^N R}(x_0)} \int_{B_{2^N R}(x_0)} \frac{|u(x) - u(y)|^2}{|x-y|^2} dy dx \geq \delta.$$

In the proof, we shall clarify the necessary requirement for N. Also, the same proof continues to hold for arbitrary bounded u with δ depending also on $\|u\|_{L^\infty}$.

Proof. Firstly, we observe that the independence of R and x_0 of δ may be obtained by rescaling and rotations, possibly after using stereographic projection. So we do not have to worry about such dependencies.

Let us argue by contradiction: Assume the statement was wrong, then there exists a sequence $u_n \in H^{1/2}(S^1)$ of bounded functions, such that:

$$\int_{B_R(x_0)} |(-\Delta)^{1/4} u_n|^2 dx \geq \varepsilon; \quad \int_{B_{2^N R}(x_0)} \int_{B_{2^N R}(x_0)} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^2} dy dx < \frac{1}{n}$$

In particular, we have (up to modifying the u_n by a constant and extracting a subsequence):

$$u_n \to 0 \quad \text{in } H^{1/2}(B_{2^N R}(x_0))$$

We emphasise that here, we use the Gagliardo-Sobolev norm on the ball. As seen in Nezza-Palatucci-Valdinoci [63], we may extend the $u_n \in H^{1/2}(B_{2^N R}(x_0))$ to $v_n \in H^{1/2}(S^1)$ which are still bounded by a common multiple of 1 and such that:

$$\|v_n\|_{H^{1/2}(S^1)} \lesssim \|u_n\|_{H^{1/2}(B_{2^N R}(x_0))} \to 0,$$

which also shows:

$$\lim_{n\to\infty} \int_{S^1} |(-\Delta)^{1/4} v_n|^2 dx = 0.$$

Thus, to arrive at a contradiction, we just need to show:

$$\liminf_{n\to\infty} \int_{B_R(x_0)} |(-\Delta)^{1/4} (u_n - v_n)|^2 dx < \varepsilon$$

This can be easily obtained by observing (due to $u_n = v_n$ on $B_{2^N R}(x_0)$):

$$\begin{aligned} &\int_{B_R(x_0)} |(-\Delta)^{1/4} (u_n - v_n)|^2 dx \\ &\leq \int_{B_R(x_0)} \left(\int_{B_{2^N R}(x_0)^c} \frac{|u_n(y) - v_n(y)|}{|x-y|^{3/2}} dy \right)^2 dx \end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_{B_R(x_0)} \left(\int_{B_{2^N R}(x_0)^c} \frac{1}{|x-y|^{3/2}} dy \right)^2 dx \\
&\leq \int_{B_R(x_0)} \frac{1}{|x \mp 2^N R|} dx \\
&\lesssim |\log\left(1-2^{-N}\right)| \lesssim 2^{-N} < \varepsilon, \qquad (2.332)
\end{aligned}
$$

provided N was chosen sufficiently large at the beginning, depending on ε. Thus the required statement follows, as this contradicts our assumptions and thus provides the desired contradiction. □

The key feature of Proposition 2.3.3.2 lies in the fact that it connects the localised (but still non-local) Gagliardo-seminorms to the concentration of energy. The power of 2 that appears is due to the non-linearity and ensures that "not too much" energy is lost by restricting to balls. Ensuring that energy is stored in balls of sufficiently small radius is crucial to obtain half-harmonic maps in the limit.

Bubbling-Analysis Having proved Lemma 2.3.3.3 as well as Proposition 2.3.3.2, we are now able to study the bubbling process in points where energy accumulates. The analysis is inspired by Struwe [89], but has to take care of the non-local behaviour associated with the fractional Laplacian:

Theorem 2.3.3.1. *Let u be a solution as in Theorem 2.3.1.2 and let $x_0 \in S^1$ be a point, such that:*

$$
\limsup_{t \to T} \int_{B_R(x_0)} |(-\Delta)^{1/4} u|^2 dx \geq \varepsilon_1, \quad \forall R > 0, \tag{2.333}
$$

where $\varepsilon_1 > 0$ is as in [103, Lemma 4.10]. Then there exists a half-harmonic map $v : \mathbb{R} \to S^{n-1}$, such that:

$$
u_n \to v \quad \text{weakly in } H^1(\mathbb{R}) \text{ and strongly in } H^{1/2}(\mathbb{R}), \tag{2.334}
$$

where u_n is a suitable rescaling and translation of u.

As stated in the introduction, an analogous result holds for any closed N instead of S^{n-1} as target manifold, up to some technical changes in the formulas. Additionally, we highlight that (2.276) implies (2.333) at a suitable point by choosing subsequences. Therefore, Theorem 2.3.3.1 actually concerns the behaviour of functions at the critical time in Theorem 2.3.1.2. It should be noted that the number of points x_0 satisfying (2.333) is finite due to the limited amount of energy available, so these points may not accumulate.

Proof. Let us argue along the lines of [89, Theorem 4.3]. The key idea is to rescale u on subintervals of $[0, T[$ and apply the results in Lemma 2.3.3.3 and Proposition 2.3.3.2 to deduce convergence. Let us always assume that N is chosen large enough to allow for $\varepsilon = \varepsilon_1/2$ in Proposition 2.3.3.2 and take $\delta > 0$ to be the associated lower bound for the Gagliardo seminorms.

We now define rescalings as follows: For each $R > 0$, we have:

$$
\varphi_R : \mathbb{R} \to S^1 \simeq \mathbb{R}/\mathbb{Z} \simeq [-\pi; \pi[, \tag{2.335}
$$

with the properties:

$$
\varphi_R(x) = R^2 x, \quad \forall x \in [-\frac{2^N}{R}, \frac{2^N}{R}]; \quad |\varphi_R'(x)| \leq R^2,
$$

and:

$$\lim_{x\to\pm\infty}\varphi_R(x)=\pm\pi.$$

The existence of such a function is clear.

By (2.333) and choosing points $(t_n,x_n)\in[0,T[\times S^1$ as in Struwe [89] with $t_n\to T, x_n\to x_0$ and such that:

$$E_{R_n}(u(t_n,\cdot),x_n)=\varepsilon_1=\sup_{0<t\le t_n, x\in B_r(x_0)}E_{R_n}(u(t,\cdot),x),$$

where $R_n\to 0$ and $r>0$ is chosen small enough that no other point with the property (2.333) is contained in $B_r(x_0)$. We shall now define:

$$u_n:[-\gamma,0]\times\mathbb{R}\to S^{n-1},\quad u_n(t,x):=u(t_n+R_n^2t,x_n+\varphi_{R_n}(x)) \tag{2.336}$$

Here, $\gamma>0$ (using Lemma 2.3.3.4) is chosen in such a way to ensure:

$$E_{2R_n}(u(t);x_n)\ge\varepsilon_1/2,\quad \forall t\in[t_n-\gamma R_n^2,t_n].$$

See also Lemma 4.9 in [103] for a justification of this fact and compare this with Struwe [89]. To define $x_n+\varphi_{R_n}(x)$, we may use the periodicity of u in the space-variable. The key properties of these functions are their boundedness properties. For example, we have:

$$\begin{aligned}
&\int_{-\gamma}^{0}\int_{\mathbb{R}}|\nabla u_n(t,x)|^2dxdt\\
&=\frac{1}{R_n^2}\int_{t_n-\gamma R_n^2}^{t_n}\int_{S^1}|\nabla u(s,y)|^2|\varphi_{R_n}'(\varphi_{R_n}^{-1}(y))|^2|(\varphi_{R_n}^{-1})'(y)|dyds\\
&=\frac{1}{R_n^2}\int_{t_n-\gamma R_n^2}^{t_n}\int_{S^1}|\nabla u(s,y)|^2|\varphi_{R_n}'(\varphi_{R_n}^{-1}(y))|dyds\\
&\le\int_{t_n-\gamma R_n^2}^{t_n}\int_{S^1}|\nabla u(s,y)|^2dyds\lesssim E(u_0),
\end{aligned} \tag{2.337}$$

where we used Lemma 2.3.3.4 as well as the choice of points (t_n,x_n) as above. Notice that the chain rule is employed at one point to simplify the expression. Similarily:

$$\begin{aligned}
&\int_{-\gamma}^{0}\int_{B_{2^N/R_n}(0)}|\partial_t u_n(t,x)|^2dxdt\\
&=R_n^2\int_{t_n-\gamma R_n^2}^{t_n}\int_{B_{2^NR_n}(x_n)}|\partial_t u(s,y)|^2|(\varphi_{R_n}^{-1})'(y)|dydt\\
&=\int_{t_n-\gamma R_n^2}^{t_n}\int_{B_{2^NR_n}(x_n)}|\partial_t u(s,y)|^2dydt\\
&\le\int_{t_n-\gamma R_n^2}^{t_n}\int_{S^1}|\partial_t u(s,y)|^2dydt\lesssim E(u_0)
\end{aligned} \tag{2.338}$$

One may now extract convergent subsequences. Thus, we end up with sequences $u_n(\tau_n,\cdot)$ which converge weakly in $H^1(S^1)$ and strongly in $H^{1/2}(\mathbb{R})$ to $v\in H^1(\mathbb{R})$. Choosing the subsequence to be pointwise convergent a.e., we may even deduce:

$$v\in S^{n-1}\quad\text{a.e.}$$

Furthermore, Proposition 2.3.3.2 shows, thanks to the concentration of energy, that:

$$\delta \leq \int_{B_{2^N R_n}(x_n)} \int_{B_{2^N R_n}(x_n)} \frac{|u(t,x)-u(t,y)|^2}{|x-y|^2} dydx,$$

for all $t \in [t_n - \gamma R_n^2, t_n]$. This also shows:

$$\delta \leq \int_{B_{2^N/R_n}(x_n)} \int_{B_{2^N/R_n}(x_n)} \frac{|u_n(\tau_n,x)-u_n(\tau_n,y)|^2}{|x-y|^2} dydx$$

Thus, by passing to the limit as $n \to \infty$:

$$E_{1/2}(v) \geq \delta > 0,$$

and so v may not be constant. It remains to check that v is actually half-harmonic. This is however an immediate consequence of the original equation:

$$\partial_t u + (-\Delta)^{1/2} u = u|d_{1/2}u|^2$$

Namely, since for τ_n, we have:

$$\partial_t u(\tau_n) \to 0,$$

as $n \to \infty$ in $L^2_{loc}(\mathbb{R})$, it remains to prove convergence of the other terms. Namely, we have for any $\varphi \in C_c^\infty(\mathbb{R})$:

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} d_{1/2}v(x,y) d_{1/2}\varphi(x,y) \frac{dydx}{|x-y|} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} d_{1/2}u_n(\tau_n)(x,y) d_{1/2}\varphi(x,y) \frac{dydx}{|x-y|} \\
&= \lim_{n\to\infty} \int_{B_{2^N/R_n}} \int_{B_{2^N/R_n}} d_{1/2}u_n(\tau_n)(x,y) d_{1/2}\varphi(x,y) \frac{dydx}{|x-y|} \\
&= \lim_{n\to\infty} \int_{B_{2^N R_n}(x_n)} \int_{B_{2^N R_n}(x_n)} d_{1/2}u(\tau_n)(x,y) d_{1/2}\left(\varphi \circ \varphi_{R_n}^{-1}\right)(x,y) \frac{dydx}{|x-y|} \\
&= \lim_{n\to\infty} \int_{S^1} \int_{S^1} d_{1/2}u(\tau_n)(x,y) d_{1/2}\left(\varphi \circ \varphi_{R_n}^{-1}\right)(x,y) \frac{dydx}{|x-y|} \\
&= \lim_{n\to\infty} \left(\int_{S^1} -\partial_t u \cdot \varphi \circ \varphi_{R_n}^{-1} dx + \int_{S^1} u(\tau_n)|d_{1/2}u(\tau_n)|^2 \cdot \varphi \circ \varphi_{R_n}^{-1} dx \right) \\
&= \lim_{n\to\infty} \int_{S^1} u(\tau_n)|d_{1/2}u(\tau_n)|^2 \cdot \varphi \circ \varphi_{R_n}^{-1} dx \\
&= \lim_{n\to\infty} \int_{B_{2^N R_n}(x_n)} u(\tau_n) \int_{B_{2^N R_n}(x_n)} |d_{1/2}u(\tau_n)(x,y)|^2 \frac{dy}{|x-y|} \cdot \varphi \circ \varphi_{R_n}^{-1} dx \\
&= \lim_{n\to\infty} \int_{B_{2^N/R_n}(x_n)} u_n(\tau_n) \int_{B_{2^N/R_n}(x_n)} |d_{1/2}u_n(\tau_n)(x,y)|^2 \frac{dy}{|x-y|} \cdot \varphi dx \\
&= \int_{\mathbb{R}} v|d_{1/2}v|^2 \varphi, \qquad (2.339)
\end{aligned}$$

which is the desired equation. Notice that throughout the computations, we used several times that appropriate terms may be omitted due to the boundedness of $u_n(\tau_n)$ and v, leading to omissions of parts of the domain of integration, switching between the distance function on S^1 and $\mathbb{R}$ and similar terms. A crucial observation is that φ is supported on a subdomain of B_{2^N/R_n} for R_n sufficiently small, so the estimates have good bounds everywhere, if n goes to ∞. So we are done, since v solves the half-harmonic map equation and thus is actually smooth, see Da Lio-Pigati [20]. In particular, v may be regarded as a 1/2-harmonic map after composition with the stereographic projection. $\square$

2.3.3.4 Existence of Global Solutions

Finally, we have all the necessary tools at our disposal to tackle the global existence problem in full generality. The main idea will be that one is easily able to extend solutions on a finite time-interval by using convergence properties as t goes to the critical time. A direct argument shows that the extension by gluing a solution at the critical time for appropriate initial data will give a global solution after at most finitely many such extensions.

Proof by "Gluing" Let us show that we may extend a solution $u : [0,T[\times S^1 \to S^{n-1}$ to be a weak solution on a slightly bigger time interval. This may be done by first observing that due to the monotone decay of energy:

$$E_{1/2}(u(t)) \leq E_{1/2}(u_0) < +\infty \tag{2.340}$$

Therefore, we may deduce that for an appropriate sequence $u(t_n) \to v \in H^{1/2}(S^1)$ with $t_n \to T$. Moreover, since $u \in H^1([0,T[;L^2(S^1))$, we must have convergence:

$$\lim_{t\to T} u(t) = v \quad \text{ in } L^2(S^1), \tag{2.341}$$

due to a standard continuity argument:

$$\|u(t) - u(t_0)\|_{L^2}^2 \leq \int_{t_0}^{t} \|\partial_t u(s)\|_{L^2}^2 ds \to 0, \quad \text{ as } t, t_0 \to T$$

This also shows uniqueness of v independent of any choice of sequence $t_n \to T$ which results in the desired L^2-convergence.

Next, we want to estimate the 1/2-energy of v. To do this, let us assume that there is just one bubbling point x_0 at time T (the general case follows analogously, losing energy in finitely many points). Then we have, using a limit to avoid the concentration at the point x_0:

$$\begin{aligned}
E_{1/2}(v) &= \int_{S^1}\int_{S^1} \frac{|v(x)-v(y)|^2}{|x-y|^2} dydx \\
&= \lim_{r\to 0} \int_{S^1\setminus B_r(x_0)} \int_{S^1\setminus B_r(x_0)} \frac{|v(x)-v(y)|^2}{|x-y|^2} dydx \\
&= \lim_{r\to 0} \liminf_{n\to\infty} \int_{S^1\setminus B_r(x_0)} \int_{S^1\setminus B_r(x_0)} \frac{|u(t_n,x)-u(t_n,y)|^2}{|x-y|^2} dydx \\
&\leq \liminf_{n\to\infty} E_{1/2}(u_n) - \varepsilon_0 = \lim_{t\to T} E_{1/2}(u(t)) - \varepsilon_0,
\end{aligned} \tag{2.342}$$

where ε_0 denotes a quantum of energy that is concentrated close to x_0. Indeed, by omitting a neighbourhood of x_0, the energy accumulated in this point is not excluded in the limit and therefore leads to a loss of energy (energy which is ultimately recovered in the form of a bubble). As ε_0 is independent of u and T, we deduce that bubbling may only occur in finitely many points, as the 1/2-energy is decreasing and bounded from below by 0. Thus, we do not have to worry about accumulations of blow-up points.

One concludes now by extending the solution u after T by the main existence result in [103], Theorem 2.3.1.2. The fact that we have obtained a weak solution is easily verified by a direct computation based on the L^2-convergence of $u(t)$ as $t \to T$, thus establishing the desired global existence result. Indeed, we assume that $u : [0, +\infty[\times S^1 \to S^{n-1}$ bubbles at time $T = 1$, the general case with finitely many times in which bubbling occur follows completely analogously. Let $\varphi \in C_c^\infty(]0, \infty[\times S^1)$, since we know that the equation holds true for sufficiently small times. Then we have:

$$
\begin{aligned}
&\int_0^\infty \int_{S^1} \partial_t u \cdot \varphi dxdt + \int_0^\infty \int_{S^1} (-\Delta)^{1/2} u \cdot \varphi dxdt \\
&= -\int_0^\infty \int_{S^1} u \cdot \partial_t \varphi dxdt + \int_0^\infty \int_{S^1} (-\Delta)^{1/4} u \cdot (-\Delta)^{1/4} \varphi dxdt \\
&= -\int_0^1 \int_{S^1} u \cdot \partial_t \varphi dxdt + \int_0^1 \int_{S^1} (-\Delta)^{1/4} u \cdot (-\Delta)^{1/4} \varphi dxdt \\
&- \int_1^\infty \int_{S^1} u \cdot \partial_t \varphi dxdt + \int_1^\infty \int_{S^1} (-\Delta)^{1/4} u \cdot (-\Delta)^{1/4} \varphi dxdt \\
&= \int_0^1 \int_{S^1} \partial_t u \cdot \varphi dxdt + \int_0^1 \int_{S^1} (-\Delta)^{1/4} u \cdot (-\Delta)^{1/4} \varphi dxdt - \int_{S^1} u(1,x)\varphi(1,x) dx \\
&+ \int_1^\infty \int_{S^1} \partial_t u \cdot \varphi dxdt + \int_1^\infty \int_{S^1} (-\Delta)^{1/4} u \cdot (-\Delta)^{1/4} \varphi dxdt + \int_{S^1} u(1,x)\varphi(1,x) dx \\
&= \int_0^1 u |d_{1/2} u|^2 \varphi dxdt - \int_{S^1} u(1,x)\varphi(1,x) dx + \int_{S^1} u(1,x)\varphi(1,x) dx + \int_1^\infty u |d_{1/2} u|^2 \varphi dxdt \\
&= \int_0^\infty u |d_{1/2} u|^2 \varphi dxdt, \qquad (2.343)
\end{aligned}
$$

which proves the fact that u extended as explained yields a global weak solution. The first line equation is just the distributional formulation, later on we use integration by parts on $[0, \tilde{t}]$ and taking limits $\tilde{t} \to T$. Naturally, similar limits are taken for $[\tilde{t}, \infty[$. Observe that the boundary terms at time $T = 1$ appear due to the previous discussion of convergence in L^2 and by the boundary value properties of the extension, see Theorem 2.3.1.2. We highlight that $u(1, x)$ is defined for the extended solution to be that limit of the $u(t, x)$ in L^2 and weak limit in $H^{1/2}$, as $t \to 1$, see (2.341). Iterating this procedure finitely many times provides therefore a global weak solution.

In conclusion, we have the following, since the argument only superficially relies on $N = S^{n-1}$:

Theorem 2.3.3.2. *Let $u_0 \in H^{1/2}(S^1; N)$, then there exists a weak solution with non-increasing 1/2-Dirichlet energy:*

$$u : [0, +\infty[\, \times S^1 \to N,$$

with $u \in L^\infty([0, +\infty[; H^{1/2}(S^1)) \cap H^1([0, +\infty[; L^2(S^1))$ such that, except for finitely many times $0 < T_1 < \ldots < T_n < T_{n+1} := +\infty$, the function u is smooth:

$$u \in C^\infty(]T_k, T_{k+1}[; N), \quad \forall k = 1, \ldots n.$$

Moreover, we may bound the number $n = n(u_0)$ as follows:

$$n(u_0) \leq \frac{E(u_0)}{\varepsilon_0},$$

where $\varepsilon_0 > 0$ is the minimum amount of $1/2$-energy a non-constant, half-harmonic map with values in N must possess.

A first uniqueness statement may also be derived from the results in [103]. However, it should be noted that uniqueness among energy class solution (weak solutions) cannot be proven by our previous arguments and thus requires further investigations. Finally, the existence of finite time bubbling is still unresolved, so the result above provides potentially a suitable regularity statement at bubbling points to help understand obstructions for bubbling or build examples in future work.

Proof by Variational Arguments In this section, we derive an alternative proof of the global weak existence of solutions to the half-harmonic gradient flow using techniques from Calculus of Variations similar to Audrito [2]. This approach does lead to existence of solutions, however, it leaves open many questions regarding the properties of the solution, most importantly regarding monotonicity of the $1/2$-Dirichlet energy. In particular, if the solutions constructed do not have monotonically decaying energy, then the solution provides an example of non-uniqueness of solutions to the half-harmonic map equation.

The definition of the energy follows Audrito [2]. Let $\varepsilon > 0$ be any positive real number. We define the following space of functions for $s \in]0,1[$ and $1 < p < +\infty$:

$$\mathcal{V}^{s,p} := H^1([0,+\infty[; L^2(S^1;\mathbb{R}^n)) \cap L^2_{loc}([0,+\infty[; W^{s,p}(S^1;\mathbb{R}^n)),$$

and use this definition to introduce for any $u_0 \in W^{s,p}(S^1;N)$, where N is a closed submanifold in $\mathbb{R}^n$:

$$\mathcal{U}^{s,p}(u_0) := \left\{ u \in \mathcal{V}^{s,p} \;\middle|\; u(t,x) \in N \text{ a.e.}, u(0) = u_0 \right\} \tag{2.344}$$

Comparing with Schikorra-Sire-Wang [77], the space (2.344) actually coincides with space in which the solutions constructed there exist. Moreover, we define the following family of energies:

$$\mathcal{E}^{s,p}_{\varepsilon}(u) := \int_0^{+\infty} \int_{S^1} e^{-t/\varepsilon} \left(\varepsilon \cdot |\partial_t u(t,x)|^2 + \frac{2}{p} \cdot \int_{S^1} \left| \frac{u(t,x) - u(t,y)}{|x-y|^s} \right|^p \frac{dy}{|x-y|} \right) dxdt, \tag{2.345}$$

for any $u \in \mathcal{U}^{s,p}(u_0)$. One notices that the energy is indeed well-defined and finite in this case. An obvious member of $\mathcal{U}^{s,p}(u_0)$ is the following map:

$$u(t,x) := u_0(x),$$

and this shows:

$$\inf_{u \in \mathcal{U}^{s,p}(u_0)} \mathcal{E}^{s,p}_{\varepsilon}(u) \leq 2E_{s,p}(u_0) \cdot \int_0^{\infty} e^{-t/\varepsilon} dt = 2\varepsilon \cdot E_{s,p}(u_0), \tag{2.346}$$

where we use the definition of $E_{s,p}$ as in Schikorra-Sire-Wang [77]. Thus, we immeidately see that if $(u_\varepsilon)_{\varepsilon \in]0,1[}$ is a sequence of minimizers, then the energies will become arbitrarily small. Additionally, existence of minimizers can easily be proven by the direct method.

Defining $v(t,x) := u(\varepsilon t, x)$, we see:

$$\begin{aligned}
\mathcal{E}_\varepsilon^{s,p}(u) &= \int_0^{+\infty}\int_{S^1} e^{-t/\varepsilon}\left(\varepsilon\cdot|\partial_t u(t,x)|^2 + \frac{2}{p}\cdot\int_{S^1}\left|\frac{u(t,x)-u(t,y)}{|x-y|^s}\right|^p\frac{dy}{|x-y|}\right)dxdt \\
&= \int_0^{+\infty}\int_{S^1} \varepsilon e^{-s}\left(\varepsilon\cdot|\partial_t u(\varepsilon s,x)|^2 + \frac{2}{p}\cdot\int_{S^1}\left|\frac{u(\varepsilon s,x)-u(\varepsilon s,y)}{|x-y|^s}\right|^p\frac{dy}{|x-y|}\right)dxds \\
&= \int_0^{+\infty}\int_{S^1} e^{-s}\left(|\partial_t v(s,x)|^2 + \frac{2\varepsilon}{p}\cdot\int_{S^1}\left|\frac{v(s,x)-v(s,y)}{|x-y|^s}\right|^p\frac{dy}{|x-y|}\right)dxds \\
&=: \mathcal{J}_\varepsilon^{s,p}(v)
\end{aligned} \tag{2.347}$$

Notice that v still lies in $\mathcal{U}^{s,p}(u_0)$ and that by computation above, we know that minimising $\mathcal{E}_\varepsilon$ and minimising $\mathcal{J}_\varepsilon$ is equivalent respecting the reparametrisation in time.

Let us now compute the Euler-Lagrange equation for $\mathcal{E}_\varepsilon^{s,p}$:

Lemma 2.3.3.5. *The Euler-Lagrange equation for minimisers $u \in \mathcal{U}^{s,p}(u_0)$ of $\mathcal{E}_\varepsilon^{s,p}$ can be stated as:*

$$-\varepsilon\partial_t^2 u(t,x) + \partial_t u(t,x) + \operatorname{div}_s\left(|d_s u(t,x,y)|^{p-2} d_s u(t,x,y)\right) \perp T_u N, \qquad \text{in } \mathcal{D}'(]0,+\infty[\times S^1) \tag{2.348}$$

Proof. We take the competitors:

$$u_\delta(t,x) := \pi(u+\delta\varphi),$$

where $\delta \in \mathbb{R}$ and $\varphi \in C_c^\infty(]0,\infty[\times S^1;\mathbb{R}^n)$. Moreover, π denotes the closest point projection onto N.

If u is a minimizer, then:

$$0 = \frac{d}{d\delta}\mathcal{E}_\varepsilon^{s,p}(u_\delta)\Big|_{\delta=0}$$

Using the explicit formula (2.345) for the energy (observe that u_δ lies in the correct space for every $\delta \in \mathbb{R}$ sufficiently small), one can differentiate immediately (we use $d_0 u(t,x,y) = u(t,x) - u(t,y)$ to simplify the terms):

$$0 = \int_0^{+\infty}\int_{S^1} e^{-t/\varepsilon}\left(2\varepsilon\partial_t u\cdot\partial_t\left(d\pi(u)\varphi\right) + 2\int_{S^1}\frac{|u(t,x)-u(t,y)|^{p-2}}{|x-y|^{1+sp}} d_0 u(t,x,y)\cdot d_0\left(d\pi(u)\varphi\right)(t,x,y)dy\right)dxdt$$

If we choose $\psi(t,x) = e^{-t/\varepsilon}\varphi(t,x)$, then:

$$0 = \int_0^{+\infty}\int_{S^1} \varepsilon\partial_t u\cdot\partial_t\left(d\pi(u)\psi\right) + \partial_t u\cdot d\pi(u)\psi + \int_{S^1}\frac{|u(t,x)-u(t,y)|^{p-2}}{|x-y|^{1+sp}} d_0 u(t,x,y)\cdot d_0\left(d\pi(u)\psi\right)(t,x,y)dydxdt$$

So the Euler-Lagrange equation is equivalent to:

$$-\varepsilon\partial_t^2 u(t,x) + \partial_t u(t,x) + \operatorname{div}_s\left(|d_s u(t,x,y)|^{p-2} d_s u(t,x,y)\right) \perp T_u N, \qquad \text{in } \mathcal{D}'(]0,+\infty[\times S^1),$$

i.e. up to the term involving the second derivative in time direction we recognise the fractional harmonic gradient flow. This proves (2.348). $\square$

In particular, if $s = 1/2, p = 2$, we find the same equation as in [103], up to the second order derivative in t. This is also the case we shall restrict our attention to for now (writing $\mathcal{J}_\varepsilon$ instead of $\mathcal{J}_\varepsilon^{1/2,2}$), the general case for arbitrary fractional harmonic flows may be treated in a completely analogous way, also extending the existence result in Schikorra-Sire-Wang [77] in a wider setting.

The ideas to complete the proof then are very similar to Audrito [2]. Namely, one may define completely analogously:

$$I(t) := \int_{S^1} |\partial_t v(t,x)|^2 dx \tag{2.349}$$

$$R(t) := \varepsilon \cdot \int_{S^1} |d_{1/2} v(t)|(x)^2 dx \tag{2.350}$$

$$E(t) := e^t \int_t^\infty e^{-s} \left(I(s) + R(s) \right) ds \tag{2.351}$$

It is easily observed that for miniizers v, we have $I, R \in L^1_{loc}([0,\infty[)$ and $e^{-s}(I(s)+R(s)) \in L^1([0,\infty[)$. Additionally, $E \in W^{1,1}_{loc}(]0,\infty[) \cap C^0([0,\infty[)$ as well as:

$$E' = E - I - R \quad \text{in } \mathcal{D}'(]0,\infty[)$$

The proof of the following lemma is an immediate adaption of the technique in Audrito [2]:

Lemma 2.3.3.6. *Assume v is a minimizer of $\mathcal{J}_\varepsilon$. Then:*

$$E'(t) = -2I(t), \qquad \textit{in } \mathcal{D}'(]0,\infty[) \tag{2.352}$$

The proof relies on suitable choices of reparametrisations in time for v and then using minimality of v. Ultimately, this allows us to show:

Lemma 2.3.3.7. *For v a minimizer of $\mathcal{J}_\varepsilon$, we have:*

$$\int_0^\infty |\partial_t v(t,x)|^2 dx dt \leq C\varepsilon, \tag{2.353}$$

as well as for any $t \geq 0$:

$$\int_t^{t+1} \int_{S^1} |d_{1/2} u|^2(t,x) dx dt \leq C, \tag{2.354}$$

for some constant $C > 0$, depending on u_0, but not ε or v.

Proof. $E(t)$ is necessarily non-increasing due to $I(t) \geq 0$, therefore:

$$E(t) \leq E(0) = \mathcal{J}_\varepsilon(v), \forall t \geq 0$$

Additionally, for any given t, we know:

$$\int_0^t I(s) ds = \frac{1}{2} \int_0^t E'(s) ds = \frac{1}{2} \left(E(0) - E(t) \right) \leq \frac{1}{2} E(0) \leq \frac{C}{2} \varepsilon,$$

by using (2.346). Letting $t \to \infty$ proves (2.353) by using (2.349).

The remaining part of the proof requires us to use (2.350) as well as:

$$\int_t^{t+1} R(s) ds = e^{t+1} \int_t^{t+1} e^{-s} R(s) ds$$

$$\leq e^{t+1} \int_{t}^{t+1} e^{-s} \left(I(s) + R(s) \right) ds \tag{2.355}$$

$$\leq e \cdot E(t) \leq Ce \cdot \varepsilon, \tag{2.356}$$

again relying on (2.346) and the bound on $E(t)$ established above. □

Thus, to obtain a solution of the half-harmonic gradient flow (which follows thanks to (2.348) after letting $\varepsilon \to 0$), one now just has to rescale the minimizer v back to u and use the following uniform bounds to extract weakly convergent subsequences. Thus, we are done, as we may extract further subsequences converging almost surely pointwise, ensuring that the limiting function assumes values only in N.

3 Compensation Phenomena [25], [26]

3.1 Improved Regularity Estimate à la Bourgain-Brezis [25]

We will now study a Bourgain-Brezis-type inequality that allows us to bound the L^2-norm of functions on $\mathbb{T}^n$, provided that suitable fractional Laplacians and their Riesz transforms all belong to $\dot{H}^{-n/2}(\mathbb{T}^n) + L^1(\mathbb{T}^n)$. Surprisingly, such a compensation result is closely connected to a characterisation of Bergmann spaces, the spaces of holomorphic functions on the disc equipped with the L^2-norm, similar to Hardy spaces. To establish the desired estimate, we define suitable operators in terms of fractional Laplacians and Riesz transforms and employ the emergence of a convolution operator which actually is bounded and thus has better regularity properties than a-priori expected.

3.1.1 Introduction

In his pioneering work [67], Riesz studied fine properties of the so-called Hardy spaces $\mathcal{H}^p(\mathbb{D})$, which are the spaces of holomorphic functions [1] $f\colon \mathbb{D} \to \mathbb{C}$ such that

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < +\infty \tag{3.1}$$

for $p > 0$. Under condition (3.1), it is known that $f(e^{i\theta})$ exists and

$$\lim_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p d\theta = 0 \ \text{ as well as } \ \lim_{r\to 1^-} f(re^{i\theta}) = f(e^{i\theta}), \tag{3.2}$$

for almost every θ, (see e.g. Section 4 [67]). For $f \in \mathcal{H}^p(\mathbb{D})$, one defines $\|f\|_{\mathcal{H}^p(\mathbb{D})} := \|f\|_{L^p(S^1)}$.

We can independently consider holomorphic functions in $L^2(\mathbb{D})$ which corresponds to the well-known Bergman space $\mathcal{A}^2(\mathbb{D})$ [2], see e.g Duren-Schuster [32].

The connection between Hardy spaces and the Bergman space $\mathcal{A}^2(\mathbb{D})$ is given by the embedding $\mathcal{H}^1(\mathbb{D}) \hookrightarrow \mathcal{A}^2(\mathbb{D})$ together with the estimate

$$\|f\|_{L^2(\mathbb{D})} \leq C\|f\|_{\mathcal{H}^1(\mathbb{D})} := \|f\|_{L^1(S^1)}. \tag{3.3}$$

In the case $\lim_{r\to 1^-} \|f(re^{i\theta})\|_{H^{-1/2}(S^1)} < +\infty$, then, by definition, the following inequality holds as well:

$$\|f\|_{L^2(\mathbb{D})} \leq C\|f\|_{H^{-1/2}(S^1)} := \lim_{r\to 1^-} \|f(re^{i\theta})\|_{H^{-1/2}(S^1)}. \tag{3.4}$$

In this note, we prove the following combination of (3.3) and (3.4):

---

[1] In 1915 Hardy observed that if f is holomorphic in $\mathbb{D}$ then $r \mapsto M(r) = \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$ is a nondcreasing function.

[2] We recall that $\mathcal{A}^2(\mathbb{D}) := \{f\colon \mathbb{D} \to \mathbb{C} :\ f \text{ holomorphic and } \|f\|_{L^2(\mathbb{D})} < +\infty\}$

Theorem 3.1.1.1. *Let $f\colon \mathbb{D} \to \mathbb{C}$ be an analytic function. Then f belongs to the Bergman space $\mathcal{A}^2(\mathbb{D})$ if and only if*

$$\|f\|_{L^1+H^{-1/2}(S^1)} := \limsup_{r\to 1^-} \|f(re^{i\theta})\|_{L^1+H^{-1/2}(S^1)} < +\infty.$$

Moreover, it holds

$$\|f\|_{L^2(\mathbb{D})} \le C\|f\|_{L^1+H^{-1/2}(S^1)}. \tag{3.5}$$

Lastly, in section 3.1.7, we provide a proof of the inequalities (3.3) and (3.4).

This type of inequalities takes its roots in the pioneering work [8], where Bourgain and Brezis proved the following striking result:

Theorem 3.1.1.2 (Lemma 1 in [8]). *Let u be a 2π-periodic function in $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} u\,dx = 0$, and let $\nabla u = f + g$, where $f \in \dot{W}^{-1,\frac{n}{n-1}}(\mathbb{R}^n)$*[3] *and $g \in L^1(\mathbb{R}^n)$ are 2π-periodic vector-valued functions. Then*

$$\|u\|_{L^{\frac{n}{n-1}}} \le c\left(\|f\|_{\dot{W}^{-1,\frac{n}{n-1}}} + \|g\|_{L^1}\right). \tag{3.6}$$

By duality, this implies the following corollary:

Corollary 3.1.1.1 (Theorem 1 in [8]). *For every 2π-periodic function $h \in L^n(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} h = 0$, there exists a 2π-periodic $v \in \dot{W}^{1,n} \cap L^\infty(\mathbb{R}^n)$ satisfying*

$$\operatorname{div} v = h \quad \text{in } \mathbb{R}^n$$

and

$$\|v\|_{L^\infty} + \|v\|_{\dot{W}^{1,n}} \le C(n)\|h\|_{L^n}. \tag{3.7}$$

One of the main result of this note is a fractional type Bourgain-Brezis inequality on the circle S^1 and on $\mathbb{T}^n$. More precisely, we have the following:

Theorem 3.1.1.3. *Let $u \in \mathcal{D}'(S^1)$ be such that $(-\Delta)^{\frac14}u, \mathcal{R}(-\Delta)^{\frac14}u \in \dot{H}^{-\frac12}(S^1) + L^1(S^1)$.*[4] *Then $u - \fint_{S^1} u \in L^2_*(S^1)$ and the following estimate holds true:*

$$\left\|u - \fint_{S^1} u\right\|_{L^2} \le C\left(\|(-\Delta)^{1/4}u\|_{\dot{H}^{-1/2}(S^1)+L^1(S^1)} + \|\mathcal{R}(-\Delta)^{1/4}u\|_{\dot{H}^{-1/2}(S^1)+L^1(S^1)}\right), \tag{3.8}$$

for some $C > 0$ independent of u.

Theorem 3.1.1.4. *Let $u \in \mathcal{D}'(\mathbb{T}^n)$ be complex-valued and such that:*

$$(-\Delta)^{\frac{n}{4}}u, \mathcal{R}_j(-\Delta)^{\frac{n}{4}}u \in (L^1 + \dot{H}^{-\frac{n}{2}})(\mathbb{T}^n), \quad \forall j \in \{1, \dots n\}.$$

Then we have $u - \fint_{\mathbb{T}^n} udx \in L^2_*(\mathbb{T}^n)$ with

$$\left\|u - \fint_{\mathbb{T}^n} udx\right\|_{L^2} \le C\left(\|(-\Delta)^{n/4}u\|_{L^1+\dot{H}^{-n/2}(\mathbb{T}^n)} + \sum_{j=1}^n \|\mathcal{R}_j(-\Delta)^{n/4}u\|_{L^1+\dot{H}^{-n/2}(\mathbb{T}^n)}\right), \tag{3.9}$$

for some $C > 0$ independent of u.

[3] For $1 < p < +\infty$, we will denote by $\dot{W}^{1,p}(\mathbb{R}^n)$ the homogeneous Sobolev space defined as the space of $f \in L^1_{loc}(\mathbb{R}^n)$ such that $\nabla f \in L^p(\mathbb{R}^n)$ and by $\dot{W}^{-1,p'}(\mathbb{R}^n)$ the corresponding dual space (p' is the conjugate of p). Every function $f \in \dot{W}^{-1,p'}(\mathbb{R}^n)$ can be represented as $f = \sum_{i=1}^n \partial_{x_i} f^j$ with $f^j \in L^{p'}(\mathbb{R}^n)$.

[4] We denote by $\mathcal{R}$ and $\mathcal{R}_j$ the Riesz transform respectively on S^1 and with respect to the x_j variable on $\mathbb{T}^n$, for $j \in \{1, \dots, n\}$ and by $\dot{H}^{-\frac{n}{2}}(\mathbb{T}^n)$ the space of $f \in \mathcal{D}'(S^1)$ such that $f = (-\Delta)^{n/4}g$, with $g \in L^2(\mathbb{T}^n)$. Recall that $L^2_*(S^1) := \{u \in L^2(S^1) : \ \fint_{S^1} u = 0\}$.

The second main result is the equivalence between Theorems 3.1.1.1 and 3.1.1.3, establishing the connection between fractional Bourgain-Brezis inequalities and Bergman spaces. It would be interesting to investigate a similar connection in dimensions $n \geq 2$.

We would like to add some comments about Bourgain-Brezis' inequality (3.6). The inequality (3.6) in its general form is of interest in the study of the PDE $\operatorname{div} Y = f$ for $f \in L^n_*(\mathbb{T}^n)$, where for finite $p \geq 1$, $L^p_*(\mathbb{T}^n)$ denotes the Banach subspace of L^p-functions with vanishing mean over the torus. Precisely, they found that Y can be chosen to be continuous and in $\dot{W}^{1,n}(\mathbb{T}^n)$, a result which is non-trivial due to the fact that $\dot{W}^{1,n}(\mathbb{T}^n)$ does not continuously embed into $L^\infty(\mathbb{T}^n)$. The key ingredient in the proof is a duality argument based on an estimate similar to (3.6) and some general results from functional analysis regarding closedness properties of the image space. This motivates the general interest in inequalities of the same type, as improved regularity results in limit cases can be invaluable. Indeed, later, such estimates have been considered and extended in different directions. In [9], Bourgain and Brezis showed how Theorem 1 in [8] is closely connected to a remarkable property concerning differential forms with coefficients in the critical Sobolev space $W^{1,n}(\mathbb{T}^n)$ and they got new regularity results for the Hodge decomposition. In [58], Maz'ya extended the inequality (3.6) on the Sobolev space $H^{1-\frac{n}{2}}(\mathbb{R}^n)$ leading to a different existence result for the PDE $\operatorname{div} Y = f$. Finally in [61], Mironescu unified in 2 dimensions the two different approaches in Bourgain-Brezis [8] and in Maz'ya [58] by a PDE-approach consisting in using elementary properties of fundamental solutions of the biharmonic operator. In Da Lio-Rivière [23], the first two authors of the current paper provide an alternative proof of (3.6) in 2 dimensions without assuming the periodicity of the function u. The proof is related to some compensation phenomena observed first in Delort [30] in the analysis of 2-dimensional perfect incompressible fluids and then also applied by Rivière in [72] in the analysis of *isothermic surfaces*. For an overview of the results in the literature regarding variations of Theorem 3.1.1.2 and Corollary 3.1.1.1, we refer for instance to very interesting paper by Van Schaftingen [99].

As seen in Da Lio-Rivière [23], the inequality (3.6) also represents the first key ingredient in the study of the regularity of $L^2(\mathbb{D}, \mathbb{R}^n)$ solutions u to a linear elliptic system of the following form

$$\operatorname{div}(S\,\nabla u) \;=\; \sum_{j=1}^{n} \operatorname{div}(S_{ij}\,\nabla u^j) = \sum_{j=1}^{n}\sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha}\left(S_{ij} u^j_{x_\alpha}\right) = 0, \tag{3.10}$$

where S is a $W^{1,2}(\mathbb{D})$ symmetric $n \times n$ matrix, such that $S^2 = id_n$.

We would like to mention some results on Riesz potentials showing that the 1-dimensional case plays a particular role in the L^1-estimates for Riesz potentials. More precisely, one can deduce from the results in Stein-Weiss [86] that for all $0 < \alpha < 1$, we have:

$$\|I_\alpha u\|_{L^{\frac{1}{1-\alpha}}} \leq C(\|\mathcal{R}u\|_{L^1} + \|u\|_{L^1}), \tag{3.11}$$

for all u in the Hardy space $\mathcal{H}^1(\mathbb{R})$. It follows in particular that:

$$\|u\|_{L^{\frac{1}{1-\alpha}}} \leq C(\|\mathcal{R}(-\Delta)^{\alpha/2} u\|_{L^1} + \|(-\Delta)^{\alpha/2} u\|_{L^1}).$$

In Schikorra-Spector-Van Schaftingen [78], the authors show that if $N \geq 2$ and $0 < \alpha < N$, then there is a constant $C = C(\alpha, N) > 0$ such that

$$\|I_\alpha u\|_{L^{\frac{N}{N-\alpha}}} \leq C\|\mathcal{R}u\|_{L^1} \tag{3.12}$$

for all $u \in C_c^\infty(\mathbb{R}^N)$, such that $\mathcal{R}u \in L^1(\mathbb{R}^N)$. The estimate (3.12) is however false in 1-D, as seen in Schikorra-Spector-Van Schaftingen [78].

The inequality (3.8) generalizes the inequality (3.11) in the case $\alpha = 1/2$ and the counter-example in Schikorra-Spector-Van Schaftingen [78] for the estimate (3.12) in 1-D shows that the estimate (3.8) is in some sense optimal.

We finally point out that it may be interesting to investigate a possible generalization of Theorem 3.1.1.4 in the framework of nonlocal operators on differential forms as it has been done in Bourgain-Brezis [9].

The paper is organized as follows: In section 3.1.2.1, we recall the definitions of the fractional Laplacian on the unit circle and on the torus. In section 3.1.3, we provide two distinct proofs of Theorem 3.1.1.3. In section 3.1.4, we establish the equivalence of Theorem 3.1.1.1 and Theorem 3.1.1.3. In section 3.1.5, we provide a short introduction of Clifford algebras and we extend the fractional Bourgain-Brezis inequality using Clifford algebras to the n-dimensional torus $\mathbb{T}^n$. In section 3.1.6, we prove existence results for certain fractional PDE-operators in the same spirit as Corollary 3.1.1.1. In section 3.1.7, we provide a proof of the inequalities (3.3) and (3.4) for the reader's convenience.

3.1.2 Preliminaries

3.1.2.1 Fractional Laplacian on the unit circle and on the torus

Before we enter the discussion and the proofs of the main results, let us recall a few notions essential in our later arguments. We mainly focus on fractional Laplacians, fractional Sobolev.

Throughout this note, we shall denote by $\mathbb{T}^n$ the torus of dimension $n \in \mathbb{N}$. This means:

$$\mathbb{T}^n = \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}} = \mathbb{R}^n/(2\pi\mathbb{Z})^n \tag{3.13}$$

where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. We denote by $\mathcal{D}(\mathbb{T}^n) := C^\infty(\mathbb{T}^n)$ the Fréchet space of smooth functions on $\mathbb{T}^n$ and by $\mathcal{D}'(\mathbb{T}^n)$ its topological dual. The natural duality paring is denoted by $\langle \cdot, \cdot \rangle$.

For $u \in \mathcal{D}'(\mathbb{T}^n)$ and $m \in \mathbb{Z}^n$, we define the Fourier coefficients of u as follows:

$$\widehat{u}(m) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} u(x) e^{-i\langle m, x\rangle} dx = \left\langle u, e^{-i\langle m, \cdot\rangle} \right\rangle. \tag{3.14}$$

The Fourier coefficients completely determine u as a distribution on $\mathbb{T}^n$ and convergence in the sense of distributions obviously implies convergence of the Fourier coefficients. Notice that, for all $u \in \mathcal{D}'(\mathbb{T}^n)$, there exists some $N > 0$ such that $|\hat{u}(m)| \lesssim (1 + |m|)^N$. Moreover, we recall that $v \in C^\infty(\mathbb{T}^n)$ if and only if the Fourier coefficients $\hat{v}(m)$ have rapid decay, i.e. $\sup_m (1 + |m|)^N |\hat{v}(m)| < \infty$ for all $N > 0$.

Given $s \in \mathbb{R}$, we define the non-homogeneous and homogeneous Sobolev spaces respectively by

$$H^s(\mathbb{T}^n) := \left\{ v \in \mathcal{D}'(\mathbb{T}^n) : \|v\|_{H^s}^2 := \sum_{k \in \mathbb{Z}^n} (1 + |k|^2)^s |\hat{v}(k)|^2 < \infty \right\},$$

and

$$\dot{H}^s(\mathbb{T}^n) := \left\{ v \in \mathcal{D}'(\mathbb{T}^n) : \|v\|_{\dot{H}^s}^2 := \sum_{k \in \mathbb{Z}^n} |k|^{2s} |\hat{v}(k)|^2 < \infty \right\},$$

where $\mathcal{D}'(\mathbb{T}^n)$ is again the space of distributions on $\mathbb{T}^n$. Notice that if $s = 0$, we have $L^2(\mathbb{T}^n) = H^0(\mathbb{T}^n)$ and $L^2_*(\mathbb{T}^n) \simeq \dot{H}^0(\mathbb{T}^n)$.

An important family of operators throughout our considerations are the so-called *fractional Laplacians.* Let $s > 0$ be real, then we define for $u : \mathbb{T}^n \to \mathbb{C}$ smooth the s-Laplacian of u by the following multiplier property:

$$\widehat{(-\Delta)^s} u(m) = |m|^{2s}\widehat{u}(m), \quad \forall m \in \mathbb{Z}^n. \tag{3.15}$$

Clearly, this definition can immediately be extended to the spaces $H^s(\mathbb{T}^n)$ or even $\mathcal{D}'(\mathbb{T}^n)$ as a multiplier operator on Fourier coefficients, possibly defining merely a distribution on $\mathbb{T}^n$. Finally, we recall the definition of the j-Riesz transform on $\mathbb{T}^n$ as a multiplier operator:

$$\mathcal{R}_j u(x) = \sum_{m\in\mathbb{Z}^n} i\frac{m_j}{|m|}\widehat{u}(m)e^{i\langle m,x\rangle}, \quad \forall x \in \mathbb{T}^n. \tag{3.16}$$

In particular, in the case $n = 1$, we have:

$$\mathcal{R}u(x) = \sum_{m\in\mathbb{Z}} i\,\mathrm{sign}(m)\widehat{u}(m)e^{im\cdot x}, \quad \forall x \in S^1. \tag{3.17}$$

3.1.3 Fractional Bourgain-Brezis inequality on the unit circle $\mathbf{S}^1$

In this section, we provide two distinct proofs of Theorem 3.1.1.3. The first proof is in the spirit of the one presented in Bourgain-Brezis [8], while the second one is inspired by that in Da Lio-Rivière [23] and is based on some particular compensation phenomena. We assume for simplicity that u is real valued (the proof for complex-valued function is completely analogous, see Remark 3.1.3.1).

First, we would like to observe that if $u \in C^\infty(S^1)$, then by definition:

$$\left\| u - \fint_{S^1} u \right\|_{L^2} \leq C\|(-\Delta)^{1/4}u\|_{\dot{H}^{-1/2}(S^1)}. \tag{3.18}$$

On the other hand, as we have already observed in the introduction, we also have[5]:

$$\left\| u - \fint_{S^1} u \right\|_{L^2(S^1)} \leq C\left(\|(-\Delta)^{1/4}u\|_{L^1(S^1)} + \|\mathcal{R}(-\Delta)^{1/4}u\|_{L^1(S^1)}\right) \simeq \|(-\Delta)^{1/4}u\|_{\mathcal{H}^1(S^1)} \tag{3.19}$$

3.1.3.1 A first proof of Theorem 3.1.1.3

Let us suppose that $u \in C^\infty(S^1)$, $\fint_{S^1} u = 0$. We assume for simplicity that u is real-valued, see Remark 3.1.3.1 for the complex-valued case. The proof below follows the main arguments of the original proof by Bourgain and Brezis. We write:

$$\begin{cases} (-\Delta)^{1/4}u = f^1 + g^1 \\ \mathcal{R}(-\Delta)^{1/4}u = f^2 + g^2 \end{cases} \tag{3.20}$$

where $f^1, f^2 \in \dot{H}^{-1/2}(S^1)$, $g^1, g^2 \in L^1(S^1)$. We set $u = \sum_{n\in\mathbb{Z}^*} u_n e^{in\theta}$. Since u is real-valued, it holds $\bar{u}_n = u_{-n}$. We see:

$$\sum_{n\in\mathbb{Z}^*} |u_n|^2 = \sum_{n\in\mathbb{Z}^*} |n|^{1/2}u_n\frac{u_{-n}}{|n|^{1/2}} = \sum_{n\in\mathbb{Z}^*} \frac{f_n^1 + g_n^1}{|n|^{1/2}}u_{-n}, \tag{3.21}$$

[5]Actually, an even sharper inequality than (3.19) holds true with $L^2(S^1)$ being replaced by the smaller Lorentz space $L^{2,1}(S^1)$.

$$\sum_{n\in\mathbb{Z}^*} \frac{f_n^1\, u_{-n}}{|n|^{1/2}} \;\leq\; \left[\sum_{n\in\mathbb{Z}^*} \frac{|f_n^1|^2}{|n|}\right]^{1/2} \left[\sum_{n\in\mathbb{Z}^*} |u_n|^2\right]^{1/2}, \tag{3.22}$$

$$\sum_{n\in\mathbb{Z}^*} \frac{g_n^1\, u_{-n}}{|n|^{1/2}} \;=\; \sum_{n>0} \frac{g_n^1\, u_{-n}}{|n|^{1/2}} + \sum_{n<0} \frac{g_n^1\, u_{-n}}{|n|^{1/2}}. \tag{3.23}$$

Observe that by definition of the Riesz transform:

$$\mathcal{R}(-\Delta)^{1/4}u \;=\; i\left[-\sum_{n<0} |n|^{1/2} u_n e^{in\theta} + \sum_{n>0} |n|^{1/2} u_n e^{in\theta}\right]. \tag{3.24}$$

Therefore:

$$u_n = \begin{cases} \frac{f_n^2+g_n^2}{-i|n|^{1/2}} & \text{if } n<0 \\ \\ \frac{f_n^2+g_n^2}{i|n|^{1/2}} & \text{if } n>0 \end{cases} \tag{3.25}$$

By combining (3.23) and (3.25), we obtain:

$$\sum_{n\in\mathbb{Z}^*} \frac{g_n^1\, u_{-n}}{|n|^{1/2}} \;=\; \sum_{n>0} g_n^1\, \frac{f_{-n}^2 + g_{-n}^2}{-i|n|} + \sum_{n<0} g_n^1\, \frac{f_{-n}^2 + g_{-n}^2}{i|n|}. \tag{3.26}$$

Let us estimate the different parts of the sum (3.26) individually:

1. We first estimate

$$\begin{aligned} \sum_{n\in\mathbb{Z}^*} \operatorname{sign}(n) \frac{g_n^1\, f_{-n}^2}{|n|} &= \sum_{n\in\mathbb{Z}^*} \operatorname{sign}(n) \frac{|n|^{1/2}u_n - f_n^1}{|n|^{1/2}} \frac{f_{-n}^2}{|n|^{1/2}} \\ &\leq \left(\sum_{n\in\mathbb{Z}^*} |u_n|^2\right)^{1/2} \left(\sum_{n\in\mathbb{Z}^*} \frac{|f_n^2|^2}{|n|}\right)^{1/2} + \left(\sum_{n\in\mathbb{Z}^*} \frac{|f_n^1|^2}{|n|}\right)^{1/2} \left(\sum_{n\in\mathbb{Z}^*} \frac{|f_n^2|^2}{|n|}\right)^{1/2} \\ &\leq \|u\|_{L^2} \|f^2\|_{\dot H^{-1/2}} + \|f^1\|_{\dot H^{-1/2}} \|f^2\|_{\dot H^{-1/2}}. \end{aligned} \tag{3.27}$$

2. It remains to estimate

$$\sum_{n\in\mathbb{Z}^*} \operatorname{sign}(n) \frac{g_n^1\, g_{-n}^2}{i|n|}.$$

For this purpose, we consider the following operator:

$$\mathbf{A}\colon L^1(S^1)\times L^1(S^1) \;\to\; \mathbb{C}, \qquad (g^1,g^2) \mapsto \sum_{n\in\mathbb{Z}^*} \operatorname{sign}(n) \frac{g_n^1\, g_{-n}^2}{i|n|}.$$

Claim 1. The operator $\mathbf{A}$ is continuous, i.e. we have the following estimate:

$$|\mathbf{A}(g^1,g^2)| \leq C\|g^1\|_{L^1} \|g^2\|_{L^1}. \tag{3.28}$$

Proof of Claim 1. It is sufficient to prove the claim in the case where g^1 and g^2 are arbitrary Dirac-delta measures.[6] Therefore, we consider $g^1 = \sum_{i\in I}\lambda_i\delta_{a_i}$ and $g^2 = \sum_{j\in J}\mu_j\delta_{b_j}$. We have $\|g^1\|_{\mathcal{M}(S^1)} = \sum_{i\in I}|\lambda_i|$, $\|g^2\|_{\mathcal{M}(S^1)} = \sum_{j\in J}|\mu_j|$. By bilinearity, we deduce:

$$
\begin{aligned}
|\mathbf{A}(g^1,g^2)| &= |\mathbf{A}(\sum_{i\in I}\lambda_i\delta_{a_i},\sum_{j\in J}\mu_j\delta_{b_j})| \\
&\leq \sum_{i\in I,j\in J}|\lambda_i||\mu_j||\mathbf{A}(\delta_{a_i},\delta_{b_j})| \\
&\leq \sup_{(a,b)\in S^1\times S^1}|\mathbf{A}(\delta_a,\delta_b)|\sum_{i\in I}|\lambda_i|\sum_{j\in J}|\mu_j| \\
&= \sup_{(a,b)\in S^1\times S^1}|\mathbf{A}(\delta_a,\delta_b)|\|g^1\|_{\mathcal{M}(S^1)}\|g^2\|_{\mathcal{M}(S^1)}. \qquad (3.29)
\end{aligned}
$$

If $\sup_{(a,b)\in S^1\times S^1}|\mathbf{A}(\delta_a,\delta_b)| < +\infty$, then the claim holds for linear combinations of Dirac measures. By a density argument, we get claim 1 for arbitrary $g^1, g^2 \in L^1(S^1)$. Hence, claim 1 is a consequence of the following:
Claim 2. $\sup_{(a,b)\in S^1\times S^1}|\mathbf{A}(\delta_a,\delta_b)| < +\infty$.
Proof of Claim 2. For $g^1 = \delta_a$ and $g^2 = \delta_b$, we have $g^1_n = e^{ina}$ and $g^2_n = e^{inb}$. In this case, we observe:

$$
\begin{aligned}
\mathbf{A}(\delta_a,\delta_b) &= \sum_{n\in Z^*}\operatorname{sign}(n)\frac{g^1_n g^2_{-n}}{i|n|} \\
&= \sum_{n\in\mathbb{Z}^*}\operatorname{sign}(n)\frac{e^{in(a-b)}}{i|n|} = 2\sum_{n>0}\frac{\sin(n(a-b))}{n} < +\infty.^7 \qquad (3.30)
\end{aligned}
$$

This proves claim 2 and from (3.30), we can deduce claim 1 as well.

By combining (3.21)-(3.29) we get

$$
\begin{aligned}
\|u\|^2_{L^2} &\lesssim \|u\|_{L^2}\left(\|f^1\|_{\dot H^{-1/2}} + \|f^2\|_{\dot H^{-1/2}}\right) + \|f^1\|_{\dot H^{-1/2}}\|f^2\|_{\dot H^{-1/2}} + C\|g^1\|_{L^1}\|g^2\|_{L^1} \\
&\lesssim \frac{1}{2}\|u\|^2_{L^2} + \frac{1}{2}\left(\|f^1\|^2_{\dot H^{-1/2}} + \|f^2\|^2_{\dot H^{-1/2}}\right) + \|f^1\|_{\dot H^{-1/2}}\|f^2\|_{\dot H^{-1/2}} + C\|g^1\|_{L^1}\|g^2\|_{L^1} \\
&\lesssim \frac{1}{2}\|u\|^2_{L^2} + \left(\|f^1\|^2_{\dot H^{-1/2}} + \|f^2\|^2_{\dot H^{-1/2}}\right) + \frac{1}{2}\left(\|g^1\|^2_{L^1} + \|g^2\|^2_{L^1}\right). \qquad (3.31)
\end{aligned}
$$

This estimate permits us to conclude the proof of Theorem 3.1.1.3. Since f^1, f^2, g^1, g^2 were arbitrary, one can deduce (3.8). In the general case where $u \in \mathcal{D}'(S^1)$, one argues by approximation (see section 3.1.3.2 for further details). $\square$

[6] We recall that the linear span of Dirac measures is dense in the space of Radon measures $\mathcal{M}(S^1)$ equipped with the weak-* topology.

[7] The value of such a series is deduced from the Fourier series of $f(x) = \frac{x}{2\pi}$ for $0 < x < 2\pi$ and $f(x+2\pi) = f(x)$.

3.1.3.2 A second proof of Theorem 3.1.1.3

As in the first proof, we will show the following: Let $u \in \mathcal{D}'(S^1)$ be such that:

$$(-\Delta)^{\frac{1}{4}} u = f_1 + g_1 \tag{3.32}$$

$$\mathcal{R}(-\Delta)^{\frac{1}{4}} u = f_2 + g_2, \tag{3.33}$$

where $f_1, f_2 \in \dot{H}^{-\frac{1}{2}}(S^1)$ and $g_1, g_2 \in L^1(S^1)$. Under these conditions, we prove:

$$u - \int_{S^1} u dx \in L^2_*(S^1) = \left\{ u \in L^2(S^1) : \ \fint_{S^1} u = 0 \right\}, \tag{3.34}$$

together with the following estimate:

$$\left\| u - \int_{S^1} u dx \right\|_{L^2} \leq C \big(\|f_1\|_{\dot{H}^{-\frac{1}{2}}} + \|f_2\|_{\dot{H}^{-\frac{1}{2}}} + \|g_1\|_{L^1} + \|g_2\|_{L^1} \big), \tag{3.35}$$

where $C > 0$ is independent of f_1, f_2, g_1, g_2 and u. We may assume for simplicity that u is real-valued (see Remark 3.1.3.1 for the complex-valued case).

Firstly, observe that it suffices to consider the case:

$$\int_{S^1} u dx = 2\pi \cdot \hat{u}(0) = 0, \tag{3.36}$$

by merely changing u by a constant. Similarly, by the conditions in (3.32) and (3.33), we see that f_j, g_j have vanishing integral over S^1 and consequently vanishing Fourier coefficient for $n = 0$.[8] For now, let us assume that u, f_j, g_j are all smooth on S^1. The general case can be dealt with using convolution with an appropriate smoothing kernel and approximation arguments as specified at the end of the proof.

First, let us define the following operators on $\mathcal{D}'(S^1)$:

$$Dv := (-\Delta)^{\frac{1}{4}} \big(Id + \mathcal{R} \big) v \tag{3.37}$$

$$\overline{D} v := (-\Delta)^{\frac{1}{4}} \big(Id - \mathcal{R} \big) v, \tag{3.38}$$

for every $v \in \mathcal{D}'(S^1)$. Consequently, using (3.32) and (3.33), we have:

$$Du = f_1 + f_2 + g_1 + g_2 = f + g \tag{3.39}$$

$$\overline{D} u = f_1 - f_2 + g_1 - g_2 = \tilde{f} + \tilde{g}. \tag{3.40}$$

Let us calculate the Fourier multipliers associated with $D, \overline{D}$. For every $n \in \mathbb{Z}$, we have:

$$\mathcal{F}\big(Dv\big)(n) = |n|^{\frac{1}{2}} (1 + i \operatorname{sign}(n)) \hat{v}(n) \tag{3.41}$$

$$\mathcal{F}\big(\overline{D} v\big)(n) = |n|^{\frac{1}{2}} (1 - i \operatorname{sign}(n)) \hat{v}(n) \tag{3.42}$$

---

[8] It would be possible to treat f_j, g_j with non-vanishing integral, i.e. treat the case $(-\Delta)^{1/4} u, \mathcal{R}(-\Delta)^{1/4} u \in L^1 + H^{-1/2}(S^1)$ by reducing to vanishing Fourier coefficient at $n = 0$: We have by the conditions $\widehat{f_j}(0) = -\widehat{g_j}(0)$. Note that $|\widehat{g_j}(0)| \lesssim \|g_j\|_{L^1}$. Note that $\|f_j\|^2_{H^{-1/2}} \simeq |\widehat{f_j}(0)|^2 + \|\tilde{f_j}\|^2_{\dot{H}^{-1/2}}$, where $\tilde{f_j}$ denotes the corrected f_j with vanishing 0th Fourier coefficient. Thus, we could reduce to the case of vanishing integral.

Claim 1: Given $f \in \dot{H}^{-1/2}(S^1)$, there is a real-valued function $F \in L^2_*(S^1)$[9], such that $DF = f$.
Proof of the Claim 1 In order to solve $DF = f$, we should have:

$$\hat{F}(n) = \frac{1}{1 + i\,\mathrm{sign}(n)} \frac{\hat{f}(n)}{\sqrt{|n|}}, \quad \text{if } n \neq 0 \tag{3.43}$$

Using the fact that the L^2-norm of F can be characterized in terms of the l^2-norm of the Fourier coefficients, we obtain:

$$\begin{aligned} \|F\|_{L^2}^2 &= \sum_{n\neq 0} \frac{1}{|1 + i\,\mathrm{sign}(n)|^2} \frac{|\hat{f}(n)|^2}{|n|} \\ &\leq \sum_{n\neq 0} \frac{|\hat{f}(n)|^2}{|n|} \\ &= \|f\|^2_{\dot{H}^{-\frac{1}{2}}}, \end{aligned} \tag{3.44}$$

where we used the definition of the $\dot{H}^{-\frac{1}{2}}$-norm. Observe that a converse inequality could be obtained along the same lines. □

Next, by defining $\tilde{u} := u - F$, we observe that due to (3.39):

$$D\tilde{u} = g. \tag{3.45}$$

Let now $w \in \mathcal{D}'(S^1)$ real-valued be such that $Dw = \tilde{u}$ and $\hat{w}(0) = 0$. Once more, existence of such a distribution w is easily deduced using Fourier coefficients. We would like to emphasise at this point that due to the assumed smoothness of u, f_j, g_j, w is smooth as well, as is F.

By (3.45), we thus notice:

$$D^2 w = g. \tag{3.46}$$

Going over to Fourier coefficients, we see that for every $n \in \mathbb{Z}^*$:

$$\mathcal{F}\big(D^2 w\big)(n) = (1 + i\,\mathrm{sign}(n))^2 |n| \hat{w}(n) = 2i\,\mathrm{sign}(n)|n|\hat{w}(n) = 2in\hat{w}(n) = \hat{g}(n), \tag{3.47}$$

or by rearranging:

$$\hat{w}(n) = -\frac{i}{2}\frac{\hat{g}(n)}{n}. \tag{3.48}$$

Next, we are going to find a suitable distribution K with coefficients $\hat{K}(n) = -\frac{i}{2n}$, in order to express w as a convolution of g with K. To this end, let us consider the function $k : [-\pi, \pi] \to \mathbb{R}$ defined by:

$$k(x) = \begin{cases} x + \pi, & \text{if } x < 0 \\ x - \pi, & \text{if } x > 0 \end{cases} \tag{3.49}$$

By slight abuse of notation, let us identify k with its 2π-periodic extension, therefore $k : S^1 \to \mathbb{R}$. We calculate the Fourier coefficients of k: If $n = 0$, it is obvious due to anti-symmetry that $\hat{k}(0) = 0$. Otherwise, we have $n \neq 0$ and so by using integration by parts:

$$\hat{k}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} k(x) e^{-inx} dx$$

[9] The observation that F may be chosen real-valued is due to $\widehat{F}(-n) = \overline{\widehat{F}(n)}$ for all n.

$$
\begin{aligned}
&= \frac{1}{2\pi}\Big(\int_{-\pi}^{0}(x+\pi)e^{-inx}dx + \int_0^{\pi}(x-\pi)e^{-inx}dx\Big)\\
&= \frac{1}{2\pi}\int_0^{\pi}(\pi-x)e^{inx} - (\pi-x)e^{-inx}dx\\
&= \frac{i}{\pi}\int_0^{\pi}(\pi-x)\sin(nx)dx\\
&= \frac{i}{n} - \frac{i}{\pi}\int_0^{\pi}\frac{\cos(nx)}{n}dx = \frac{i}{n}.
\end{aligned}
\tag{3.50}
$$

Consequently, observe that $K = -\frac{1}{2}k$ precisely yields the desired distribution. Let us notice that K is therefore bounded and measurable on S^1, thanks to the explicit formula for k.

Using (3.48) and the convolution formula for Fourier coefficients, namely:

$$
\widehat{f * g}(n) = 2\pi \hat{f}(n)\hat{g}(n), \quad \forall n \in \mathbb{Z}, \tag{3.51}
$$

it is clear that:

$$
w = \frac{1}{2\pi}K * g. \tag{3.52}
$$

From (3.52), we obtain by using Young's inequality on S^1:

$$
\|w\|_{L^\infty} \leq \frac{1}{2\pi}\|K\|_{L^\infty}\|g\|_{L^1} = \frac{1}{4}\|g\|_{L^1}. \tag{3.53}
$$

To conclude the first part of the proof, let us observe the following[10]:

$$
\begin{aligned}
\int_{S^1}(u-F)^2dx &= \int_{S^1} Dw(u-F)dx\\
&\simeq \sum_{n\in\mathbb{Z}} \widehat{Dw}(n)\widehat{u-F}(-n)\\
&= \sum_{n\in\mathbb{Z}} |n|^{\frac{1}{2}}(1+i\,\mathrm{sign}(n))\widehat{w}(n)\widehat{u-F}(-n)\\
&= \sum_{n\in\mathbb{Z}} \widehat{w}(n)\cdot|n|^{\frac{1}{2}}(1-i\,\mathrm{sign}(-n))\widehat{u-F}(-n)\\
&\simeq \int_{S^1} w\overline{D}(u-F)dx\\
&= \int_{S^1} w\overline{D}udx - \int_{S^1} w\overline{D}Fdx\\
&= \int_{S^1} w\tilde{g}dx + \int_{S^1} w\tilde{f}dx - \int_{S^1} w\overline{D}Fdx
\end{aligned}
\tag{3.54}
$$

where we used the Fourier representation of the distribution $u - F$ to justify the second equation. Observe that this enables us to estimate:

$$
\Big|\int_{S^1} w\tilde{g}dx\Big| \leq \|w\|_{L^\infty}\|\tilde{g}\|_{L^1} \leq \frac{1}{4}\big(\|g_1\|_{L^1} + \|g_2\|_{L^1}\big)^2, \tag{3.55}
$$

[10] This will actually be the first and only point in the proof where we use the fact that u is real-valued in a meaningful way. See Remark 3.1.3.1 for an extension to complex-valued distributions.

and:

$$\Big|\int_{S^1} w\overline{D}F dx\Big| \leq \|w\|_{\dot{H}^{\frac{1}{2}}} \|\overline{D}F\|_{\dot{H}^{-\frac{1}{2}}} \leq C\|u-F\|_{L^2}\|f\|_{\dot{H}^{-\frac{1}{2}}}. \tag{3.56}$$

The remaining summand may be estimated completely analogous to (3.56). Notice that we used the explicit definition of the norms of Sobolev spaces with negative exponents and the Fourier multipliers to obtain (3.56), see (3.44) for the main ideas. Using (3.55) and (3.56) yields:

$$\begin{aligned}\|u-F\|_{L^2}^2 &\leq \frac{1}{4}\big(\|g_1\|_{L^1} + \|g_2\|_{L^1}\big)^2 + 2C\|u-F\|_{L^2}\|f\|_{\dot{H}^{-\frac{1}{2}}} \\ &\leq \frac{1}{4}\big(\|g_1\|_{L^1} + \|g_2\|_{L^1}\big)^2 + \frac{1}{2}\|u-F\|_{L^2}^2 + \frac{4C^2}{2}\|f\|_{\dot{H}^{-\frac{1}{2}}}^2, \end{aligned} \tag{3.57}$$

using the arithmetic-geometric mean inequality. Note that the factor $2C$ is due to estimate (3.56) also applying to the integral of $w\tilde{f}$. By absorbing the L^2-norm of $u-F$, we arrive at:

$$\begin{aligned}\|u-F\|_{L^2}^2 &\leq \frac{1}{2}\big(\|g_1\|_{L^1} + \|g_2\|_{L^1}\big)^2 + 4C^2\|f\|_{\dot{H}^{-\frac{1}{2}}}^2 \\ &\leq \max\{\frac{1}{2}, 4C^2\}\big(\|g_1\|_{L^1} + \|g_2\|_{L^1} + \|f_1\|_{\dot{H}^{-\frac{1}{2}}} + \|f_2\|_{\dot{H}^{-\frac{1}{2}}}\big)^2. \end{aligned} \tag{3.58}$$

Consequently, by estimating the L^2-norm of F by the $\dot{H}^{-\frac{1}{2}}$-norm of f_1, f_2 using (3.44), we immediately conclude:

$$\|u\|_{L^2} \leq \tilde{C}\big(\|g_1\|_{L^1} + \|g_2\|_{L^1} + \|f_1\|_{\dot{H}^{-\frac{1}{2}}} + \|f_2\|_{\dot{H}^{-\frac{1}{2}}}\big). \tag{3.59}$$

The constant $\tilde{C} > 0$ appearing in the estimate is independent of u, f_j, g_j.

Now, for a general distribution $u \in \mathcal{D}'(S^1)$ with $\hat{u}(0) = 0$, let us observe that if we convolute u with a smooth function φ, the resulting distribution $\varphi * u$ will be a smooth function as well (in the sense of regular distributions). By a direct computation, (3.32) and (3.33) will continue to hold true if we replace u, f_j, g_j by their corresponding convolutions with φ. This is an immediate consequence of the fact that the operators $(-\Delta)^{\frac{1}{4}}, \mathcal{R}$ are Fourier multipliers as well as the linearity of convolutions. Choosing φ to be supported on arbitrarily small neighbourhoods of the neutral element in S^1 (i.e. an approximation of the identity φ_ε) ensures that the convolutions of φ with f_j, g_j converge in the respective norms as we collapse the support of φ (i.e. let the parameter ε in φ_ε tend to 0) and the approximations of u converge in the distributional sense. As a result, we obtain uniform bounds in the respective spaces. This results in an uniform L^2-bound for u convoluted with φ_ε independent of ε, which can be seen to imply $u \in L^2_*(S^1)$ by using a weak-L^2-convergent subsequence. The estimate follows by the lower semi-continuity of the norm. This concludes our proof. □

Remark 3.1.3.1. *Before we enter the discussion of applications and later a generalisation of Theorem 3.1.1.3, let us quickly discuss the assumption that u is real-valued. In fact, this is merely used at a single point in the proof, namely in* (3.54). *However, if we proceed similar to the proof of the generalised result in section 3.1.5, i.e. we use:*

$$\int_{S^1} |u-F|^2 dx = \int_{S^1} (u-F)\cdot\overline{u-F}dx = \int_{S^1} Dw\cdot\overline{u-F}dx \sim \sum_{n\in\mathbb{Z}} \widehat{Dw}(n)\cdot\overline{\widehat{u-F}(n)}, \tag{3.60}$$

we can easily avoid the use of properties of real-valued distributions. The remainder of the proof follows then completely analogous, i.e. we can remove the assumption of u being real-valued effortlessly. Indeed, this slight generalization will be key to our applications to Bergman spaces below.

3.1.4 Fractional Bourgain-Brezis inequality in the Bergman space $\mathcal{A}^2(\mathbb{D})$

We start with the **Proof of Theorem 3.1.1.1.**

Let us consider an analytic function $f\colon \mathbb{D} \to \mathbb{C}$ such that $\limsup_{r\to 1^-} \|f(re^{i\theta})\|_{L^1+H^{-1/2}(S^1)} < +\infty$.
1. Now let us write $f(z) = \sum_{n\geq 0} f_n z^n$ and $u(e^{i\theta}) = \sum_{n\geq 1} \frac{f_n}{\sqrt{n}} e^{in\theta}$. We first observe that $f - f(0) = \sum_{n\geq 1} f_n z^n$ and if $f(e^{i\theta}) = g(e^{i\theta}) + h(e^{i\theta})$ with $g \in L^1(S^1)$ and $h \in H^{-1/2}(S^1)$, then:

$$f(e^{i\theta}) - f(0) = g - \fint_{S^1} g + h - \widehat{h}(0).$$

Note that $h - \widehat{h}(0) \in \dot{H}^{-1/2}(S^1)$ with the norm being controlled by $\|h\|_{H^{-1/2}(S^1)}$. We observe that, using the explicit definitions of the norm:

$$\left\| g - \fint_{S^1} g \right\|_{L^1(S^1)} + \|h - \widehat{h}(0)\|_{\dot{H}^{-1/2}(S^1)} \lesssim \|g\|_{L^1(S^1)} + \|h\|_{H^{-1/2}(S^1)}$$

Therefore, we may conclude by taking the infimum over all such g, h:

$$\|f - f(0)\|_{L^1+\dot{H}^{-1/2}(S^1)} \leq C\|f\|_{L^1+H^{-1/2}(S^1)}. \tag{3.61}$$

Assume therefore first that

$$f(e^{i\theta}) - f(0) = \sum_{n\geq 1} f_n e^{in\theta} \in L^1 + \dot{H}^{-1/2}(S^1).$$

In this case, we get $(-\Delta)^{1/4} u = f - f(0) \in L^1 + \dot{H}^{-1/2}(S^1)$. Additionally, we observe that since u contains only positive frequencies, we trivially have $\mathcal{R}(-\Delta)^{1/4} u \in L^1 + \dot{H}^{-1/2}(S^1)$ as well with

$$\|\mathcal{R}(-\Delta)^{1/4} u\|_{L^1+\dot{H}^{-1/2}(S^1)} = \|(-\Delta)^{1/4} u\|_{L^1+\dot{H}^{-1/2}(S^1)}.$$

From the inequality (3.8), observing that $\fint_{S^1} u = 0$, we deduce that

$$\|u\|_{L^2(S^1)} \leq C\|(-\Delta)^{1/4} u\|_{(L^1+\dot{H}^{-1/2})(S^1)} = C\|f - f(0)\|_{L^1+\dot{H}^{-1/2}(S^1)} \leq C'\|f\|_{L^1+H^{-1/2}(S^1)},$$

where we used (3.61). Hence $\sum_{n>0} \frac{f_n}{n} e^{in\theta} \in H^{1/2}(S^1)$ and $g(z) = \sum_{n>0} \frac{f_n}{n} z^n \in H^1(\mathbb{D})$. We have $g'(z) = \sum_{n\geq 0} f_{n+1} z^n \in L^2(\mathbb{D})$ and

$$\begin{aligned}
\|f(z) - f(0)\|_{L^2(\mathbb{D})} &= \|z g'(z)\|_{L^2(\mathbb{D})} \\
&\leq C\|g\|_{H^{1/2}(S^1)} = C\|u\|_{L^2(S^1)} \\
&\leq C\|f\|_{L^1+H^{-1/2}(S^1)}.
\end{aligned} \tag{3.62}$$

The desired estimate follows by the triangle inequality, if we can show:

$$|f(0)| \leq C\|f\|_{L^1+H^{-1/2}(S^1)}$$

To achieve this, let us decompose $f = g + h$ with $g \in L^1(S^1)$ as well as $h \in H^{-1/2}(S^1)$. Then we denote as usual the Fourier coefficients of g, h by g_n, h_n for all $n \in \mathbb{Z}$ and define:

$$G(z) := \sum_{n\geq 0} g_n z^n + \sum_{n<0} g_n \overline{z}^{|n|}, \quad H(z) := \sum_{n\geq 0} h_n z^n + \sum_{n<0} h_n \overline{z}^{|n|}.$$

By the summability properties, these define harmonic functions on $\mathbb{D}$ having boundary values g, h respectively. By comparison of the coefficients, we also observe:

$$f(z) = G(z) + H(z),$$

in particular for $z = 0$. Moreover, by the mean value property of harmonic functions over the boundary of the disc, we can deduce:

$$|G(0)| \lesssim \|g\|_{L^1(S^1)}.$$

Using the mean value property over the entire disc, we similarily see by Hölder's inequality:

$$|H(0)| \lesssim \|H\|_{L^1(\mathbb{D})} \lesssim \|H\|_{L^2(\mathbb{D})}.$$

It is easy to verify by a direct computation analogous to the same characterisation of the norm in $\mathcal{A}^2(\mathbb{D})$ that:

$$\|H\|^2_{L^2(\mathbb{D})} \sim \sum_{n\in\mathbb{Z}} \frac{|h_n|^2}{|n|+1} \leq \|h\|_{H^{-1/2}(S^1)}.$$

In conclusion, we have:

$$|f(0)| \leq C\left(\|g\|_{L^1(S^1)} + \|h\|_{H^{-1/2}(S^1)}\right).$$

By taking the infimum over g, h such that $f = g + h$ we get

$$|f(0)| \leq C\|f\|_{L^1+H^{-1/2}(S^1)} \tag{3.63}$$

By combining (3.62) and (3.63), we obtain the desired estimate:

$$\|f\|_{L^2(\mathbb{D})} \leq C\|f\|_{L^1+H^{-1/2}(S^1)}. \tag{3.64}$$

2. In the general case when $\limsup_{r\to 1^-} \|f(re^{i\theta})\|_{L^1+H^{-1/2}(S^1)} < +\infty$, we consider for every $0 < r < 1$ the function $f_r(z) = f(rz) \in C^\infty(\bar{B}(0,1))$. We can apply (3.62) to f_r and obtain that

$$\|f_r\|_{L^2(\mathbb{D})} \leq C\|f_r\|_{L^1+H^{-1/2}(S^1)}. \tag{3.65}$$

Since by assumption $\limsup_{r\to 1^-} \|f(re^{i\theta})\|_{L^1+H^{-1/2}(S^1)} < +\infty$, we deduce that

$$\sup_{0<r<1} \|f_r\|_{L^2(\mathbb{D})} < +\infty. \tag{3.66}$$

The inequality implies that actually $f \in L^2(\mathbb{D})$ as well as[11] and

$$\|f\|_{L^2(\mathbb{D})} \leq C\|f\|_{L^1+H^{-1/2}(S^1)}. \tag{3.69}$$

[11] Let $f(z) = \sum_{n\geq 0} f_n z^n$ We observe that

$$\begin{aligned}\|f_r\|^2_{L^2(\mathbb{D})} &= \int_0^1\int_0^{2\pi} |f(\rho re^{i\theta})|^2 \rho d\theta d\rho \\ &= 2\pi\int_0^1 \sum_{n=0}^\infty |a_n|^2 r^{2n}\rho^{2n+1} d\rho = 2\pi\sum_{n=0}^\infty \frac{|a_n|^2 r^{2n}}{2n+2}.\end{aligned} \tag{3.67}$$

and similarily

$$\|f\|^2_{L^2(\mathbb{D})} = 2\pi\sum_{n=0}^\infty \frac{|a_n|^2}{2n+2}. \tag{3.68}$$

From (3.66) and extracting a weakly convergent subsequence which by convergence of the Fourier coefficients must have limit f, it follows that $\|f\|^2_{L^2(\mathbb{D})} < +\infty$ and Abel's Theorem on power series yields that

$$\lim_{r\to 1^-} \|f_r\|^2_{L^2(\mathbb{D})} = \|f\|^2_{L^2(\mathbb{D})}.$$

Conversely, let $f\colon \mathbb{D}\to\mathbb{C}$ be in $\mathcal{A}^2(\mathbb{D}^2)$. We write $f(z)=\sum_{n=0}^{\infty} a_n z^n$. We prove the following:
Claim: $\limsup_{r\to 1^-}\|f(re^{i\theta})\|_{L^1+H^{-1/2}(S^1)}<+\infty$.
Proof of the claim. We show that $\limsup_{r\to 1^-}\|f(re^{i\theta})\|_{H^{-1/2}(S^1)}<+\infty$. For every $0<r<1$, we set $f_r(z)=f(rz)\in C^\infty(\bar{B}(0,1))$. Since $f\in L^2(\mathbb{D})$, we have

$$\limsup_{r\to 1^-}\|f_r\|_{L^2(\mathbb{D})}=\|f\|_{L^2(\mathbb{D})}. \tag{3.70}$$

Moreover

$$\|f_r\|^2_{H^{-1/2}(S^1)}=\sum_{n\geq 0}\frac{|f_n|^2}{1+n}r^{2n} \tag{3.71}$$

and

$$\begin{aligned}\int_{\mathbb{D}}|f_r|^2 &= 2\pi\int_0^1\sum_{n\geq 0}|f_n|^2 r^{2n}s^{2n}s ds\simeq\sum_{n\geq 0}\frac{|f_n|^2}{2n+2}r^{2n}\\ &\simeq \frac{1}{2}\sum_{n\geq 0}\frac{|f_n|^2}{n+1}r^{2n}\simeq\|f_r\|_{H^{-1/2}(S^1)}.\end{aligned} \tag{3.72}$$

By combining (3.70) and (3.72), we get that

$$\limsup_{r\to 1^-}\|f_r\|_{H^{-1/2}(S^1)}\lesssim\|f\|_{L^2(\mathbb{D})}<+\infty. \tag{3.73}$$

We conclude the proof. □

Next we show that Theorem 3.1.1.1 is actually equivalent to Theorem 3.1.1.3.

Proposition 3.1.4.1. *Theorem 3.1.1.1 implies Theorem 3.1.1.3. Therefore, they are equivalent.*

Proof. We have already seen in the proof of Theorem 3.1.1.1 that Theorem 3.1.1.3 implies the fact that a holomorphic function with the property that $\limsup_{r\to 1^-}\|f(re^{i\theta})\|_{L^1+H^{-1/2}(S^1)}<+\infty$ is in $L^2(\mathbb{D})$, namely it belongs to the Bergman space $\mathcal{A}^2(\mathbb{D})$.

Conversely, let us consider $u\in C^\infty(S^1)$ such that $(-\Delta)^{1/4}u,\mathcal{R}(-\Delta)^{1/4}u\in L^1+\dot{H}^{-1/2}(S^1)$. We assume that $\int_0^{2\pi}u(e^{i\theta})d\theta=0$. We decompose $u=u^++u^-$, where

$$u^+=\sum_{n>0}u_n e^{in\theta},\quad u^-=\sum_{n<0}u_n e^{in\theta}.$$

Let us first consider u^+. By assumption we have $\sum_{n\geq 1}n^{1/2}u_n e^{in\theta}=1/2((-\Delta)^{1/4}u-i\mathcal{R}(-\Delta)^{1/4}u)\in L^1+\dot{H}^{-1/2}(S^1)$. Let $f(z)=\sum_{n\geq 1}n^{1/2}u_n z^n$ be the harmonic extension of $v=(-\Delta)^{1/4}u^+$ in $\mathbb{D}$. From Theorem 3.1.1.1, it follows that $f^+=\sum_{n>0}n^{1/2}u_n z^n\in L^2(\mathbb{D})$ and

$$\|f^+\|_{L^2(\mathbb{D})}\leq C\|f^+\|_{L^1+H^{-1/2}(S^1)}\leq\|(-\Delta)^{1/4}u^+\|_{L^1+\dot{H}^{-1/2}(S^1)}.$$

Switching to the homogeneous Sobolev space is possible, as we have $\dot{H}^{-1/2}(S^1)\subset H^{-1/2}(S^1)$ continuously embedded. Since $f^+(z)=\sum_{n>0}n^{1/2}u_n z^n\in L^2(\mathbb{D})$, it follows that $\sum_{n>0}\frac{u_n}{n^{1/2}}z^n\in H^1(\mathbb{D})$ and therefore $\sum_{n>0}\frac{u_n}{n^{1/2}}e^{in\theta}\in\dot{H}^{1/2}(S^1)$. Hence $u^+\in L^2(S^1)$ with

$$\begin{aligned}\|u^+\|_{L^2(S^1)} &\leq C\|(-\Delta)^{1/4}u^+\|_{L^1+\dot{H}^{-1/2}(S^1)}\\ &\leq C\left(\|(-\Delta)^{1/4}u\|_{L^1+\dot{H}^{-1/2}(S^1)}+\|\mathcal{R}(-\Delta)^{1/4}u\|_{L^1+\dot{H}^{-1/2}(S^1)}\right).\end{aligned} \tag{3.74}$$

The same arguments hold for u^-. We conclude the proof. □

3.1.5 The Bourgain-Brezis Inequality on the Torus $\mathbb{T}^n, n \geq 2$

In this section, we are going to prove Theorem 3.1.1.4 which generalises the result from Theorem 3.1.1.3 to domains of dimension $n \geq 2$. To achieve this while retaining the general structure of the proof, we first have to determine the right set of conditions and the appropriate domain. Observe that it is clear, due to the proof for S^1 heavily relying on Fourier series, that the natural domain for such a generalisation is the torus $\mathbb{T}^n$. In investigating generalisations of the proof, we have to focus on two aspects: Clifford algebras and boundedness of the kernel. In the first part of the proof we introduce complex Clifford algebras and show how to generalize the argument presented in section 3.1.3.2. The results and properties of Clifford algebras are due to Gilbert-Murray [38] and Hamilton [42] and are briefly discussed in section 3.1.5.1 below. In the second part of the proof we show that the kernel used is actually bounded, following an argument presented in [8, p.405-406]. In the case $n = 1$, we have seen that k has an explicit description as a sawtooth function. In higher dimensions, unfortunately, we are not aware of an explicit formula for the kernel. However, due to some estimates on alternating sums, we can remedy this lack of explicit representation and derive the crucial properties abstractly.

3.1.5.1 A short introduction to Clifford algebras

The material covered here is due to Gilbert-Murray [38] and Hamilton [42] and we refer to them for further details on the topics introduced. For the remainder of this subsection, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ denote a scalar field and V a finite dimensional $\mathbb{K}$-vector space. Let $Q : V \to \mathbb{K}$ be a map, such that:

1.) For all $\lambda \in \mathbb{K}$ and $v \in V$, we have: $Q(\lambda v) = \lambda^2 \cdot Q(v)$.

2.) The map $B(v, w) := \frac{1}{2}\big(Q(v + w) - Q(v) - Q(w)\big)$ defines a $\mathbb{K}$-bilinear map on $V \times V$.

Such a Q will be called a *quadratic form* and the pair (V, Q) a *quadratic space*. Standard examples include real vector spaces equipped with scalar products, but not complex vector spaces with scalar products due to complex anti-linearity in the second argument. Inspired by this example, we say that a basis $e_1, \ldots, e_n$ of a quadratic space (V, Q) is *B-orthonormal*, if for all $j \in \{1, \ldots, n\}$, we have $|Q(e_j)| = 1$ as well as:

$$B(e_j, e_k) = 0, \quad \forall j \neq k \in \{1, \ldots, n\}. \tag{3.75}$$

Given such a quadratic space (V, Q), we call a pair $(\mathcal{A}, \nu)$ a *Clifford algebra* for (V, Q), if the following holds, see [38, p.8, (2.1)]:

i.) $\mathcal{A}$ is an associative algebra with unit 1 and $\nu : V \to \mathcal{A}$ is $\mathbb{K}$-linear and injective.

ii.) $\mathcal{A}$ is generated as an algebra by $\nu(V)$ and $\mathbb{K} \cdot 1$.

iii.) For every $v \in V$, we have: $\nu(v)^2 = -Q(v) \cdot 1$

An important immediate corollary of the definition is the following commutation relation:

$$\nu(v)\nu(w) + \nu(w)\nu(v) = -2B(v, w) \cdot 1, \quad \forall v, w \in V. \tag{3.76}$$

Thus, pairs of orthogonal vectors with respect to B anti-commute as elements in $\mathcal{A}$. We usually omit explicitly mentioning ν and therefore identify v with $\nu(v)$, which is justified due to ν being injective.

For the remainder of the section, let us focus on (V,Q) non-degenerate, i.e. for all $v \in V$, there is a $w \in V$, such that $B(v,w) \neq 0$. In this case, there actually exists a basis $e_1,\dots,e_n$, where $n = \dim_{\mathbb{K}} V$, orthonormal with respect to B and, consequently, such that:

$$e_j e_k + e_k e_j = \pm 2\delta_{jk} \cdot 1, \quad \forall j,k \in \{1,\dots,n\}, \tag{3.77}$$

(see e.g. Theorem 1.5 in Gilbert-Murray [38]). The signs are determined by the signature of the quadratic form Q and may vary for different choices j,k. Provided $\mathbb{K} = \mathbb{C}$, we may assume that all signs are the same, see Gilbert-Murray [38].

It can be shown that every Clifford algebra has $\mathbb{K}$-dimension at most 2^n. If the dimension is equal to 2^n, the Clifford algebra is called *universal*.[12] An important result in [38, Thm. 2.7] states that there always exists a universal Clifford algebra for any given quadratic space. Moreover, there exist explicit descriptions of all universal Clifford algebras up to isomorphisms in terms of matrices, see [38].

To conclude this brief treatment of Clifford algebras, let us provide an explicit example: Let $V = \mathbb{C}^n$, $\mathbb{K} = \mathbb{C}$ and define Q as follows:

$$Q(z_1,\dots,z_n) := \sum_{j=1}^{n} z_j^2, \quad \forall (z_1,\dots,z_n) \in \mathbb{C}^n. \tag{3.78}$$

It is clear that (V,Q) is a non-degenerate quadratic space, as B is the standard scalar product up to a complex conjugation in the second argument. In this case, the standard basis $e_1,\dots,e_n$ already is B-orthonormal. Thus, we have:

$$e_j e_k + e_k e_j = -2\delta_{jk} \cdot 1, \quad \forall j,k \in \{1,\dots,n\}. \tag{3.79}$$

The universal Clifford algebra is then spanned by the finite products e_α of the basis elements, where $\alpha \subset \{1,\dots,n\}$ is an ordered subset and we define:

$$e_\alpha = \prod_{j \in \alpha} e_j$$

In particular, $e_\emptyset = 1$ by definition. It can be seen that every complex universal Clifford algebra associated with a non-degenerate quadratic space of dimension n is isomorphic to this one, see Gilbert-Murray [38] and the definition of universal Clifford algebra presented there.

Lastly, let us introduce a few definitions from Chapter 1, Section 7 in Gilbert-Murray [38]: We may identify the universal Clifford algebra $\mathcal{A}$ as a vector space with $\mathbb{K}^{2^n}$, if $\dim_{\mathbb{K}} V = n$. This allows us to generalize the natural scalar product-induced norm on $\mathbb{K}^{2^n}$ to the Clifford algebra and we shall denote this norm by $\|\cdot\|$. Moreover, there is a notion of conjugation on Clifford algebras defined by:

$$\overline{e_{j_1}\dots e_{j_k}} := (-1)^k Q(e_{j_1})\dots Q(e_{j_k}) \cdot e_{j_k}\dots e_{j_1} = (-1)^{\frac{k(k+1)}{2}} Q(e_{j_1})\dots Q(e_{j_k}) \cdot e_{j_1}\dots e_{j_k}, \tag{3.80}$$

see [38, (3.80)], and extending linearly. If $\mathbb{K} = \mathbb{C}$, we also conjugate the complex coefficients in the usual manner, i.e. we extend complex anti-linearly. We highlight the following key property of the conjugation:

$$\overline{xy} = \overline{y} \cdot \overline{x}, \quad \forall x,y \in \mathcal{A}. \tag{3.81}$$

[12]This definition is justified, as universal Clifford algebras $\mathcal{A}$ have an extension property for linear maps from V to any Clifford algebra respecting the characteristic multiplication relation in $\mathcal{A}$, see Gilbert-Murray [38].

This is due to the inversion of factors in (3.80). We emphasise that the definition in (3.80) is precisely made with the identity below in mind:

$$\overline{e_{j_1}\dots e_{j_k}}\cdot e_{j_1}\dots e_{j_k}=1. \tag{3.82}$$

The following property will be useful later as well: Let $x \in \mathcal{A}$ be given and denote by P_0 the linear projection of an element in the Clifford algebra to the coefficient associated with the neutral element 1. More precisely, $P_0 : \mathcal{A} \to \mathbb{K}$ is the following linear map:

$$P_0\Big(\sum_{\alpha} x_\alpha e_\alpha\Big) = x_\emptyset$$

We have by a direct computation:

$$\begin{aligned} P_0(\overline{x}x) &= \sum_{\alpha\subset\{1,\dots,n\}} \overline{x_\alpha}x_\alpha \\ &= \|x\|^2, \end{aligned} \tag{3.83}$$

where we wrote explicitly $x = \sum_{\alpha\subset\{1,\dots,n\}} x_\alpha e_\alpha$ with $x_\alpha \in \mathbb{K}$. It suffices to observe that $e_\alpha \cdot e_\beta$ has non-vanishing contribution in the $e_\emptyset = 1$-direction, if and only if $\alpha = \beta$. The formula then follows.

3.1.5.2 Proof of Theorem 3.1.1.4.

Let us first note that, if we take $\mathbb{T}^n$ with $n \geq 2$, there are n different Riesz transforms, one for each basis direction. This suggests that the right conditions should involve some restriction on each of the Riesz transforms. In addition, considering the symbol of D^2, we see that we rely some cancellation property stemming from the complex nature of i. [13] Therefore, a natural way to obtain a generalisation would involve Clifford algebras to include sufficiently many anticommuting complex units.

Firstly, it is immediate that the same simplifications as in the case $n = 1$ apply here. So we may assume $\hat{u}(0) = 0$. Throughout most of this proof, the coefficient $m = 0$ will be implicitly omitted, as it will be vanishing for all functions/distributions considered. Moreover, the reduction to smooth functions applies equally well in this case. Therefore, we may assume without loss of generality that u, f_j, g_j are all smooth.

The heart of the argument lies in the correct definition of D and $\overline{D}$ on $\mathbb{T}^n$. As mentioned in the introduction of the current section, Clifford algebras and their set of complex units actually provide the desired framework. Let $\mathbb{C}_n$ denote the universal complex Clifford algebra associated with the quadratic space $(\mathbb{C}^n, Q)$, where:

$$Q(z_1,\dots,z_n) := -\sum_{j=1}^{n} z_j^2, \quad \forall (z_1,\dots,z_n)\in\mathbb{C}^n. \tag{3.84}$$

We emphasise that the particular choice of Q is at odds with usual conventions for complex Clifford algebras, but using our quadratic form, we obtain the appropriate basis commutation relations while remaining isomorphic to the usual convention. One could reduce to the usual defining quadratic form by choosing $i \cdot e_j$ instead of the standard basis e_j throughout our proof. In fact, the main reason why

[13] This refers to the property $i^2 = -1$ which was key to reduce the multiplier of D^2 to a simpler form.

we decided to use our convention is to use the Riesz operators in their usual form.
Observe that we then have, for the standard basis denoted by $e_1, \dots, e_n$:

$$e_j e_k + e_k e_j = 2\delta_{jk}, \quad \forall j,k \in \{1,\dots,n\}, \tag{3.85}$$

simply by the definition of Clifford algebras and the quadratic form Q. We define now for any $v \in C^\infty(\mathbb{T}^n, \mathbb{C})$:

$$Dv = \Delta^{\frac{n}{4}}(Id + \sum_{j=1}^n e_j \mathcal{R}_j)v \tag{3.86}$$

$$\overline{D}v = \Delta^{\frac{n}{4}}(Id - \sum_{j=1}^n e_j \mathcal{R}_j)v. \tag{3.87}$$

We emphasise the similarity with [38, (5.14)] used in the context of Hardy spaces. The crucial observation for our purposes is the following multiplier property for Fourier series for every $m \in \mathbb{Z}^n$:

$$\mathcal{F}(Dv)(m) = |m|^{\frac{n}{2}}\big(1 + \sum_{j=1}^n e_j \cdot i\frac{m_j}{|m|}\big)\mathcal{F}(v)(m) \tag{3.88}$$

$$\mathcal{F}(\overline{D}v)(m) = |m|^{\frac{n}{2}}\big(1 - \sum_{j=1}^n e_j \cdot i\frac{m_j}{|m|}\big)\mathcal{F}(v)(m), \tag{3.89}$$

where $|m|$ denotes the Euclidean norm on $\mathbb{Z}^n$. We highlight that at this point, we know that Du and $\overline{D}u$ are functions in $L^1 + \dot{H}^{-\frac{n}{2}}(\mathbb{T}^n, \mathbb{C}_n)$. Completely analogous to the proof of Theorem 3.1.1.3, we may find $F \in L^2$ (due to the invertibility of non-zero vectors $v \in \mathbb{R}^n$ in $\mathbb{C}_n$[14]). To be precise, observe that if $DF = f$, f and g are defined to satisfy $Du = f + g$ by splitting the terms f_j, g_j in the natural way, then:

$$\forall m \neq 0: \quad |m|^{\frac{n}{2}}\big(1 + \sum_{j=1}^n e_j \cdot i\frac{m_j}{|m|}\big)\widehat{F}(m) = \widehat{f}(m), \tag{3.90}$$

which may be rewritten as:

$$\widehat{F}(m) = \frac{1}{2|m|^{\frac{n}{2}}}\Big(1 - \sum_{j=1}^n e_j \cdot i\frac{m_j}{|m|}\Big)\widehat{f}(m), \tag{3.91}$$

by using the multiplication relations and associativity on $\mathbb{C}_n$. To conclude that $F \in L^2$, it suffices to check summability of the Fourier coefficients:

$$\begin{aligned}\sum_{m \in \mathbb{Z}^n \setminus \{0\}} \|\widehat{F}(m)\|^2 &= \sum_{m \neq 0} \Big\| \frac{1}{2|m|^{\frac{n}{2}}}\big(1 - \sum_{j=1}^n e_j \cdot i\frac{m_j}{|m|}\big)\widehat{f}(m)\Big\|^2 \\ &\lesssim \sum_{m\neq 0} \frac{1}{|m|^n}\|\widehat{f}(m)\|^2\end{aligned}$$

[14]Observe that for real vectors in $\mathbb{R}^n$, we find $m^2 = |m|^2$. For general vectors in $\mathbb{C}^n$, this fails, as can be seen in the counterexample:

$$(e_1 + ie_2)^2 = 0$$

$$\lesssim \|f\|^2_{\dot{H}^{-\frac{n}{2}}} < +\infty. \tag{3.92}$$

We mention here that the characterisations for regularity and integrability carry over without problem, even if we use Clifford algebra-valued functions by verifying componentwise regularity.

Consequently, as in the case $n = 1$, we may define $\tilde{u} = u - F$ and observe that $D\tilde{u} =: g \in L^1$. Solving $Dw = \tilde{u}$ in the sense of distributions leaves us with $D^2 w = g$.

The key point behind the second proof of Theorem 3.1.1.3 lies in the fact, that D^2 has an inverse given by the convolution with a bounded function. By a direct computation, we arrive at the following expression for the multiplier associated with D^2:

$$\mathcal{F}(D^2 w)(m) = |m|^n \big(1 + \sum_{j=1}^n e_j \cdot i \frac{m_j}{|m|}\big)^2 \mathcal{F}(w)(m) = 2i \cdot |m|^n \big(\sum_{j=1}^n e_j \frac{m_j}{|m|}\big) \mathcal{F}(w)(m), \tag{3.93}$$

for every $m \in \mathbb{Z}^n$. Observe that we used the fact that the complex unit i of $\mathbb{C}$ commutes with all e_j (as the Clifford algebra is a complex algebra) and that:

$$(i \cdot e_j)^2 = i^2 \cdot e_j^2 = i^2 = -1. \tag{3.94}$$

Let us identify $m = \sum m_j e_j$, i.e. we consider the vector $m \in \mathbb{Z}^n \subset \mathbb{C}^n$ as an element in $\mathbb{C}_n$. Therefore, (3.93) becomes:

$$\mathcal{F}(D^2 w)(m) = 2i |m|^{n-1} \cdot m \mathcal{F}(w)(m), \quad \forall m \in \mathbb{Z}^n. \tag{3.95}$$

As stated before, all vectors $\mathbb{R}^n \subset \mathbb{C}_n$ are invertible due to:

$$z^2 = -Q(z), \quad \forall z \in \mathbb{C}^n. \tag{3.96}$$

So, for the real vector m, we have due to $m \cdot m = -Q(m)$:

$$m^{-1} = \frac{m}{|m|^2}, \quad \forall 0 \neq m \in \mathbb{Z}^n. \tag{3.97}$$

This means that $D^2 w = g$ can be restated as:

$$\mathcal{F}(w)(m) = \frac{1}{2i} \cdot \frac{m}{|m|^{n+1}} \mathcal{F}(g)(m), \tag{3.98}$$

for every $0 \neq m \in \mathbb{Z}^n$.

For now, let us assume that a bounded function K on the torus exists, such that:

$$\widehat{K}(m) = \frac{1}{2i} \cdot \frac{m}{|m|^{n+1}}, \quad \forall m \in \mathbb{Z}^n \setminus \{0\}. \tag{3.99}$$

In this case, we may check using Fourier coefficients that (keeping in mind that the order of factors in the convolution matters for products in Clifford algebras):

$$w = \frac{1}{(2\pi)^n} K * g \tag{3.100}$$

Thus, we have the following inequality:

$$\|w\|_{L^\infty} \lesssim \|K\|_{L^\infty} \|g\|_{L^1}. \tag{3.101}$$

This is an immediate consequence of the definition, Minkowski's inequality and continuity of the Clifford multiplication in the Clifford algebra norm.
Moreover, we may deduce:

$$\begin{aligned}
\|u-F\|_{L^2}^2 &= \int_{\mathbb{T}^n} P_0\big(\overline{(u-F)}\cdot(u-F)\big)dx \\
&= P_0\Big(\int_{\mathbb{T}^n} \overline{(u-F)}\cdot(u-F)dx\Big) \\
&\leq \Big\| \int_{\mathbb{T}^n} \overline{(u-F)}\cdot(u-F)dx\Big\| \\
&= \Big\| \int_{\mathbb{T}^n} \overline{Dw}\cdot(u-F)dx\Big\| \\
&= \Big\| \int_{\mathbb{T}^n} \Big(\sum_m \overline{\widehat{Dw}(m)} e^{-i\langle m,x\rangle}\Big)\cdot\Big(\sum_{\tilde{m}} \widehat{u-F}(m) e^{i\langle \tilde{m},x\rangle}\Big)dx\Big\| \\
&\simeq \Big\| \sum_m \overline{\widehat{Dw}(m)}\cdot\widehat{u-F}(m)\Big\| \\
&= \Big\| \sum_m \overline{|m|^{\frac{n}{2}}\big(1+\sum_{j=1}^n e_j\cdot i\frac{m_j}{|m|}\big)\widehat{w}(m)}\cdot\widehat{u-F}(m)\| \\
&= \Big\| \sum_m \overline{\widehat{w}(m)}\cdot\overline{|m|^{\frac{n}{2}}\big(1+\sum_{j=1}^n e_j\cdot i\frac{m_j}{|m|}\big)}\widehat{u-F}(m)\| \\
&= \Big\| \sum_m \overline{\widehat{w}(m)}\cdot|m|^{\frac{n}{2}}\big(1-\sum_{j=1}^n e_j\cdot i\frac{m_j}{|m|}\big)\widehat{u-F}(m)\| \\
&= \Big\| \sum_m \overline{\widehat{w}(m)}\cdot\overline{D}\widehat{(u-F)}(m)\| \\
&\simeq \Big\| \int_{\mathbb{T}^n} \overline{w}\cdot\overline{D}(u-F)dx\Big\|.
\end{aligned} \tag{3.102}$$

Observe that in the first inequality, we used that the norm squared of $u-F$ actually appears as the coefficient associated with 1 in the product $\overline{u-F}\cdot(u-F)$. In addition, the conjugation in the ninth line can easily deduced from our definition in the preliminary section of the paper, see (3.80). The remainder of the argument then follows completely analogous to the $1D$-proof, up to the obvious modifications. Again, simple considerations show that we even have the following inequality:

$$\Big\|u-\fint_{\mathbb{T}^n} udx\Big\|_{L^2} \lesssim \sum_{j=0}^n \big\|(-\Delta)^{\frac{n}{2}}\mathcal{R}_j u\big\|_{\dot{H}^{-\frac{n}{2}}+L^1}, \tag{3.103}$$

where $\mathcal{R}_0 = Id$.

To complete the proof in the same way as for Theorem 3.1.1.3, we still need to find a bounded kernel K satisfying:

$$\widehat{K}(m) = \frac{1}{2i}\cdot\frac{m}{|m|^{n+1}}, \quad \forall 0 \neq m \in \mathbb{Z}^n. \tag{3.104}$$

This is the purpose of the next subsection, so we may conclude the proof of Theorem 3.1.1.4 at this point. □

3.1.5.3 Boundedness of the Kernel

Lastly, let us find an appropriate kernel. We first notice that due to linearity, symmetry and the splitting into different directions, it is enough to find a bounded function k, such that:

$$\widehat{k}(m) = \frac{m_1}{|m|^{n+1}}, \quad \forall 0 \neq m \in \mathbb{Z}^n. \tag{3.105}$$

Consequently, we want to study the boundedness of the following conditionally convergent series:

$$k(x) = \sum_{m \in \mathbb{Z}^n \setminus \{0\}} \frac{m_1}{|m|^{n+1}} e^{i\langle m,x\rangle}. \tag{3.106}$$

Let us fix some notation. We usually identify $m \in \mathbb{Z}^n$ with $m = (m_1, \tilde{m})$, where $\tilde{m} \in \mathbb{Z}^{n-1}$. We will sometimes use the same notation for $x \in \mathbb{R}^n$. Moreover, for any m, we define $m' = (-m_1, \tilde{m})$. This allows us to immediately see:

$$\widehat{k}(m') = -\widehat{k}(m), \quad \forall m \in \mathbb{Z}^n \setminus \{0\}. \tag{3.107}$$

This observation enables us to rewrite (3.106) as follows:

$$k(x) = 2i \cdot \sum_{m_1 > 0} \sum_{\tilde{m} \in \mathbb{Z}^{n-1}} \frac{m_1}{|m|^{n+1}} \sin(m_1 x_1) e^{i\langle \tilde{m}, \tilde{x}\rangle}. \tag{3.108}$$

The strategy of the proof is based on [8, p.405-406]. Thus, the main point is to split the sum into partial sums involving m_1 and $|\tilde{m}|$ being comparable to some dyadic 2^{k_1} and $2^{\tilde{k}}$ respectively. Then, we distinguish $k_1 \leq \tilde{k}$ and $k_1 \geq \tilde{k}$ to conclude. Thus, we consider the following sum derived from (3.108):

$$|k(x)| \leq \sum_{k_1 \geq 0} \sum_{\tilde{k} \geq 0} \Big| \sum_{m_1 \sim 2^{k_1}} \sum_{|\tilde{m}| \sim 2^{\tilde{k}}} \frac{m_1}{|m|^{n+1}} \sin(m_1 x_1) e^{i\langle \tilde{m}, \tilde{x}\rangle} \Big|. \tag{3.109}$$

Let us mention an uniform estimate for fixed $k_1, \tilde{k}$. To achieve this, we distinguish two cases: $k_1 \geq \tilde{k}$ and $k_1 < \tilde{k}$. We shall need the following estimate that can be found in [8, (4.22)]:

$$\Big| \sum_{\ell \in I} \sin(\ell x) \Big| \lesssim 4^k |x| \wedge \frac{1}{|x|}, \tag{3.110}$$

for every $k \in \mathbb{N}$, $x \in S^1$ and subinterval $I \subset [2^{k-1}, 2^k]$. Here, $\wedge$ denotes the minimum of two functions. Let us provide the argument in a more abstract manner: Consider a finite sum of the form:

$$\sum_{m_1} \sum_{\tilde{m}} a_{m_1} b_{\tilde{m}} c_{m_1, \tilde{m}}. \tag{3.111}$$

Observe that the summands in (3.109) inside the absolute value clearly have this form. Let us denote by A_{m_1} the partial sum of all a_l up to the m_1-th element. In the case of (3.109), this would be a sum

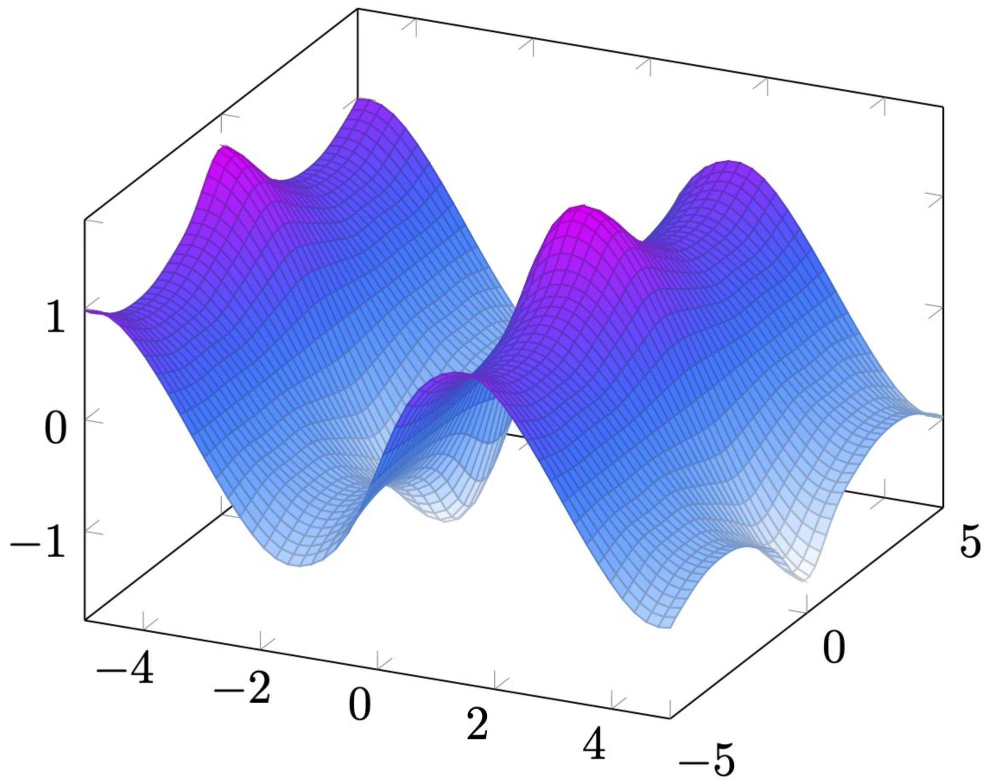

Figure 3.1.1: Plot of an Approximation of k in Dimension 2

of $\sin(lx)$ over an interval with l comparable to 2^{k_1}, hence we may use the bound (3.110). Therefore, we may rewrite (3.111) as:

$$\begin{aligned}\sum_{m_1}\sum_{\tilde{m}} a_{m_1} b_{\tilde{m}} c_{m_1,\tilde{m}} &= \sum_{m_1}\sum_{\tilde{m}} (A_{m_1} - A_{m_1-1}) b_{\tilde{m}} c_{m_1,\tilde{m}} \\ &= \sum_{m_1}\sum_{\tilde{m}} A_{m_1} b_{\tilde{m}} (c_{m_1,\tilde{m}} - c_{m_1+1,\tilde{m}}), \end{aligned} \tag{3.112}$$

which, in the case of (3.109), can be estimated using the bound on sums of sinus functions in (3.110), the boundedness of the $b_{\tilde{m}}$ which are merely $e^{i\langle \tilde{m},\tilde{x}\rangle}$ and finally the estimate:

$$\Big|\frac{m_1}{|m|^{n+1}} - \frac{m_1+1}{((m_1+1)^2 + |\tilde{m}|^2)^{\frac{n+1}{2}}}\Big| \lesssim \frac{1}{|m|^{n+1}}. \tag{3.113}$$

We mention the slight imprecision, as in (3.112), the extremal partial sums A_l require further attention. However, in the case we are considering, similar techniques can be applied (since we no longer sum over m_1) and we omit further details.

Therefore, we arrive at the following estimate:

$$\Big|\sum_{m_1\sim 2^{k_1}}\sum_{|\tilde{m}|\sim 2^{\tilde{k}}} \frac{m_1}{|m|^{n+1}} \sin(m_1 x_1) e^{i\langle \tilde{m},\tilde{x}\rangle}\Big| \lesssim 2^{k_1}\big(2^{k_1}|x_1| \wedge \frac{1}{2^{k_1}|x_1|}\big)\Big\|\frac{1}{|m|^{n+1}}\Big\|_{l^1(m_1\sim 2^{k_1},|\tilde{m}|\sim 2^{\tilde{k}})} \tag{3.114}$$

If $k_1 \geq \tilde{k}$, we may simplify (3.109) using (3.114) as follows:

$$\begin{aligned} |k(x)| &\lesssim \sum_{k_1 \geq 1} 2^{\tilde{k}(n-1)} 2^{k_1} \frac{1}{2^{k_1(n+1)}} \cdot 2^{k_1} \big(2^{k_1}|x_1| \wedge \frac{1}{2^{k_1}|x_1|}\big) \\ &\leq \sum_{k_1 \geq 0} 2^{k_1}|x_1| \wedge \frac{1}{2^{k_1}|x_1|} \lesssim C < \infty, \end{aligned} \tag{3.115}$$

which can be easily bounded by the definition of the minimum.

If $\tilde{k} > k_1$, we find:

$$\begin{aligned} &\sum_{k_1 \geq 0} \sum_{\tilde{k} \geq 0} \Big| \sum_{m_1 \sim 2^{k_1}} \sum_{|\tilde{m}| \sim 2^{\tilde{k}}} \frac{m_1}{|m|^{n+1}} \sin(m_1 x_1) e^{i\langle \tilde{m}, \tilde{x}\rangle} \Big| \\ &\lesssim \sum_{k_1} \sum_{\tilde{k} > k_1} 4^{k_1} \big(2^{k_1}|x_1| \wedge \frac{1}{2^{k_1}|x_1|}\big) \cdot \frac{1}{2^{k_1(n+1)}} \sum_{|\tilde{m}| \sim 2^{\tilde{k}}} \frac{1}{(1 + \frac{|\tilde{m}|^2}{2^{2k_1}})^{\frac{n+1}{2}}} \\ &\lesssim \sum_{k_1 \geq 0} \frac{2^{k_1(n-1)} 4^{k_1}}{2^{k_1(n+1)}} \big(2^{k_1}|x_1| \wedge \frac{1}{2^{k_1}|x_1|}\big) \\ &\leq \sum_{k_1 \geq 0} 2^{k_1}|x_1| \wedge \frac{1}{2^{k_1}|x_1|} \leq C < \infty, \end{aligned} \tag{3.116}$$

where we estimated the sum over $\tilde{m}, \tilde{k}$ by a dominating integral. This shows that $k(x)$ is actually bounded and possesses the required Fourier coefficients, hence adding the last ingredient missing in our proof of Theorem 3.1.1.4.

3.1.6 Existence Result for a certain Fractional PDE

Similar to Bourgain-Brezis [8], the estimates in Theorem 3.1.1.3 and 3.1.1.4 may be used to derive existence results for a particular differential operator. However, before turning to the PDE itself, let us briefly provide an alternative formulation of our main theorems for a more general class of distributions:

Theorem 3.1.6.1. *Let $u \in \mathcal{D}'(\mathbb{T}^n, \mathbb{C}_n)$ be $\mathbb{C}_n$-valued and assume that:*

$$Du, \overline{D}u \in \dot{H}^{-\frac{n}{2}} + L^1(\mathbb{T}^n, \mathbb{C}_n). \tag{3.117}$$

Here, D and $\overline{D}$ are the operators defined in the proof of Theorem 3.1.1.4. Then $u \in L^2(\mathbb{T}^n, \mathbb{C}_n)$ and we have the following estimate:

$$\Big\| u - \int_{\mathbb{T}^n} u dx \Big\|_{L^2} \lesssim \|Du\|_{\dot{H}^{-\frac{n}{2}} + L^1} + \|\overline{D}u\|_{\dot{H}^{-\frac{n}{2}} + L^1}. \tag{3.118}$$

This result is an immediate corollary of the proof of Theorem 3.1.1.4, as we always work with Du and $\overline{D}u$ rather than the $\mathcal{R}_j(-\Delta)^{\frac{n}{2}}$. The possibility to generalise to Clifford algebra-valued distributions follows directly, as all arguments involved behave well with respect to the Clifford algebra product. One could also rewrite the estimate by separating the identity operator from the Riesz operators.

Let us now turn to the existence result. We would like to consider the following problem:

$$g = (-\Delta)^{\frac{n}{4}} f_0 + \sum_{j=1}^{n} (-\Delta)^{\frac{n}{4}} \bar{\mathcal{R}}_j f_j, \tag{3.119}$$

where $g \in L^2_*(\mathbb{T}^n) = \{u \in L^2(\mathbb{T}^n) : \ \fint_{\mathbb{T}^n} u = 0\}$.[15] Obviously, the PDE admits solutions $f_0, \dots, f_n$ in $\dot{H}^{\frac{n}{2}}(\mathbb{T}^n)$. Again, using Sobolev embeddings, it is also clear that there is a-priori no way to deduce that the f_j may be chosen to be bounded or even continuous. We shall remedy this apparent lack of regularity:

Corollary 3.1.6.1. *Let $g \in L^2_*(\mathbb{T}^n)$. Then there exist $f_0, \dots, f_n \in \dot{H}^{\frac{n}{2}} \cap C^0(\mathbb{T}^n)$, such that* (3.119) *holds.*

Proof of Corollary 3.1.6.1. The proof is completely analogous to the one in [8, Proof of Theorem 1]: Let us define the following operator:

$$T : \bigoplus_{j=0}^{n} \dot{H}^{\frac{n}{2}} \cap C^0(\mathbb{T}^n) \to L^2_*(\mathbb{T}^n), \quad T(u_0, \dots, u_n) := (-\Delta)^{\frac{n}{4}} u_0 + \sum_{j=1}^{n} (-\Delta)^{\frac{n}{4}} \bar{\mathcal{R}}_j u_j. \tag{3.120}$$

It is clear that T is a bounded, linear operator. Moreover, we have that its dual operator is given by:

$$T^* : L^2_*(\mathbb{T}^n) \to \bigoplus_{j=0}^{n} \dot{H}^{-\frac{n}{2}} + \mathcal{M}(\mathbb{T}^n), \quad T^*(v) := \big((-\Delta)^{\frac{n}{4}} v, \mathcal{R}_1 (-\Delta)^{\frac{n}{4}} v, \dots, \mathcal{R}_n (-\Delta)^{\frac{n}{4}} v\big). \tag{3.121}$$

Here, $\mathcal{M}(\mathbb{T}^n)$ denotes the collection of Radon measures on $\mathbb{T}^n$. As in [8, (4.3)], it can be easily seen (using convolutions) that:

$$\| \cdot \|_{\dot{H}^{-\frac{n}{2}} + \mathcal{M}} = \| \cdot \|_{\dot{H}^{-\frac{n}{2}} + L^1} \quad \text{on } \dot{H}^{-\frac{n}{2}} + L^1(\mathbb{T}^n). \tag{3.122}$$

Therefore, we know by (3.103) that:

$$\|u\|_{L^2} \lesssim \|T^* u\|_{\bigoplus \dot{H}^{-\frac{n}{2}} + \mathcal{M}(\mathbb{T}^n)}. \tag{3.123}$$

This implies that T is surjective (see Theorem 2.20 in Brezis [10]). The open mapping Theorem yields that there is $C > 0$ such that $B^{L^2_*}(0, C) \subseteq T(B^E(0,1))$, where $E = \bigoplus_{j=0}^n \dot{H}^{\frac{n}{2}} \cap C^0(\mathbb{T}^n)$. Therefore, for every $g \in L^2_*(S^1)$, there are $(f_0, \dots, f_n) \in E$ such that $(-\Delta)^{1/4} f_0 + \sum_{i=1}^n (-\Delta)^{1/4} \bar{\mathcal{R}}_j f_j = g$ and

$$\sum_{j=0}^{n} \|f_j\|_{\dot{H}^{\frac{n}{2}} \cap L^\infty} \leq C \|g\|_{L^2}, \tag{3.124}$$

for some fixed $C > 0$. This concludes the proof. □

Using Corollary 3.1.6.1, we may derive the following simple result:

[15] The conjugate operator $\bar{\mathcal{R}}_j$ appears due to the duality used in the proof. This ensures, that we can apply the result in Theorem 3.1.1.4. It is simpel to see that by suitably exchanging $\mathcal{R}_j$ by $\bar{\mathcal{R}}_j$ throughout the proof of Theorem 3.1.1.4, the same inequality can be obtained for the dual operators and thus yields the same result as in Corollary 3.1.6.1 for the usual Riesz operators.

Corollary 3.1.6.2. *Let $f \in \dot{H}^{\frac{n}{2}}(\mathbb{T}^n)$. Then there exist $f_0, \dots, f_n \in \dot{H}^{\frac{n}{2}} \cap C^0(\mathbb{T}^n)$ as well as a smooth function $\varphi \in C^\infty(\mathbb{T}^n)$, such that:*

$$f = \varphi + \sum_{j=0}^{n} \mathcal{R}_j f_j. \tag{3.125}$$

Proof of Corollary 3.1.6.2. Take $g = (-\Delta)^{\frac{n}{4}} f \in L^2_*(\mathbb{T}^n)$. By Corollary 3.1.6.1, we see that there exist $f_0, \dots, f_n \in \dot{H}^{\frac{n}{2}} \cap C^0(\mathbb{T}^n)$, such that (3.119) is satisfied. Therefore, we know:

$$(-\Delta)^{\frac{n}{4}} \Big(f - \sum_{j=0}^{n} \mathcal{R}_j f_j \Big) = 0. \tag{3.126}$$

But this implies that the difference lies in the kernel of $(-\Delta)^m$, where m is the smallest integer larger or equal than $\frac{n}{4}$. Thus the difference is smooth, leading to the desired decomposition. $\square$

3.1.7 Appendix

In this section, we provide for the reader's convenience a proof of the two inequalities (3.3) and (3.4), since the authors have not found a precise reference in the literature.

1. Assume first that $f(z) = \sum_{n \geq 0} a_n z^n$ is an analytic function such that $\lim_{r \to 1^-} \|f(re^{i\theta})\|_{L^1(S^1)} < +\infty$. Let $h \in L^1(S^1)$ be such that $\lim_{r \to 1^-} \|f(re^{i\theta}) - h\|_{L^1(S^1)} = 0$. We set $g(z) = \sum_{n \geq 0} \frac{a_n}{n+1} z^{n+1}$. We observe that $g'(z) = f(z)$. From our hypothesis, we have $\lim_{r \to 1^-} \|g'(re^{i\theta})\|_{L^1(S^1)} < +\infty$. Observe that this implies that $\lim_{r \to 1^-} (\|\partial_\theta g(re^{i\theta})\|_{L^1(S^1)} + \|\partial_r g(re^{i\theta})\|_{L^1(S^1)}) < +\infty$. Define $g_r(z) = g(rz)$ for $0 < r < 1$. Since g is harmonic in $\mathbb{D}$, we have

$$\begin{aligned} 0 &= \int_{\mathbb{D}} (\Delta g_r \bar{g}_r + g_r \Delta \bar{g}_r) dx = \int_{\partial \mathbb{D}} (\partial_r g_r \cdot \bar{g}_r + \partial_r \bar{g}_r g_r) d\sigma - 2 \int_{\mathbb{D}} |\nabla g_r|^2 dx \\ &= \int_{\partial \mathbb{D}} (\partial_r g \cdot \bar{g} + \partial_r \bar{g} g) d\sigma - \int_{\mathbb{D}} |g_r'|^2 dx. \end{aligned} \tag{3.127}$$

We first have (observe that $\fint_{S^1} g_r = 0$)

$$\|g_r\|_{L^\infty(S^1)} \lesssim \|\partial_\theta g_r\|_{L^1(S^1)} \tag{3.128}$$

and from (3.127) it follows that

$$\|f_r\|_{L^2(\mathbb{D})} \simeq \|g_r'\|_{L^2(\mathbb{D})} \lesssim \|g_r\|_{L^\infty(S^1)} \|\partial_r g\|_{L^1(S^1)} \lesssim \|g_r'\|^2_{L^1(S^1)}. \tag{3.129}$$

We let $r \to 1$ in (3.129) and get

$$\|f\|_{L^2(\mathbb{D})} \lesssim \|h\|_{L^1(S^1)}. \tag{3.130}$$

2. Assume now that $f(z) = \sum_{n \geq 0} a_n z^n$ is an analytic function such that:

$$\lim_{r \to 1^-} \|f(re^{i\theta})\|_{H^{-1/2}(S^1)} < +\infty.$$

Claim. Assume $a_0 = 0$ in the power series above. Then the series $\sum_{n \geq 1} \frac{|a_n|^2}{n} < +\infty$ and

$$\sum_{n \geq 1} \frac{|a_n|^2}{n} = \lim_{r \to 1^-} \|f(re^{i\theta})\|^2_{\dot{H}^{-1/2}(S^1)}.$$

Proof of the claim. We set $A = \lim_{r\to 1^-} \|f(re^{i\theta})\|^2_{H^{-1/2}(S^1)}$. We observe that:

$$\|f(re^{i\theta})\|^2_{\dot{H}^{-1/2}(S^1)} = \sum_{n>0} \frac{|a_n|^2 r^{2n}}{n}.$$

For every $N > 1$, we have

$$\begin{aligned} A \geq \lim_{r\to 1^-} \sum_{n=1}^{N} \frac{|a_n|^2 r^{2n}}{n} &= \sum_{n=1}^{N} \lim_{r\to 1^-} \frac{|a_n|^2 r^{2n}}{n} \\ &= \sum_{n=1}^{N} \frac{|a_n|^2}{n}. \end{aligned} \tag{3.131}$$

By letting $N \to +\infty$, we get $\sum_{n=1}^{\infty} \frac{|a_n|^2}{n} < +\infty$ and by Abel's theorem on power series, we deduce that the norms converge

$$\lim_{r\to 1-} \sum_{n>0} \frac{|a_n|^2 r^{2n}}{n} = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n}.$$

Therefore, $f(e^{i\theta}) \in \dot{H}^{-1/2}(S^1)$ and $\lim_{r\to 1^-} \|f(re^{i\theta}) - f(e^{i\theta})\|_{\dot{H}^{-1/2}(S^1)} = 0$, by observing that the convergence holds weakly and the norms converge, which is an equivalent characterisation for convergence with respect to the norm in Hilbert spaces. This proves the claim.

Consider the function $g_r(z) = \sum_{n\geq 0} \frac{a_n}{n+1}(rz)^{n+1}$. In this case we have $g_r \in \dot{H}^{1/2}(S^1)$. We have $r \cdot f_r(z) = g_r'(z)$. Since g is harmonic in $\mathbb{D}$ we have

$$\|f_r\|_{L^2(\mathbb{D})} \simeq \|\nabla g_r\|_{L^2(\mathbb{D})} \lesssim \|g_r\|_{\dot{H}^{1/2}} \equiv \|f_r\|_{H^{-1/2}} \tag{3.132}$$

We let $r \to 1$ in (3.132) and get

$$\|f\|_{L^2(\mathbb{D})} \lesssim \|f\|_{H^{-1/2}(S^1)}. \tag{3.133}$$

Both inequalities (3.3) and (3.4) have been proved.

3.2 Integrability by Compensation Results for the Dirac Equation [26]

Building on the work in Da Lio-Rivière [23] that exposed suitable compensation phenomena not involving antisymmetric potentials, a more general kind of regularity result for Dirac equations is established by revealing compensation properties that are a-priori hidden. Among the most significant ideas is the introduction of Clifford algebras which naturally allows one to imitate the step from $\mathbb{C}$ to $\mathbb{H}$ as in Da Lio-Rivière [23] that ultimately provided access to a suitable gauge in a compact Lie group. Due to technical limitations, unfortunately, we will be restricted to domains of dimension ≤ 8 in general, the main points behind this being discussed in the next section.

3.2.1 Introduction

The present paper is a new contribution to the study of linear critical systems with special structures enjoying integrability by compensation properties.

In [70], the first author proved the sub-criticality of local *a-priori* critical Schödinger systems in 2 dimensions of the form

$$-\Delta u = \Omega \cdot \nabla u \quad \text{in } \mathcal{D}'(B^2), \tag{3.134}$$

where $u = (u^1, \cdots, u^n) \in W^{1,2}(B^2, \mathbb{R}^n)$ and $\Omega \in L^2(B^2, \mathbb{R}^2 \otimes \mathfrak{so}(n))$, ($\mathfrak{so}(n)$ is the Lie algebra of antisymmetric $n \times n$ matrices). Systems of the form (3.134) are related to concentration compactness and regularity results of Euler-Lagrange equations of conformal invariant functionals in 2-D, such as for instance the *harmonic map equation*.

Following Rivière [70], in a series of works, various critical local and non local systems with antisymmetric potentials, often related to geometric variational problems, have been singled out as enjoying compactness properties similar to the ones of (3.134). Successively the following systems for the corresponding critical regimes[16] and where Ω denotes an antisymmetric potential have been proven to have subcritical behaviour below a threshold of energy

$$\Delta^2 u = \Delta(V \cdot \nabla u) + \operatorname{div}(w \, \nabla u) + \Omega \cdot \nabla u \tag{3.135}$$

in Lamm-Rivière [53],

$$-\Delta v = \Omega \, v \tag{3.136}$$

in Rivière [71].

In the nonlocal framework, denoting for $\sigma \in (0,1)$

$$(-\Delta)^\sigma u(x) = PV \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x-y|^{1+2\sigma}} \, dy \ ,$$

similar sub-critical behaviour have been proven to hold for systems of the form respectively

$$(-\Delta)^{1/4} v = \Omega \cdot v \,, \tag{3.137}$$

in Da Lio-Rivière [22] , as well as

$$(-\Delta)^{1/2} u = \Omega \cdot d_{1/2}(u) \ , \tag{3.138}$$

in Mazowiecka-Schikorra [57] where $d_{1/2}$ is the the half gradient given by

$$d_{1/2}\varphi(x,y) = \frac{\varphi(x) - \varphi(y)}{|x-y|^{1/2}} \ ,$$

as well as

$$(-\Delta)^{1/4} u = \int_{\mathbb{R}} K(x,y) \; u(y) dy \ , \tag{3.139}$$

where $K(x,y) = -\, K^t(y,x)$ (see Da Lio-Rivière [24]).

In all the above examples the antisymmetry (3.137), (3.138) or the anti-self adjoint duality (3.139) of the potential appearing in the equation are responsible for the regularity of the solutions or for the stability under weak convergence as in the original work Rivière [70]. Recently in Da Lio-Rivière [23] the first and the second authors have discovered new integrability by compensation phenomena for linear systems in 2-D where the antisymmetry is not directly involved. They are systems of the form

$$\operatorname{div}(S \, \nabla u) = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^2), \tag{3.140}$$

[16] In the function spaces which makes them critical.

where $u \in L^2(\mathbb{C}), \mathbb{R}^n)$ and $S \in W^{1,2}(\mathbb{C}, Sym(n))$ where $Sym(n)$ denotes the set of symmetric $n \times n$-matrices over $\mathbb{R}$ and where the crucial *involution* assumption is made

$$S^2 = Id_n \ . \tag{3.141}$$

In the case of 2-D codomains ($n = 2$) the resolution of (3.140) required a different formulation of the equation in the form

$$\partial_z \mathfrak{f} = \Omega \cdot \bar{\mathfrak{f}} \ \text{ in } \mathcal{D}'(\mathbb{C}), \tag{3.142}$$

where $\Omega \in L^2(\mathbb{C}, \mathfrak{so}(2) \otimes \mathbb{C})$ is given by

$$\Omega = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}, \tag{3.143}$$

for some $\beta \in L^2(\mathbb{C}, \mathbb{C})$ and $\mathfrak{f} \in L^2(\mathbb{C}, \mathbb{C}^2)$, $\bar{\mathfrak{f}} = (\bar{\mathfrak{f}}^1, \bar{\mathfrak{f}}^2)$, and $\bar{\mathfrak{f}}^i$ is the complex conjugate of $\mathfrak{f}^i$, (see Proposition III.2 in Da Lio-Rivière [23]). We observe that in this context the Lie Algebra $\mathfrak{so}(2) \otimes \mathbb{C}$ does not generate a compact Lie group. This differs completely from all the previously mentioned results above where the compactness of the underlying Lie Group, $SO(n)$, was the crucial assumption allowing the construction of suitable gauge transformations *à la Uhlenbeck* [98] in order to "absorb" the potential in the left-hand-side of the system.

The main result of Da Lio-Rivière [23] leading to the regularity of solutions to (3.140) for $n = 2$ is the following theorem.

Theorem 3.2.1.1 (Theorem III.7 in [23]). *Let $\beta \in L^2(\mathbb{C}, \mathbb{C})$ with*

$$\partial_{x_1} \beta_2 - \partial_{x_2} \beta_1 = 0 \ .$$

Let $\mathfrak{f} \in L^2(\mathbb{C}, \mathbb{C} \times \mathbb{C})$ be a solution of

$$\partial_z \mathfrak{f} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \bar{\mathfrak{f}} \tag{3.144}$$

Then $\mathfrak{f} \in L^{\mathfrak{q}}_{loc}(\mathbb{C})$ for all $q < \infty$. □

We observe that actually the system (3.142) is critical in the sense that if we start with a $L^2(\mathbb{C})$ solution $\mathfrak{f}$ then from the fact that $\partial_L \mathfrak{f} \in L^1$ we get that $\mathfrak{f} \in L^{2,\infty}(\mathbb{C})$ namely we return almost to the starting point. The new integrability by compensation results discovered in Da Lio-Rivière [23] are related to Wente's inequality for 2-D Jacobians.

The purpose of the present work is to extend the integrability by compensation result given by Theorem 3.2.1.1 to higher dimensions. For this purpose we need to recall the fundamental notions related to Clifford Algebras:

For every $m \geq 0$, we denote by $C\ell_m$ the universal Clifford algebra on $\mathbb{R}^m$ (sometimes also denoted $C\ell(0, m)$). $C\ell_m$ is a real associative algebra with identity containing linearly a copy of $\mathbb{R}^m$, such that for any orthonormal basis $(e_1, \ldots, e_m)$ of $\mathbb{R}^m$, it holds

$$e_i e_j + e_j e_i = -\delta_{ij},$$

for $1 \leq i, j \leq m-1$ and the reduced products $e_I = e_{i_1} \cdots e_{i_k}$, $1 \leq i_1 \leq \cdots \leq e_k \leq m$ and $e_0 = 1$ are a basis for $C\ell_m$. [17] Any $f \in C\ell_m$ can be decomposed as follows:

$$f = \sum_I f_I \, e_I, \qquad \text{where } \; e_I = e_{i_0} e_{i_1} \cdots e_{i_k} \;,\; I = \{i_0, \ldots, i_k\} \quad , \quad 0 \leq i_1 \leq \cdots \leq i_k \leq m \;.$$

Let $\sigma \colon C\ell_m \to C\ell_m$ be the unique involutive automorphism such that $\sigma(e_i) = -e_i$ for every $i = 1, \ldots, m$ and $\sigma|_{\mathbb{R}} = Id$. it is called the *principal automorphism* on $C\ell_m$ in mathematics and *grade involution* or *grade automorphism* in physics[18] . For $\mathfrak{f} \in C\ell_m$ we also denote

$$\hat{\mathfrak{f}} := \sigma(\mathfrak{f}) \;.$$

Observe for instance that by definition

$$\hat{e_0} = e_0 \;,\; \hat{e_i} = -e_i \;\; \widehat{e_i e_j} = \hat{e_i}\hat{e_j} = e_i e_j \;\; \cdots$$

We point out that the principal automorphism σ is the only involution which is compatible with the Clifford Algebra structure[19]. We will refer for instance to Gilbert-Murray [38] and Hamilton [42] for a presentation of Clifford Algebras.

Finally for $\mathfrak{f} \colon \mathbb{R}^m \to (C\ell_m)^2$ we consider the Dirac operator $\partial_L \mathfrak{f}$ defined by

$$\partial_L \mathfrak{f} = e_0 \cdot \partial_{x_0} \mathfrak{f} - e_1 \cdot \partial_{x_1} \mathfrak{f} - \ldots - e_{m-1} \cdot \partial_{x_{m-1}} \mathfrak{f} \;, \tag{3.145}$$

Our main result in the present work is the following integrability by compensation theorem which is the 3 and 4 dimensional counterpart of theorem 3.2.1.1

Theorem 3.2.1.2. *Let $m = 3, 4$. $\beta = (\beta_0, \cdots \beta_{m-1}) \in W^{1,m/2}_{loc}(\mathbb{R}^m, \mathrm{span}_{\mathbb{R}}\{e_0, \cdots, e_{m-1}\})$ with*

$$\forall \; i, j = 1 \cdots m-1 \qquad \partial_{x_i} \beta_j - \partial_{x_j} \beta_i = 0 \;. \tag{3.146}$$

[17] If $m = 0, 1, 2$ then $C\ell_0 \simeq \mathbb{R}$, $C\ell_1 \simeq \mathbb{C}$ and $C\ell_2 \simeq \mathbb{H}$ respectively, where

$$\mathbb{H} := \{a + b \cdot i + c \cdot j + d \cdot k, \quad (a, b, c, d) \in \mathbb{R}^4\},$$

is the algebra of quaternions.

$C\ell_3$ is a real 8 dimensional space with a basis given by the following *paravectors*

$$\begin{cases} e_0 \quad \textit{Scalar} \\ e_1,\, e_2,\, e_3 \quad \textit{Vectors} \\ e_1e_2,\; e_2e_3,\; e_3e_1 \quad \textit{Bivectors} \\ e_1e_2e_3 \quad \textit{Trivectors} \end{cases}$$

[18] https://en.wikipedia.org/wiki/Paravector

[19] In the case $m = 1$ then the principal automorphism coincides with the complex conjugation: $\hat{f} = \bar{f}$. While in the case of $m = 2$ with $C\ell_2 \simeq \mathbb{H}$ the automorphism σ does not coincide with the other involution on $\mathbb{H}$ given by the conjugation operation on quaternions :

$$\bar{1} = 1 \;,\; \bar{i} = -i \;,\; \bar{j} = -j \;\text{ and }\; \bar{k} = -k$$

while

$$\hat{1} = 1 = e_0 \;,\; \hat{i} = -i = e_1 \;,\; \hat{j} = -j = e_2 \;,\; \hat{k} = k = e_1\, e_2$$

Let $\mathfrak{f} \in L^{m/m-1}(\mathbb{R}^m, C\ell_{m-1} \times C\ell_{m-1})$ *be a solution of*

$$\partial_L \mathfrak{f} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \hat{\mathfrak{f}} \tag{3.147}$$

Then $\mathfrak{f} \in L^q_{loc}(\mathbb{R}^m, C\ell_{m-1} \times C\ell_{m-1})$ *for all* $q < \infty$. □

Remark 3.2.1.1. *Observe that the system (3.147) is critical in the sense that the r-h-s is a-priori only*[20] *in* L^1_{loc} *which is preventing a direct use of Calderon Zygmund theory. Any direct attempt to bootstrap is blocked by the fact that* $\partial_L^{-1} L^1_{loc} \hookrightarrow L^{m/m-1,\infty}_{loc}(\mathbb{R}^m)$. *Which means that a-priori integrability information on* $\mathfrak{f}$ *is lost from the first iteration on. It is only because of its very peculiar structure that, thanks to some "hidden" compensation, a gain of integrability and local compactness holds. In fact a quantitative version of the theorem 3.2.1.2 can bee formulated in the form of an* ϵ*-regularity.* □

Remark 3.2.1.2. *Some gain of integrability still holds when instead of assuming (3.146) one assumes that* $\partial_{x_i}\beta_j - \partial_{x_j}\beta_i \in L^p_{loc}(\mathbb{R}^m)$ *(*$m = 3, 4$*) for some* $p > 2$. □

Remark 3.2.1.3. *It would be interesting to study the possibility whether or not theorem 3.2.1.2 continues to hold if instead of assuming* β *to be in* $W^{1,m/2}_{loc}$ *one would make the milder hypothesis* $\beta \in L^m_{loc}$. *In fact, we are proving theorem 3.2.1.2 under the assumption that* β *belongs to the Lorentz space* $L^{(m,2)}_{loc}$ *in which* $W^{1,m/2}_{loc}$ *embeds in* m *dimensions for* $m = 3, 4$*(see Rivière [73]).* □

Remark 3.2.1.4. *The investigations made by the authors is leading them to the conclusion that the theorem does not generalise to arbitrary* m *in a straightforward way and the proofs given below for the cases* $m = 3, 4$ *is very much "dimension depending". Some results have been obtained by the third author in dimensions* $m \leq 8$ *in Section 3.3.2.*

Similarly to the 2-dimensional case the resolution of Theorem 3.2.1.2 for $m = 4$ for instance goes through the canonical inclusion of $C\ell_3$ into $C\ell_4$ (i.e. $C\ell_4 \simeq C\ell_3 \oplus C\ell_3\, e_4$) and the introduction of the new variable $\mathfrak{g} = \mathfrak{f}^1 + \mathfrak{f}^2 e_4 \in C\ell_4$. The equation satisfied by $\mathfrak{g}$ is then[21]

$$\partial_L \mathfrak{g} = -(\beta e_4) \cdot \mathfrak{g} \quad \text{in } \mathcal{D}'(\mathbb{R}^4). \tag{3.149}$$

[20] Indeed we have $W^{1,m/2}_{loc}(\mathbb{R}^m) \hookrightarrow L^m(\mathbb{R}^m)$

[21] In this form the equation identifies to the **covariant Dirac** equation commonly written as follows

$$\sum_{\mu=0}^{3} \gamma_\mu (\partial_{x_\mu} - \mathbf{i} A_\mu)\psi = 0 \tag{3.148}$$

where γ_0 is the 2×2 identity matrix, $\gamma_\mu = -e_\mu$, $\mathbf{i} = e_4$, the connection components are given by $A_\mu = \beta_\mu$ and the group representing on the spinor space $C\ell_4 \simeq C\ell_3 \oplus C\ell_3\, e_4$ is the abelian group $\exp(e_4\mathbb{R})$. With this identification at hand one could then imagine that, for instance assuming $\beta_0 = 0$ and $\partial_{x_0}\beta = 0$ the flatness of the connection A implied by (3.146) would make the absorption of the r-h-s of (3.149) trivial by multiplying on the left (3.148) by $\exp(e_4\varphi)$ where $d\varphi = A$. However we have

$$\forall\ l = 1, 2, 3 \qquad \exp(e_4\,\varphi) e_l = e_l\, \exp(-e_4\,\varphi)$$

and this multiplication would then give

$$\partial_{x_0}(\exp(e_4\,\varphi)\mathfrak{g}) - \sum_{i=1}^{3} e_l\ \partial_{x_i}\,(\exp(-e_4\,\varphi)\mathfrak{g})) = 0\ ,$$

which is not easily invertible neither unless in the very particular case where $\mathfrak{g}$ is known to be independent of x_0 (that we are not assuming a-priori).

The "absorption" of the right hand side of this equation by the left-hand-side will be achieved through the construction *à la Uhlenbeck* of a *Coulomb type Gauge* in the Lie group whose Lie algebra is given by

$$\mathcal{E}_4 = \{e_4, e_1e_4, e_2e_4, e_3e_4, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3\} \ .$$

This Lie group happens to be isomorphic to $Spin(5)$ (see Appendix for a presentation of $Spin(m)$) and is hence compact which is crucial for the gain of integrability similarly to the seminal work Rivière [70].

We also would like to stress that the linearized natural *Coulomb type condition* in the present framework is given by the **Lorenz gauge equation** for an electric potential φ and magnetic vector potential A (see (3.212) and (3.213)) :

$$\left\{ \begin{array}{rcl} \mathbf{curl}(A) &=& \mathbf{B} \\ -\partial_t A - \nabla\varphi &=& \mathbf{E} \\ \partial_t\varphi + \mathbf{div}(A) &=& \beta_0 \end{array} \right. \tag{3.150}$$

where $\mathbf{B}$, and $\mathbf{E}$ represent respectively the **electric** and the **magnetic fields** and are taken in our case to be $\mathbf{B} = 0$ and $\mathbf{E} = (\beta_1, \beta_2, \beta_3)$ while assuming (3.146) and x_0 is the time variable.

Finally it is legitimate to ask if the resolution of theorem 3.2.1.2 leads to any new result regarding solutions in $L^{m/m-1}$ of real Elliptic Systems in Divergence form (3.140) involving *critical chirality operator* $S \in W^{2,m/2}_{loc}(\mathbb{R}^m, Sym(m))$ with $S^2 = Id_m$. We have not been able to establish this connection so far[22].

3.2.2 Bootstrap Test for (3.149) in 4-D

We first start with the dimension $m = 4$ that looks more natural to us. We consider systems of the form

$$\partial_L \mathfrak{f} \quad = \quad \beta e_4 \cdot \mathfrak{f}\,. \tag{3.153}$$

where $\mathfrak{f} = \mathfrak{f}^1 + \mathfrak{f}^2\, e_4$, and $\mathfrak{f}^1, \mathfrak{f}^2 \colon \mathbb{R}^4 \to C\ell_3$. The function $\mathfrak{f}$ assumes values in the Clifford algebra $C\ell_4$.

In this context ∂_L and ∂_R denote respectively the the left and right Dirac operator in $\mathbb{R}^4$ defined by

$$\partial_L \mathfrak{f} \quad := \quad \partial_{x_0}\mathfrak{f} - \sum_{i=1}^{3} e_i \partial_{x_i}\mathfrak{f} \tag{3.154}$$

[22] Actually in dimension $m = 3$ $L^{3/2}$ solutions of

$$\mathrm{div}(S\,\nabla u) = 0 \ \text{ in } \mathcal{D}'(\mathbb{R}^3) \tag{3.151}$$

happen to be rather related to a solution of a system of the type:

$$\partial_L \mathfrak{f} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \bar{\mathfrak{f}} \tag{3.152}$$

where $\mathfrak{f} = (\mathfrak{f}^1, \mathfrak{f}^2) \in \mathbb{H} \times \mathbb{H}$ and $\overline{\mathfrak{f}^i}$ denotes the conjugate of $\mathfrak{f}$ in $\mathbb{H}$ which does not coincide with the involution σ in $C\ell_2 \simeq \mathbb{H}$ given by the principal automorphism we are considering in (3.147). Up to our knowledge, the question of a possible higher integrability for $L^{3/2}$ solutions of systems of the form (3.152) for $\beta \in L^3$ or even in $W^{1,3/2}$ as well as the question of the possible higher integrability for $L^{3/2}$ solutions of (3.151) with $W^{2,3/2}_{loc}$ Involution operator S that would naturally extend to 3D the theorems in 2 dimension obtained in Da Lio-Rivière [23] are still open.

$$\partial_R \mathfrak{f} \;:=\; \partial_{x_0}\mathfrak{f} - \sum_{i=1}^{3} \partial_{x_i}\mathfrak{f} e_i. \tag{3.155}$$

The first main result is the following Theorem which corresponds to a bootstrap test for the equation (3.153):

Theorem 3.2.2.1. *There exists $\varepsilon_0 > 0$ such that for every $\beta \in L^{(4,2)}(\mathbb{R}^4, V_3)$ satisfying $\|\beta\|_{L^{(4,2)}(\mathbb{R}^4)} \leq \varepsilon_0$ as well as*

$$\forall i,j \quad \partial_{x_i}\beta^j - \partial_{x_j}\beta^i = 0 \ ,$$

and where $V_3 = \mathrm{span}_{\mathbb{R}}\{e_0, e_1, e_2, e_3\}$ and every $\mathfrak{f} \in L^{4/3}(\mathbb{R}^4, C\ell_4)$ solving

$$\partial_L \mathfrak{f} = \beta e_4 \cdot \mathfrak{f} \ , \tag{3.156}$$

we have $\mathfrak{f} \equiv 0$. □

In order to prove Theorem 3.2.2.1, we first perform the construction of a suitable gauge. This relies on the use of certain projections to render the emerging gauge equations elliptic and therefore enabling direct existence and regularity arguments. The arguments are given in the following subsections and we will make use of a new compensation phenomenon linked to the appearance of Maxwell-type equations for our changes of gauge.

3.2.2.1 Construction of a Gauge

In order to employ an absorption argument by a change of gauge, we consider the compact Lie algebra generated by $\{e_4, e_1e_4, e_2e_4, e_3e_4\}$. Such an algebra is isomorphic to $\mathfrak{spin}(5)$ and it is given by:

$$E = \mathrm{span}_{\mathbb{R}}\{e_4, e_1e_4, e_2e_4, e_3e_4, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3\}.$$

We may split

$$E = E_4 \oplus E_6,$$

where

$$E_4 := \mathrm{span}_{\mathbb{R}}\{e_4, e_1e_4, e_2e_4, e_3e_4\},$$

and

$$E_6 := \mathrm{span}_{\mathbb{R}}\{e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3\}.$$

Note that E_6 is also a compact Lie algebra of dimension 6 isomorphic to $\mathfrak{spin}(4)$. Let us introduce the following projections:

$$\Pi_6 \colon C\ell_4 \to E_6 \tag{3.157}$$
$$\Pi_4 \colon C\ell_4 \to E_4 \tag{3.158}$$
$$\Pi_3 \colon C\ell_4 \to E_3 := \mathrm{span}_{\mathbb{R}}\{e_2e_3e_4, e_3e_1e_4, e_1e_2e_4\} \tag{3.159}$$
$$\mathcal{P} \colon E_3 \to E_4, \quad e_{k+1}e_{k-1}e_4 \mapsto e_ke_4, \quad k = 1,2,3. \tag{3.160}$$

In the projection $\mathcal{P}$, we use the indexing in $\mathbb{Z}/3\mathbb{Z}$. This means for example that we identify 4 with 1 in (3.160).

We multiply both sides of the equation (3.156) from the left by $\mathfrak{q}$ belonging to the compact Lie group corresponding to the Lie algebra E. Such a Lie group is isomorphic to $Spin(5) \simeq Sp(2)$[23]. We obtain

$$\mathfrak{q}\partial_L \mathfrak{f} = \partial_{x_0}(\mathfrak{q}f) - (\partial_{x_0}\mathfrak{q})\mathfrak{f} - \sum_{i=1}^{3} \partial_{x_i}(\mathfrak{q}e_i\mathfrak{f}) + \sum_{i=1}^{3} \partial_{x_i}\mathfrak{q}e_i\mathfrak{f} \tag{3.161}$$

We define the following expression:

$$\mathcal{D}(\mathfrak{q}) := \mathfrak{q}^{-1}\partial_{x_0}\mathfrak{q} - \sum_{i=1}^{3} \mathfrak{q}^{-1}\partial_{x_i}\mathfrak{q}e_i = \mathfrak{q}^{-1}\partial_R\mathfrak{q}.$$

Let us observe that

$$\beta e_4 = \beta^0\ e_4 - \sum_{i=1}^{3} \beta^i\ e_i e_4 \quad \in \quad E_4 \tag{3.162}$$

$$\mathcal{D}(\mathfrak{q}) \quad \in \quad E_3 \oplus E_4 \oplus E_6 \oplus \mathbb{R}\cdot e_1e_2e_3. \tag{3.163}$$

By combining (3.156), (3.161) and (3.163) we get

$$\partial_{x_0}(\mathfrak{q}\mathfrak{f}) - \sum_{i=1}^{3} \partial_{x_i}(\mathfrak{q}e_i\mathfrak{f}) \quad = \quad \mathfrak{q}(\beta\, e_4 + \mathcal{D}(\mathfrak{q}))\mathfrak{f}. \tag{3.164}$$

We notice that (3.164) is a system of 15 equations in 10 unknowns, if we split the PDE according to the basis in $C\ell_4$. If we try to directly solve:

$$\beta\, e_4 + \mathcal{D}(\mathfrak{q}) = 0,$$

this will have therefore little to no chance of success. Instead, let us try and approximately solve this equation.

More precisely, our main goal is to find $\mathfrak{q} \in \dot{W}^{1,4}(\mathbb{R}^4, Spin(5))$[24] such that $\mathcal{D}(\mathfrak{q}) = -\beta e_4 + \mathcal{V}(x)$, where $\mathcal{V}$ is a more regular potential than β, namely $\mathcal{V} \in L^{(4,1)}(\mathbb{R}^4)$.

To achieve this, we introduce the following operator:

$$\mathbf{N}\colon \dot{W}^{1,4}(\mathbb{R}^4, Spin(5)) \to W^{-1,4}(\mathbb{R}^4, E_6) \times L^4(\mathbb{R}^4, E_4) \times L^4(\mathbb{R}^4, E_3) \tag{3.165}$$

$$\mathfrak{q} \mapsto \left(\Pi_6 \left(\sum_{i=0}^{3} (\partial_{x_i}((\mathfrak{q})^{-1}\partial_{x_i}\mathfrak{q})) \right), \Pi_4(\mathcal{D}(\mathfrak{q})), \Pi_3(\mathcal{D}(\mathfrak{q})) \right)$$

We observe that $\mathbf{N}$ is an operator from $Spin(5)$-valued maps, i.e. functions whose range has dimension 10, to $E_6 \oplus E_4 \oplus E_3$-valued functions, namely functions taking values in a space of dimension 13. Therefore, there is no hope of proving that it is an isomorphism. This suggests that we have to further reduce dimensionality to arrive at an operator which takes all values sufficiently close to 0.

Indeed, we would like to prove the following result (which, as we stated before, is a-priori impossible in the generality presented):

[23] see Thm. 9.11.(iii.) in Gilbert-Murray [38]. The symplectic group, $Sp(n)$ is the subgroup of $Gl(n, \mathbb{H})$, the invertible $n \times n$ quaternionic matrices A satisfying $\overline{A}^t A = A\overline{A}^t = Id$.

[24] $\dot{W}^{1,4}(\mathbb{R}^4, Spin(5))$ denotes the space of functions $u \in L^1_{loc}(\mathbb{R}^4, Spin(5))$ such that $\nabla u \in L^4(\mathbb{R}^4)$

Lemma 3.2.2.1. *There exists* $\varepsilon_0 > 0$ *such that for every* $\beta \in L^4(\mathbb{R}^4, V_3)$ *and* $\|\beta\|_{L^4(\mathbb{R}^4)} \leq \varepsilon_0$, *there exists* $\mathfrak{q} \in \dot{W}^{1,4}(\mathbb{R}^4, Spin(5))$ *such that*

$$\mathbf{N}(\mathfrak{q}) = (0, \beta e_4, 0). \tag{3.166}$$

together with an estimate:

$$\|\nabla \mathfrak{q}\|_{L^4} \lesssim \|\beta\|_{L^4(\mathbb{R}^4)}. \tag{3.167}$$

Unfortunately, this strong form of a gauge is not possible. However, if we allow for an error term of slightly better integrability, which will suffice for the regularity result we are trying to establish, we can actually achieve a suitable change of gauge by using a slightly weaker gauge operator.

In order to prove a weaker analogue of Lemma 3.2.2.1, we first consider a different nonlinear operator:

$$\mathcal{N}\colon \dot{W}^{1,4}(\mathbb{R}^4, Spin(5)) \to W^{-1,4}(\mathbb{R}^4, E_6) \times L^4(\mathbb{R}^4, E_4) \tag{3.168}$$

$$\mathfrak{q} \mapsto \left(\Pi_6 \left(\sum_{i=0}^{3} (\partial_{x_i}((\mathfrak{q})^{-1} \partial_{x_i}(\mathfrak{q})) \right), (\Pi_4 + \mathcal{P})(\mathcal{D}(\mathfrak{q})) \right)$$

Observe that

$$(\Pi_4 + \mathcal{P})(\mathcal{D}(\mathfrak{q})) \quad = \quad \Pi_{e_4}(\mathcal{D}(\mathfrak{q})) + \sum_{i=1}^{3} (\Pi_{e_i e_4} + \Pi_{\mathcal{P}(e_{i+1}, e_{i-1} e_4)})(\mathcal{D}(\mathfrak{q})) \tag{3.169}$$

We shall construct a gauge for the nonlinear operator in (3.168):

Lemma 3.2.2.2. *There are constants* $\varepsilon_0 > 0$ *and* $C > 0$ *such that for any choice of* $\omega \in W^{-1,4}(\mathbb{R}^4, E_6)$ *and* $\mathfrak{g} \in L^4(\mathbb{R}^4, E_4)$ *satisfying*

$$\|\omega\|_{W^{-1,4}} \leq \varepsilon_0, \quad \|\mathfrak{g}\|_{L^4} \leq \varepsilon_0 \tag{3.170}$$

there exists $\mathfrak{q} \in \dot{W}^{1,4}(\mathbb{R}^4, Spin(5))$ *such that*

$$\mathcal{N}(\mathfrak{q}) = (\omega, \mathfrak{g}) \tag{3.171}$$

and

$$\|\nabla \mathfrak{q}\|_{L^4} \leq C(\|\omega\|_{W^{-1,4}} + \|\mathfrak{g}\|_{L^4}). \tag{3.172}$$

In order to prove Lemma 3.2.2.2, we shall need to introduce some notation and establish a few intermediate results: As in [22, 27], by an approximation argument, it suffices to prove Lemma 3.2.2.2 under the assumption that ω and $\mathfrak{g}$ are slightly more regular. More precisely, we first prove Lemma 3.2.2.2 in the case $\omega \in (W^{-1,p} \cap W^{-1,p'})(\mathbb{R}^4, E_6)$ and $\mathfrak{g} \in (L^p \cap L^{p'})(\mathbb{R}^4, E_4)$ for some fixed $4 < p$ and its Hölder-dual $p' = \frac{p}{p-1}$:

For $\varepsilon > 0$, let us now introduce:

$$\mathcal{U}_\varepsilon := \left\{ \begin{array}{c} (\omega, \mathfrak{g}) \in (W^{-1,p} \cap W^{-1,p'})(\mathbb{R}^4, E_6) \times (L^p \cap L^{p'})(\mathbb{R}^4, E_4) \\ \\ \|\omega\|_{W^{-1,4}} + \|\mathfrak{g}\|_{L^4} \leq \varepsilon \end{array} \right\} \tag{3.173}$$

For constants $\varepsilon, \Theta > 0$, let $\mathcal{V}_{\varepsilon,\Theta} \subseteq \mathcal{U}_\varepsilon$ denote the set of pairs $(\omega, \mathfrak{g})$ for which we have a decomposition as in (3.171) and which are satisfying the following estimates:

$$\|\nabla \mathfrak{q}\|_{L^4} \leq \Theta(\|\omega\|_{W^{-1,4}} + \|\mathfrak{g}\|_{L^4}) \tag{3.174}$$

$$\|\nabla \mathfrak{q}\|_{p} \leq \Theta(\|\omega\|_{W^{-1,p}} + \|\mathfrak{g}\|_{L^p})\,, \tag{3.175}$$

$$\mathcal{U}_\varepsilon\|\nabla \mathfrak{q}\|_{p'} \leq \Theta(\|\omega\|_{W^{-1,p'}} + \|\mathfrak{g}\|_{L^{p'}})\,. \tag{3.176}$$

That is:

$$\mathcal{V}_{\varepsilon,\Theta} := \left\{ (\omega, \mathfrak{g}) \in \mathcal{U}_\varepsilon : \begin{array}{c} \text{there exists } \mathfrak{q} \in (\dot{W}^{1,p} \cap \dot{W}^{1,p'})(\mathbb{R}^4, Spin(5)), \text{ so that} \\ \mathfrak{q} - \mathfrak{I} \in L^{4p/3p-4}(\mathbb{R}^4, Spin(5)) \\ \text{and } \ (3.171), (3.174), (3.175), (3.176) \ \text{ hold.} \end{array} \right\}$$

The strategy to prove Lemma 3.2.2.2 follows the one introduced by K. Uhlenbeck in [98] in order to construct Coulomb gauges in critical dimensions. In fact, Lemma 3.2.2.2 is going to be a consequence of the following proposition, which will establish our Lemma by using a standard connectedness argument on suitable spaces.

Proposition 3.2.2.1. *There exist $\Theta > 0$ and $\varepsilon > 0$, such that $\mathcal{V}_{\varepsilon,\Theta} = \mathcal{U}_\varepsilon$.* □

Proof of Proposition 3.2.2.1. Proposition 3.2.2.1 will follow, once we have shown the following four properties:

(i.) $\mathcal{U}_\varepsilon$ is connected.

(ii.) $\mathcal{V}_{\varepsilon,\Theta}$ is nonempty.

(iii.) For any $\varepsilon, \Theta > 0$, $\mathcal{V}_{\varepsilon,\Theta}$ is a relatively closed subset of $\mathcal{U}_\varepsilon$.

(iv.) There exist $\Theta > 0$ and $\varepsilon > 0$, such that $\mathcal{V}_{\varepsilon,\Theta}$ is a relatively open subset of $\mathcal{U}_\varepsilon$.

Property (i.) is obvious, since $\mathcal{U}_\varepsilon$ is clearly starshaped with center 0 and hence path-connected. Property (ii.) is also evident, since $(0,0) \in \mathcal{V}_{\varepsilon,\Theta}$ follows by choosing the constant map $\mathfrak{q} = \mathfrak{I}$. Consequently, it remains to verify the latter two:

The closedness property (iii.) follows almost verbatim from those in Da Lio-Rivière [22] and Da Lio-Schikorra [28]: Assume that $(\omega_n, \mathfrak{g}_n), (\omega, \mathfrak{g}) \in \mathcal{V}_{\varepsilon,\Theta}$ and moreover, $(\omega_n, \mathfrak{g}_n) \to (\omega, \mathfrak{g})$ and let $\mathfrak{q}_n$ be as in the definition of $\mathcal{V}_{\varepsilon,\Theta}$, i.e. $\mathcal{N}(\mathfrak{q}_n) = (\omega_n, \mathfrak{g}_n)$ and satisfying (3.174), (3.175), (3.176). Observe that $\nabla \mathfrak{q}_n$ is bounded in L^p and $L^{p'}$. Therefore, we can extract weakly converging subsequences with limit $\mathfrak{p}$. Furthermore, we may extract another subsequence of $\mathfrak{q}_n - \mathfrak{I}$ converging locally in L^q for some $q < \frac{4p}{3p-4}$ we may choose, due to the $\dot{W}^{1,p'}$-boundedness of $\mathfrak{q}_n$. The limit $\mathfrak{q} - \mathfrak{I}$ satisfies $\nabla \mathfrak{q} = \mathfrak{p}$ and $\mathfrak{q}$ assumes values in $Spin(5)$ a.e.. This can be seen by extracting another subsequence of $\mathfrak{q}_n - \mathfrak{I}$ converging a.e. pointwise and using the closedness of $Spin(5)$. Due to the weak lower semi-continuity of the norms, we immediately obtain that (3.174), (3.175) and (3.176) hold. Finally, observe that, in the distributional sense, we have the convergence:

$$\Pi_6\left(\sum_{i=0}^{3}(\partial_{x_i}((\mathfrak{q}_n)^{-1}\partial_{x_i}(\mathfrak{q}_n))\right) \to \Pi_6\left(\sum_{i=0}^{3}(\partial_{x_i}((\mathfrak{q})^{-1}\partial_{x_i}(\mathfrak{q}))\right),$$

as well as

$$(\Pi_4 + \mathcal{P})(\mathcal{D}(\mathfrak{q}_n)) \to (\Pi_4 + \mathcal{P})(\mathcal{D}(\mathfrak{q})).$$

This shows $\mathcal{N}(\mathfrak{q}) = (\omega, \mathfrak{g})$ and therefore relative closedness is established. This takes care of (iii.).

Lastly, we verify the openness property (iv.). For this let $\omega_0, \mathfrak{g}_0$ be arbitrary in $\mathcal{V}_{\varepsilon,\Theta}$, for some $\varepsilon, \Theta > 0$ determined later on in an appropriate manner: Let $\mathfrak{q}_0 \in \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{R}^4, \mathfrak{spin}(5))$, $\mathfrak{q}_0 - \mathfrak{I} \in L^{4p/3p-4}(\mathbb{R}^4)$, such that the decomposition (3.171) as well as the estimates (3.174), (3.175) and (3.176) are satisfied for ω_0 and $\mathfrak{g}_0$.

Let us consider perturbations of $\mathfrak{q}_0$ of the form $\mathfrak{q} = \mathfrak{q}_0 e^{\mathfrak{u}}$ where $\mathfrak{u} \in \dot{W}^{1,p} \cap \dot{W}^{1,p'} \cap L^{4p/3p-4}(\mathbb{R}^4, \mathfrak{spin}(5))$. Observe that the exponent $p > 4$ has been chosen in particular to ensure $\mathfrak{u} \in C^0 \cap L^\infty(\mathbb{R}^4)$ and $\mathfrak{q}_0 e^{\mathfrak{u}} - \mathfrak{I} \in L^{\frac{4p}{3p-4}}$. Indeed for a Schwartz function $\mathfrak{u}$, one has

$$\mathfrak{u}(x) = C \int_{\mathbb{R}^4} \nabla_x |x-y|^{-2} \cdot \nabla \mathfrak{u}(y)\, dy \quad \Rightarrow \quad \|\mathfrak{u}\|_\infty \lesssim \|\nabla_x |x-y|^{-2}\|_{L^{(4/3,\infty)}} \|\nabla \mathfrak{u}\|_{L^{(4,1)}} \tag{3.177}$$

The generalized Hölder inequality (see Grafakos [39]) yields the required estimate of the Lorentz norm:

$$\|\nabla \mathfrak{u}\|_{L^{(4,1)}} \leq C \|\nabla \mathfrak{u}\|_{L^p}^{\alpha} \|\nabla \mathfrak{u}\|_{L^{p'}}^{1-\alpha}.$$

where $4^{-1} = \alpha p^{-1} + (1-\alpha) p'^{-1}$. The statement $\mathfrak{u} \in L^\infty$, and thus continuity by approximation, follows due to the density of Schwartz functions in $\dot{W}^{1,p} \cap \dot{W}^{1,p'} \cap L^{\frac{4p}{3p-4}}$. It can be easily seen, that the argument carries over to domains of arbitrary dimension m, if $m < p$, as the density result and the interpolation identity do not critically depend on $m = 4$ in any significant way. This observation ensures that the argument presented could be generalised up to this point to higher-dimensional domains without issues.

Study of the linearized operator The key idea is that we can deduce general global properties of the gauge operator by considering its differential at the element 0. This is in line with the usual local inversion theorem, which again reduces local invertibility to a question about invertibility of the differential which is its local linearisation.

We compute $D\mathcal{N}(\mathfrak{q}_0)$ as follows:

$$D\mathcal{N}(\mathfrak{q}_0) = \frac{d}{dt} \mathcal{N}(\mathfrak{q}_0 e^{t\mathfrak{u}}) \Big|_{t=0} =: \mathcal{L}_{\mathfrak{q}_0}(\mathfrak{u}),$$

where $\mathfrak{u} \in (\dot{W}^{1,p} \cap \dot{W}^{1,p'} \cap L^{4p/3p-4})(\mathbb{R}^4, \mathfrak{spin}(5))$. Let us write this in components:

$$\mathcal{L}_{\mathfrak{q}_0}(\mathfrak{u}) = (\mathcal{L}^6_{\mathfrak{q}_0}(\mathfrak{u}), \mathcal{L}^4_{\mathfrak{q}_0}(\mathfrak{u}))$$

where we have:

$$\begin{aligned}
\mathcal{L}^6_{\mathfrak{q}_0}(\mathfrak{u}) &:= \Pi_6 \left[\Delta \mathfrak{u} + \sum_{i=0}^{3} \partial_{x_i} \left(\mathfrak{q}_0^{-1} (\partial_{x_i} \mathfrak{q}_0) \mathfrak{u} - \mathfrak{u} \mathfrak{q}_0^{-1} \partial_{x_i} \mathfrak{q}_0 \right) \right] \\
\mathcal{L}^4_{\mathfrak{q}_0}(\mathfrak{u}) &:= (\Pi_4 + \mathcal{P})[\partial_{x_0} \mathfrak{u} - \sum_{i=1}^{3} \partial_{x_i} \mathfrak{u} e_i]
\end{aligned}$$

$$+(\Pi_4+\mathcal{P})[\mathfrak{q}_0^{-1}(\partial_{x_0}\mathfrak{q}_0)\mathfrak{u}-\mathfrak{u}\mathfrak{q}_0^{-1}\partial_{x_0}\mathfrak{q}_0-\sum_{j=1}^{3}(\mathfrak{q}_0^{-1}(\partial_{x_j}\mathfrak{q}_0)\mathfrak{u}-\mathfrak{u}\mathfrak{q}_0^{-1}\partial_{x_j}\mathfrak{q}_0)e_j]$$

and for $i=1,2,3$. First, we investigate the invertibility of $\mathcal{L}_{\mathfrak{q}_0}(\mathfrak{u})$ in the special case $\mathfrak{q}_0=\mathfrak{I}$.

Invertibility of $\mathcal{L}_{\mathfrak{I}}(\mathfrak{u})$ If $\mathfrak{q}_0=\mathfrak{I}$, we obviously have $d\mathfrak{q}=0$ and as a result, the operator $\mathcal{L}_{\mathfrak{I}}(\mathfrak{u})=(\mathcal{L}^6_{\mathfrak{I}}(\mathfrak{u}),\mathcal{L}^4_{\mathfrak{I}}(\mathfrak{u}))$ has the following simpler form:

$$\begin{aligned}\mathcal{L}^6_{\mathfrak{I}}(\mathfrak{u}) &= \Pi_6\,[\Delta\mathfrak{u}]\\ \mathcal{L}^4_{\mathfrak{I}}(\mathfrak{u}) &= (\Pi_4+\mathcal{P})[\partial_{x_0}\mathfrak{u}-\sum_{i=1}^{3}\partial_{x_i}\mathfrak{u}e_i]\end{aligned}\tag{3.178}$$

The following will suffice to prove existence of solutions and regularity:

Proposition 3.2.2.2. *The operator $\mathcal{L}_{\mathfrak{I}}(\mathfrak{u})$ is elliptic.*

We mention at this point that this will be the only point where we crucially use the dimension of the domain, as we shall observe the Riemann-Fueter operator on $\mathbb{R}^4$ emerging in our computations.

Proof of Proposition 3.2.2.2. We write $\mathfrak{u}=w+v$ where $w\in E_6$ and $v=v^0e_4+v^1e_1e_4+v^2e_2e_4+v^3e_3e_4\in E_4$. We observe that

$$\begin{aligned}\mathcal{L}^6_{\mathfrak{I}}(\mathfrak{u}) &= \Pi_6\,[\Delta w+\Delta v]=\Delta w\\ \mathcal{L}^4_{\mathfrak{I}}(\mathfrak{u}) &= (\Pi_4+\mathcal{P})\left[\partial_{x_0}v-\sum_{i=1}^{3}\partial_{x_i}ve_i\right]\end{aligned}$$

We explicitly compute $\mathcal{L}^4_{\mathfrak{I}}(\mathfrak{u})$:

$$\begin{aligned}(\Pi_4+\mathcal{P})\left[\partial_{x_0}v-\sum_{i=1}^{3}\partial_{x_i}ve_i\right] &= \Pi_{e_4}\left[\partial_{x_0}v-\sum_{i=1}^{3}\partial_{x_i}ve_i\right]\\ &+ \sum_{i=1}^{3}(\Pi_{e_ie_4}+\Pi_{\mathcal{P}[e_{i+1}e_{i-1}e_4]})\left[\partial_{x_0}v-\sum_{j=1}^{3}\partial_{x_j}ve_j\right]\\ &= \left(\partial_{x_0}v^0-\sum_{j=1}^{3}\partial_{x_i}v^i\right)e_4\\ &+ \sum_{i=1}^{3}\left(\partial_{x_0}v^i+\partial_{x_i}v^0-\partial_{x_{i+1}}v^{i-1}+\partial_{x_{i-1}}v^{i+1}\right)e_ie_4.\end{aligned}\tag{3.179}$$

We can associate to the operator $(\Pi_4+\mathcal{P})$ the following symbol:

$$\sigma(\xi)=\begin{pmatrix}\xi_0 & -\xi_1 & -\xi_2 & -\xi_3\\ \xi_1 & \xi_0 & \xi_3 & -\xi_2\\ \xi_2 & -\xi_3 & \xi_0 & \xi_1\\ \xi_3 & \xi_2 & -\xi_1 & \xi_0\end{pmatrix}\tag{3.180}$$

It can be easily seen that the columns of the symbol form an orthogonal system. Therefore, we know $\det(\sigma(\xi)) = \pm(\sum_{i=1}^{4}\xi_i^2)^2$ due to the multilinearity of the determinant coupled with the determinant of real orthogonal matrices being either 1 or -1. This implies that the symbol is invertible for all $\xi \neq 0$ and as a result, the differential operator is elliptic by definition. Due to the connectedness of $\mathbb{R}^4 \setminus \{0\}$ and the continuity of the determinant, we may even conclude that the sign of the determinant has to be constant and by noticing $\det(\sigma((1,0,0,0))) = 1$, we deduce $\det(\sigma(\xi)) = (\sum_{i=1}^{4}\xi_i^2)^2$ for all ξ. Combining the ellipticity of the Laplacian with the ellipticity of $\sigma(\xi)$, we deduce that $\mathcal{L}_{\mathfrak{I}}(\mathfrak{u})$ is elliptic as well. This concludes the proof of Proposition 3.2.2.2. □

We may now prove the following result:

Lemma 3.2.2.3. *For any $\Theta > 0$, there exists $\varepsilon > 0$ so that the following holds for any $\omega_0, \mathfrak{g}_0$ and $\mathfrak{q}_0$ satisfying* (3.171)*,* (3.174)*,* (3.175)*,* (3.176)*:*

For any $\omega \in (W^{-1,p} \cap W^{-1,p'})(\mathbb{R}^4, E_6)$ and $\mathfrak{g} \in (L^p \cap L^{p'})(\mathbb{R}^4, E_4)$, there exists a unique $\mathfrak{u} \in \dot{W}^{1,p} \cap \dot{W}^{1,p'} \cap L^{4p/3p-4}(\mathbb{R}^4, \mathfrak{spin}(5))$, such that

$$(\omega, \mathfrak{g}) = \mathcal{L}_{\mathfrak{q}_0}(\mathfrak{u})$$

and for some constant $C = C(\omega_0, \mathfrak{g}_0, \Theta) > 0$, it holds

$$\begin{aligned} \|\nabla\mathfrak{u}\|_{L^p(\mathbb{R}^4)} + \|\nabla\mathfrak{u}\|_{L^{p'}(\mathbb{R}^4)} \lesssim\ & \|\omega\|_{W^{-1,p}(\mathbb{R}^4)} + \|\omega\|_{W^{-1,p'}(\mathbb{R}^4)} \\ & + \|\mathfrak{g}\|_{L^p(\mathbb{R}^4)} + \|\mathfrak{g}\|_{L^{p'}(\mathbb{R}^4)}. \end{aligned} \tag{3.181}$$

Proof of Lemma 3.2.2.3.
Claim 1. $\mathcal{L}_{\mathfrak{I}}(\mathfrak{u})$ is invertible as a map between the space of functions $\mathfrak{u} \in \dot{W}^{1,p'} \cap L^{4p/3p-4}(\mathbb{R}^4, \mathfrak{spin}(5))$ and the space $W^{-1,p'}(\mathbb{R}^4, E_6) \times L^{p'}(\mathbb{R}^4, E_4)$
Proof of the Claim 1. We have seen that $\mathcal{L}_{\mathfrak{I}}(\mathfrak{u})$ is elliptic and therefore a Caldéron-Zygmund operator. More precisely, let Γ_4 denote the fundamental solution of Δ on $\mathbb{R}^4$. Using the decomposition $\mathfrak{u} = w + v$ as before, we have:

$$\Delta w = \omega \Longrightarrow w = \Gamma_4 * \omega.$$

Similarily, we write $v = v^0 e_4 + v^1 e_1 e_4 + v^2 e_2 e_4 + v^3 e_3 e_4$ and up to replacing e_4, e_1e_4, e_2e_4 and e_3e_4 by the quaternionic basis $1, i, j$ and k respectively, we see:

$$(\Pi_4 + \mathcal{P})[\partial_R v] = \mathfrak{g} \Longleftrightarrow D_R^{RF} v = \mathfrak{g},$$

where $D_R^{RF} = \partial_{x_0} + \partial_{x_1}\cdot i + \partial_{x_2}\cdot j + \partial_{x_3}\cdot k$ is the quaternionic Riemann-Fueter operator in 4D. Observe that this emergence crucially limits the dimension of the domains to which this very argument could be applied. A simple calculation as outlined in the Appendix enables us to see:

$$\overline{D}_R^{RF} D_R^{RF} = \Delta,$$

where $\overline{D}_R^{RF} = \partial_{x_0} - \partial_{x_1}\cdot i - \partial_{x_2}\cdot j - \partial_{x_3}\cdot k$ is the conjugate operator. Therefore, we have:

$$\Delta v = \overline{D}_R^{RF}\mathfrak{g}.$$

As a result, we deduce:

$$v = \Gamma_4 * \overline{D}_R^{RF}\mathfrak{g} = \overline{D}_R^{RF}\mathfrak{g} * \Gamma_4 = \overline{D}_R^{RF}(\mathfrak{g} * \Gamma_4) = \mathfrak{g} * \overline{D}_R^{RF}\Gamma_4.$$

We highlight that the change of order in the convolution is made to emphasise explicitly the non-commutativity of elements in the Clifford algebra. Using standard Caldéron-Zygmund estimates for the Laplacian, we obtain:

$$\begin{aligned} \|\nabla w\|_{L^p} &\lesssim \|\omega\|_{W^{-1,p'}} \\ \|\nabla v\|_{L^p} &\lesssim \|\mathfrak{g}\|_{L^{p'}}. \end{aligned}$$

Consequently, given $\omega \in W^{-1,p'}(\mathbb{R}^4, E_6)$, $\mathfrak{g} \in L^{p'}(\mathbb{R}^4, E_4)$, there exists a unique $\mathfrak{u} \in \dot{W}^{1,p'} \cap L^{4p/3p-4}(\mathbb{R}^4, \mathfrak{spin}(5))$ such that:

$$\mathcal{L}_{\mathfrak{I}}(\mathfrak{u}) = (\omega, \mathfrak{g}).$$

The elliptic estimates above yield in combination:

$$\|\nabla \mathfrak{u}\|_{L^{p'}} \lesssim \|\omega\|_{W^{-1,p'}} + \|\mathfrak{g}\|_{L^{p'}}.$$

The claim is therefore proved. □

Estimate for $\mathcal{L}_{\mathfrak{q}_0}(\mathfrak{u}) - \mathcal{L}_{\mathfrak{I}}(\mathfrak{u})$
To generalize the invertibility to arbitrary $\mathfrak{q}_0$, let us consider $\mathcal{L}_{\mathfrak{q}_0}$ as a perturbation of $\mathcal{L}_{\mathfrak{I}}$. Invertibility is ensured, if the operators are close enough by the usual perturbation-type argument. Thus, it suffices to estimate using Hölder's inequality, boundedness/compactness of the Spin-groups, Sobolev-embeddings and the L^4-estimate for $\nabla \mathfrak{q}$:

$$\begin{aligned} \|\mathfrak{q}_0^{-1}(\partial_{x_i}\mathfrak{q}_0)\mathfrak{u} - \mathfrak{u}\mathfrak{q}_0^{-1}\partial_{x_i}\mathfrak{q}_0\|_{L^{p'}} &\lesssim \|\mathfrak{q}_0^{-1}\|_{L^\infty}\|\nabla \mathfrak{q}_0\|_{L^4}\|\mathfrak{u}\|_{L^{4p/3p-4}} \\ &\lesssim \|\nabla \mathfrak{q}_0\|_{L^4}\|\nabla \mathfrak{u}\|_{L^{p'}} \\ &\lesssim \Theta\varepsilon \cdot \|\nabla \mathfrak{u}\|_{L^{p'}}. \end{aligned}$$

Using this inequality, we conclude:

$$\begin{aligned} \|\partial_{x_i}\left(\mathfrak{q}_0^{-1}(\partial_{x_i}\mathfrak{q}_0)\mathfrak{u} - \mathfrak{u}\mathfrak{q}_0^{-1}\partial_{x_i}\mathfrak{q}_0\right)\|_{W^{-1,p'}} &\le \|\mathfrak{q}_0^{-1}(\partial_{x_i}\mathfrak{q}_0)\mathfrak{u} - \mathfrak{u}\mathfrak{q}_0^{-1}\partial_{x_i}\mathfrak{q}_0\|_{L^{p'}} \\ &\lesssim \Theta\varepsilon \cdot \|\nabla \mathfrak{u}\|_{L^{p'}}. \end{aligned}$$

Choosing $\varepsilon > 0$ small enough (depending on Θ), we obtain that $\mathcal{L}_{\mathfrak{q}_0}(\mathfrak{u})$ is an invertible map from $\dot{W}^{1,p'}(\mathbb{R}^4, \mathfrak{spin}(5))$ to $W^{-1,p'}(\mathbb{R}^4, E_6) \times L^{p'}(\mathbb{R}^4, E_4)$.

Claim 2. Assuming now $\omega \in (W^{-1,p} \cap W^{-1,p'})(\mathbb{R}^4, E_6)$ as well as $\mathfrak{g} \in (L^p \cap L^{p'})(\mathbb{R}^4, E_4)$, we show that the unique solution $\mathfrak{u}$ of $(\omega, \mathfrak{g}) = \mathcal{L}_{\mathfrak{q}_0}(\mathfrak{u})$ lies in $\dot{W}^{1,p}(\mathbb{R}^4)$.
Proof of Claim 2: Firstly, due to $\nabla \mathfrak{u} \in L^{p'}$, we know that we may choose $\mathfrak{u}$ by Sobolev-embeddings and the density of Schwartz functions in the following way:

$$\mathfrak{u} \in L^{4p/3p-4}(\mathbb{R}^4)$$

We have been using this observation implicitly before. As previously, in order to bootstrap, it suffices to deduce improved integrability of $\mathfrak{q}_0^{-1}\partial_{x_l}\mathfrak{q}_0\mathfrak{u}$, as this implies improved integrability of $\nabla \mathfrak{u}$ by means of elliptic estimates. The same estimates immediately apply to $\mathfrak{u}\mathfrak{q}_0^{-1}\partial_{x_l}\mathfrak{q}_0$, meaning that there is no issue in merely establishing estimates for $\mathfrak{q}_0^{-1}\partial_{x_l}\mathfrak{q}_0\mathfrak{u}$ for brevity's sake. By the considerations in (3.177), it suffices to show that $\nabla \mathfrak{u} \in L^q$ for some $q > 4$, because then, by interpolation, $\nabla \mathfrak{u} \in L^{p'} \cap L^q$ and we could thus conclude that $\mathfrak{u} \in L^\infty$ completely analogous to (3.177) leading to $\mathfrak{q}_0^{-1}\partial_{x_l}\mathfrak{q}_0\mathfrak{u} \in L^p$, which immediately establishes $\nabla \mathfrak{u} \in L^p$. Therefore, Claim 2 would be proven in the process.

We argue by a bootstrap argument: Assume that $\mathfrak{u} \in L^r$ for some $4 > r \geq \frac{4p}{3p-4}$. In this case, Hölder's inequality implies:

$$\|\mathfrak{q}_0^{-1}\partial_{x_l}\mathfrak{q}_0\mathfrak{u}\|_{L^t} \lesssim \|\nabla\mathfrak{q}_0\|_{L^p}\|\mathfrak{u}\|_{L^r},$$

for $\frac{1}{t} = \frac{1}{p} + \frac{1}{r} > \frac{1}{p} + \frac{1}{4} > \frac{1}{4}$. Observe that $\frac{4}{3} \leq t < 4 < p$ by the inequalities satisfied by r. We conclude due to the elliptic estimates as in Claim 1 and the identity $\mathcal{L}_{\mathfrak{q}_0}(\mathfrak{u}) = (\omega, \mathfrak{g})$:

$$\nabla\mathfrak{u} \in L^t.$$

This implies by Sobolev-embeddings that $\mathfrak{u} \in L^{4t/4-t}$. Thus, if we define $\tilde{r} = \frac{4t}{4-t}$, we observe:

$$\frac{1}{\tilde{r}} = \frac{1}{t} - \frac{1}{4} = \frac{1}{r} + \frac{1}{p} - \frac{1}{4},$$

which implies that the reciprocal values are decreasing by a constant amount with each iterating step, due to $p > 4$. Therefore, after finitely many steps (the number of which depends only on p), we have:

$$\frac{1}{\tilde{r}} < \frac{1}{4} \;\Rightarrow\; \tilde{r} > 4$$

This implies, by the previously outlined argument, that $\nabla\mathfrak{u} \in L^p(\mathbb{R}^4, \mathfrak{spin}(5))$, finishing the proof of Claim 2. Observe that by keeping track of the estimates, we may deduce the L^p-part of the inequality (3.287). Therefore, the Lemma is proven. □

Proof of Proposition 3.2.2.1 continued.
For $\varepsilon = \varepsilon(\Theta) > 0$ chosen small enough and for any $(\omega_0, \mathfrak{g}_0) \in \mathcal{V}_{\varepsilon,\Theta}$, the local inversion theorem applied to $\mathcal{N}_{\mathfrak{q}_0}$ implies the existence of some $\delta > 0$ (that might depend on $(\omega_0, \mathfrak{g}_0)$) such that, for every $(\omega, \mathfrak{g}) \in \mathcal{U}_\varepsilon$ with

$$\|\omega - \omega_0\|_{W^{-1,p}(\mathbb{R}^4)} + \|\omega - \omega_0\|_{W^{-1,p'}(\mathbb{R}^4)} < \delta \tag{3.182}$$

$$\|\mathfrak{g} - \mathfrak{g}_0\|_{L^p(\mathbb{R}^4)} + \|\mathfrak{g} - \mathfrak{g}_0\|_{L^{p'}(\mathbb{R}^4)} < \delta, \tag{3.183}$$

we surely find $\mathfrak{q} = \mathfrak{q}_0 e^{\mathfrak{u}} \in \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{R}^4)$, such that $\mathfrak{q} - \mathfrak{I} \in L^{4p/3p-4}(\mathbb{R}^4)$ and (3.171) is satisfied.

It remains to prove the estimates (3.174), (3.175) and (3.176). They will be an immediate consequence of the following lemma, together with sufficiently small chosen $\varepsilon, \delta > 0$:

Lemma 3.2.2.4. *There exist $\Theta > 0$ and $\sigma > 0$, such that, whenever $\mathfrak{q} \in \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{R}^4)$ with $\mathfrak{q} - \mathfrak{I} \in L^{4p/3p-4}(\mathbb{R}^4)$ satisfying* (3.171) *is given, and it holds:*

$$\|\nabla\mathfrak{q}\|_{L^4(\mathbb{R}^4)} \leq \sigma, \tag{3.184}$$

then the estimates in (3.174), (3.175) *and* (3.176) *hold true as well.*

Proof of Lemma 3.2.2.4. Let $(\omega, \mathfrak{g}) \in \mathcal{U}_\varepsilon$ satisfy (3.182) and (3.183) and let $\mathfrak{q} = \mathfrak{q}_0 e^{\mathfrak{u}} \in \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{R}^4)$, such that $\mathfrak{q} - \mathfrak{I} \in L^{4p/3p-4}(\mathbb{R}^4)$ and (3.171) is satisfied. We first consider the following Hodge decomposition of $\mathfrak{q}^{-1}d\mathfrak{q}$:

$$\mathfrak{q}^{-1}d\mathfrak{q} = d\Gamma_{\mathfrak{q}} + d^*Y_{\mathfrak{q}} \tag{3.185}$$

where $Y_{\mathfrak{q}} \in \Omega^2(\mathbb{R}^4)$, $Y_{\mathfrak{q}} = \sum_{0\leq i<j\leq 3} Y_{\mathfrak{q}}^{ij} dx_i \wedge dx_j$ is a differential 2-form and $\Gamma_{\mathfrak{q}}$ a 0-form, i.e. a function. We denote $d^* Y_{\mathfrak{q}} = \sum_{i=0}^{3} y_{\mathfrak{q}}^i dx_i$[25] for brevity. We may choose $\Gamma_{\mathfrak{q}}$ and $Y_{\mathfrak{q}}$ as follows:

$$\Gamma_{\mathfrak{q}} = (-\Delta)^{-1} d^*(\mathfrak{q}^{-1} d\mathfrak{q}) \tag{3.186}$$

$$Y_{\mathfrak{q}} = (-\Delta)^{-1} d(\mathfrak{q}^{-1} d\mathfrak{q}) \tag{3.187}$$

In particular, we then have $dY_{\mathfrak{q}} = 0$ and $d^*\Gamma_{\mathfrak{q}} = 0$, i.e. exactness and coexactness respectively.

Due to (3.186), it follows that:

$$(-\Delta)\Pi_6(\Gamma_{\mathfrak{q}}) = \Pi_6(-\Delta\Gamma_{\mathfrak{q}}) = \Pi_6(d^*(\mathfrak{q}^{-1}d\mathfrak{q})) = -\omega. \tag{3.188}$$

Therefore for every $r \in [p', p]$ we have:

$$\|\Pi_6(\nabla\Gamma_{\mathfrak{q}})\|_{L^r} \lesssim \|\omega\|_{W^{-1,r}} \tag{3.189}$$

From (3.185), it follows that

$$-\Delta Y_{\mathfrak{q}} = d(\mathfrak{q}^{-1}d\mathfrak{q}) = d\mathfrak{q}^{-1} \wedge d\mathfrak{q}. \tag{3.190}$$

Using (3.190), it follows that $\nabla Y_q \in L^r(\mathbb{R}^4)$ and due to the compensation result in Lemma 3.2.7.1:

$$\|\nabla Y_{\mathfrak{q}}\|_{L^r} \lesssim \|d\mathfrak{q}\|_{L^4(\mathbb{R}^4)} \|d\mathfrak{q}\|_{L^r(\mathbb{R}^4)} \leq \sigma \|d\mathfrak{q}\|_{L^r(\mathbb{R}^4)}. \tag{3.191}$$

By inserting (3.185), it follows that (we write $D = \partial_R$ for the moment for brevity's sake):

$$\begin{aligned} \mathfrak{g} = (\Pi_4 + \mathcal{P})\mathcal{D}(\mathfrak{q}) &= (\Pi_4 + \mathcal{P})(D\Gamma_{\mathfrak{q}}) \\ &+ (\Pi_4 + \mathcal{P})(y_{\mathfrak{q}}^0 - \sum_{i=1}^{3} y_{\mathfrak{q}}^i e_i). \end{aligned} \tag{3.192}$$

Therefore:

$$(\Pi_4 + \mathcal{P})(D\Gamma_{\mathfrak{q}}) = \mathfrak{g} - (\Pi_4 + \mathcal{P})(y_{\mathfrak{q}}^0 - \sum_{i=1}^{3} y_{\mathfrak{q}}^i e_i) \tag{3.193}$$

Observe that $d\Gamma_{\mathfrak{q}} \in \mathfrak{spin}(5)$, since $\mathfrak{q}^{-1}d\mathfrak{q} \in \mathfrak{spin}(5)$. Therefore:

$$d\Gamma_{\mathfrak{q}} = (\Pi_4 + \Pi_6)(d\Gamma_{\mathfrak{q}}).$$

Hence:

$$(\Pi_4 + \mathcal{P})(D\Pi_4(d\Gamma_{\mathfrak{q}})) = d\mathfrak{g} - d(\Pi_4 + \mathcal{P})(y_{\mathfrak{q}}^0 - \sum_{i=1}^{3} y_{\mathfrak{q}}^i e_i) \tag{3.194}$$

[25] We recall that $d^* = (-1)^{n(k-1)+1} * d *$, $*$ is the Hodge operator. If $\xi = \sum_{0\leq i<j\leq 3} \xi_{ij} dx_i \wedge dx_j$ then $d^*\xi = -(\alpha_0 dx_0 + \alpha_1 dx_1 + \alpha_2 dx_2 + \alpha_3 dx_3)$ where:

$$\begin{aligned} \alpha_0 &= \partial_{x_1}\xi_{01} + \partial_{x_2}\xi_{02} + \partial_{x_3}\xi_{03} \\ \alpha_1 &= -\partial_{x_0}\xi_{01} + \partial_{x_2}\xi_{12} + \partial_{x_3}\xi_{13} \\ \alpha_2 &= -\partial_{x_0}\xi_{02} - \partial_{x_1}\xi_{12} + \partial_{x_3}\xi_{23} \\ \alpha_3 &= -\partial_{x_2}\xi_{23} - \partial_{x_1}\xi_{13} - \partial_{x_0}\xi_{03} \end{aligned}$$

$$- \quad (\Pi_4 + \mathcal{P})(D\Pi_6(d\Gamma_{\mathfrak{q}})) \tag{3.195}$$

Since the operator $(\Pi_4 + \mathcal{P}) \circ D$ is invertible by the arguments in Claim 1 of the proof of Lemma 3.2.2.3 above, we find:

$$\begin{aligned}
\Pi_4(d\Gamma_{\mathfrak{q}}) &= ((\Pi_4 + \mathcal{P}) \circ D)^{-1} d\mathfrak{g} \\
&+ ((\Pi_4 + \mathcal{P}) \circ D)^{-1} d(\Pi_4 + \mathcal{P}) \left(y^0_{\mathfrak{q}} - \sum_{i=1}^{3} y^i_{\mathfrak{q}} e_i \right) \\
&+ ((\Pi_4 + \mathcal{P}) \circ D)^{-1} \left[(\Pi_4 + \mathcal{P})(D\Pi_6(d\Gamma_{\mathfrak{q}})) \right]. \qquad (3.196)
\end{aligned}$$

By using (3.196), we get:

$$\begin{aligned}
\|\Pi_4(d\Gamma_{\mathfrak{q}})\|_{L^r} &\lesssim \|\mathfrak{g}\|_{L^r} + \|d^* Y_{\mathfrak{q}}\|_{L^r} + \|\omega\|_{W^{-1,r}} \\
&\lesssim \|\mathfrak{g}\|_{L^r} + \sigma \|d\mathfrak{q}\|_{L^r} + \|\omega\|_{W^{-1,r}} \qquad (3.197)
\end{aligned}$$

Combining (3.185), (3.189) and (3.197), we get the following estimate:

$$\begin{aligned}
\|d\mathfrak{q}\|_{L^r} &\lesssim \|d\Gamma_{\mathfrak{q}}\|_{L^r} + \|d^* Y_{\mathfrak{q}}\|_{L^r} \qquad (3.198) \\
&\lesssim \|\Pi_4(d\Gamma_{\mathfrak{q}})\|_{L^r} + \|\Pi_6(d\Gamma_{\mathfrak{q}})\|_{L^r} + \|d^* Y_{\mathfrak{q}}\|_{L^r} \\
&\leq C(\|\mathfrak{g}\|_{L^r} + \sigma \|d\mathfrak{q}\|_{L^r} + 2\|\omega\|_{W^{-1,r}} + \sigma \|d\mathfrak{q}\|_{L^r}). \qquad (3.199)
\end{aligned}$$

Choosing $\Theta := \frac{C}{1-2C\sigma}$, we finally arrive at the desired inequality:

$$\|d\mathfrak{q}\|_{L^r} \leq \Theta(\|\omega\|_{W^{-1,r}} + \|\mathfrak{g}\|_{L^r}).$$

This concludes the proof of lemma 3.2.2.4. □

End of the proof of Proposition 3.2.2.1
Thanks to Lemma 3.2.2.4, the openness property (iv.) is proven. Proposition 3.2.2.1 is thus established. □

3.2.2.2 Improved Integrability

We are now going to finish the proof of Theorem 3.2.2.1. Before we start, however, let us briefly recall the definition of the gauge operator and the conditions: Let $\mathfrak{f} \in L^{4/3}(\mathbb{R}^4)$ be a solution of

$$\partial_{x_0}(\mathfrak{q}\mathfrak{f}) - \sum_{i=1}^{3} \partial_{x_i}(\mathfrak{q} e_i \mathfrak{f}) = \mathfrak{q}\left(\beta e_4 + \mathcal{D}(\mathfrak{q})\right)\mathfrak{f}. \tag{3.200}$$

If $\|\beta\|_{L^{(4,2)}(\mathbb{R}^4)} \leq \varepsilon$ (this is the required regularity assumption for our arguments, the corresponding L^4-estimate follows immediately) for some $\varepsilon > 0$ sufficiently small, then there exists $\mathfrak{q} \in \dot{W}^{1,4}(\mathbb{R}^4)$ such that

$$\mathcal{N}(\mathfrak{q}) = (0, -\beta e_4) \tag{3.201}$$

with

$$\|\nabla \mathfrak{q}\|_{L^4(\mathbb{R}^4)} \leq \Theta \|\beta\|_{L^4(\mathbb{R}^4)} \tag{3.202}$$

This is what we have proven in the last subsection. Here, $\mathcal{N}$ denotes the following gauge operator:

$$\mathcal{N}\colon \dot{W}^{1,4}(\mathbb{R}^4, Spin(5)) \to W^{-1,4}(\mathbb{R}^4, E_6) \times L^4(\mathbb{R}^4, E_4) \tag{3.203}$$

$$\mathfrak{q} \mapsto \left(\Pi_6\left(\sum_{i=0}^{3}(\partial_{x_i}(\mathfrak{q}^{-1}\partial_{x_i}\mathfrak{q})\right), (\Pi_4 + \mathcal{P})(\mathcal{D}(\mathfrak{q}))\right)$$

In order to avoid worrying about signs, we shall from now on work with βe_4 instead of $-\beta e_4$. This can be achieved by replacing β by $-\beta$ and does not affect the argument in any meaningful way.
In particular, it follows from (3.201) that:

$$(\Pi_4 + \mathcal{P})\mathcal{D}(\mathfrak{q}) = \beta e_4. \tag{3.204}$$

Namely, if $\beta = (\beta^0, \beta^1, \beta^2, \beta^3)$:

$$\Pi_{e_4}(\mathcal{D}(\mathfrak{q})) = \beta^0 \tag{3.205}$$

$$(\Pi_{e_ie_4} + \Pi_{\mathcal{P}(e_{i+1}e_{i-1}e_4)}(\mathcal{D}(\mathfrak{q})) = \beta^i e_i e_4 \tag{3.206}$$

An important step in the proof of our regularity result stems from the observation that the solution of this type of problem can be easily computed directly. This can be exploited to obtain further information and stronger integrability properties as seen below:

Lemma 3.2.2.5. *Under the above assumptions, we have*

$$\Pi_6(\mathcal{D}(\mathfrak{q})), \quad \Pi_{e_{i+1}e_{i-1}e_4}(\mathcal{D}(\mathfrak{q})) \in L^{(4,1)}(\mathbb{R}^4). \tag{3.207}$$

We first prove a related result concerning the linearized operator $\mathcal{L}_{\mathfrak{J}}$. For convenience's sake, given $\mathfrak{u} = w + v \in E_6 \oplus E_4$, we set:

$$Dv := \partial_R v = \partial_{x_0} v - \sum_{i=1}^{3} \partial_{x_i} v e_i. \tag{3.208}$$

The result in Lemma 3.2.2.5 has an infinitesimal analogue for the differential which is in fact the key element required to prove it:

Lemma 3.2.2.6. *Let $\mathfrak{u} = w + v \in E_6 \oplus E_4$ be such that*

$$\mathcal{L}_{\mathfrak{J}}(\mathfrak{u}) = (\Delta w, \Pi_{e_4}(Dv), (\Pi_{e_ie_4} + \Pi_{\mathcal{P}(e_{i+1}e_{i-1}e_4)}(Dv)) = (0, \beta e_4) \tag{3.209}$$

Then for all $i = 1, 2, 3$ we have

$$\Pi_{e_{i+1}e_{i-1}e_4}(Dv) = 0, \tag{3.210}$$

and therefore:

$$\Pi_{\mathcal{P}(e_{i+1}e_{i-1}e_4)}(Dv) = 0.$$

The key idea behind the proof is the use of explicit representations of the solution $\mathfrak{u}$.

Proof of Lemma 3.2.2.6.

We write $v = v^0 e_4 + v^1 e_1 e_4 + v^2 e_2 e^4 + v^3 e_3 e_4$ as $v = (v^0, v')$ where $v' = (v^1, v^2, v^3)$ and similarily $x = (x_0, x')$, where $x' = (x_1, x_2, x_3)$. We observe that Dv can be computed as follows:

$$\begin{aligned} Dv &= (\partial_{x_0} v^0 - \operatorname{div}_{x'} v') e_4 + \sum_{i=1}^{3} (\partial_{x_i} v^0 + \partial_{x_0} v^i) e_i e_4 \\ &+ (\partial_{x_3} v^2 - \partial_{x_2} v^3) e_2 e_3 e_4 + (\partial_{x_1} v^3 - \partial_{x_3} v^1) e_3 e_1 e_4 + (\partial_{x_2} v^1 - \partial_{x_1} v^2) e_1 e_2 e_4 \end{aligned} \tag{3.211}$$

Therefore, we may express Dv in the following form:

$$Dv = \begin{pmatrix} \partial_{x_0} v^0 - \operatorname{div}_{x'} v' \\ \nabla_{x'} v^0 + \partial_{x_0} v' \\ -\operatorname{curl}_{x'} v' \end{pmatrix} \tag{3.212}$$

We want to find the solution $v \in \dot{W}^{1,4}(\mathbb{R}^4)$ of the following system of equations:

$$Dv = \begin{pmatrix} \beta^0 \\ \beta^1 \\ \beta^2 \\ \beta^3 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{3.213}$$

1) Assume that $\beta \in \mathcal{S}(\mathbb{R}^4)$. We show the existence of a smooth solution v and look for a-priori estimates.

First of all, we notice that:

$$\begin{aligned} \Delta v^0 &= \partial_{x_0} \left(\partial_{x_0} v^0 \right) + \partial_{x_1} \left(\partial_{x_1} v^0 \right) + \partial_{x_2} \left(\partial_{x_2} v^0 \right) + \partial_{x_3} \left(\partial_{x_3} v^0 \right) \\ &= \partial_{x_0} \left(\partial_{x_0} v^0 \right) + \operatorname{div}_{x'} \nabla_{x'} v^0 \\ &= \partial_{x_0} \left(\beta^0 + \operatorname{div}_{x'} v' \right) + \operatorname{div}_{x'} \left(\beta' - \partial_{x_0} v' \right) \\ &= \operatorname{div} \beta, \end{aligned} \tag{3.214}$$

and thus:

$$v^0(x) := (-\Delta)^{-1} (\operatorname{div} \beta)(x) = -\int_{\mathbb{R}^4} \operatorname{div}(\beta)(y) |x-y|^{-2} dy$$

Our goal is now to arrive at similar expressions for v^j for all $j = 1, 2, 3$. To achieve this, we observe that for any such j:

$$\begin{aligned} \partial_{x_0} \beta^j - \partial_{x_j} \beta^0 &= \partial_{x_0} \left(\partial_{x_j} v^0 + \partial_{x_0} v^j \right) - \partial_{x_j} \left(\partial_{x_0} v^0 - \operatorname{div}_{x'} v' \right) \\ &= \partial^2_{x_0} v^j + \sum_{k \neq j} \partial_{x_j} \partial_{x_k} v^k \\ &= \partial^2_{x_0} v^j + \sum_{k \neq j} \partial_{x_k} \partial_{x_j} v^k \\ &= \partial^2_{x_0} v^j + \sum_{k \neq j} \partial_{x_k} \partial_{x_k} v^j \end{aligned}$$

$$= \Delta v^j, \tag{3.215}$$

where we used that $\partial_{x_j} v^k = \partial_{x_k} v^j$ for all $j \neq k$ should hold by the third set of equations in (3.212) (namely $\mathrm{curl}_{x'} v' = 0$). Thus, we also know:

$$v^j(x) := (-\Delta)^{-1} \left(\partial_{x_j}\beta^0 - \partial_{x_0}\beta^j\right)(x) = -\int_{\mathbb{R}^4} \left(\partial_{x_j}\beta^0(y) - \partial_{x_0}\beta^j(y)\right)|x-y|^{-2} dy$$

We observe that v obtained this way clearly satisfies the desired L^4-estimate by the usual Calderon-Zygmund inequality. Consequently, we merely have to verify that this solution does indeed solve the equation (3.213). Since this is done by direct computations, let us only present the computations in the case of the second set of equations in (3.212):

$$\begin{aligned}
\partial_{x_j} v^0 + \partial_{x_0} v^j &= -\partial_{x_j}(\Gamma * \operatorname{div}\beta) + \partial_{x_0}\left(\Gamma * \left(\partial_{x_j}\beta^0 - \partial_{x_0}\beta^j\right)\right) \\
&= \Gamma * \left(-\partial_{x_j}\partial_{x_0}\beta 0 - \partial_{x_j}^2\beta^j - \sum_{k\neq j}\partial_{x_j}\partial_{x_k}\beta^k + \partial_{x_0}\partial_{x_j}\beta^0 - \partial_{x_j}^2\beta^j\right) \\
&= \Gamma * \left(-\Delta\beta^j - \sum_{k\neq j}\partial_{x_k}\left(\partial_{x_j}\beta^k - \partial_{x_k}\beta^j\right)\right) \\
&= \Gamma * \left(-\Delta\beta^j\right) = \beta^j,
\end{aligned} \tag{3.216}$$

where we denote by Γ the fundamental solution of the Laplacian $-\Delta$ in 4D and we used $\mathbf{curl}_{x'}\beta' = 0$. This computation is valid for any $j = 1, 2, 3$. This shows that the second set of equations in (3.212) holds true and the other two sets of equations may be checked completely analogously and are omitted here.

2) The general case, i.e. the case of $\beta \in L^4(\mathbb{R}^4; \mathrm{span}_{\mathbb{R}}\{e_0, e_1, e_2, e_3\})$ satisfying the vanishing curl condition, can be dealt with by approximation. Notice that any such β can be approximated by Schwartz functions or smooth, compactly supported functions for which the previous computations hold. Then, the uniformity of the estimates on the gradient of v leads to the desired conclusion.

A particular special case is when $\beta = \partial_L\alpha$ for some $\alpha \in \dot{W}^{1,(4,2)}(\mathbb{R}^4)$ real-valued. Keep in mind that:

$$\partial_L\alpha = \partial_{x_0}\alpha - \partial_{x_1}\alpha\cdot e_1 - \partial_{x_2}\alpha\cdot e_2 - \partial_{x_3}\alpha\cdot e_3. \tag{3.217}$$

In fact, in this case, we may find an even more explicit representation of the solution v. Indeed, by the vanishing curl assumption on v', it is natural to look for solutions:

$$v' = \nabla_{x'}\varphi \tag{3.218}$$

Inserting this expression into the second set of equations in (3.212), we find:

$$\nabla_{x'} v^0 + \partial_{x_0}\nabla_{x'}\varphi = -\nabla_{x'}\alpha,$$

where we remember that we currently assume $\beta = \partial_L\alpha$. This immediately yields:

$$\nabla_{x'}\left(v^0 + \partial_{x_0}\varphi + \alpha\right) = 0,$$

which would be satisfied, if for instance:

$$v^0 + \partial_{x_0}\varphi + \alpha = 0. \tag{3.219}$$

It remains to check whether the first equation in (3.212) can hold true. Inserting yields:

$$\partial_{x_0} v^0 - \operatorname{div}_{x'} v' = \partial_{x_0}\alpha,$$

which, by using the identity from (3.219) in the following form:

$$v^0 = -\alpha - \partial_{x_0}\varphi,$$

further reduces to:

$$-\partial_{x_0}^2\varphi - \operatorname{div}_{x'}\nabla_{x'}\varphi = -\Delta\varphi = 2\partial_{x_0}\alpha. \tag{3.220}$$

Therefore:

$$\varphi := 2(-\Delta)^{-1}\partial_{x_0}\alpha,$$

and v^0, v' can now be computed from (3.218) and (3.219). The desired estimates are evident from our computations and using that $\nabla\alpha \in L^{(4,2)}$, i.e. the gradient of α possesses L^4-integrability. Notice that the formula provides the same result as in the previous computation for general β. $\square$

It should be noted that the arguments in the previous section do not make use of the dimension of the domain being 4 in any meaningful way, besides ensuring that a connection to the gauge operator $\mathcal{N}$ exists. Indeed, the very same arguments could be applied in other dimensions, in particular the construction of a curl-free solution of a system of PDEs.

Proof of Lemma 3.2.2.5. We argue in different steps:

Step 1. We consider the Hodge decomposition of

$$\mathfrak{q}^{-1}d\mathfrak{q} = d\Gamma_{\mathfrak{q}} + d^*Y_{\mathfrak{q}} \tag{3.221}$$

where $Y_{\mathfrak{q}} \in \Omega^2(\mathbb{R}^4)$, $Y_{\mathfrak{q}} = \sum_{0\leq i<j\leq 3} Y_{\mathfrak{q}}^{ij}\, dx_i \wedge dx_j$ is a differential 2-form and $\Gamma_{\mathfrak{q}}$ is a 0-form or function. We denote as before $d^*Y_{\mathfrak{q}} = \sum_{i=0}^3 y_{\mathfrak{q}}^i dx_i$. Notice that once again, we can choose $\Gamma_{\mathfrak{q}}$ and $Y_{\mathfrak{q}}$ as follows:

$$\Gamma_{\mathfrak{q}} = (-\Delta)^{-1}d^*(\mathfrak{q}^{-1}d\mathfrak{q}) \tag{3.222}$$
$$Y_{\mathfrak{q}} = (-\Delta)^{-1}d(\mathfrak{q}^{-1}d\mathfrak{q}) \tag{3.223}$$

In particular, we have $dY_{\mathfrak{q}} = 0$ and $d^*\Gamma_{\mathfrak{q}} = 0$, i.e. exactness and coexactness respectively. Moreover $\nabla Y_{\mathfrak{q}}, \nabla\Gamma_{\mathfrak{q}} \in L^4(\mathbb{R}^4)$.
Due to (3.221), it follows that

$$\begin{aligned} -\Delta Y_{\mathfrak{q}} &= d(\mathfrak{q}^{-1}d\mathfrak{q}) = d\mathfrak{q}^{-1}\wedge d\mathfrak{q} \\ &= d\mathfrak{q}^{-1}\mathfrak{q}\wedge\mathfrak{q}^{-1}d\mathfrak{q} = -(\mathfrak{q}^{-1}d\mathfrak{q}\wedge\mathfrak{q}^{-1}d\mathfrak{q}) \\ &= -(d\Gamma_{\mathfrak{q}} + d^*Y_{\mathfrak{q}})\wedge(d\Gamma_{\mathfrak{q}} + d^*Y_{\mathfrak{q}}) \in L^4\cdot L^4 \hookrightarrow L^2(\mathbb{R}^4) \end{aligned} \tag{3.224}$$

From (3.224), it follows that $\nabla^2 Y_{\mathfrak{q}} \in L^2$ and by generalized Sobolev embeddings therefore $\nabla Y_{\mathfrak{q}} \in L^{(4,2)}(\mathbb{R}^4)$.

Since $\Pi_6(d^*(\mathfrak{q}^{-1}d\mathfrak{q}))=0$ by the choice of $\mathfrak{q}$ using (3.201), we deduce from (3.222) that $\Pi_6(\Delta\Gamma_{\mathfrak{q}})=0$ and since $\nabla\Gamma_{\mathfrak{q}}\in L^4(\mathbb{R}^4)$, this leads us to:

$$\Pi_6(\nabla\Gamma_{\mathfrak{q}})=0 \tag{3.225}$$

Step 2. Next, we have, by using D as in Lemma 3.2.2.6:

$$\begin{aligned}\beta e_4 &= (\Pi_4+\mathcal{P})\mathcal{D}(\mathfrak{q})=(\Pi_4+\mathcal{P})(D\Gamma_{\mathfrak{q}})\\ &+ (\Pi_4+\mathcal{P})(y^0_{\mathfrak{q}}-\sum_{i=1}^3 y^i_{\mathfrak{q}}e_i).\end{aligned} \tag{3.226}$$

Since $\mathfrak{q}^{-1}d\mathfrak{q}$ is *purely imaginary*, namely it is a linear combination of elements in $\mathfrak{spin}(5)$, and $\Pi_6(d\Gamma_{\mathfrak{q}})=0$ due to (3.226), we find:

$$(\Pi_4+\mathcal{P})(Dd\Gamma_{\mathfrak{q}}) \;=\; d\beta e_4-d(\Pi_4+\mathcal{P})(y^0_{\mathfrak{q}}-\sum_{i=1}^3 y^i_{\mathfrak{q}}e_i) \tag{3.227}$$

From (3.227) and the invertibility of $(\Pi_4+\mathcal{P})\circ D$, it follows that

$$\begin{aligned}d\Gamma_{\mathfrak{q}} &= ((\Pi_4+\mathcal{P})\circ D)^{-1}(d\beta e_4) \\ &- ((\Pi_4+\mathcal{P})\circ D)^{-1}d(\Pi_4+\mathcal{P})(y^0_{\mathfrak{q}}-\sum_{i=1}^3 y^i_{\mathfrak{q}}e_i).\end{aligned} \tag{3.228}$$

Now we set $\tilde{Y}_{\mathfrak{q}}:=((\Pi_4+\mathcal{P})\circ D)^{-1}(\Pi_4+\mathcal{P})(y^0_{\mathfrak{q}}-\sum_{i=1}^3 y^i_{\mathfrak{q}}e_i)$ and let v be such that $(\Pi_4+\mathcal{P})Dv=\beta e_4$. Existence is justified by ellipticity and using the connection to the Riemann-Fueter operator introduced in the previous subsection. Observe that by elliptic estimates, we have $\nabla v\in L^{(4,2)}$ since $\beta\in L^{4,2}$. This is the key-point where we need that $\beta\in L^{4,2}$. Therefore $\nabla\Gamma_{\mathfrak{q}}\in L^{(4,2)}$ as well with

$$\|\nabla\Gamma_{\mathfrak{q}}\|_{L^{(4,2)}}\lesssim\|\nabla\mathfrak{q}\|^2_{L^4}+\|\beta\|_{L^{(4,2)}}. \tag{3.229}$$

We estimate:

$$\begin{aligned}(d\Gamma_{\mathfrak{q}})\wedge(d\Gamma_{\mathfrak{q}}) &= dv\wedge dv+dv\wedge d\tilde{Y}_{\mathfrak{q}}\\ &+ d\tilde{Y}_{\mathfrak{q}}\wedge dv+d\tilde{Y}_{\mathfrak{q}}\wedge d\tilde{Y}_{\mathfrak{q}}\end{aligned} \tag{3.230}$$

Now observe that all terms are products of functions in $L^{(4,2)}$. Therefore, the product lies in $L^{2,1}$ by the Lorentz-Hölder inequality. Similarly, we can easily see that:

$$-(d\Gamma_{\mathfrak{q}}+d^*Y_{\mathfrak{q}})\wedge(d\Gamma_{\mathfrak{q}}+d^*Y_{\mathfrak{q}})\in L^{2,1}(\mathbb{R}^4),$$

with

$$\|d\Gamma_{\mathfrak{q}}+d^*Y_{\mathfrak{q}})\wedge(d\Gamma_{\mathfrak{q}}+d^*Y_{\mathfrak{q}}\|_{L^{2,1}(\mathbb{R}^4)}\lesssim(\|\nabla\mathfrak{q}\|^2_{L^4}+\|\beta\|_{L^{(4,2)}})^2. \tag{3.231}$$

From (3.224), (3.229) and (3.231) it follows that $\nabla Y_{\mathfrak{q}}\in L^{(4,1)}$ with

$$\|\nabla Y_{\mathfrak{q}}\|_{L^{(4,1)}}\lesssim(\|\nabla\mathfrak{q}\|^2_{L^4}+\|\beta\|_{L^{(4,2)}})^2,$$

Step 3. We may write:

$$\mathcal{D}(\mathfrak{q}) \;=\; D\Gamma_{\mathfrak{q}}+\psi_{\mathfrak{q}} \tag{3.232}$$

where $\psi_{\mathfrak{q}} \in L^{(4,1)}(\mathbb{R}^4)$ and

$$\|\psi_{\mathfrak{q}}\|_{L^{(4,1)}} \lesssim (\|\nabla \mathfrak{q}\|^2_{L^4} + \|\beta\|_{L^{(4,2)}})^2.$$

This is simply due to (3.221) and the explicit formula for $\mathcal{D}(\mathfrak{q})$. It follows by direct evaluation of the term that

$$(\Pi_4 + \mathcal{P})\mathcal{D}(\mathfrak{q}) = (\Pi_4 + \mathcal{P})(D\Gamma_{\mathfrak{q}}) + (\Pi_4 + \mathcal{P})\psi_{\mathfrak{q}} \tag{3.233}$$

Next, we notice that $(\Pi_4 + \mathcal{P})\mathcal{D}(\mathfrak{q}) = \beta e_4$ if and only if:

$$\begin{aligned}(\Pi_4 + \mathcal{P})(D\Gamma_{\mathfrak{q}}) &= \beta e_4 - (\Pi_4 + \mathcal{P})\psi_{\mathfrak{q}} \\ &= (\Pi_4 + \mathcal{P})(\beta e_4) - (\Pi_4 + \mathcal{P})\psi_{\mathfrak{q}}.\end{aligned} \tag{3.234}$$

We have seen that the linear operator $(\Pi_4 + \mathcal{P}) \circ D$ (which in fact corresponds to the differential $\mathcal{L}_{\mathfrak{J}}$ computed in the previous subsection) is an elliptic operator and if $w = ((\Pi_4 + \mathcal{P}) \circ D)^{-1}(\beta e_4)$ and $\tilde{\psi}_{\mathfrak{q}} = -((\Pi_4 + \mathcal{P}) \circ D)^{-1}(\Pi_4 + \mathcal{P})\psi_{\mathfrak{q}}$, then by Lemma 3.2.2.6:

$$\Pi_{e_{i+1}e_{i-1}e_4}(Dw) = 0.$$

From (3.233) and $w = \Gamma_{\mathfrak{q}} - \tilde{\psi}_{\mathfrak{q}}$, it follows that

$$\Pi_{e_{i+1}e_{i-1}e_4}(D\Gamma_{\mathfrak{q}}) = \Pi_{e_{i+1}e_{i-1}e_4}(D\tilde{\psi}_{\mathfrak{q}}) \in L^{(4,1)}(\mathbb{R}^4)$$

with by elliptic estimates:

$$\|\nabla \tilde{\psi}_{\mathfrak{q}}\|_{L^{(4,1)}(\mathbb{R}^4)} \lesssim \|\nabla Y_{\mathfrak{q}}\|_{L^{(4,1)}(\mathbb{R}^4)}. \tag{3.235}$$

It follows now:

$$\Pi_{e_{i+1}e_{i-1}e_4}\left(\mathcal{D}(\mathfrak{q})\right) \in L^{(4,1)}(\mathbb{R}^4) \quad \text{for all } i = 1, 2, 3. \tag{3.236}$$

This shows the desired improved regularity result. $\square$

3.2.2.3 Conclusion of the Bootstrap Test

Let $\mathfrak{f} \in L^{4/3}(\mathbb{R}^4)$ be a solution of (3.200). By choosing $\mathfrak{q}$ as with our gauge operator, we find:

$$\partial_{x_0}[\mathfrak{q}\mathfrak{f}] - \sum_{i=1}^{3}[\partial_{x_i}(\mathfrak{q}e_i\mathfrak{f})] \quad = \quad \mathfrak{q}V(x)\mathfrak{f}. \tag{3.237}$$

where $V(x) \in L^{(4,1)}$ by our investigation in the previous subsection ($V(x) = \psi_q + \Pi_{e_{i+1}e_{i-1}e_4}(D\Gamma_{\mathfrak{q}})$). Indeed, observe that this is a consequence of the choice of gauge and the improved integrability we have established. Furthermore, by the estimate proven before:

$$\|V(x)\|_{L^{(4,1)}} \lesssim (\|\nabla \mathfrak{q}\|^2_{L^4} + \|\beta\|_{L^{(4,2)}})^2.$$

Since From Lemma 3.2.2.1 we can get rid of the power 2 by choosing $\varepsilon > 0$ possibly slightly smaller. Indeed, we can show using the estimate (3.167) for $\mathfrak{q}$:

$$\|V\|_{L^{(4,1)}} \lesssim \|\beta\|_{L^{(4,2)}}$$

We set

$$F = \begin{pmatrix} \mathfrak{q}\mathfrak{f} \\ -\mathfrak{q}e_1\mathfrak{f} \\ -\mathfrak{q}e_2\mathfrak{f} \\ -\mathfrak{q}e_3\mathfrak{f} \end{pmatrix}$$

Our goal is to prove Morrey estimates just like in Da Lio-Rivière [23]. In order to achieve this, we will use a non-linear Hodge decomposition. The reason behind this is, that Wente's inequality is no longer at our disposal and therefore, we need a suitable replacement, see Lemma 3.2.7.1.
Claim 1: There are $A, B \in \dot{W}^{1,(4/3,\infty)}(\mathbb{R}^4)$, where B is differential 2-form, such that:

$$F = dA + \mathfrak{q}d^*B \tag{3.238}$$

Proof of the Claim 1. We argue by induction:

Step 1. We find A_0, B_0 such that

$$\begin{aligned} -\Delta A_0 &= -\operatorname{div}(F) && (3.239)\\ -\Delta B_0 &= d(\mathfrak{q}^{-1}F) && (3.240) \end{aligned}$$

Then for $k \geq 1$ we solve:

$$\begin{aligned} -\Delta A_k &= -d^*(qd^*B_{k-1}) = *(dq \wedge d * B_{k-1}) && (3.241)\\ -\Delta B_k &= -d(\mathfrak{q}^{-1}) \wedge dA_{k-1} && (3.242) \end{aligned}$$

We set $A = \sum_{i=0}^{\infty} A_k$ and $B = \sum_{i=0}^{\infty} B_k$. We then have:

$$\begin{aligned} -\Delta A &= -d^*(\mathfrak{q}d^*B) - \operatorname{div}(F) && (3.243)\\ -\Delta B &= -d(\mathfrak{q}^{-1}) \wedge dA_{k-1} + d(\mathfrak{q}^{-1}F). && (3.244) \end{aligned}$$

From (3.243) and (3.244), we deduce the following estimates:

$$\begin{aligned} & d^*(F - dA - qd^*B) = 0 && (3.245)\\ & d(\mathfrak{q}^{-1}F - d^*B - \mathfrak{q}^{-1}dA) = 0 && (3.246) \end{aligned}$$

From (3.246), it follows there exists a function $\gamma \in \dot{W}^{1,4/3}(\mathbb{R}^4)$ such that

$$\mathfrak{q}^{-1}F - d^*B - \mathfrak{q}^{-1}dA = d\gamma. \tag{3.247}$$

By combining (3.245) and (3.247) we get

$$\begin{aligned} & d^*(\mathfrak{q}d\gamma) = 0 && (3.248)\\ & d(\mathfrak{q}d\gamma) = d\mathfrak{q} \wedge d\gamma && (3.249)\\ & \|\mathfrak{q}d\gamma\|_{L^{(4/3,\infty)}} \lesssim \|d\mathfrak{q}\|_{L^4}\|d\gamma\|_{L^{(4/3,\infty)}} \leq \varepsilon_0\|d\gamma\|_{L^{(4/3,\infty)}} && (3.250) \end{aligned}$$

Notice that in the last line, we used the compensation result in Lemma 3.2.7.1.

It follows that, if $\varepsilon > 0$ is chosen small enough, $d\gamma = 0$ and therefore

$$F = dA + \mathfrak{q}d^*B.$$

We conclude the **proof the claim 1**. $\square$

We continue with the **proof of Theorem 3.2.2.1**: From (3.238), it follows that

$$-\Delta A = \mathfrak{q}V(x)\mathfrak{f} + d^*(\mathfrak{q}d^*B) = \mathfrak{q}V(x)\mathfrak{f} + *(d\mathfrak{q}\wedge d*B). \tag{3.251}$$

Then, by using the fundamental solution, we see:

$$\begin{aligned}\|\nabla A\|_{L^{(4/3,\infty)}} &\lesssim \|-\Delta A\|_{L^1} \lesssim \|V\|_{L^{(4,1)}}\|\mathfrak{q}\ \mathfrak{f}\|_{L^{(4/3,\infty)}} + \|\nabla\mathfrak{q}\|_{L^4}\|d^*B\|_{L^{(4/3,\infty)}}\\ &\lesssim \|\beta\|_{L^{(4,2)}}\ \|\mathfrak{f}\|_{L^{(4/3,\infty)}} + \|\nabla\mathfrak{q}\|_{L^4}\|d^*B\|_{L^{(4/3,\infty)}}\\ &\lesssim \varepsilon\|\mathfrak{q}\ \mathfrak{f}\|_{L^{(4/3,\infty)}} + \|\beta\|_{L^{(4,2)}}\|d^*B\|_{L^{(4/3,\infty)}}\\ &\lesssim \varepsilon(\|\mathfrak{q}\ \mathfrak{f}\|_{L^{(4/3,\infty)}} + \|d^*B\|_{L^{(4/3,\infty)}})\end{aligned} \tag{3.252}$$

Computing ΔB using exactness, we find:

$$-\Delta B = d(\mathfrak{q}^{-1}F) + d(\mathfrak{q}^{-1}dA) = d(\mathfrak{q}^{-1}F) + d\mathfrak{q}^{-1}\wedge dA \tag{3.253}$$

From (3.253), it follows as above that

$$\|\nabla B\|_{L^{(4/3,\infty)}} \lesssim \|d\mathfrak{q}^{-1}\|_{L^4}\|\nabla A\|_{L^{(4/3,\infty)}} + \|\mathfrak{q}\mathfrak{f}\|_{L^{(4/3,\infty)}} \tag{3.254}$$

By plugging (3.254) into (3.252), we get for $\varepsilon > 0$ sufficiently small:

$$\|\nabla A\|_{L^{(4/3,\infty)}} \lesssim \varepsilon\|\mathfrak{q}\mathfrak{f}\|_{L^{(4/3,\infty)}} \tag{3.255}$$

We set $d^*B = \sum_{i=0}^3 b_i dx_i$. By definition, it holds $d^*d^*B = \sum_{i=0}^3 \partial_{x_i} b_i = 0$. Moreover, by comparison of the entries in F, we observe:

$$\begin{aligned}\mathfrak{q}b_0 &= \mathfrak{q}\mathfrak{f} - \partial_{x_0}A\\ \mathfrak{q}b_i &= -\mathfrak{q}\ e_i\mathfrak{f} - \partial_{x_i}A\end{aligned}$$

These can be slightly rearranged to express b_j in terms of $\mathfrak{f}$:

$$\begin{aligned}b_0 &= \mathfrak{f} - \mathfrak{q}^{-1}\partial_{x_0}A\\ b_i &= -\ e_i\mathfrak{f} - \mathfrak{q}^{-1}\partial_{x_i}A\end{aligned}$$

Hence, if we solve the equations above for $\mathfrak{f}$:

$$\mathfrak{f} = b_0 + \mathfrak{q}^{-1}\partial_{x_0}A = e_i(b_i + \mathfrak{q}^{-1}\partial_{x_i}A) \tag{3.256}$$

Then it is now clear:

$$\partial_{x_i}b_i = -e_i\partial_{x_i}b_0 - e_i\partial_{x_i}\left(\mathfrak{q}^{-1}\partial_{x_0}A\right) - \partial_{x_i}\left(\mathfrak{q}^{-1}\partial_{x_i}A\right).$$

Using the previously established fact that $\sum_{i=0}^3 \partial_{x_i}b_i = 0$, we note:

$$\partial_{x_0}b_0 - \sum_{i=1}^3 e_i\partial_{x_i}b_0 = \sum_{i=1}^3 \partial_{x_i}\left(e_i\mathfrak{q}^{-1}\partial_{x_0}A + \mathfrak{q}^{-1}\partial_{x_i}A\right) \in W^{-1,(4/3,\infty)} \tag{3.257}$$

As a result, using ellipticity and the corresponding estimates:

$$\|b_0\|_{L^{(4/3,\infty)}} \lesssim \|\nabla A\|_{L^{(4/3,\infty)}}$$

From (3.256), this estimate easily generalises to all b_j. Namely, it follows that

$$\|b_i\|_{L^{(4/3,\infty)}} \lesssim \|\nabla A\|_{L^{(4/3,\infty)}}, \quad \forall i = 1,2,3.$$

Consequently, recalling the definition of the b_j, we arrive at the desired estimate for d^*B:

$$\|d^*B\|_{L^{(4/3,\infty)}} \lesssim \|\nabla A\|_{L^{(4/3,\infty)}} \tag{3.258}$$

From (3.238), it finally follows that:

$$\|\mathfrak{q}\mathfrak{f}\|_{L^{(4/3,\infty)}} \lesssim \|\nabla A\|_{L^{(4/3,\infty)}} \lesssim \varepsilon_0 \|\mathfrak{q}\mathfrak{f}\|_{L^{(4/3,\infty)}}.$$

If $\varepsilon > 0$ is chosen small enough, then $\mathfrak{q}\mathfrak{f} = 0$ is an immediate corollary, thus establishing the bootstrap lemma.

3.2.3 The Proof of the Main Theorem 3.2.1.2 in 4-D

We observe that Theorem 3.2.1.2 follows similar to Theorem 3.2.2.1 by using localization arguments analogous to Proposition III.4 in Da Lio-Rivière [23]. We provide here a sketch of proof in the 4-D case, most of the ideas rely on just applying a local change of gauge, modifying the result presented in the previous subsections by adding zero boundary values whenever necessary:

First, we will briefly explain how to obtain an appropriate version of the non-linear Hodge decomposition on balls $B_r(x)$. For simplicity's sake, let us assume $x = 0$, the general case is obtained by translation. Let for this G be an arbitrary 1-form in $W^{1,\frac{4}{3}}(B_r(0))$ as obtained in the proof. Then, by classical Hodge decomposition, there exist a function A on $B_r(0)$ vanishing along the boundary and a 2-form $\tilde{A}$, such that:

$$dA + d^*\tilde{A} = G \tag{3.259}$$

Next, we consider the Hodge decomposition in the same manner of $\mathfrak{q}^{-1}d^*A$, again obtaining zero boundary conditions for the function $\tilde{B}$:

$$d\tilde{B} + d^*B = \mathfrak{q}^{-1}d^*\tilde{A} \tag{3.260}$$

Thus, we have:

$$G = dA + d^*A = dA + \mathfrak{q}d^*B + \mathfrak{q}d\tilde{B} \Rightarrow G - dA - \mathfrak{q}d^*B = \mathfrak{q}d\tilde{B} \tag{3.261}$$

We observe that on $B_r(0)$:

$$\Delta\tilde{B} = d^*d\tilde{B} = d^*\left(\mathfrak{q}^{-1}d^*\tilde{A}\right) = - * \left(d\mathfrak{q}^{-1} \wedge d(*\tilde{A})\right) \tag{3.262}$$

Due to the zero boundary condition, we can therefore deduce by similar arguments as in our compensation result in Lemma 3.2.7.1:

$$\|\nabla B\|_{L^{\frac{4}{3}}(B_r(0))} \lesssim \|d\mathfrak{q}\|_{L^4(B_r(0))} \|d^*\tilde{A}\|_{L^{\frac{4}{3}}(B_r(0))} \lesssim \varepsilon \|G\|_{L^{\frac{4}{3}}(B_r(0))} \tag{3.263}$$

So, if $\varepsilon > 0$ is sufficiently small, we can argue by iteration that there exists a solution to the non-linear Hodge decomposition as in the case of codomains of dimension 2, such that A has boundary value 0.

Now, to deduce local regularity, we merely have to establish slightly improved regularity and hence Morrey estimates as in Da Lio-Rivière [23], the full regularity as in Theorem 3.2.1.2 follows by Morrey-bootstrapping going over to possibly smaller balls to obtain uniform powers in the Morrey estimates. Therefore, let us just point out the differences to Da Lio-Rivière [23] and our considerations in connection with the bootstrap lemma: Namely, we can estimate A as in the bootstrap lemma, if we find A, B for a given $B_r(x)$. More precisely, due to the boundary conditions, we will find:

$$\begin{aligned}\|\nabla A\|_{L^{(4/3,\infty)}(B_r(x))} &\lesssim \varepsilon\|\mathfrak{q}\mathfrak{f}\|_{L^{(4/3,\infty)}(B_r(x))} + \varepsilon\|d^*B\|_{L^{(4/3,\infty)}(B_r(x))}\\ &\lesssim \varepsilon\|\mathfrak{q}\mathfrak{f}\|_{L^{(4/3,\infty)}(B_r(x))} + \varepsilon\|\nabla A\|_{L^{(4/3,\infty)}(B_r(x))},\end{aligned} \tag{3.264}$$

by using the same arguments as before and using $F = dA + \mathfrak{q}d^*B$. So if ε is sufficiently small, we arrive at:

$$\|\nabla A\|_{L^{(4/3,\infty)}(B_r(x))} \lesssim \varepsilon\|\mathfrak{q}\mathfrak{f}\|_{L^{(4/3,\infty)}(B_r(x))}, \tag{3.265}$$

Then, it remains to obtain appropriate estimates for d^*B. For this, write $d^*B = \sum_j b_j dx_j$ and we can deduce completely analogous to (3.257) in the proof of the bootstrap lemma:

$$\partial_L b_0 = \sum_{j\geq 2} \partial_{x_j} R_j,$$

where R_j is an expression depending on $\mathfrak{q}$ and ∇A as already found in the proof of Theorem 3.2.2.1. So we can now split b_0 into a Clifford analytic and thus harmonic part, which can be estimated by means of Campanato-estimates as in Da Lio-Rivière [23] and the convolution of the RHS in the equation above with the fundamental solution of ∂_L on $\mathbb{R}^4$. This second summand can be estimated by usual estimates for the fundamental solution of the Laplacian. Therefore, we arrive at the desired estimates for d^*B by completely the same means as in Da Lio-Rivière [23] once we use the link between b_j and b_0 established in the bootstrap lemma.

3.2.4 The 3-D Case

Before we briefly discuss the general case, let us provide another example on how to construct an appropriate gauge operator. More precisely, we shall consider the case of 3D-domains. This will illustrate that the result we have obtained will not generalise in an "easy" manner to arbitrary dimensions $m \geq 3$, but one has to take some care when investigating the gauge operators involved:

Let us consider the following equation:

$$\partial_L \mathfrak{f} = \beta e_3 \cdot \mathfrak{f}, \tag{3.266}$$

where $\mathfrak{f} : \mathbb{R}^3 \to C\ell_3$ is in $L^{3/2}$. Let us assume that

$$\beta = \beta^0 + \beta^1 e_1 + \beta^2 e_2 \in L^{(3,2)}(\mathbb{R}^3; \mathrm{span}_{\mathbb{R}}\{e_0, e_1, e_2\}),$$

as well as:

$$\mathbf{curl}_{x_1,x_2}\beta = \partial_2\beta^1 - \partial_1\beta^2 = 0. \tag{3.267}$$

We will sketch the proof of the following Theorem which is along the same lines as the proof of Theorem 3.2.1.2:

Theorem 3.2.4.1. *Let* $\beta = (\beta_0, \beta_1, \beta_2) \in W^{1,2}(\mathbb{R}^3, \mathrm{span}_{\mathbb{R}}\{e_0, e_1, e_2\})$ *with*

$$\partial_{x_2}\beta_1 - \partial_{x_1}\beta_2 = 0 \ . \tag{3.268}$$

Let $\mathfrak{f} \in L^{3/2}(\mathbb{R}^3, C\ell_2^2)$ *be a solution of*

$$\partial_L \mathfrak{f} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \hat{\mathfrak{f}} \tag{3.269}$$

Then $\mathfrak{f} \in L^q_{loc}(\mathbb{R}^3)$ *for all* $q < \infty$. □

It is clear that we may reformulate (3.269) into an equation of the following form:

$$\partial_L \mathfrak{g} = \beta e_3 \cdot \mathfrak{g},$$

for $\mathfrak{g} = \mathfrak{f}^1 + \mathfrak{f}^2 e_3$. Moreover, there is also the following bootstrap test:

Theorem 3.2.4.2. *There exists* $\varepsilon_0 > 0$ *such that for every* $\beta \in L^{(3,2)}(\mathbb{R}^3, \mathrm{span}_{\mathbb{R}}\{e_0, e_1, e_2\})$ *satisfying* $\|\beta\|_{L^{(3,2)}(\mathbb{R}^3)} \leq \varepsilon_0$ *as well as:*

$$\partial_2\beta^1 - \partial_1\beta^2 = 0,$$

and every $\mathfrak{f} \in L^{3/2}(\mathbb{R}^3, C\ell_3)$ *solving:*

$$\partial_L \mathfrak{f} = \beta e_3 \cdot \mathfrak{f} \ , \tag{3.270}$$

we have $\mathfrak{f} \equiv 0$. □

In our current discussion, we focus on Theorem 3.2.4.2, see the discussion in the previous section regarding the proof of Theorem 3.2.2.1 for a sketch on how to apply Morrey-estimates and the 4D case treated before.For later convenience, let us introduce the following spaces:

$$V_3 := \mathrm{span}_{\mathbb{R}}\{e_3, e_1e_3, e_2e_3\} \tag{3.271}$$
$$V_2 := \mathrm{span}_{\mathbb{R}}\{e_1, e_2\} \tag{3.272}$$
$$V_1 := \mathbb{R} \cdot e_1e_2, \tag{3.273}$$

and denote by Π_3, Π_2 and Π_1 the projections of $\mathcal{U}_3$ onto the respective subspaces.

As in Da Lio-Rivière [23] and previously seen in the case of 4-dimensional domains, let us multiply both sides of (3.266) by a function $\mathfrak{q} : \mathbb{R}^3 \to Spin(4)$ to reveal a slight gain in integrability after a change of gauge. We obtain by using Leibniz' rule:

$$\mathfrak{q}\partial_L \mathfrak{f} = \partial_{x_0}(\mathfrak{q}f) - (\partial_{x_0}\mathfrak{q})\mathfrak{f} - \sum_{i=1}^{2} \partial_{x_i}(\mathfrak{q}e_i\mathfrak{f}) + \sum_{i=1}^{2} \partial_{x_i}\mathfrak{q}e_i\mathfrak{f} \tag{3.274}$$

We denote by:

$$\mathcal{D}(\mathfrak{q}) := \mathfrak{q}^{-1}\partial_{x_0}\mathfrak{q} - \sum_{i=1}^{2} \mathfrak{q}^{-1}\partial_{x_i}\mathfrak{q}e_i = \mathfrak{q}^{-1}\partial_R\mathfrak{q}.$$

Observe that

$$\beta e_3 = \beta^0 \cdot e_3 - \sum_{i=1}^{2} \beta^i \cdot e_ie_3 \in V_3 \tag{3.275}$$

By using (3.274) and rearranging, we get:

$$\partial_{x_0}(\mathfrak{q}f) - \sum_{i=1}^{2} \partial_{x_i}(\mathfrak{q}e_i\mathfrak{f}) \quad = \quad \mathfrak{q}(\beta\, e_3 + \mathcal{D}(\mathfrak{q}))\mathfrak{f}. \tag{3.276}$$

We notice that in (3.276), the absorption of βe_3 by $\mathcal{D}(\mathfrak{q})$ leads to a system of 8 equations in merely 6 unknowns, which is overdetermined much like in the 4-dimensional case. Therefore, there is generally no hope of completely absorbing the "bad term", however, inspired by our proof in 4D, we hope to absorb βe_3 up to a term of higher integrability as before.

The main aim is to find $\mathfrak{q} \in \dot{W}^{1,3}(\mathbb{R}^3, Spin(4))$ such that $\mathcal{D}(\mathfrak{q}) = -\beta e_3 + V(x)$ where V is a more regular potential than βe_3, namely $V \in L^{(3,1)}(\mathbb{R}^3)$. To do this, let us introduce the following non-linear operator reminiscent of (3.168):

$$\mathcal{N}\colon \dot{W}^{1,3}(\mathbb{R}^3, Spin(4)) \to W^{-1,3}(\mathbb{R}^3, V_2) \times L^3(\mathbb{R}^3, \mathcal{U}_2) \tag{3.277}$$

$$\mathfrak{q} \mapsto \left(\Pi_2\left(\sum_{i=0}^{3} (\partial_{x_i}(\mathfrak{q}^{-1}\partial_{x_i}\mathfrak{q}) \right), -\Pi_3(\mathcal{D}(\mathfrak{q}))e_3 + \Pi_1(\mathfrak{q}^{-1}\partial_{x_0}\mathfrak{q}) - \sum_{j=1}^{2} \Pi_1(\mathfrak{q}^{-1}\partial_{x_j}\mathfrak{q})e_j \right)$$

We notice that the first component is analogous to (3.168), while the second component of $\mathcal{N}$ looks more complicated than before. As we shall see later, this definition neatly connects the differential of $\mathcal{N}$ to the Riemann-Fueter operator once again. Indeed, analogous to our previous discussion for $\mathbb{R}^4$, we have the following as a main result:

Lemma 3.2.4.1. *There exists $\varepsilon_0 > 0$ and $C > 0$ such that for any choice $\omega \in W^{-1,3}(\mathbb{R}^3, V_2)$ and $\mathfrak{g} \in L^3(\mathbb{R}^3, C\ell_2)$ satisfying*

$$\|\omega\|_{W^{-1,3}} \leq \varepsilon_0, \|\mathfrak{g}\|_{L^3} \leq \varepsilon_0, \tag{3.278}$$

there is $\mathfrak{q} \in \dot{W}^{1,3}(\mathbb{R}^3, Spin(4))$ such that

$$\mathcal{N}(\mathfrak{q}) = (\omega, \mathfrak{g}) \tag{3.279}$$

as well as

$$\|\nabla \mathfrak{q}\|_{L^3} \leq C(\|\omega\|_{W^{-1,3}} + \|\mathfrak{g}\|_{L^3}). \tag{3.280}$$

The proof essentially proceeds as in Da Lio-Rivière [23] and the case of domains of dimension 4, so let us introduce the analogous simplifications: Again similar to Da Lio-Rivière [22], Da Lio-Schikorra [27], using an approximation argument similar to the our closedness argument later on, it suffices to prove Lemma 3.2.4.1 for ω and $\mathfrak{g}$ slightly more integrable, namely under the assumption $\omega \in (W^{-1,p} \cap W^{-1,p'})(\mathbb{R}^3, V_2)$ and $\mathfrak{g} \in (L^p \cap L^{p'})(\mathbb{R}^3, \mathcal{U}_2)$ for some $3 < p$, $p' = \frac{p}{p-1}$. For the remainder of our discussion, we fix some $3 < p$. Given $\varepsilon > 0$, we again define as previously:

$$C\ell_\varepsilon := \left\{ \begin{array}{c} (\omega, \mathfrak{g}) \in (W^{-1,p} \cap W^{-1,p'})(\mathbb{R}^2, V_2) \times (L^p \cap L^{p'})(\mathbb{R}^3, \mathcal{U}_2) \\ \\ \|\omega\|_{W^{-1,3}} + \|\mathfrak{g}\|_{L^3} \leq \varepsilon \end{array} \right\} \tag{3.281}$$

For constants $\varepsilon, \Theta > 0$, let $\mathcal{V}_{\varepsilon,\Theta} \subseteq C\ell_\varepsilon$ be the set where we have the decomposition (3.279) with the estimates

$$\|\nabla \mathfrak{q}\|_{L^3} \leq \Theta(\|\omega\|_{W^{-1,3}} + \|\mathfrak{g}\|_{L^3}) \tag{3.282}$$

$$\|\nabla \mathfrak{q}\|_p \le \Theta(\|\omega\|_{W^{-1,p}} + \|\mathfrak{g}\|_{L^p})\,, \tag{3.283}$$

$$\|\nabla \mathfrak{q}\|_{p'} \le \Theta(\|\omega\|_{W^{-1,p'}} + \|\mathfrak{g}\|_{L^{p'}})\,. \tag{3.284}$$

That is

$$\mathcal{V}_{\varepsilon,\Theta} := \left\{ \omega, \mathfrak{g} \in C\ell_\varepsilon : \begin{array}{c} \text{there exists } \mathfrak{q} \in (\dot{W}^{1,p} \cap \dot{W}^{1,p'})(\mathbb{R}^3, Spin(4)), \text{ so that} \\ \mathfrak{q} - \mathfrak{I} \in L^{3p/2p-3}(\mathbb{R}^3, Spin(4)) \\ \text{and } \ (3.279), (3.282), (3.283), (3.284) \ \text{ hold.} \end{array} \right\}$$

The strategy to prove Lemma 3.2.4.1 is precisely the same as for Lemma 3.2.2.2 and it is a corollary of the following:

Proposition 3.2.4.1. *There exist $\Theta > 0$ and $\varepsilon > 0$, such that $\mathcal{V}_{\varepsilon,\Theta} = C\ell_\varepsilon$.* □

Proof of Proposition 3.2.4.1. Proposition 3.2.4.1 follows, once we show the following four properties

(i.) $C\ell_\varepsilon$ is connected.

(ii.) $\mathcal{V}_{\varepsilon,\Theta}$ is nonempty.

(iii.) For any $\varepsilon, \Theta > 0$, $\mathcal{V}_{\varepsilon,\Theta}$ is a relatively closed subset of $C\ell_\varepsilon$.

(iv.) There exist $\Theta > 0$ and $\varepsilon > 0$ so that $\mathcal{V}_{\varepsilon,\Theta}$ is a relatively open subset of $C\ell_\varepsilon$.

As in Da Lio-Rivière [23], property (i.) and (ii.) are obvious and (iii.) follows as in the case of 4-dimensional domains. For further details, we refer to our discussion of the 4D-case.

It remains to show the openness property (iv.). For this let $(\omega_0, \mathfrak{g}_0)$ be arbitrary in $\mathcal{V}_{\varepsilon,\Theta}$. Let $\mathfrak{q}_0 \in \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{R}^3, Spin(4))$, $\mathfrak{q}_0 - \mathfrak{I} \in L^{3p/2p-3}(\mathbb{R}^3)$ so that the decomposition (3.279) as well as the estimates (3.282), (3.283) and (3.284) are satisfied for ω_0 and $\mathfrak{g}_0$. The idea is to study perturbations of $\mathfrak{q}_0$ of the form $\mathfrak{q} = \mathfrak{q}_0 e^{\mathfrak{u}}$, where $\mathfrak{u} \in \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{R}^3, \mathfrak{spin}(4)) \cap L^{3p/2p-3}(\mathbb{R}^3)$. Completely analogous to before, the exponent $p > 3$ has been chosen in particular to ensure $p' < 3$ and, as a result, $\mathfrak{u} \in C^0 \cap L^\infty(\mathbb{R}^3)$ and $\mathfrak{q}_0 e^{\mathfrak{u}} - \mathfrak{I} \in L^{\frac{3p}{2p-3}}$. This follows precisely the same way as in the 4-dimensional case treated previously, where we mentioned that the main estimate is independent of the dimension of the underlying space.

Attentive readers know what comes next: We compute the differential $D\mathcal{N}(\mathfrak{q}_0)$ as

$$D\mathcal{N}(\mathfrak{q}_0) = \frac{d}{dt}\mathcal{N}(\mathfrak{q}_0 e^{t\mathfrak{u}})\Big|_{t=0} =: \mathcal{L}_{\mathfrak{q}_0}(\mathfrak{u}),$$

where $\mathfrak{u} \in (\dot{W}^{1,p} \cap \dot{W}^{1,p'} \cap L^{3p/2p-3})(\mathbb{R}^3, \mathfrak{spin}(4))$. We write

$$\mathcal{L}_{\mathfrak{q}_0}(\mathfrak{u}) = (\mathcal{L}^2_{\mathfrak{q}_0}(\mathfrak{u}), \mathcal{L}^3_{\mathfrak{q}_0}(\mathfrak{u}))$$

where

$$\begin{aligned} \mathcal{L}^2_{\mathfrak{q}_0}(\mathfrak{u}) \quad &:= \quad \Pi_2 \left[\Delta \mathfrak{u} + \sum_{j=0}^{2} \partial_{x_j} \left(\mathfrak{q}_0^{-1}(\partial_{x_j}\mathfrak{q}_0)\mathfrak{u} - \mathfrak{u}\mathfrak{q}_0^{-1}\partial_{x_j}\mathfrak{q}_0 \right) \right] \\ \mathcal{L}^3_{\mathfrak{q}_0}(\mathfrak{u}) \quad &= \quad -\Pi_3(\partial_R \mathfrak{u})e_3 + \partial_R \Pi_1(\mathfrak{u}) \end{aligned}$$

$$-\sum_{j=0}^{2}(-1)^{\delta_{0j}}\Pi_3\left((\mathfrak{q}_0^{-1}\partial_{x_j}\mathfrak{q}_0\mathfrak{u}-\mathfrak{u}\mathfrak{q}_0^{-1}\partial_{x_j}\mathfrak{q}_0)e_j\right)e_3$$

$$+\sum_{j=0}^{2}(-1)^{\delta_{0j}}\Pi_1(\mathfrak{q}_0^{-1}\partial_{x_j}\mathfrak{q}_0\mathfrak{u}-\mathfrak{u}\mathfrak{q}_0^{-1}\partial_{x_j}\mathfrak{q}_0)e_j$$

The essential property we will be using is the invertibility of $\mathcal{L}_{\mathfrak{q}_0}(\mathfrak{u})$ in the special case $\mathfrak{q}_0 = \mathfrak{I}$. If $\mathfrak{q}_0 = \mathfrak{I}$, we have $d\mathfrak{q}_0 = 0$ and therefore the differential simplifies significantly:

$$\begin{aligned}\mathcal{L}^2_{\mathfrak{I}}(\mathfrak{u}) &= \Pi_2\left[\Delta\mathfrak{u}\right]\\ \mathcal{L}^3_{\mathfrak{I}}(\mathfrak{u}) &= -\Pi_3(\partial_R\mathfrak{u})e_3+\partial_R\Pi_1(\mathfrak{u})\end{aligned}\tag{3.285}$$

Proposition 3.2.4.2. *The operator $\mathcal{L}_{\mathfrak{I}}(\mathfrak{u})$ is elliptic.*

Proof of Proposition 3.2.4.2. We write $\mathfrak{u} = w + v$ where $w \in V_2$ and $v = v^0e_3 + v^1e_1e_3 + v^2e_2e_3 + v^3e_1e_2 \in V_1 \oplus V_3$. We observe that

$$\begin{aligned}\mathcal{L}^2_{\mathfrak{I}}(\mathfrak{u}) &= \Pi_2\left[\Delta w+\Delta v\right]=\Delta w\\ \mathcal{L}^3_{\mathfrak{I}}(\mathfrak{u}) &= -\Pi_3(\partial_R v)e_3+\partial_R\Pi_1(v)\end{aligned}$$

Computing $\mathcal{L}^3_{\mathfrak{I}}(\mathfrak{u})$ explicitly, we find:

$$\begin{aligned}-\Pi_3(\partial_R\mathfrak{u})e_3+\partial_R\Pi_1(\mathfrak{u}) &= (\partial_{x_0}v^0-\partial_{x_1}v^1-\partial_{x_2}v^2)+(\partial_{x_1}v^0+\partial_{x_0}v^1-\partial_{x_2}v^3)e_1\\ &+ (\partial_{x_2}v^0+\partial_{x_0}v^2+\partial_{x_1}v^3)e_2+(\partial_{x_2}v^1-\partial_{x_1}v^2+\partial_{x_0}v^3)e_1e_2\\ &= D_R^{RF}(v^0+v^1i+v^2j+v^3k)\end{aligned}$$

We can associate to this operator the following symbol:

$$\sigma(\xi)=\begin{pmatrix}\xi_0 & -\xi_1 & -\xi_2 & 0\\ \xi_1 & \xi_0 & 0 & -\xi_2\\ \xi_2 & 0 & \xi_0 & \xi_1\\ 0 & \xi_2 & -\xi_1 & \xi_0\end{pmatrix}\tag{3.286}$$

It is immediately clear that this is now the Riemann-Fueter operator applied to functions depending only on the first 3 variables. Therefore, one may argue as in 4D that the symbol is everywhere invertible. In fact, this is an immediate corollary of the computations in 4D. This concludes the proof of Proposition 3.2.4.2. □

We can prove the following result, which we only state, since the proof is now more or less a copy of the corresponding result in 4D:

Lemma 3.2.4.2. *For any $\Theta > 0$, there exists $\varepsilon > 0$ so that the following holds for any $\omega_0, \mathfrak{g}_0$ and $\mathfrak{q}_0$ as above:*

For any $\omega \in (W^{-1,p} \cap W^{-1,p'})(\mathbb{R}^3, V_2)$ *and* $\mathfrak{g} \in (L^p \cap L^{p'})(\mathbb{R}^3, \mathcal{U}_2)$ *there exists a unique* $\mathfrak{u} \in \dot{W}^{1,p} \cap \dot{W}^{1,p'} \cap L^{3p/2p-3}(\mathbb{R}^3, \mathfrak{spin}(4))$ *so that*

$$(\omega, \mathfrak{g}) = \mathcal{L}_{\mathfrak{q}_0}(\mathfrak{u})$$

and for some constant $C = C(\omega_0, \mathfrak{g}_0, \Theta) > 0$ *it holds*

$$\begin{aligned}\|\nabla \mathfrak{u}\|_{L^p(\mathbb{R}^3)} + \|\nabla \mathfrak{u}\|_{L^{p'}(\mathbb{R}^3)} \quad \lesssim \quad & \|\omega\|_{W^{-1,p}(\mathbb{R}^3)} + \|\omega\|_{W^{-1,p'}(\mathbb{R}^3)} \\ & + \|\mathfrak{g}\|_{L^p(\mathbb{R}^3)} + \|\mathfrak{g}\|_{L^{p'}(\mathbb{R}^3)}.\end{aligned} \tag{3.287}$$

Proof of Proposition 3.2.4.1 continued.
For $\varepsilon = \varepsilon(\Theta) > 0$ chosen small enough and for any $(\omega_0, \mathfrak{g}_0) \in \mathcal{V}_{\varepsilon,\Theta}$, the local inversion theorem applied to $\mathcal{N}_{\mathfrak{q}_0}$ gives the existence of some $\delta > 0$ (that might depend on $(\omega_0, \mathfrak{g}_0)$) such that, for every $(\omega, \mathfrak{g}) \in C\ell_\varepsilon$ with

$$\|\omega - \omega_0\|_{W^{-1,p}(\mathbb{R}^3)} + \|\omega - \omega_0\|_{W^{-1,p'}(\mathbb{R}^3)} < \delta \tag{3.288}$$

$$\|\mathfrak{g} - \mathfrak{g}_0\|_{L^p(\mathbb{R}^3)} + \|\mathfrak{g} - \mathfrak{g}_0\|_{L^{p'}(\mathbb{R}^3)} < \delta, \tag{3.289}$$

we find $\mathfrak{q} = \mathfrak{q}_0 e^{\mathfrak{u}} \in \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{R}^3, Spin(4))$, so that $\mathfrak{q} - \mathfrak{I} \in L^{3p/2p-3}(\mathbb{R}^3)$ and (3.279) is satisfied. It remains to prove (3.282), (3.283) and (3.284). This will be implied by the following lemma, whose proof is again analogous to the 4D-case and therefore omitted:

Lemma 3.2.4.3. *There exists* $\Theta > 0$ *and* $\sigma > 0$*, such that whenever* $\mathfrak{q} \in \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{R}^3)$ *with* $\mathfrak{q} - \mathfrak{I} \in L^{3p/2p-3}(\mathbb{R}^3)$ *satisfying* (3.279) *and it holds*

$$\|\nabla \mathfrak{q}\|_{L^3(\mathbb{R}^3)} \leq \sigma, \tag{3.290}$$

then (3.282)*,* (3.283) *and* (3.284) *hold true as well.* □

Thanks to Lemma 3.2.4.3, the openness property (iv.) is proven. Proposition 3.2.4.1, and as a corollary also Lemma 3.2.4.1, is now established. □

In order to establish the bootstrap lemma and Morrey estimates, one can now proceed completely analogous to the case of domains of dimension 4. Indeed, the arguments for improved regularity of the potential carry over immediately and the non-linear Hodge decomposition works equally well in this case. We refer to our discussion for $\mathbb{R}^4$, the modifications should be self-explanatory.

3.2.5 Perspectives for Domains of Dimension $5 \leq m \leq 8$

Finally, let us briefly discuss the possibility to extend the results presented to domains of arbitrary dimensions ≤ 8. There in fact is a way to generalise the construction of the gauge operator in these cases and we refer to Section 3.3 for the details and some motivation for the necessity of such a condition. The key is that in the cases $m = 3$ and $m = 4$, the gauge operator relies on the ellipticity of the Riemann-Fueter operator to show existence and appropriate estimates. For $5 \leq m \leq 8$, we may substitute the Riemann-Fueter operator by the octonionic derivative in a suitable sense, which allows us to conclude in much the same way. This is not very surprising, considering that the Riemann-Fueter operator is indeed the same as the quaternionic derivative. In some sense, the main property we use is the existence of an orthogonal frame which happens to parallelize the sphere, a property closely linked to the existence of normed division algebras and thus to quaternions and octonions. In fact, considering the gauge operator we constructed, the existence of an invertible gauge operator was realised by means of using an elliptic operator of first order whose symbol has columns forming an orthogonal frame. Since this is only possible for the spheres in dimension $0, 1, 3, 7$, we are thus restricted by our technique to $m \leq 8$. If one manages to find a sufficiently nice elliptic, first order operator having some additional properties to ensure that it is related to the change of gauge as in (3.165), the range of dimensions m to which our proof applies could be extended.

3.2.6 Appendix A: Riemann-Fueter and Dirac operators

In this appendix, we introduce and define the most important notions that have been used in this note. We mostly limit ourselves to stating the definitions and main properties and refer to the literature for further details as well as the corresponding proofs.

The reduction from a system of divergence PDE to a linear one will be greatly simplified by introducing a family of important first order differential operators, the so-called Dirac operators. In one of the final sections, we shall consider a variation of the definition here which retains most of the same properties, but is slightly better behaved with respect to the change of gauge we envision.

3.2.6.1 Riemann-Fueter Operator on $\mathbb{H}$

Let $f : \mathbb{H} \to \mathbb{H}$ be a quaternion-valued function over $\mathbb{H} \simeq \mathbb{R}^4$. The 4D-Riemann-Fueter operator D_R^{RF} acting from the right is defined by:

$$\begin{aligned} D_R^{RF} f := &\big(\partial_{x_0} f_0 - \partial_{x_1} f_1 - \partial_{x_2} f_2 - \partial_{x_3} f_3\big) \\ &+ \big(\partial_{x_0} f_1 + \partial_{x_1} f_0 - \partial_{x_2} f_3 + \partial_{x_3} f_2\big) i \\ &+ \big(\partial_{x_0} f_2 + \partial_{x_1} f_3 + \partial_{x_2} f_0 - \partial_{x_3} f_1\big) j \\ &+ \big(\partial_{x_0} f_3 - \partial_{x_1} f_2 + \partial_{x_2} f_1 + \partial_{x_3} f_0\big) k, \end{aligned} \tag{3.291}$$

where $f = f_0 + f_1 \cdot i + f_2 \cdot j + f_3 \cdot k$, or abbreviated:

$$D_R^{RF} f = \partial_{x_0} f + \partial_{x_1} f \cdot i + \partial_{x_2} f \cdot j + \partial_{x_3} f \cdot k.$$

The conjugated differential operator $\overline{D}_R^{RF}$ is similarily defined:

$$\overline{D}_R^{RF} f = \partial_{x_0} f - \partial_{x_1} f \cdot i - \partial_{x_2} f \cdot j - \partial_{x_3} f \cdot k.$$

It is easy to see by a direct calculation:

$$\overline{D}_R^{RF} D_R^{RF} f = D_R^{RF} \overline{D}_R^{RF} f = \Delta f.$$

This can for instance be proven by considering the symbol $\sigma_{D_R^{RF}}$ of the differential operator D_R^{RF}:

$$\sigma_{D_R^{RF}}(\xi) = \begin{pmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_1 & \xi_0 & \xi_3 & -\xi_2 \\ \xi_2 & -\xi_3 & \xi_0 & \xi_1 \\ \xi_3 & \xi_2 & -\xi_1 & \xi_0 \end{pmatrix} \tag{3.292}$$

We emphasize that the connection between D_R^{RF} and the Laplacian mirrors the same relation between the complex derivative ∂_z and the Laplacian. In particular, we have access to regularity results by using the Laplacian as an intermediate step. In particular, deriving a fundamental solution is greatly simplified and many results from complex analysis can be carried over to Riemann-Fueter operators, see Gilbert-Murray [38]. As a simple example, if $D_R^{RF} f = 0$, then f is automatically harmonic and thus smooth.

Naturally, analogous operators D_L^{RF} and $\overline{D}_L^{RF}$ using multiplication from the left rather than from the right can be defined and satisfies similar properties. However, it should be noted, that the two pairs of operators are not the same due to the non-commutativity of the quaternions. This is in stark contrast with the situation on $\mathbb{C}$, which is a commutative field, and already hints at possible difficulties that might arise in our arguments later on.

3.2.6.2 General Dirac Operators on Clifford Algebras

Let now $m \in \mathbb{N}$ be given and we define for functions $f : U \subset \mathbb{R}^{m+1} \to C\ell_m$ the Dirac operator ∂_L in the following way:

$$\partial_L f = \partial_{x_0} f - e_1 \cdot \partial_{x_1} f - \ldots - e_m \cdot \partial_m f. \tag{3.293}$$

We refer to Gilbert-Murray [38] for details on properties of this kind of operator. Once again, we can easily generalise this definition by changing signs to obtain $\overline{\partial}_L$ or by moving the multiplications to the other side to arrive at ∂_R and $\overline{\partial}_R$ respectively.

By a direct computation, we can easily deduce that:

$$\partial_L \overline{\partial}_L f = \overline{\partial}_L \partial_L f = \Delta f,$$

extending the connection between the Laplacian and complex differentiation or the Riemann-Fueter operator to arbitrary Clifford algebras. We emphasise that the Riemann-Fueter operator is not a special case of the Dirac operators, although they share a lot of common features, see Gilbert-Murray [38]. In addition, observe the different conventions regarding the signs associated with the partial derivatives. As earlier, this enables us to easily extend regularity results for the Laplacian to the Dirac operators.

For completeness' sake, let us introduce the following notion as in Gilbert-Murray [38]: A function f is called Clifford-analytic, if $\partial_L f = 0$. By our previous elaborations, such functions are harmonic and thus smooth. A theory of such functions in analogy to complex analysis can be built up from scratch, see Gilbert-Murray [38] as well as the theory of Hardy spaces by using Clifford analytic functions.

3.2.6.3 Spin Groups

An important subset of $C\ell_m$ is the so-called Spin-group: For a fixed $m \in \mathbb{N}$, we define:

$$\mathrm{Spin}(m) = \{v_1 \cdot \ldots \cdot v_{2k} \mid k \in \mathbb{N}, v_j \in C\ell_m^{(1)} \simeq \mathbb{R}^m \text{ and } \|v_j\| = 1 \text{ for all } j\} \subset C\ell_m$$

These groups are actually compact Lie groups and provide a natural two-fold covering of $\mathfrak{so}(m)$. Their Lie algebras are given by:

$$\mathfrak{spin}(m) = C\ell_m^{(2)}.$$

Observe that $\dim_{\mathbb{R}} \mathrm{Spin}(m) = \frac{1}{2}m(m-1)$. In a similar manner, we can introduce the compact Lie groups $\mathrm{Spoin}(m)$, see Gilbert-Murray [38]:

$$\mathrm{Spoin}(m) = \{v_1 \cdot \ldots \cdot v_k \mid k \in \mathbb{N}, v_j \in C\ell_m^{(0)} \oplus C\ell_m^{(1)} \simeq \mathbb{R}^{m+1} \text{ and } \|v_j\| = 1 \text{ for all } j\} \subset C\ell_m$$

This group provides another two-fold covering, this time one for $\mathfrak{so}(m+1)$. As a result, it is easy to deduce that $\mathrm{Spoin}(m) \simeq \mathrm{Spin}(m+1)$ due to the uniqueness of the universal covering of $\mathfrak{so}(m+1)$. The Lie algebra $spoin(m)$ is given by:

$$spoin(m) = C\ell_m^{(1)} \oplus C\ell_m^{(2)} \simeq \mathfrak{spin}(m+1).$$

As a simple, explicit example, we have:

$$\mathfrak{spin}(4) \simeq spoin(3) \simeq \mathrm{span}\{e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3\}.$$

In what follows, we will usually denote $Spoin(m)$ in $C\ell_m$ by $Spin(m+1)$ in order to adhere to common terminology. We refer to Theorems 6.3, 6.8, 6.12, 7.26, 7.27 and 8.10 in Gilbert-Murray [38] for further details regarding these groups.

3.2.6.4 Hodge Decomposition and Hodge $*$-Operator

Let us briefly recall the Hodge $*$-operator on $\mathbb{R}^m$ with respect to the standard basis. On $\mathbb{R}^m$, we use the standard basis $b_0, \ldots, b_{m-1}$ and we have for the standard euclidean inner product:

$$\langle b_i, b_j \rangle = \delta_{ij}, \quad \forall i, j \in \{0, \ldots, m-1\}.$$

Denote by $b_0^*, \ldots, b_{m-1}^*$ the dual basis. Then $b_{i_1}^* \wedge \ldots \wedge b_{i_k}^*$ for $0 \le k \le m$ and $0 \le i_1 < \ldots < i_k \le m-1$ form a basis for $\bigwedge(\mathbb{R}^m)^*$. We may now define a scalar product $\langle \cdot, \cdot \rangle_{\bigwedge(\mathbb{R}^m)^*}$ on $\bigwedge(\mathbb{R}^m)^*$ by declaring the collection of all $b_{i_1}^* \wedge \ldots \wedge b_{i_k}^*$ to be an orthonormal basis. The scalar product can also be defined, and actually is, independent of the choice of orthonormal basis $b_0, \ldots, b_{m-1}$, even for arbitrary k-forms as well as arbitrary Riemannian metrics g, by using local g-orthonormal frames. From now on, we shall write dx_j instead of b_j^*, following the usual convention.

The Hodge $*$-operator is then defined for all η, ω k-forms by the following formula:

$$\eta \wedge *\omega = \langle \eta, \omega \rangle_{\bigwedge \mathbb{R}^m} \mu,$$

where $\mu = dx_0 \wedge \ldots \wedge dx_{m-1}$ is the standard volume form on $\mathbb{R}^m$. Using this operator, we can introduce the codifferential d^* of a k-differential form ω on $\mathbb{R}^m$ by the following formula:

$$d^*\omega = (-1)^{m(k-1)+1} * d * \omega,$$

where d denotes the usual exterior derivative on differential forms. The Laplacian of a form ω is then defined as follows:

$$-\Delta\omega = (dd^* + d^*d)\omega.$$

Let us provide a computation of d^* in the special case $m = 4$: Assume $\omega = \omega^0 dx_0 + \ldots + \omega^3 dx_3$ is a 1-form. Direct considerations show that:

$$\begin{aligned} d^*\omega &= - \star d\big(\omega^0 dx_1 \wedge dx_2 \wedge dx_3 - + \ldots - \omega^3 dx_0 \wedge dx_1 \wedge dx_2\big) \\ &= - \star \big(\partial_{x_0}\omega^0 + \partial_{x_1}\omega^1 + \partial_{x_2}\omega^2 + \partial_{x_3}\omega^3\big)\mu = -\big(\partial_{x_0}\omega^0 + \partial_{x_1}\omega^1 + \partial_{x_2}\omega^2 + \partial_{x_3}\omega^3\big) \end{aligned}$$

This formula will be used later. In addition, it can be easily shown that the Laplacian on 0- and 1-forms actually agrees with the usual componentwise Laplacian up to a sign.

3.2.7 Appendix B: A Result in Integrability by Compensation

Later, we shall make repeated use of the following compensation result:

Lemma 3.2.7.1. *Let $da \in L^{m,\infty}(\mathbb{R}^m)$, $db \in L^{p,r}(\mathbb{R}^m)$ for $1 < p < +\infty$ and $1 \le r \le +\infty$. Then, we have $da \wedge db \in W^{-1,(p,r)}(\mathbb{R}^m)$ together with the following estimate:*

$$\|da \wedge db\|_{W^{-1,(p,r)}} \le C\|da\|_{L^{m,\infty}}\|db\|_{L^{p,r}}, \tag{3.294}$$

for a constant $C > 0$.

Proof of Lemma 3.2.7.1.
By density, we may assume $a, b \in \mathcal{S}(\mathbb{R}^m)$, the general case follows by approximation. Let now u be a solution of the following equation:

$$\Delta u = da \wedge db \text{ in } \mathcal{D}'(\mathbb{R}^m). \tag{3.295}$$

We will show that $\nabla u \in L^p$ as well as:

$$\|\nabla u\|_{L^p} \le C\|da\|_{L^{m,\infty}}\|db\|_{L^p}, \tag{3.296}$$

the general case is a direct consequence of real interpolation (consider da fixed to obtain the required linear operator in the interpolation argument). We distinguish two cases:

Case 1: If $p > \frac{m}{m-1}$, we know by the general Hölder inequality:

$$da \wedge db \in L^{q,r}, \tag{3.297}$$

where:

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{m}, \quad r = p.$$

By elliptic regularity, we deduce that $u \in W^{2,(q,r)}$ and by Sobolev embeddings:

$$\nabla u \in L^{p,p} = L^p,$$

together with the estimate:

$$\|\nabla u\|_{L^p} \lesssim \|u\|_{W^{2,(q,r)}} \lesssim \|\Delta u\|_{L^{q,r}} \lesssim \|da\|_{L^{m,\infty}}\|db\|_{L^p}.$$

Case 2: If $p < \frac{m}{m-1}$, we take $\bar{b} \in \mathbb{R}$ such that $b - \bar{b} \in L^{p^*,p}$. Here, we denote by p^* the parameter determined by:

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{m}.$$

Observe that $da \wedge db = d\left(da \wedge (b - \bar{b})\right)$. Hölder's inequality immediately shows:

$$da \wedge (b - \bar{b}) \in L^{q,p}, \tag{3.298}$$

where $1/q = 1/p^* + 1/m = 1/p$. Thus $q = p$. We therefore conclude:

$$\|da \wedge db\|_{W^{-1,p}} \lesssim \|da \wedge (b - \bar{b})\|_{L^p} \lesssim \|da\|_{L^{m,\infty}}\|b - \bar{b}\|_{L^{p^*,p}} \lesssim \|da\|_{L^{m,\infty}}\|db\|_{L^p}. \tag{3.299}$$

Thus, we may deduce:

$$\|\nabla u\|_{L^p} \lesssim \|da\|_{L^{m,\infty}}\|db\|_{L^p}. \tag{3.300}$$

This finishes our proof. We emphasize that, in particular, the "critical" case $p = \frac{m}{m-1}$ is obtained by interpolation. $\square$

3.3 Various Further Directions

In the current section, we shall explore extensions of the results in [26]. In particular, we address the approach required to treat general codomains by means of introducing suitable matrix Lie groups as well as how to generalise the previous results to domains of dimension ≤ 8 and why this is optimal for the technique we use. These are computations not contained in previous submissions or publications.

3.3.1 The Case of Domains of Dimension 4 - General Codomains

Before treating the general case of domains with dimension at most 8, let us provide the discussion in the exemplary case of domains of dimension 4. The idea behind this is that the main features associated with the general case become apparent in this special case.

After having dealt with the case of codomains of dimension 2 in [26], we would like to expand the ideas used to general codomains. This is the goal of the current subsection. An important step in the proof will be the construction of an appropriate compact Lie group for the change of gauge. Notice that for $C\ell_3$, there do no longer exist immediate candidates like the (hyper-)unitary matrices used in Da Lio-Rivière [23].

First, let us state again the PDE of interest to our discussion:

$$\partial_L \mathfrak{f} = \Omega^+ \cdot \mathfrak{f} + \Omega^- \cdot \hat{\mathfrak{f}}, \tag{3.301}$$

where $\mathfrak{f} \in L^{\frac{4}{3}}$. $\Omega^\pm$ are anti-symmetric and assume values in $\mathrm{span}_\mathbb{R}\{1, e_1, e_2, e_3\}$. This is in complete analogy to Da Lio-Rivière [23]. If we proceed as in Da Lio-Rivière [23], we can basically eliminate Ω^+ by a change of gauge using orthogonal matrices. The only point that really differs is the application of Wente's inequality which can be replaced by Lemma 3.2.7.1 using the higher intergrability assumption $\nabla S \in L^{(4,2)}$ which also shows $Q \in L^{(4,2)}$ and therefore $\Omega^+ \in L^{(4,2)}$. By a careful examination of the proof of the change of gauge theorem, we can see that $L^{(4,2)}$-integrability can be established. Thus, Ω^+ can be dealt with in analogy to Da Lio-Rivière [23] by making use of the stricter regularity assumptions. One needs to have in mind, however, that dealing with Ω^+ introduces

Therefore, the only part of the equation that is relevant to our considerations is the following: We consider the equation:

$$\partial_L F = \Gamma_0 \cdot F, \tag{3.302}$$

where $\overline{\Gamma}_0^T + \Gamma_0 = 0$ is as in the paper Da Lio-Rivière [23] a matrix-valued Clifford-antihermitian function with $\|\Gamma_0\|_{L^{(4,2)}} < \varepsilon$ and $f \in L^{\frac{4}{3}}(\mathbb{R}^4, C\ell_4^{2k})$. In fact, Γ_0 has the following structure:

$$\Gamma_0 := \begin{pmatrix} 0 & -\Gamma e_4 \\ \Gamma e_4 & 0 \end{pmatrix} \tag{3.303}$$

Γ is anti-symmetric and given analogous to Da Lio-Rivière [23]. In particular, Γ_0 assumes values in:

$$E_4 := \mathrm{span}\left\{e_4, e_1e_4, e_2e_4, e_3e_4\right\}$$

For later, we also define:

$$E_c := \mathrm{span}\left\{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3, e_1e_2e_4, e_1e_2e_4, e_2e_3e_4, e_1e_2e_3e_4\right\}$$

In fact, we shall see that we want Γ_0 to have an even more specific form to ensure the increased regularity argument from the case of codomains of dimension 2 carries over. This means, that our result only applies to S, such that for example $Q = e^U$, if we use the notation as in the previous subsections to denote the change of basis as well as the formulas found in Da Lio-Rivière [23].

The main idea is now to proceed as in Da Lio-Rivière [23] and our previous considerations: Let P be a map defined on $\mathbb{R}^n$ assuming values in an appropriate compact Lie group G to be specified in the next subsection. For the moment, we only know it is a compact matrix subgroup of $C\ell_4^{2k\times 2k}$ with Lie algebra $\mathfrak{g}$. Then we define the following vector-valued function:

$$G := \begin{pmatrix} PF \\ -Pe_1F \\ -Pe_2F \\ -Pe_3F \end{pmatrix} \tag{3.304}$$

It then holds:

$$\begin{aligned} \operatorname{div} G &= \partial_0\big(PF\big) + \partial_1\big(- Pe_1F\big) + \partial_2\big(- Pe_2F\big) + \partial_3\big(- Pe_3F\big) \\ &= \partial_R P \cdot F + P \cdot \partial_L F \\ &= \partial_R P \cdot F + P\Gamma_0 \cdot F \\ &= P\Big(P^{-1}\partial_R P + \Gamma_0\Big)F \end{aligned} \tag{3.305}$$

We now try to find a suitable P assuming values in a compact Lie group, such that:

$$P^{-1}\partial_R P + \Gamma_0 \in L^{(4,1)} \tag{3.306}$$

In this case, if we have appropriate $L^{(4,1)}$-estimates, we could argue completely analogous to [26] in the case of codomains of dimension 2 to show that if $\varepsilon > 0$ was small enough, then $G = 0$ and thus $F = 0$. This is clear, as all ideas used in the non-linear Hodge decomposition as well as the estimates following it are not dependent on the dimension of the codomain in any meaningful way.

To arrive at the required improvement in regularity, let us introduce the following gauge operator:

$$N : \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{R}^4, G) \to W^{-1,p} \cap W^{-1,p'}(\mathbb{R}^4, \Pi_c(\mathfrak{g})) \times L^p \cap L^{p'}(\mathbb{R}^4, \Pi_4(\mathfrak{g})) \tag{3.307}$$

where we impose implicitly some integrability condition on $P - Id$ as in the case of codomains of dimension 2. The formula for N is the following:

$$N(P) := \Big(\Pi_c\big(\sum_{j=0}^{3} \partial_j(P^{-1}\partial_j P)\big), \Pi_{Sym} \circ \big(\Pi_4 + \mathcal{P}\big)\big(P^{-1}\partial_R P\big)\Big) \tag{3.308}$$

where Π_4 denotes the componentwise (i.e. for each matrix entry individually) projection to the subspace E_4 and Π_c denotes the componentwise projection onto E_c. By abuse of notation, we shall also refer to $\Pi_4(\mathfrak{g})$ as E_4 and to $\Pi_c(\mathfrak{g})$ as E_c. Π_{Sym} refers to the projection onto the symmetric part of the matrices. We shall further explain the operator as well as give a definition of N and provide some justification for the subspaces used after the next subsection.

3.3.1.1 Lie Group for a Change of Gauge

The goal of the current subsection is to find a compact Lie group suited for a change of gauge. Ideally, this group should have a Lie algebra containing Γ_0. Since Γ_0 is Clifford-antihermitian and this condition is quite similar to the defining property of the Lie algebra of the unitary groups, the following definition is intuitively the right way to proceed:

$$\tilde{U}(n,k) := \left\{ P \in C\ell_n^{k\times k} \;\middle|\; P\overline{P}^T = Id \right\}, \tag{3.309}$$

where $\overline{P}$ is the componentwise Clifford conjugation. In addition, we define:

$$U(n,k) := \left\{ P \in C\ell_n^{k\times k} \;\middle|\; P\overline{P}^T = \overline{P}^T P = Id \right\} \tag{3.310}$$

We have the following observation:

Lemma 3.3.1.1. *The set $\tilde{U}(n,k)$ is a smooth manifold.*

It is improbable that the set $\tilde{U}(n,k)$ is actually a Lie group, as $\overline{P}^T P$ cannot be determined from $P\overline{P}^T = Id$ in general due to the lack of commutativity of the Clifford multiplication. This means that it is unclear, whether $\tilde{U}(n,k)$ is closed under inverses, whereas closedness under the natural matrix product is straighforward. On the other hand, it is easy to see that $U(n,k)$ is a group, but the same result as in Lemma 3.3.1.1 is not as easy to obtain a-priori.

Proof. It suffices to show that $\{Id\}$ is a regular value of the map:

$$\varphi : C\ell_n^{k\times k} \to \left\{ A \in C\ell_n^{k\times k} \mid \overline{A}^T = A \right\}, \quad \varphi(P) := P\overline{P}^T \tag{3.311}$$

Let thus $P \in \varphi^{-1}\big(\{Id\}\big)$. Computing the differential leads us to:

$$d\varphi(P)H = H\overline{P}^T + P\overline{H}^T, \quad \forall H \in C\ell_n^{k\times k} \tag{3.312}$$

Let us choose $H = \frac{1}{2}\tilde{H}P$, where $\overline{\tilde{H}}^T = \tilde{H}$ is arbitrary. Then we have:

$$\begin{aligned} d\varphi(P)H &= H\overline{P}^T + P\overline{H}^T \\ &= \frac{1}{2}\tilde{H}P\overline{P}^T + \frac{1}{2}P\overline{P}^T\overline{\tilde{H}}^T \\ &= \frac{1}{2}\tilde{H} + \frac{1}{2}\overline{\tilde{H}}^T \\ &= H, \end{aligned} \tag{3.313}$$

which shows surjectivity of the differential. Therefore, as P was arbitrary and $\tilde{U}(n.k) = \varphi^{-1}(\{Id\})$, $\tilde{U}(n,k)$ is a smooth manifold. ☐

We define $u(n,k) := \{H \in C\ell_n^{k\times k} | \overline{H}^T + H = 0\}$. This is the kind of Lie algebra we want our Lie group to possess. The key point to show lies in the following:

Lemma 3.3.1.2. *The matrix exponential defined as usual leads to a map:*

$$\exp : u(n,k) \to U(n,k) \subset \tilde{U}(n,k), \tag{3.314}$$

which is a diffeomorphism on a neighbourhood of 0 *and thus maps an open neighbourhood of* 0 *bijectively to an open neighbourhood of* Id *in* $\tilde{U}(n,k)$*. The image of the open neighbourhood is then also a submanifold.*

Proof. It is easy to see:

$$\begin{aligned}\overline{\exp(H)}^T &= \overline{\sum_{l=0}^{\infty} \frac{1}{l!} H^l}^T \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} \left(\overline{H}^T\right)^l \\ &= \exp\left(\overline{H}^T\right) \\ &= \exp(-H) = \exp(H)^{-1},\end{aligned} \tag{3.315}$$

where in the last line, we actually mean the inverse as matrices. Indeed, it can be directly checked:

$$\exp(H)\exp(-H) = \exp(-H)\exp(H) = Id, \tag{3.316}$$

and therefore showing that the map in Lemma 3.3.1.2 is well-defined. Smoothness of the map is immediate, as the matrix product is bilinear and due to the manifold property in Lemma 3.3.1.1. The local diffeomorphism statement follows, as:

$$d\exp(0)H = H, \quad \forall H \in u(n,k), \tag{3.317}$$

which obviously is invertible. Note that in Lemma 3.3.1.1, we actually implicitely showed that the tangent space at the identity of $\tilde{U}(n,k)$ is $u(n,k)$, which we use here to conclude bijectivity of the differential. So the desired result is proven. □

Let us now denote by V the open image in $\tilde{U}(n,k)$ of some open neighbourhood of 0 under exp, on which exp defines a diffeomorphism. Without any loss of generality, we may assume that V is the image of a symmetric open neighbourhood of 0, i.e. a neighbourhood containing H if and only if it contains $-H$. This property is related to V being closed under inverses which is crucial for our later considerations. We have the following:

Lemma 3.3.1.3. *The set:*

$$H := \bigcup_{l=1}^{\infty} V^l \tag{3.318}$$

is an open subset of $\tilde{U}(n,k)$ *and an open subset of* $U(n,k)$*, as well. Moreover,* H *is even a Lie group.*

Proof. This result is immediate, as any $P \in V$ is invertible and thus induces a homeomorphism when multiplied from the left, therefore showing that $V^2 = \bigcup_{P \in V} P \cdot V$ is open as a union of open subsets (the product of sets is naturally defined in the usual manner evoking the product structure provided for elements). The openness of V^l follows by induction and the openness of the union by general properties of a topology. The openness in $U(n,k)$ is clear due to the definition of the subspace-topology and the group property is immediate, if we assume from the beginning that V is the image of a symmetric neighbourhood of 0 under exp to ensure it is closed under inverses. □

We now want to show that H actually agrees with the connected component $U(n,k)_0$ of the identity in $U(n,k)$, which would show that the connected component is a Lie group with $u(n,k)$ as its Lie algebra. This will be the Lie group we shall be working with. We highlight that one can easily show that $\tilde{U}(n,k)$ is compact by using the definition as well as general properties of the Clifford product, in particular its connection to the norm. This also enables one to show that $U(n,k)$ is a Lie group as well.

To arrive at this conclusion, let us observe the following: $H \subset U(n,k)_0$ is an open subset which is also a subgroup. This immediately implies that H is also relatively closed, as H splits $U(n,k)_0$ into disjoint equivalence classes in the usual way, each of which is homeomorphic to H and thus open. Therefore, H is clopen in $U(n,k)_0$, which by connectedness shows $H = U(n,k)_0$, since $Id \in H$ excludes the scenario $H = \emptyset$. This proves our desired assertion. Thus we have shown, as H is a Lie group and $U(n,k)$ is compact:

Lemma 3.3.1.4. *The connected component of the identity matrix $U(n,k)_0$ in $U(n,k)$ is a compact Lie group with Lie algebra $u(n,k)$.*

3.3.1.2 Gauge Operator and its Differential

In the definition of the operator N, we shall from now on use $G = U(4,2k)_0$ and $\mathfrak{g} = u(4,2k)$. The precise definition of the operator is as follows:

$$N(P) := \Big(\Pi_c\big(\sum_{j=0}^{3} \partial_j(P^{-1}\partial_j P)\big), \Pi_{Sym} \circ \big(\Pi_4 + \mathcal{P}\big)\big(P^{-1}\partial_R P\big)\Big) \tag{3.319}$$

Here, $\mathcal{P}$ is defined as in the case of 2-dimensional codomains before and Π_{Sym} denotes the projection onto the symmetric part (i.e. the part with $H^T = H$), more precisely:

$$\Pi_{Sym}(H) = \frac{H + H^T}{2}, \quad \forall H \in C\ell_4^{2k\times 2k} \tag{3.320}$$

The reason for the inclusion of Π_{Sym} lies in the need to ensure that the second component of N maps to the image of $\mathfrak{g}$ under Π_4 to be able to require:

$$N(P) = (0, \Gamma_0). \tag{3.321}$$

To do this, we observe that projecting $H \in \mathfrak{g}$ to E_4 componentwise, the resulting matrix is symmetric due to:

$$\overline{e_j} = -e_j, \quad \overline{e_j e_k} = -e_j e_k, \tag{3.322}$$

for all $j \neq k \in \{1,2,3,4\}$. So symmetry of this projection follows from $\overline{H}^T = -H$. We shall see that the inclusion of Π_{Sym} will not obstruct our arguments in any way and we may argue completely analogous to the case of 2D-codomains. In addition, the gauge operator generalises to the case of domains of dimension ≤ 8 in the same way, as we shall see later on.

Let now $P_0 \in \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{R}^4, G)$ with $P_0 - Id \in L^{p'^*}(\mathbb{R}^4)$ be given and we compute the differential of N at P_0 in the following way (just as in Da Lio-Rivière [23]), using $U \in \dot{W}^{1,p} \cap \dot{W}^{1,p'} \cap L^{p'^*}(\mathbb{R}^4, \mathfrak{g})$, if we consider $N = (N_1, N_2)$ in components:

$$dN_1(P_0)U = \frac{d}{dt} N_1\big(P_0 \exp(tU)\big)$$

$$
\begin{aligned}
&= \Pi_c\Big(\Delta U + \sum_{j=0}^{3}\big(-UP_0^{-1}\partial_j P_0 + P_0^{-1}\partial_j P_0 U\big)\Big) \\
dN_2(P_0)U &= \frac{d}{dt}N_2\big(P_0\exp(tU)\big) \\
&= \Pi_{Sym}\circ\big(\Pi_4+\mathcal{P}\big)\Big(\partial_R U - \sum_{j=0}^{3}(-1)^{\delta_{0j}}\big(-UP_0^{-1}\partial_j P_0 + P_0^{-1}\partial_j P_0 U\big)\Big) \qquad (3.323)
\end{aligned}
$$

If we choose $P_0 = Id$, the expression simplifies to:

$$
dN(Id)U = \Big(\Pi_c\big(\Delta U\big), \Pi_{Sym}\circ\big(\Pi_4+\mathcal{P}\big)\big(\partial_R U\big)\Big) \qquad (3.324)
$$

We shall use the notation $\mathcal{L}_P(U) := dN(P)U$ later. We show the following:

Lemma 3.3.1.5. *The differential $dN(Id)$ defines an elliptic operator.*

Proof. Let us split $U \in \dot{W}^{1,p}\cap\dot{W}^{1,p'}\cap L^{p'^*}(\mathbb{R}^4,\mathfrak{g})$ into $U_c = \Pi_c(U), U_4 = \Pi_4(U)$. It is clear that:

$$
U = U_c + U_4
$$

Observe:

$$
\begin{aligned}
dN(Id)U &= \Big(\Pi_c\big(\Delta U\big), \Pi_{Sym}\circ\big(\Pi_4+\mathcal{P}\big)\big(\partial_R U\big)\Big) \\
&= \Big(\Delta U_c, \big(\Pi_4+\mathcal{P}\big)\big(\partial_R U_4\big) + \Pi_{Sym}\circ\big(\Pi_4+\mathcal{P}\big)\big(\partial_R U_c\big)\Big) \qquad (3.325)
\end{aligned}
$$

Notice that $U_4^T = U_4$ by direct computation, so the projection to the symmetric part can be omitted for this summand. It is obvious that $dN_1(Id)$ is elliptic, since the Laplacian is elliptic. Moreover, this means that we can restrict our attention to the part involving U_4 of $dN_2(Id)$, as this will be the only part that impacts the invertibility of the symbol of $dN(Id)$. However, as before, it can be easily shown that $\big(\Pi_4+\mathcal{P}\big)\circ\partial_R$ actually corresponds to the Riemann-Fueter operator acting from the right, thus it must also be elliptic. This proves the claim. ☐

We emphasise that we could even omit the contribution of U_c in $dN_2(Id)$ simply due to the fact that at worst, the projection of U to the directions in the Clifford algebra which include e_4 and have length smaller or equal to 4 could contribute. However, for lengths 3 and 4, the associated projections are skew-symmetric and thus get mapped to 0 under Π_{Sym}. On the other hand, the elements of length 1 and 2 are precisely the ones contained in E_4. So U_4 is the sole contributor to $dN_2(Id)$.

A close examination of the proof presented for codomains of dimension 2 as provided in [26] shows that the existence of the change of gauge is now immediate from the very same proof as before in the case of codomains of dimension 2. So there exists P, such that:

$$
N(P) = (0, \Gamma_0), \qquad (3.326)
$$

where P satisfies the same estimates as in the case of codomains of dimension 2. We shall present some of the details in the next subsection, in order to convince the reader that the changes are not affecting the argument in a meaningful way:

3.3.1.3 The Gauge Lemma for the compact Matrix Lie group

The main goal of this section is to prove the following assertion:

Theorem 3.3.1.1. *There exist constants $C, \varepsilon > 0$, such that for any $\omega \in W^{-1,4}(\mathbb{R}^4, E_c)$ and $g \in L^4(\mathbb{R}^4, E_4)$ satisfying:*

$$\|\omega\|_{W^{-1,4}}, \|g\|_{L^4} < \varepsilon,$$

there exists a $P \in \dot{W}^{1,4}(\mathbb{R}^4, G)$ such that:

$$N(P) = (\omega, g), \tag{3.327}$$

and the following estimate holds true:

$$\|\nabla P\|_{L^4} \leq C\big(\|\omega\|_{W^{-1,4}} + \|g\|_{L^4}\big) \tag{3.328}$$

In order to prove Theorem 3.3.1.1, we shall need to introduce some notations and establish some intermediate results completely the same way a for codomains of dimension 2: By approximation, it suffices to prove Theorem 3.3.1.1 assuming that ω and g are more regular, see also the argument in Da Lio-Rivière [23]. More precisely, we first prove Theorem 3.3.1.1 in the case $\omega \in (W^{-1,p} \cap W^{-1,p'})(\mathbb{R}^4, E_c)$ and $g \in (L^p \cap L^{p'})(\mathbb{R}^4, E_4)$ for some fixed $4 < p$ and its Hölder-dual $p' = \frac{p}{p-1}$.

For $\varepsilon > 0$, we now introduce:

$$C\ell_\varepsilon := \left\{ \begin{array}{c} (\omega, g) \in (W^{-1,p} \cap W^{-1,p'})(\mathbb{R}^4, E_c) \times (L^p \cap L^{p'})(\mathbb{R}^4, E_4) \\ \\ \|\omega\|_{W^{-1,4}} + \|g\|_{L^4} < \varepsilon \end{array} \right\} \tag{3.329}$$

For constants $\varepsilon, \Theta > 0$, let $\mathcal{V}_{\varepsilon,\Theta} \subseteq C\ell_\varepsilon$ denote the set of pairs (ω, g) for which we have a decomposition as in (3.327) satisfying the following estimates:

$$\|\nabla P\|_{L^4} \leq \Theta(\|\omega\|_{W^{-1,4}} + \|g\|_{L^4}) \tag{3.330}$$

$$\|\nabla P\|_p \leq \Theta(\|\omega\|_{W^{-1,p}} + \|g\|_{L^p}), \tag{3.331}$$

$$\|\nabla P\|_{p'} \leq \Theta(\|\omega\|_{W^{-1,p'}} + \|g\|_{L^{p'}}). \tag{3.332}$$

That is:

$$\mathcal{V}_{\varepsilon,\Theta} := \left\{ (\omega, g) \in C\ell_\varepsilon : \begin{array}{c} \text{there exists } P \in (\dot{W}^{1,p} \cap \dot{W}^{1,p'})(\mathbb{R}^4, G), \text{ so that} \\ \\ P - Id \in L^{4p/3p-4}(\mathbb{R}^4, G) \\ \\ \text{and } \ (3.327), (3.330), (3.331), (3.332) \ \text{ hold.} \end{array} \right\}$$

The strategy to prove Theorem 3.3.1.1 follows the one introduced by K. Uhlenbeck in order to construct Coulomb gauges in critical dimensions. In fact, Theorem 3.3.1.1 is going to be a consequence of the following proposition:

Proposition 3.3.1.1. *There exist $\Theta > 0$ and $\varepsilon > 0$, such that $\mathcal{V}_{\varepsilon,\Theta} = C\ell_\varepsilon$.*

Proof of Proposition 3.3.1.1. Proposition 3.3.1.1 will follow, once we have shown the following four properties:

(i.) $C\ell_\varepsilon$ is connected.

(ii.) $\mathcal{V}_{\varepsilon,\Theta}$ is nonempty.

(iii.) For any $\varepsilon, \Theta > 0$, $\mathcal{V}_{\varepsilon,\Theta}$ is a relatively closed subset of $C\ell_\varepsilon$.

(iv.) There exist $\Theta > 0$ and $\varepsilon > 0$, such that $\mathcal{V}_{\varepsilon,\Theta}$ is a relatively open subset of $C\ell_\varepsilon$.

Property (i.) is obvious, since $C\ell_\varepsilon$ is obviously starshaped with center 0, rendering it even path-connected. Property (ii.) is also clear, because $(0,0) \in \mathcal{V}_{\varepsilon,\Theta}$ can be checked by using $P = Id$. Consequently, it remains to verify the latter two:

The closedness property (iii.) follows almost verbatim as in the case of physical Dirac operators in the previous section: Assume that $(\omega_n, g_n), (\omega, g) \in \mathcal{V}_{\varepsilon,\Theta}$ and moreover, $(\omega_n, g_n) \to (\omega, g)$ and let P_n be as in the definition of $\mathcal{V}_{\varepsilon,\Theta}$. Observe that ∇P_n is bounded in L^p and $L^{p'}$. Therefore, we can extract weakly converging subsequences with limit $\tilde{P}$. Furthermore, we may extract another subsequence of $P_n - Id$ converging locally in L^q for some $q < \frac{4p}{3p-4}$ we may choose, due to the $\dot{W}^{1,p'}$-boundedness of P_n. The limit $P - Id$ satisfies $\nabla P = \tilde{P}$ and P assumes values in G a.e.. This can be seen by extracting another subsequence of $P_n - Id$ converging a.e. pointwise and using the closedness of G. Due to the weak lower semi-continuity of the norms, we immediately obtain that (3.330), (3.331) and (3.332) hold. Finally, observe that, in the distributional sense, we have the convergence:

$$\Pi_c\left(\sum_{i=0}^{3}(\partial_{x_i}((P_n)^{-1}\partial_{x_i}(P_n))\right) \to \Pi_c\left(\sum_{i=0}^{3}(\partial_{x_i}((P)^{-1}\partial_{x_i}(P))\right),$$

as well as

$$\Pi_{Sym} \circ (\Pi_4 + \mathcal{P})(\mathcal{D}(P_n)) \to \Pi_{Sym} \circ (\Pi_4 + \mathcal{P})(\mathcal{D}(P)).$$

This shows $N(P) = (\omega, g)$ and therefore relative closedness.

Lastly, we show the openness property (iv). For this let ω_0, g_0 be arbitrary in $\mathcal{V}_{\varepsilon,\Theta}$, for some $\varepsilon, \Theta > 0$ chosen below and let $P_0 \in \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{R}^4, \mathfrak{g})$, $P_0 - Id \in L^{4p/3p-4}(\mathbb{R}^4)$, such that the decomposition (3.327) as well as the estimates (3.330), (3.331) and (3.332) are satisfied for ω_0 and g_0. The argument now relies on the reduction to a special subclass of perturbations provided by the exponential:

Let us consider perturbations of P_0 of the form $P = P_0 \exp(U)$ where $U \in \dot{W}^{1,p} \cap \dot{W}^{1,p'} \cap L^{4p/3p-4}(\mathbb{R}^4, \mathfrak{g})$. Observe that the exponent $p > 4$ has been chosen in particular to ensure $U \in C^0 \cap L^\infty(\mathbb{R}^4)$ and $P_0 \exp(U) - Id \in L^{\frac{4p}{3p-4}}$. Indeed, as argued in Da Lio-Rivière [23], for a Schwartz function U, one has

$$U(x) = C \int_{\mathbb{R}^4} \nabla_x |x-y|^{-2} \cdot \nabla U(y)\, dy \quad \Rightarrow \quad \|U\|_\infty \lesssim \|\nabla_x |x-y|^{-2}\|_{L^{4/3,\infty}} \|\nabla U\|_{L^{4,1}}$$

The generalized Hölder-Lorentz inequality yields moreover:

$$\|\nabla U\|_{L^{4,1}} \leq C \|\nabla U\|_{L^p}^{\alpha} \|\nabla U\|_{L^{p'}}^{1-\alpha}.$$

where $4^{-1} = \alpha p^{-1} + (1-\alpha)p'^{-1}$. The statement $U \in L^\infty$, and thus continuity by approximation, follows due to the density of Schwartz functions in $\dot{W}^{1,p} \cap \dot{W}^{1,p'} \cap L^{\frac{4p}{3p-4}}$. It can be easily seen, that the argument carries over to domains of arbitrary dimension m, if $m < p$, as the density result and the interpolation identity do not critically depend on $m = 4$. This is in preparation for extensions to domains of arbitrary dimension ≤ 8 later.

We now aim to prove the following:

Lemma 3.3.1.6. *For any $\Theta > 0$ there exists $\varepsilon > 0$ so that the following holds for any ω_0, g_0 and P_0 as above: For any $\omega \in W^{-1,p} \cap W^{-1,p'}(\mathbb{R}^4, E_c)$ and $g \in L^p \cap L^{p'}(\mathbb{R}^4, E_4)$, there exists a unique $U \in \dot{W}^{1,p} \cap \dot{W}^{1,p'} \cap L^{4p/3p-4}(\mathbb{R}^4, \mathfrak{g})$, such that*

$$(\omega, g) = \mathcal{L}_{P_0}(U)$$

and for some constant $C = C(\omega_0, g_0, \Theta) > 0$, it holds

$$\begin{aligned} \|\nabla U\|_{L^p(\mathbb{R}^4)} + \|\nabla U\|_{L^{p'}(\mathbb{R}^4)} \quad &\leq \quad C\big(\|\omega\|_{W^{-1,p}(\mathbb{R}^4)} + \|\omega\|_{W^{-1,p'}(\mathbb{R}^4)} \\ &\qquad + \|g\|_{L^p(\mathbb{R}^4)} + \|g\|_{L^{p'}(\mathbb{R}^4)}\big). \end{aligned} \tag{3.333}$$

Proof of Lemma 3.3.1.6.
Claim 1. $\mathcal{L}_{Id}(U)$ is invertible as a map between the space of functions $U \in \dot{W}^{1,p'} \cap L^{p'^*}(\mathbb{R}^4, \mathfrak{g})$ and the space $W^{-1,p'}(\mathbb{R}^4, E_c) \times L^{p'}(\mathbb{R}^4, E_4)$
Proof of the Claim 1. We have seen that $\mathcal{L}_{Id}(U)$ is elliptic and therefore a Caldéron-Zygmund operator. More precisely, let Γ_4 denote the fundamental solution of Δ. Using the decomposition $U = U_c + U_4$ as introduced before, we have:

$$\Delta U_c = \omega \Longrightarrow U_c = \Gamma_4 * \omega.$$

Similarily, we write $U_4 = U_4^0 e_4 + U_4^1 e_1 e_4 + U_4^2 e_2 e_4 + U_4^3 e_3 e_4$ and up to replacing e_4, e_1e_4, e_2e_4 and e_3e_4 by the quaternionic basis $1, i, j$ and k respectively, we see:

$$(\Pi_4 + \mathcal{P})[\partial_R U_4] = g \Longleftrightarrow D_R^{RF} U_4 = g,$$

where $D_R^{RF} = \partial_0 + \partial_1 \cdot i + \partial_2 \cdot j + \partial_3 \cdot k$ is the quaternionic Riemann-Fueter operator in 4D acting from the right. We highlight that there is no need to include any contributions of U_c here, as Π_{Sym} cancels them out. A simple calculation shows:

$$\overline{D}_R^{RF} D_R^{RF} = \Delta,$$

where $\overline{D}_R^{RF} = \partial_0 - \partial_1 \cdot i - \partial_2 \cdot j - \partial_3 \cdot k$ is the conjugate operator. Therefore, we have:

$$\Delta U_4 = \overline{D}_R^{RF} g.$$

As a result, we deduce:

$$U_4 = \Gamma_4 * \overline{D}_R^{RF} g = \overline{D}_R^{RF} \Gamma_4 * g.$$

Using Caldéron-Zygmund estimates for the Laplacian, we obtain:

$$\begin{aligned} \|\nabla U_c\|_{L^{p'}} &\lesssim \|\omega\|_{W^{-1,p'}} \\ \|\nabla U_4\|_{L^{p'}} &\lesssim \|g\|_{L^{p'}}. \end{aligned}$$

Consequently, given $\omega \in W^{-1,p'}(\mathbb{R}^4, E_6)$, $\mathfrak{g} \in L^{p'}(\mathbb{R}^4, E_4)$, there exists a unique $U \in \dot{W}^{1,p'}(\mathbb{R}^4, \mathfrak{g})$ such that:

$$\mathcal{L}_{Id}(U) = (\omega, g)\,.$$

The elliptic estimates above yield:

$$\|\nabla U\|_{L^{p'}} \lesssim \|\omega\|_{W^{-1,p'}} + \|\mathfrak{g}\|_{L^{p'}}.$$

The claim is therefore proved. $\square$

Estimate for $\mathcal{L}_{P_0}(U) - \mathcal{L}_{Id}(U)$
It suffices to estimate using Hölder's inequality, boundedness of G due to compactness, the Sobolev-embeddings and the L^4-estimate for ∇P:

$$\begin{aligned}
\|P_0^{-1}(\partial_{x_i}P_0)U - UP_0^{-1}\partial_{x_i}P_0\|_{L^{p'}} &\lesssim \|P_0^{-1}\|_{L^\infty}\|\nabla P_0\|_{L^4}\|U\|_{L^{4p/3p-4}} \\
&\lesssim \|\nabla P_0\|_{L^4}\|\nabla U\|_{L^{p'}} \\
&\lesssim \Theta\varepsilon \cdot \|\nabla U\|_{L^{p'}}.
\end{aligned}$$

Using this inequality, we conclude:

$$\begin{aligned}
\|\partial_{x_i}\left(P_0^{-1}(\partial_{x_i}P_0)U - UP_0^{-1}\partial_{x_i}P_0\right)\|_{W^{-1,p'}} &\leq \|P_0^{-1}(\partial_{x_i}P_0)U - UP_0^{-1}\partial_{x_i}P_0\|_{L^{p'}} \\
&\lesssim \Theta\varepsilon \cdot \|\nabla U\|_{L^{p'}}.
\end{aligned}$$

Choosing $\varepsilon > 0$ small enough (depending on Θ), we obtain that $\mathcal{L}_{P_0}(\mathfrak{u})$ is an invertible map from $\dot{W}^{1,p'} \cap L^{p'^*}(\mathbb{R}^4, \mathfrak{g})$ to $W^{-1,p'}(\mathbb{R}^4, E_c) \times L^{p'}(\mathbb{R}^4, E_4)$ by general results from Functional Analysis.

Claim 2. Assuming now $\omega \in W^{-1,p} \cap W^{-1,p'}(\mathbb{R}^4, E_c)$, $g \in L^p \cap L^{p'}(\mathbb{R}^4, E_4)$, we show that the unique solution U of $(\omega, g) = \mathcal{L}_{P_0}(U)$ lies in $\dot{W}^{1,p}(\mathbb{R}^4)$.
Proof of Claim 2: Firstly, due to $\nabla U \in L^{p'}$, we know that we may choose U by Sobolev-embeddings and the density of Schwartz functions in the following way:

$$U \in L^{4p/3p-4}.$$

As previously, in order to bootstrap, it suffices to deduce improved integrability of $P_0^{-1}\partial_{x_l}P_0U$, as this implies improved integrability of ∇U. The same estimates immediately apply to $UP_0^{-1}\partial_{x_l}P_0$. By a previous comment, it suffices to show that $\nabla U \in L^q$ for some $q > 4$, because then, by interpolation, $\nabla U \in L^{p'} \cap L^q$ and we could thus conclude that $U \in L^\infty$ leading to $P_0^{-1}\partial_{x_l}P_0U \in L^p$, which immediately shows $\nabla U \in L^p$. Therefore, Claim 2 is proven in the process.

We argue by a bootstrap argument: Assume that $U \in L^r$ for some $4 > r \geq \frac{4p}{3p-4} = p'^*$. In this case, Hölder's inequality implies:

$$\|P_0^{-1}\partial_{x_l}P_0U\|_{L^t} \lesssim \|\nabla P_0\|_{L^p}\|U\|_{L^r},$$

for $\frac{1}{t} = \frac{1}{p} + \frac{1}{r} \geq \frac{1}{p} + \frac{1}{4}$. Observe that $\frac{4}{3} \leq t < 4 < p$. We conclude due to elliptic estimates as in Claim 1 and the identity $\mathcal{L}_{P_0}(U) = (\omega, g)$:

$$\nabla U \in L^t.$$

This implies by Sobolev-embeddings that $U \in L^{4t/4-t}$. Thus, if we define $r' = \frac{4t}{4-t}$, we observe:

$$\frac{1}{r'} = \frac{1}{t} - \frac{1}{4} = \frac{1}{r} + \frac{1}{p} - \frac{1}{4},$$

which implies that the reciprocal values are decreasing by a constant amount due to $p > 4$ with each iterating step. Therefore, after finitely many steps, we have:

$$r' > 4.$$

This implies, by the previously outlined argument, that $\nabla U \in L^p(\mathbb{R}^4, \mathfrak{g})$, finishing the proof of Claim 2. $\square$

Proof of proposition 3.3.1.1 continued.

For $\varepsilon = \varepsilon(\Theta) > 0$ chosen small enough and for any $(\omega_0, g_0) \in \mathcal{V}_{\varepsilon,\Theta}$, the local inversion theorem applied to N implies the existence of some $\delta > 0$ (that might depend on (ω_0, g_0)) such that, for every $(\omega, g) \in C\ell_\varepsilon$ with

$$\|\omega - \omega_0\|_{W^{-1,p}(\mathbb{R}^4)} + \|\omega - \omega_0\|_{W^{-1,p'}(\mathbb{R}^4)} < \delta \tag{3.334}$$

$$\|g - g_0\|_{L^p(\mathbb{R}^4)} + \|g - g_0\|_{L^{p'}(\mathbb{R}^4)} < \delta, \tag{3.335}$$

we surely find $P = P_0 e^U \in \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{R}^4)$, so that $P - Id \in L^{4p/3p-4}(\mathbb{R}^4)$ and (3.327) is satisfied. It remains to prove (3.330), (3.331) and (3.332). This will be a consequence of the following Lemma:

Lemma 3.3.1.7. *There exists a $\Theta > 0$ and a $\sigma > 0$, such that, whenever $P \in \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{R}^4)$ with $P - Id \in L^{4p/3p-4}(\mathbb{R}^4)$ satisfying* (3.327) *is given, and it holds*

$$\|\nabla P\|_{L^4(\mathbb{R}^4)} \leq \sigma, \tag{3.336}$$

then (3.330), (3.331) *and* (3.332) *hold true as well.*

Proof of Lemma 3.3.1.7.

Let $(\omega, g) \in C\ell_\varepsilon$ satisfy (3.334) and (3.335) and let $P = P_0 e^U \in \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{R}^4)$, such that $P - Id \in L^{4p/3p-4}(\mathbb{R}^4)$ and (3.327) is satisfied. We first consider the following Hodge decomposition of $P^{-1}dP$:

$$P^{-1}dP = d\Gamma_P + d^* Y_P \tag{3.337}$$

where $Y_P \in \Omega^2(\mathbb{R}^4)$, $Y_P = \sum_{0\leq i<j\leq 3} Y_P^{ij} dx_i \wedge dx_j$ is a differential 2-form. We denote $d^*Y_P = \sum_{i=0}^3 y_P^i dx_i$[26] for brevity. We may choose Γ_P and Y_P as follows:

$$\Gamma_P = \Delta^{-1} d^*(P^{-1}dP) \tag{3.338}$$

$$Y_P = \Delta^{-1} d(P^{-1}dP) \tag{3.339}$$

In particular, we then have $dY_P = 0$ and $d^*\Gamma_P = 0$.

Due to (3.338), it follows that:

$$\Pi_c(-\Delta\Gamma_P) = \Pi_c(d^*(P^{-1}dP)) = \omega. \tag{3.340}$$

[26]We recall that $d^* = (-1)^{n(k-1)+1} * \ d \ *$, $*$ is the Hodge operator. If $\xi = \sum_{0\leq i<j\leq 3} \xi_{ij} dx_i \wedge dx_j$ then $d^*\xi = -(\alpha_0 dx_0 + \alpha_1 dx_1 + \alpha_2 dx_2 + \alpha_3 dx_3)$ where:

$$\begin{aligned}
\alpha_0 &= \partial_{x_1}\xi_{01} + \partial_{x_2}\xi_{02} + \partial_{x_3}\xi_{03} \\
\alpha_1 &= -\partial_{x_0}\xi_{01} + \partial_{x_2}\xi_{12} + \partial_{x_3}\xi_{13} \\
\alpha_2 &= -\partial_{x_0}\xi_{02} - \partial_{x_1}\xi_{12} + \partial_{x_3}\xi_{23} \\
\alpha_3 &= -\partial_{x_2}\xi_{23} - \partial_{x_1}\xi_{13} - \partial_{x_0}\xi_{03}
\end{aligned}$$

Therefore, for every $r \in [p', p]$ we have:

$$\|\Pi_c(\nabla\Gamma_P)\|_{L^r} \lesssim \|\omega\|_{W^{-1,r}} \tag{3.341}$$

From (3.337), it follows that

$$-\Delta Y_P \;=\; d(P^{-1}dP) = dP^{-1} \wedge dP. \tag{3.342}$$

From (3.342), it follows that $\nabla Y_q \in L^r(\mathbb{R}^4)$ and due to the compensation result in Lemma 3.2.7.1:

$$\|\nabla Y_P\|_{L^r} \lesssim \|dP\|_{L^4(\mathbb{R}^4)}\|dP\|_{L^r(\mathbb{R}^4)} \leq \sigma\|dP\|_{L^r(\mathbb{R}^4)}. \tag{3.343}$$

From (3.337) it follows now:

$$\begin{aligned} g = (\Pi_4 + \mathcal{P})\mathcal{D}(P) \;&=\; (\Pi_4 + \mathcal{P})(D\Gamma_P) \\ &+\; (\Pi_4 + \mathcal{P})(y_P^0 - \sum_{i=1}^{3} y_P^i e_i). \end{aligned} \tag{3.344}$$

Therefore:

$$(\Pi_4 + \mathcal{P})(D\Gamma_P) = g - (\Pi_4 + \mathcal{P})(y_P^0 - \sum_{i=1}^{3} y_P^i e_i) \tag{3.345}$$

Observe that $d\Gamma_P \in \mathfrak{g}$, since $P^{-1}dP \in \mathfrak{g}$. Therefore:

$$d\Gamma_P = (\Pi_4 + \Pi_c)(d\Gamma_P).$$

Hence:

$$(\Pi_4 + \mathcal{P})(D\Pi_4(d\Gamma_P)) \;=\; dg - d(\Pi_4 + \mathcal{P})(y_P^0 - \sum_{i=1}^{3} y_P^i e_i) \tag{3.346}$$

$$-\; (\Pi_4 + \mathcal{P})(D\Pi_6(d\Gamma_P)) \tag{3.347}$$

Since the operator $(\Pi_4 + \mathcal{P}) \circ D$ is invertible, we find:

$$\Pi_4(d\Gamma_P) \;=\; ((\Pi_4 + \mathcal{P}) \circ D)^{-1} dg \tag{3.348}$$

$$+\; ((\Pi_4 + \mathcal{P}) \circ D)^{-1} d(\Pi_4 + \mathcal{P})(y_P^0 - \sum_{i=1}^{3} y_P^i e_i) \tag{3.349}$$

$$+\; ((\Pi_4 + \mathcal{P}) \circ D)^{-1}[(\Pi_4 + \mathcal{P})(D\Pi_6(d\Gamma_P))].$$

By using (3.348), we get:

$$\begin{aligned} \|\Pi_4(d\Gamma_P)\|_{L^r} \;&\lesssim\; \|g\|_{L^r} + \|d^*Y_P\|_{L^r} + \|\omega\|_{W^{-1,r}} \\ &\lesssim\; \|g\|_{L^r} + \sigma\|dP\|_{L^r} + \|\omega\|_{W^{-1,r}} \end{aligned} \tag{3.350}$$

Combining (3.337), (3.341) and (3.350), we get the following estimate:

$$\|dP\|_{L^r} \;\lesssim\; \|d\Gamma_P\|_{L^r} + \|d^*Y_P\|_{L^r} \tag{3.351}$$

$$
\begin{aligned}
&\lesssim \|\Pi_4(d\Gamma_P)\|_{L^r} + \|\Pi_6(d\Gamma_P)\|_{L^r} + \|d^*Y_P\|_{L^r} \\
&\leq C(\|g\|_{L^r} + \sigma\|dP\|_{L^r} + 2\|\omega\|_{W^{-1,r}} + \sigma\|dP\|_{L^r}). \qquad (3.352)
\end{aligned}
$$

Choosing $\Theta = \frac{C}{1-2C\sigma}$, we finally arrive at the inequality:

$$\|dP\|_{L^r} \leq \Theta(\|\omega\|_{W^{-1,r}} + \|g\|_{L^r}).$$

This concludes the proof of Lemma 3.3.1.7. □

End of the proof of Proposition 3.3.1.1
Thanks to Lemma 3.3.1.7, the openness property (iv.) is proven. Proposition 3.3.1.1 is thus established. □

3.3.1.4 Improved Regularity

The argument for improved integrability in the case of codomains of dimension 2 relies on the fact, that $\partial_L\alpha$ appears and is in $L^{(4,2)}$ with an appropriate smallness condition satisfied. If we assume that:

$$\Gamma_0 = \partial_L W,$$

with an estimate of the $L^{(4,2)}$-norm (which is for example the case if $Q = e^{\tilde{W}}$ for some antisymmetric $\tilde{W}$), we could reiterate the very same argument and arrive at the same conclusion. It is important that we assume that W is symmetric and takes values in $\mathbb{R}\cdot e_4$, to ensure that $\partial_L W$ assumes values in $\mathfrak{g}$. The argument for the bootstrap test carries over as well.

Indeed, let us assume that $\|\nabla W\|_{L^{(4,2)}} < \varepsilon$ and $\partial_L W = \Gamma_0$. If ε is sufficiently small, we can solve:

$$N(P) = (0, -\partial_L W),$$

by using Theorem 3.3.1.1. We have the following Lemma:

Lemma 3.3.1.8. *Let $U = U_4 + U_c$ be a splitting as previously and assume that:*

$$\mathcal{L}_{Id}(U) = (\Delta U_c, \Pi_{Sym}\circ(\Pi_4 + \mathcal{P})(DU_4)) = (0, -\partial_L W),$$

then:

$$\Pi_{e_{i+1}e_{i-1}e_4}(DU_4) = 0, \forall i = 1,2,3$$

The proof is an exact copy of the argument in the case of codomains of dimension 2. This result in turn allows us to establish the following:

Lemma 3.3.1.9. *Under the assumptions in this section, we have:*

$$\Pi_c(P^{-1}\partial_R P), \Pi_{e_{i+1}e_{i-1}e_4}(P^{-1}\partial_R P) \in L^{(4,1)}.$$

There exist corresponding estimates for the $L^{(4,1)}$-norms depending on the $L^{(4,2)}$-norms of ∇P and ∇W.

This follows again by the very same argument as in the case of codomains of dimension 2, see [26]. The result also generalises to situations where we have a rotation-condition analogous to the codimension 2 case, see [26], as the argument still goes through without issues, but the naturality of this kind of condition is less apparent than the one presented. Therefore, we can conclude the bootstrap lemma as in the previous subsection.

3.3.1.5 Morrey-type Estimates

In this subsection, we work under the assumption that Proposition III.4 from Da Lio-Rivière [23] extends to our current scenario. Therefore, we merely have to prove that an appropriate decomposition exists and that Morrey-type estimates can be established. For all further steps, we refer to Da Lio-Rivière [23]. So the general regularity result follows, as soon as we have the desired Morrey estimates:

Non-linear Hodge Decomposition First, we will show how to obtain an appropriate version of the non-linear Hodge decomposition on balls $B_r(x)$. Let for this G be an arbitrary 1-form in $W^{1,\frac{n}{n-1}}(B_r(0))$ as obtained in the proof. Then, by classical Hodge decomposition, there exist a function A on $B_r(0)$ vanishing along the boundary and a 2-form $\tilde{A}$, such that:

$$dA + d^*\tilde{A} = G \tag{3.353}$$

Next, we consider the Hodge decomposition in the same manner of $P^{-1}d^*A$, again obtaining zero boundary conditions for the function $\tilde{B}$:

$$d\tilde{B} + d^*B = P^{-1}d^*\tilde{A} \tag{3.354}$$

Thus, we have:

$$G = dA + d^*A = dA + Pd^*B + Pd\tilde{B} \Rightarrow G - dA - Pd^*B = Pd\tilde{B} \tag{3.355}$$

We observe that on $B_r(0)$:

$$\Delta\tilde{B} = d^*dB = d^*\big(P^{-1}d^*\tilde{A}\big) = \pm * \big(dP^{-1} \wedge d(*\tilde{A})\big) \tag{3.356}$$

Due to the zero boundary condition, we can therefore deduce by similar arguments as in our compensation result in Lemma 3.2.7.1:

$$\|\nabla B\|_{L^{\frac{n}{n-1}}(B_r(x))} \lesssim \|dP\|_{L^n(B_r(x))}\|d^*\tilde{A}\|_{L^{\frac{n}{n-1}}(B_r(x))} \lesssim \varepsilon\|G\|_{L^{\frac{n}{n-1}}(B_r(x))} \tag{3.357}$$

So, if $\varepsilon > 0$ is sufficiently small, we can argue by iteration that there exists a solution to the non-linear Hodge decomposition as in the case of codomains of dimension 2, such that A has boundary value 0.

Morrey-Estimates To deduce local regularity, we merely have to establish slightly improved regularity and hence Morrey-estimates as in Da Lio-Rivière [23], the full regularity as in Theorem 3.2.1.2 follows by Morrey-bootstrapping going over to possibly smaller balls to obtain uniform powers in the Morrey estimates. Therefore, let us point out the differences to Da Lio-Rivière [23] and our considerations in connection with the bootstrap lemma: Namely, we can estimate A as in the bootstrap lemma, if we find A, B for a given $B_r(x)$. More precisely, due to the boundary conditions, we will find:

$$\|\nabla A\|_{L^{\frac{n}{n-1},\infty}(B_r(x))} \lesssim \varepsilon\|PF\|_{L^{\frac{n}{n-1},\infty}(B_r(x))} + \varepsilon\|A\|_{L^{\frac{n}{n-1},\infty}(B_r(x))}, \tag{3.358}$$

so if ε is sufficiently small, we arrive at:

$$\|\nabla A\|_{L^{\frac{n}{n-1},\infty}(B_r(x))} \lesssim \varepsilon\|PF\|_{L^{\frac{n}{n-1},\infty}(B_r(x))}, \tag{3.359}$$

Then, it remains to obtain appropriate estimates for d^*B. For this, write $d^*B = \sum_j b_j dx_j$ and we can deduce completely analogous to the bootstrap lemma:

$$\partial_L(e_1 b_1) = -\sum_{j\geq 2} \partial_j R_j,$$

where R_j is an expression depending on P and ∇A. So we can now split $e_1 b_1$ into a Clifford analytic and thus harmonic part, which can be estimated by means of Campanato-estimates as in Da Lio-Rivière [23] and the convolution of the RHS in the equation above with the fundamental solution of ∂_L on the entire $\mathbb{R}^m$. This second summand can be estimated by usual estimates for the fundamental solution of the Laplacian. Therefore, we arrive at the desired estimates by completely the same means as in Da Lio-Rivière [23] once we use the link between b_j and b_1 established in the bootstrap lemma.

Naturally, all arguments extend in the same spirit to domains of dimension 3 and, of course, of dimension ≤ 8. The existence of suitable gauge groups has been established, so it merely remains to modify the gauge lemma as needed.

3.3.2 Domains of Dimension ≤ 8 and Octonionic Differentiation

As promised at the beginning of the current section, we shall explain how the results from [26] generalise to domains of dimension ≤ 8.

3.3.2.1 General case

Using the operators outlined in the current section, we can construct similar changes of gauge and arguments leading higher regularity in the same spirit as in the case of domains of dimension 3 and 4 in the paper [26] as well as the previous subsection. The restriction to dimensions ≤ 8 is binding, as we make in some sense use of a complete orthonormal frame on a sphere of higher dimension by using a suitable first order operator.

3.3.2.2 Gauge Operators

Let us consider a domain of dimension $n \leq 8$. In the cases $n = 3, 4$, we were able to reduce the linearised gauge operator to a Laplacian on a subspace as well as a Riemann-Fueter operator on $\mathbb{H}$ together up to perturbation. This procedure is limited to these cases, as $\mathbb{H}$ is merely 4-dimensional and thus cannot accommodate similar procedures for domains of dimension > 4. In order to extend these considerations to higher dimensional domains, we have to use a larger space than the quaternions and there are two natural candidates: the Clifford algebra $C\ell_3$ and the octonions $\mathbb{O}$[27] with the respective Riemann-Fueter-type differential operators.

Let us start by considering the ramifications of using $C\ell_3$. The natural extension of the Riemann-Fueter operator would be the Clifford derivative:

$$\tilde{D}_R^{RF} = \partial_0 + \partial_1 \cdot e_1 + \partial_2 \cdot e_2 + \partial_3 \cdot e_3 + \partial_4 \cdot e_1 e_2 + \partial_5 \cdot e_2 e_3 + \partial_6 \cdot e_3 e_1 + \partial_7 \cdot e_1 e_2 e_3$$

There appears a major issue in this case: The resulting operator is no longer elliptic. We highly emphasise that the operator considered is no standard Dirac operator. This is reflected in the symbol

[27] The octonions form a non-commutative, non-associative normed division algebra. We refer the interested reader to Baez [3].

matrix no longer consisting of pairwise orthogonal columns, which in the case $\mathbb{H} = C\ell_2$ was reflected in the identity:

$$D_R^{RF}\overline{D_R^{RF}} = \overline{D_R^{RF}}D_R^{RF} = \Delta. \tag{3.360}$$

So we are required to work over the octonions, if we want to have any hope of generalising the previous considerations in an immediate manner. Also, this immediately provides an explanation for the remaining restriction $n \leq 8$, as in higher dimensions, we merely have Clifford algebras $C\ell_k$ as a possible candidates to construct a Riemann-Fueter-type differential operator.

Fortunately, if we choose to work with octonions, we can remedy this issue. The use of the octonions is motivated by the fact that they introduce a parallelisation on S^7 which essentially boils down, by going over from symbols to differential operators, to an identity similar to (3.360). For the remainder of this subsection, we shall work with symbols of linear differential operators and show how to deduce the general operator $\mathcal{N}$ (see (3.168) for its definition in the case of 4D and (3.277) for 3D domains[28]), as the proof previously given will generalise immediately to these situations as well, without major modifications.

The symbol of the general first order differential operator is the following:

$$\sigma(\xi) = \begin{pmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 & -\xi_4 & -\xi_5 & -\xi_6 & -\xi_7 \\ \xi_1 & \xi_0 & -\xi_3 & \xi_2 & -\xi_5 & \xi_4 & \xi_7 & -\xi_6 \\ \xi_2 & \xi_3 & \xi_0 & -\xi_1 & -\xi_6 & -\xi_7 & \xi_4 & \xi_5 \\ \xi_3 & -\xi_2 & \xi_1 & \xi_0 & -\xi_7 & \xi_6 & -\xi_5 & \xi_4 \\ \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_5 & -\xi_4 & \xi_7 & -\xi_6 & \xi_1 & \xi_0 & \xi_3 & -\xi_2 \\ \xi_6 & -\xi_7 & -\xi_4 & \xi_5 & \xi_2 & -\xi_3 & \xi_0 & \xi_1 \\ \xi_7 & \xi_6 & -\xi_5 & -\xi_4 & \xi_3 & \xi_2 & -\xi_1 & \xi_0 \end{pmatrix} \tag{3.361}$$

One easily notices that the columns are pairwise orthogonal and therefore that $\det\sigma(\xi) = \pm|\xi|^8$. Inserting $\xi = (1, 0, 0, 0, 0, 0, 0, 0)$, we deduce immediately:

$$\det\sigma(\xi) = |\xi|^8,$$

due to the connectedness of $\mathbb{R}^n \setminus \{0\}$, if $n > 1$. Thus the octonionic Riemann-Fueter operator is elliptic.

In order to illustrate how we construct the general $\mathcal{N}$, let us consider the case $n = 5$. All other cases follow completely analogously. Observe that in this case, as only derivatives $\partial_0, \dots, \partial_4$ appear, we have $\xi_5 = \xi_6 = \xi_7 = 0$, resulting in the symbol:

$$\sigma(\xi) = \begin{pmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 & -\xi_4 & 0 & 0 & 0 \\ \xi_1 & \xi_0 & -\xi_3 & \xi_2 & 0 & \xi_4 & 0 & 0 \\ \xi_2 & \xi_3 & \xi_0 & -\xi_1 & 0 & 0 & \xi_4 & 0 \\ \xi_3 & -\xi_2 & \xi_1 & \xi_0 & 0 & 0 & 0 & \xi_4 \\ \xi_4 & 0 & 0 & 0 & \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ 0 & -\xi_4 & 0 & 0 & \xi_1 & \xi_0 & \xi_3 & -\xi_2 \\ 0 & 0 & -\xi_4 & 0 & \xi_2 & -\xi_3 & \xi_0 & \xi_1 \\ 0 & 0 & 0 & -\xi_4 & \xi_3 & \xi_2 & -\xi_1 & \xi_0 \end{pmatrix} \tag{3.362}$$

Let us now compute the symbol of the Dirac operator involved in the gauge arguments ∂_R on $u = u^0 e_5 + u^1 e_1 e_5 u^2 e_2 e_5 + u^3 e_3 e_5 + u^4 e_4 e_5$. In this case, we easily find:

[28] In particular, we remind the reader that the second component of $\mathcal{N}$ is controlling the operator at hand, thus requiring special care in our considerations.

Basis direction	$u^0 e_5$	$u^1 e_1 e_4$	$u^2 e_2 e_5$	$u^3 e_3 e_5$	$u^4 e_4 e_5$
e_5	ξ_0	$-\xi_1$	$-\xi_2$	$-\xi_3$	$-\xi_4$
$e_1 e_5$	ξ_1	ξ_0			
$e_2 e_5$	ξ_2		ξ_0		
$e_3 e_5$	ξ_3			ξ_0	
$e_4 e_5$	ξ_4				ξ_0
$e_1 e_2 e_5$		ξ_2	$-\xi_1$		
$e_1 e_3 e_5$		ξ_3		$-\xi_1$	
$e_1 e_4 e_5$		ξ_4			$-\xi_1$
$e_2 e_3 e_5$			ξ_3	$-\xi_2$	
$e_2 e_4 e_5$			ξ_4		$-\xi_2$
$e_3 e_4 e_5$				ξ_4	$-\xi_3$

The columns contains the derivatives of the term at the top under ∂_R split into the different directions. We notice that the first 5 rows of the table above appear the same way in $\sigma(\xi)$. Therefore, we now merely want to define operators that exchange the base directions in $C\ell_5$ to arrive at the differential symbol $\sigma(\xi)$ (observe that the last three columns in the symbol are similar to the 3D-case, where we used an additional degree of freedom in $Spin(4)$ remedy the non-ellipticity). In this case, it is straightforward to check that the following works:

- Map $e_1 e_3 e_5$ to $e_2 e_5$
- Map $-e_1 e_2 e_5$ to $e_3 e_5$
- Map $-e_2 e_3 e_5$ to $e_1 e_5$

The remaining directions will be kept and the additional degrees of freedom of $\mathfrak{q}$ we use will as in the 4D-case to include a Laplacian, once linearised. Similar considerations apply in all other cases $5 \leq n \leq 8$. This in turn enables us to formulate a gauge Lemma similar to the ones for $n = 3, 4$ and prove the corresponding assertions.

The reader is also invited to compare the current restrictions with Gilbert-Murray [38, p.205-206], where it is shown that the ellipticity of a first order, constant coefficient differential operator on a vectorbundle over a domain of dimension n requires the fibers to be of dimension k, such that:

$$\rho(k) \geq n, \tag{3.363}$$

with $\rho(k)$ being the Radon-Hurwitz number. If $k = (2a + 1)16^b 2^c$ with $0 \leq c \leq 3$, then:

$$\rho(k) = 2^c + 8b, \tag{3.364}$$

see Gilbert-Murray [38]. It is related to the number of linearly independent tangent vector fields over $S^{k-1} \subset \mathbb{R}^k$ and therefore to the parallelisability. The condition may be rephrased in terms of k. In our current situation, it may be observed that k necessarily satisfies $k \leq n(n+1)/2$ by the restriction imposed by the dimension of the Lie algebra of the Spin group. It is simple to see, following the argument in Gilbert-Murray [38], that for $n = 8m + r + 1$ with $0 \leq r \leq 7$, we have:

$$\rho(k) \geq n \;\Leftrightarrow\; k \geq \begin{cases} 16^m 2^{\lceil r/2 \rceil}, & \text{if } r = 0, 6, 7 \\ 16^m 2^{\lceil r/2 \rceil + 1}, & \text{if } r = 1, 2, 3, 4, 5 \end{cases}, \tag{3.365}$$

for some k in the range previously outlined holds only if $n \leq 16$[29], showing already that the technique is limited to a finite range of dimensions of the domain. Indeed, the only n in question besides $n \leq 8$ would be $n \in \{9, 10, 11, 12, 16\}$. Accounting for other factors like a required relationship between the first order differential operator and the Dirac operator encountered in the change of gauge as well as the way we create gauge operators, the condition becomes reasonably motivated, since the construction of gauge operators also does not generalise in a straightforward manner to values of n that are > 8 (the still potentially admissible values above).

3.3.2.3 Compensation by Means of Explicit Solvability

In this subsection, we mimic the argument in the 4D-case in order to deduce improved regularity of the solution modulo absorption of the α-term in our considerations. We will do so in full generality, extending even to domains of dimension $n \geq 9$. All considerations merely imitate the arguments in the case $n = 4$.

We want to consider the equation:

$$\Pi_n(\partial_R \mathfrak{u}) = \partial_L \alpha e_n,$$

where $\mathfrak{u}$ takes values in $spin(n+1)$ and ∂_L, ∂_R are the generalized Dirac operators in dimension n. Here, Π_n denotes the projection in $C\ell_n$ on the subspace $C\ell_{n-1} \cdot e_n$.

It should be noted that $\partial_R \Pi_n(\mathfrak{u}) = -\Pi_n(\partial_R \mathfrak{u})$, therefore it suffices to merely assume that $\mathfrak{u} = u^0 e_n + \dots u^n e_{n-1} e_n \in C\ell_{n-1} \cdot e_n$ (the remaining part will be controlled by its Laplacian being 0, see the case $n = 4$). We observe that $\partial_R \mathfrak{u}$ has three different kinds of terms:

- In the direction e_n, we find the following term $\partial_0 u^0 - \partial_1 u^1 - \dots \partial_{n-1} u^{n-1}$.
- In the directions $e_j e_n$, we have contributions of the form $\partial_0 u^j + \partial_j u^0$.
- In the directions $e_i e_j e_n$ where $i < j$, we have contributions of the form $\partial_j u^i - \partial_i u^j$.

Therefore, the system of equations we want to study becomes:

$$\partial_0 u^0 - \sum_{j=1}^{n-1} \partial_j u^j = \partial_0 \alpha$$
$$\partial_j u^0 + \partial_0 u^j = -\partial_j \alpha$$

[29] This is easily seen by a direct computation and noticing that our vector spaces are subspaces of the Spin-Lie algebras. This is due to the gauge group being the Spin group and leads to $k \leq \frac{n(n+1)}{2}$.

$$\partial_j u^i - \partial_i u^j = 0,$$

for all $i \neq j \in \{1, \dots, n-1\}$. The third set of PDEs implies that the vector $(u^1, \dots, u^{n-1})$ should be a gradient of some function β, i.e. of the following form:

$$\nabla_{x'}\beta = \begin{pmatrix} u^1 \\ u^2 \\ \vdots \\ u^{n-1} \end{pmatrix},$$

where $x' = (x_1, \dots x_{n-1})$. Therefore, the second set of equations turns into:

$$\nabla_{x'}u^0 + \nabla_{x'}\partial_0\beta = -\nabla_{x'}\alpha.$$

A possible solution of this would be:

$$u^0 = -\partial_0\beta - \alpha.$$

Inserting now the previously found expressions into the first equation, we arrive at the following PDE for β:

$$-\Delta\beta - \partial_0\alpha = \partial_0\alpha.$$

This can now be solved for β, meaning that we found a solution where we do not need further terms or rely on the contributions of the projections used to create an elliptic operator. This implies that the result in Lemma 3.2.2.5 generalizes to all $n \leq 8$ and consequently, so does Lemma 3.2.2.6 as well. Notice that our assumption of slightly higher integrability does not affect this argument. This however yields the major tools required to prove the bootstrap test, so we may conclude that the bootstrap test continues to hold for domains of dimension $n \leq 8$.

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- 2007 - 2013: *Grammar School, KZO Wetzikon*
- 2001 - 2007: *Primary School, Obermatt Pfäffikon*

Research

My research concerns mainly questions from Geometric Analysis in the non-local realm, i.e. geometric variational problems involving non-local energies and/or operators such as fractional Laplacians.

1. *Bergmann-Bourgain-Brezis Inequality* (with F.Da Lio, T.Rivière)
   Journal of Functional Analysis, Vol. 291, Issue 9, 01.November 2021
   DOI: 10.1016/j.jfa.2021109201
2. *Uniqueness and Regularity of the Fractional Harmonic Gradient Flow in S^{n-1}*
   Nonlinear Analysis, Vol. 214, January 2022
   DOI: 10.1016/j.na.2021.112592

3. *Integrability by compensation for Dirac Equation* (with F.Da Lio, T.Rivière)
Transactions of the American Mathematical Society, Vol. 375, No. 6, June 2022, 4477-4511
DOI: 10.1090/tran/8656

4. *Existence, Uniqueness and Regularity of the Fractional Harmonic Gradient Flow in General Target Manifolds*
Preprint, submitted
arXiv: https://arxiv.org/abs/2109.11458

5. *Half-Harmonic Gradient Flow: Aspects of a Non-Local Geometric PDE*
Preprint, submitted
arXiv: https://arxiv.org/abs/2112.08846